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OPTIMAL GROWTH IN A LINEAR-LOGARITHMIC ECONOMY*

BY ROY RADNER[1]

1. INTRODUCTION

Purpose of the Study

THE GOAL OF THIS study of optimal growth in a linear-logarithmic economy has been to provide an example of a multisector model of economic growth in which optimal programs can be explicitly calculated for several different formulations of the criterion of optimality. With such explicit calculations one can illustrate various propositions and test conjectures about the properties of optimal growth. If the model studied here turns out to be sufficiently flexible, the relative simplicity of the required calculations may also make the model attractive for empirical applications.

The general concept of optimality that I have adopted is that of maximizing some function of the sequence of outputs and/or consumption of the various goods and services. This maximization is subject to technological constraints, which are to be interpreted as excluding, as far as possible, any effects of organizational constraints. For example, I will not discuss whether or not the calculated programs can be achieved by some particular free-market private-property system of economic organization.

The central assumptions that facilitate the calculations are: (1) currently produced goods and services are produced according to linear-logarithmic (Cobb-Douglas) functions of the inputs; (2) second-hand goods disappear (are used up) according to arbitrary distributions of life length, which distributions are independent, however, of the uses made of these goods in production; and (3) the criterion to be maximized is a function of a sequence of measures of "one-period-welfare", the welfare in any single time period being a linear function of the logarithms of the quantities of various goods and services that are

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produced and/or consumed.

A General Problem Formulation

I think that the reader will find it easier to follow a summary of the problem formulations and solutions if these are described without reference to the special linear-logarithmic assumptions just mentioned. Suppose that there is a fixed list of *commodities*, where the term "commodity" is to be interpreted very broadly, including goods and services, distinguished in all respects that are important from a technological or welfare point of view. In particular, commodities may be distinguished according to age. Let time be a discrete variable, and let

$y_j(t)$ denote the output of commodity j during period t, and

$q_j(t)$ denote the supply of commodity j that is made available to the economy exogenously during period t.

The total stock $z_j(t)$ of commodity j at the end of period t is the sum of the two, i.e.,

$$z_j(t) = y_j(t) + q_j(t) . \tag{1.1}$$

The technological possibilities will be described by saying that the output vector $y(t+1)$ for the next period $(t+1)$ must be in some set $\mathscr{P}(z[t])$; that is to say, $\mathscr{P}(z)$ is the set of all output vectors y that can be produced in one period from the vector z of stocks. Here the term "output" is to be interpreted broadly, including the "output" of second-hand commodities. In this paper, the production possibility set $\mathscr{P}$ will be defined in terms of the special production functions and lifetime distribution functions mentioned above.

Given an initial stock vector $z(0)$, a horizon T, and a sequence $q(1), \cdots, q(T)$ of vectors of exogenous supplies, a program $y(1), z(1), \cdots, y(T), z(T)$ will be said to be feasible if

$$\begin{aligned} &y(t) \text{ is in } \mathscr{P}[z(t-1)] , \\ &z(t) = y(t) + q(t) , \end{aligned} \tag{1.2}$$

for $t = 1, \cdots, T$. (The horizon T may be infinite.)

The problems of choosing a best feasible program will be formulated in terms of choosing a best sequence of "one-period-welfares" as follows. Let u be a real-valued function defined on non-negative (M-dimensional) vectors, and for any given program define the one-period-welfare v_t at period t by

$$v_t = u[z(t)] . \tag{1.3}$$

A sequence of one-period-welfares is feasible if it corresponds to a feasible program. In this paper it is assumed that $u(z)$ is a linear function of the logarithms of the coordinates of z.

I shall now formulate four different problems of optimal growth.

Problem I. Let δ be a positive number and define

$$v = \sum_{t=1}^{T} \delta^{t-1} v_t \ . \tag{1.4}$$

A feasible program is optimal in the sense of Problem I if it maximizes the total discounted welfare v (given $z(0)$). In the special cases studied in this paper, if the horizon T is infinite then a feasible program that is optimal in this sense typically exists only for $\delta < 1$. However, as δ approaches 1 (from below) the corresponding optimal programs will approach a limit program that is feasible.

Problem II. A feasible program is optimal in the sense of Problem II if it maximizes the one-period-welfare v_T of the final stock vector $z(T)$, given $z(0)$ and T.

Problem III. Suppose that the horizon T is infinite and consider a class of feasible programs for which in the long run one-period-welfare grows linearly with time; i.e., such that for each program in the class

$$S = \lim_{t \to \infty} \frac{v_t}{t} \tag{1.5}$$

exists. For such a class, a program is optimal in the sense of Problem III if it maximizes the long-run rate of change, S, of welfare with respect to time.

Problem IV. Consider a class of feasible programs of the type considered in Problem III, with the further properties that all programs in the class have the *same* long run rate of change, S, of welfare with respect to time, as defined in (1.5), and that for each program

$$H = \lim_{t \to \infty} (v_t - tS) \tag{1.6}$$

exists. For such a class, a program is optimal in the sense of Problem IV if it maximizes H.

A closely related problem is the following. Consider a class of feasible programs with the same long-run growth rate g; i.e., for every program in the class and every commodity

$$\lim_{t \to \infty} \frac{z_i(t)}{(1+g)^t} = h_i$$

exists. One may now pose the problem of choosing a program that maximizes

$$u(h_1, \cdots, h_M) \ .$$

This appears to me to be one way of interpreting J. Robinson's problem of the "golden age" (see [11]). Phelps has analyzed a different version of the "golden age" problem in the context of a 2-commodity model [7], with results similar to those of Section 5 of this paper.

A solution to Problem IV may also be interpreted as illustrating "capital saturation" [3], and Allais' concept of a "capitalistic optimum" [1].

Classification of Commodities

In order to describe the solutions of Problems I–IV, I must first define certain classes of commodities.

A commodity is a *primary resource* if it cannot be produced, but can only be obtained exogenously, i.e.

$$z_j(t) \equiv q_j(t)$$

for all feasible programs. I will assume in this paper that *there is no exogenous supply of commodities that are not primary resources*; i.e., if commodity i is not a primary resource, then

$$q_i(t) = 0 .$$

It will be seen that the long-run properties of optimal programs depend in important ways on whether or not there are primary resources in the economy that are necessary to production.

A commodity j is *non-productive* if the production possibility set $\mathscr{P}(z)$ does not depend upon the quantity z_j.

Consumption

So far nothing has been said about consumption, and it may have seemed odd to the reader that one-period-welfare was defined as a function of total stocks $z_j(t)$ rather than as a function of consumption. Formally, however, the description of consumption has not been excluded. First, certain commodities may be "consumption" goods or services by nature. Second, a quantity of a good or service that has been earmarked for consumption may be distinguished as a different commodity from the quantity that has been earmarked as an input into production. Combining these two points, one may designate certain commodities as "consumption commodities", and assume that the one-period-welfare $u(z)$ is independent of the quantities z_j of all commodities j that are not consumption commodities. In keeping with the usual formulation of consumption, a consumption commodity would be classified as non-productive. In other words, in the language of the present paper, the usual treatment of consumption amounts to the assumption that only non-productive commodities enter directly as arguments in the welfare function. This will be called here the assumption of the *separability of consumption and production.* Such an assumption seems to be unwarranted in general; for example, a beautiful building may simultaneously give direct pleasure to those who see it and be a factor in the production of other goods and services. In the context of the present study the separability assumption is unnecessarily strong, for it leads to no simplification of the analysis. However, it is interesting to analyze how the output of non-productive commodities depends upon the parameters of the problem.

It should be emphasized that from a purely formal point of view one can maintain the separability of consumption and production by defining new commodities that are to be interpreted as "consumer services", available jointly with the services of the commodity as a factor of production. I have not used such a device here because with the special production functions I am using, joint production is ruled out, except insofar as the use of durable goods represents a type of joint production.

Calculation of Optimal Programs

Under the assumptions to be made about production possibilities, a program is determined by determining in every period $t = 1, \cdots, T$ the quantity $x_{ij}(t)$ of commodity i that is to be devoted to the production of commodity j. One may write

$$x_{ij}(t) = f_{ij}(t)z_i(t-1) , \tag{1.7}$$

where $f_{ij}(t)$ is the fraction of the final stock of commodity i at the end of period $(t-1)$ that is allocated to the production of commodity j in period t. The fractions $f_{ij}(t)$ will be called the allocation coefficients; they may be taken to be the decision variables, instead of the $x_{ij}(t)$.

Under the linear-logarithmic assumptions, the optimal allocation coefficients for Problems I and II do not depend upon the initial stock vector $z(0)$. Furthermore, for a fixed period t, as the horizon T increases without limit, the optimal allocation coefficient $f_{ij}(t)$ approaches a value ϕ_{ij} that does not depend upon t. In particular, in Problem I with an infinite horizon, one may say that a constant fraction of the national product is "invested" each period, i.e., allocated to the production of productive commodities, and the rest is "consumed", i.e., allocated to the production of non-productive commodities. The explicit formulas for the optimal allocation coefficients for Problems I and II are given in Sections 3, 4 and 6.

The Long-Run Behavior of Optimal Programs for Problem I

Consider Problem I with an infinite horizon. I shall show that in the case in which there are no primary resources (Section 3), the optimal output sequence $z(t)$ approaches proportional (balanced) growth in the long run, i.e.,

$$\lim_{t\to\infty} \frac{z(t)}{s^t} = h \tag{1.8}$$

exists for some positive number s, provided the production functions exhibit constant returns to scale and satisfy a certain condition of acyclicity. If, moreover, the set of production functions satisfies an additional regularity condition, then the limit h is independent of the initial stock vector $z(0)$, except through a multiplicative factor. On the other hand, if there are strictly decreasing returns to scale, then output will approach a constant. From here on I will refer to the combined assumption of constant returns to scale, acyclicity and the additional regularity condition alluded to above by saying that *production is fully regular.*

In the case in which primary resources are present (and necessary for production), the long-run behavior of output depends upon the long-run behavior of the sequence $q(t)$ of supplies of primary resources. (Section 4.) If each primary resource grows at a constant rate, i.e.,

$$q_j(t) = q_j(0)q_j^t\,, \tag{1.9}$$

then (with constant returns to scale) in the long run the stock of each produced commodity will grow at a constant rate, i.e.,

$$\lim_{t\to\infty} \frac{z_j(t)}{s_j^t} = h_j\,; \tag{1.10}$$

furthermore, the growth factor s_j will be a weighted geometric mean of the growth factors q_j of the primary resources. In particular, if all the primary resources grow at the same constant rate, then in the long run the stocks of produced commodities will also tend to grow at that rate.

Relations among the Solutions of Problems I–IV

If, in Problem I, the discount factor δ approaches unity, then the total discounted welfare will typically increase without limit. If production is fully regular, however, the corresponding optimal programs approach a well-defined program as a limit. This limit program, corresponding to $\delta = 1$, is related to the solutions of Problems II–IV in a way that I will now describe.

In the case of no primary resources, all programs with *fixed* allocation coefficients tend towards proportional growth under the full regularity condition. In this case the limit program for Problem I corresponding to $\delta = 1$ is the program with the largest asymptotic rate of growth among all programs with fixed allocation coefficients; hence it is a solution to a problem of type III.

In the case of primary resources, all programs with fixed allocation coefficients have the same asymptotic growth rate for any given produced commodity, but such programs may differ in the magnitudes h_j of (1.10). In this case the limit program for Problem I corresponding to $\delta = 1$ is the fixed allocation coefficient program that has the largest value of

$$\sum_j \omega_j \log h_j \; .$$

Hence this limit program is the solution of a problem of type IV. These versions of Problems III and IV are treated in Section 5.

In Problem II (maximizing final welfare) with no primary resource, the maximum final welfare typically diverges as the horizon T increases without limit. Again, however, there is a limit solution if production is fully regular. Indeed, this limit solution is the same as the limit solution for $\delta = 1$ in Problem I with no primary resources; hence also the solution of Problem III (maximal rate of growth). This last provides an illustration of the "turnpike theorem" (see [8], [5] and [6]).

The situation in Problem II *with* primary resources is different, and in a sense anomalous. The limit solution as the horizon T increases without limit is well defined, but it does not correspond to any solution of Problem I, nor to the limit solution for $\delta = 1$. Furthermore, this limit solution for Problem II can be inefficient.

Shadow Prices

To every optimal program for Problem I corresponds a sequence of shadow prices of inputs and outputs such that the production plan in each period for that program is profit maximizing. In addition to calculating the shadow prices for Problem I, I will show (Section 7) that in the special case of separability of consumption and production, the own-rate of interest $d_i(t)$ for any commodity i that is both produced and productive is related to the growth rate $g_i(t)$ of that commodity and the discount factor δ in the welfare function, by

$$1 + d_i(t) = \left(\frac{1}{\delta}\right)[1 + g_i(t)] \; .$$

Hence the own-rate of interest $d_i(t)$ is greater than the growth rate $g_i(t)$. In particular, if the optimal program has the property of proportional growth at a constant rate g (as any optimal program will tend to have in the long run, under the full regularity condition), then there is a naturally defined rate of interest d given by

$$1 + d = \left(\frac{1}{\delta}\right)(1 + g) ,$$

with, of course, $d > g$. Notice that as the discount factor δ approaches 1, the interest rate and the growth rate approach equality.

Changing Technology and Tastes

The analysis of a linear-logarithmic economy is easily extended to the case in which technology and tastes change in time in any way that can be described in terms of changes in the parameters of the production and welfare functions. The appropriate formulas are given in [10], but the study of the effects of particular patterns of parameter changes is a project in itself—and is not undertaken here.

2. FORMULATION OF PROBLEM I: CASE OF NO PRIMARY RESOURCES

The Production Possibilities

Suppose that there are M commodities, labelled $1, \cdots, M$, and let $z_i(t)$ denote the stock of commodity i at the end of period t ($t = 0, 1, 2, \cdots, T$).

At the beginning of period t, one allocates a part, $f_{ij}(t)z_i(t-1)$, of the stock $z_i(t-1)$ of commodity i carried over from the end of period $(t-1)$, to the production of commodity j ($i, j = 1, \cdots, M$). The output $z_j(t)$ of commodity j at the end of period t is assumed to be determined by a linear-logarithmic (or "Cobb-Douglas") production function[2]

$$\log z_j(t) = \beta_j + \sum_{i=1}^{M} \alpha_{ij} \log [f_{ij}(t) z_i(t-1)] , \qquad j = 1, \cdots, M; \quad t = 1, \cdots, T . \tag{2.1}$$

By definition

$$f_{ij}(t) \geqq 0, \qquad \sum_{j=1}^{M} f_{ij}(t) \leqq 1 . \tag{2.2}$$

I assume that production of each commodity exhibits non-negative, non-increasing marginal productivity and non-increasing returns to scale, i.e.,

$$\alpha_{ij} \geqq 0 , \qquad \sum_{i=1}^{M} \alpha_{ij} \leqq 1 . \tag{2.3}$$

Notice that no commodities are supplied that are not produced, i.e., there are no primary resources.

The Welfare Function

In this section and the next I discuss the problem of finding a

[2] Unless otherwise noted, the convention $0 \log 0 = 0$ is to be understood.

sequence of outputs $z(1), \cdots, z(T)$ that maximizes *welfare* defined as

$$v = \sum_{t=1}^{T} \delta^{t-1} \sum_{j=1}^{M} \omega_j \log [z_j(t)] , \tag{2.4}$$

given the initial stocks $z_j(0)$, and subject to the technological and accounting constraints (2.1) and (2.2), where $\omega_1 \cdots, \omega_M$ and δ are parameters satisfying

$$\omega_j \geqq 0 , \quad \sum_{j=1}^{M} \omega_j = 1 , \tag{2.5}$$

$$0 < \delta < 1 . \tag{2.6}$$

Condition (2.5) says that no commodities are undesirable, and some commodities are strictly desirable. The condition that the sum of the ω_j be 1 is purely conventional; the sum could as well be any other positive number.

The restriction of the discount factor δ to values less than 1 is required to ensure convergence of the welfare (2.4) in the case of an infinite horizon. I will, however, discuss the behavior of the solution as $\delta \to 1$.

In expression (2.4) the term

$$\sum_{j=1}^{M} \omega_j \log [z_j(t)]$$

may be interpreted as the welfare at period t, and the parameter δ as indicating the relative preference for future as against present welfare.

Consumption

The following formulation of the problem might appear closer to conventional treatments.

The stock $z_i(t-1)$ of commodity i is divided into two parts. The first part, $c_i(t)$, is consumed, and the second part, $x_i(t)$, is used as an input into the production process. A fraction, $f_{ij}(t)x_i(t)$, of the total input of commodity is allocated to the production of commodity j; the output $z_j(t)$ of commodity j is assumed to be determined by

$$\log z_j(t) = \beta_j + \sum_{i=1}^{M} \alpha_{ij} \log [f_{ij}(t)x_i(t)] . \tag{2.6}$$

The above remarks imply the following accounting conditions:

$$\begin{aligned} &c_i(t) + x_i(t) = z_i(t-1) , \\ &c_i(t) \geqq 0 , \quad x_i(t) \geqq 0 , \\ &f_{ij}(t) \geqq 0 , \quad \sum_k f_{ik}(t) \leqq 1 . \end{aligned} \tag{2.7}$$

In this formulation, welfare would be defined as

$$v = \sum_{t=1}^{T} \delta^{t-1} \sum_{j=1}^{M} \omega_j \log [c_j(t)] . \tag{2.8}$$

Since many of the coefficients ω_j would typically be zero, there would be correspondingly many commodities for which consumption would be zero. These commodities would be produced, if at all, only because of their role in the production of commodities for which $\omega_j > 0$.

There are two reasons for preferring the original formulation (2.1)–(2.4) to the one just outlined in (2.6)–(2.8). First, from a purely formal point of view, the second (while notationally more complicated) can be regarded as a special case of the first, as follows. Corresponding to each commodity i in the list of M commodities in the second formulation define a new commodity, say $M + i$; this commodity $(M + i)$ is to be interpreted as commodity i after being earmarked for consumption. The "transformation" of commodity i into commodity $(M + i)$ has the same form as the production relations (2.1)–(2.3), in a trivial way; furthermore, commodity $(M + i)$ will not enter into the production of any other commodity.[3]

Thus

$$\begin{aligned} \alpha_{i,M+i} &= 1 , & i &= 1, \cdots, M , \\ \alpha_{M+i,j} &= 0 , & j &= 1, \cdots, 2M . \end{aligned} \tag{2.9}$$

Finally, the assumption that welfare is derived only from "consumption" would be expressed by

$$\omega_i = 0 , \qquad i = 1, \cdots, M . \tag{2.10}$$

A second reason for preferring the first formulation to the second is that it allows for the possibility of expressing more general welfare functions. It is commonplace that a physical commodity may *simultaneously* serve as a factor of production and produce immediate welfare (e.g., a beautiful building). This point has already been discussed in Section 1.

Capital and Investment

Returning now to the original problem formulation (2.1)–(2.4), one may define a commodity i as *productive* if it enters into the production function of some commodity, i.e., if $\alpha_{ij} > 0$ for some j. *Capital* may be interpreted here as the stocks of all productive commodities. The

[3] The idea of formulating consumption in this way, as a special case of the situation in which the welfare function is defined on output, is due to C. B. McGuire. The earlier study in [9] used the formulation (2.6)–(2.8).

formulation used here permits (but does not require) one to postulate that the existence of stocks of certain capital goods produces welfare directly. Investment is the allocation of current resources to the production of productive commodities.

Durability

The durability of commodities can be expressed in terms of the same formal model of production, but not without some attendant difficulties.

Let commodities be distinguished according to age, and for any commodity j that is not new, let $p(j)$ denote the index of the corresponding commodity that is one period younger. Assume that the stock of commodity j at any period is a given fraction of the stock of commodity $p(j)$ at the previous period, with this fraction possibly depending on j, thus

$$z_j(t) = e^{\beta_j} z_{p(j)}(t-1) , \tag{2.11}$$

or

$$\log z_j(t) = \beta_j + \log [z_{p(j)}(t-1)] . \tag{2.12}$$

This means that I assume that each commodity disappears (is used up) at a rate that depends upon the commodity and its age, but not upon the use to which it is put.

Note that (2.12) is a special case of the production function (2.1), with $\alpha_{p(j),j} = 1$. However, no resources are devoted to this production. Hence, if the newly produced resources are denoted by $j = 1, \cdots, N$, and the rest by $j = N+1, \cdots, M$, then the allocation coefficients $f_{ij}(t)$ are defined only for $j = 1, \cdots, N$.

One difficulty with the proposed treatment of durable goods is that the linear-logarithmic production function does not allow the expression of perfect substitutability among different commodities. Thus, if commodities 2 and 3 were perfect substitutes in the production of commodity 1, then one would have to modify the production function (2.1) to admit terms of the form

$$\alpha_{21} \log [f_{21}(t) z_2(t-1) + f_{31}(t) z_3(t-1)] .$$

Of course, such a model could be considered, but unfortunately this modification appears to destroy the simplicity of the results reported in this paper.

Reformulation in Vector-Matrix Notation

Define

$$Z_j = \log z_j , \qquad j = 1, \cdots, M ,$$
$$f = ((f_{ij})) , \quad A = ((\alpha_{ij})) ,$$

$$\eta_j(f) = \begin{cases} \sum_i \alpha_{ij} \log f_{ij} \,, & j = 1, \cdots, N \,, \\ 0 & j = N+1, \cdots, M \,, \end{cases}$$

$$z = \begin{pmatrix} z_1 \\ \vdots \\ z_M \end{pmatrix}, \quad Z = \begin{pmatrix} Z_1 \\ \vdots \\ Z_M \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_M \end{pmatrix}, \quad \eta = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_M \end{pmatrix}, \quad \omega = \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_M \end{pmatrix}. \tag{2.13}$$

Make the convention that for $j = N + 1, \cdots, M$,

$$\alpha_{ij} = \begin{cases} 1 \,, & \text{for } i = p(j) \,, \\ 0 \,, & \text{otherwise} \,. \end{cases}$$

The production function (2.1) can now be rewritten as

$$\begin{aligned} Z(t) &= \beta + \eta[f(t)] + A'Z(t-1) \,, & t &= 1, \cdots, T \,, \\ f_{ij}(t) &\geqq 0 \,, \quad \sum_j f_{ij}(t) \leqq 1 \,, & i &= 1, \cdots, M \,, \\ & & j &= 1, \cdots, N \,, \end{aligned} \tag{2.14}$$

and the welfare function as

$$v = \sum_{t=1}^{T} \delta^{t-1} \omega' Z(t) = \omega' \sum_{t=1}^{T} \delta^{t-1} Z(t) \,. \tag{2.15}$$

Unless otherwise noted, it will be assumed that the initial stock vector $z(0)$ is positive, so that $Z(0)$ is finite.

3. OPTIMAL GROWTH FOR THE CASE OF NO PRIMARY RESOURCES

Solution for a Finite Horizon T

The difference equation (2.14) can be solved to give output at t as a function of initial stocks and the sequence of allocation coefficients:

$$Z(t) = (A')^t Z(0) + \sum_{k=0}^{t-1} (A')^k \beta + \sum_{k=0}^{t-1} (A')^k \eta[f(t-k)] \,. \tag{3.1}$$

Welfare over T periods can be expressed as follows: first define

$$\omega(t) = \sum_{k=0}^{t} (\delta A)^k \omega \,; \tag{3.2}$$

then from (2.15) and (3.1), with a little calculation, one obtains the welfare

$$\begin{aligned} v = \omega(T-1)' A' Z(0) &+ \sum_{1}^{T} \delta^{t-1} \omega(T-t)' \beta \\ &+ \sum_{1}^{T} \delta^{t-1} w(T-t)' \eta[f(t)] \,. \end{aligned} \tag{3.3}$$

Notice that v depends upon initial stocks through the first term of

(3.3), and on the allocation coefficients through the third term.

The partial derivative of welfare with respect to the allocation coefficient $f_{ij}(t)$ is

$$\frac{\partial v}{\partial f_{ij}(t)} = \frac{\delta^{t-1}\alpha_{ij}\omega_j(T-t)}{f_{ij}(t)} . \tag{3.4}$$

From this it is routine to determine the optimal value of the allocation coefficients, for all productive commodities i, as

$$f_{ij}(t) = \frac{\alpha_{ij}\omega_j(T-t)}{\sum_{k=1}^{N}\alpha_{ik}\omega_k(T-t)} \qquad \begin{array}{l} i \text{ productive},\\ j = 1, \cdots, N,\\ t = 1, \cdots, T. \end{array} \tag{3.5}$$

For i non-productive, $f_{ij}(t) = 0$.

Solution for an Infinite Horizon

For the rest of this section, I shall consider the case of an infinite horizon.

Keeping t fixed, let the horizon T increase without limit in expression (3.5) for the optimal allocation coefficients. Define

$$\tilde{A} = \sum_{k=0}^{\infty}(\delta A)^k , \qquad \tilde{\omega} = \tilde{A}\omega . \tag{3.6}$$

Convergence is assured by the assumption of non-increasing returns to scale (2.3), if $0 < \delta < 1$. Examination of (3.2) and (3.5) shows that

$$\lim_{t\to\infty}\omega(t) = \tilde{\omega} \tag{3.7}$$

and that the allocation coefficients $f_{ij}(t)$ in (3.5) approach the limits

$$\phi_{ij} = \frac{\alpha_{ij}\tilde{\omega}_j}{\sum_{k=1}^{N}\alpha_{ik}\tilde{\omega}_k} \qquad \begin{array}{l} i \text{ productive},\\ j = 1, \cdots, N. \end{array} \tag{3.8}$$

These are the *optimal*[4] *allocation coefficients in every period* t *(the same for all* t*) for the problem with an infinite horizon.* Of course, for i non-productive, $\phi_{ij} = 0$.

The maximum welfare is, from (3.3) and (3.8),

$$v = (A\tilde{\omega})'Z(0) + \left(\frac{1}{1-\delta}\right)\tilde{\omega}'\zeta , \tag{3.9}$$

where

$$\zeta \equiv \beta + \eta(\phi) . \tag{3.10}$$

[4] This does not constitute a rigorous proof of optimality for the case $T = \infty$; see however [10].

Optimal Growth is Asymptotically Proportional for a "Fully Regular" Economy

Continuing the discussion of the case of an infinite horizon, the optimal path can be described by

$$Z(t) = (A')^t Z(0) + \sum_{k=0}^{t-1} (A')^k \zeta , \tag{3.11}$$

where ζ is given by (3.10).

If there are *strictly decreasing* returns to scale in every production function, i.e., $\sum_i \alpha_{ij} < 1$ for every j, then the sequence of vectors $Z(t)$ of logarithms of outputs will converge to

$$\lim_{t\to\infty} Z(t) = \left(\sum_{k=0}^{\infty} A^k\right)' \zeta . \tag{3.12}$$

Thus the optimal path will approach a long-run stationary state.

On the other hand, if there are *constant returns* to scale, the sequence of vectors $Z(t)$ will typically diverge, with the consequence that the output $z_i(t)$ of each commodity will typically either diverge or converge to zero.

Nevertheless, if the matrix A is *fully regular*[5] then, as will now be shown, output grows asymptotically at a constant rate (the same rate for all commodities); i.e., the *optimal path exhibits proportional growth in the long run.* Furthermore, *the relative long-run proportions* are independent of the initial stock vector $z(0)$. (Recall that this last was assumed to be positive.)

The economic interpretation of the assumption that A is fully regular is roughly speaking:

(a) The list of commodities cannot be partitioned into two or more sub-lists, each corresponding to an independent *self-sustaining* sub-economy.

(b) Production is acyclic in the sense that one cannot partition the commodities into groups $B_1, \cdots, B_K$ such that commodities in group B_2 can be produced from commodities in group B_1 only, commodities in B_3 produced from those in B_2 only, etc., $\cdots$, and commodities in B_1 produced from those in B_K only.

To prove the proposition about long-run proportional growth, first observe that since A is fully regular, the limit

$$\lim_{t\to\infty} A^t = \bar{A} \tag{3.13}$$

[5] A non-negative matrix A with column sums equal to 1 is called *fully regular* if A has no characteristic values of modulus 1 other than 1 itself, and 1 is a simple root of the characteristic equation of A.

exists, and the columns of $\bar{A}$ are identical, say equal to the vector $\bar{a}$. Hence the coordinates of the vector $\bar{A}'\zeta$ are identical, and equal to $\bar{a}'\zeta$, which will be seen to be the asymptotic growth factor. Now note that

$$Z(t) - t\bar{A}'\zeta = (A')^t Z(0) + \sum_{k=0}^{t-1} (A^k - \bar{A})'\zeta . \tag{3.14}$$

Since A is fully regular, the convergence of A^t to $\bar{A}$ is geometric (see [2, p. 89]), so that the sum on the right side of (3.14) converges as $t \to \infty$. Hence

$$\lim_{t\to\infty} [Z(t) - t\bar{A}'\zeta] = \bar{A}'Z(0) + \sum_{k=0}^{\infty} (A^k - \bar{A})'\zeta . \tag{3.15}$$

Note that $\bar{A}'Z(0)$ is a vector with identical coordinates equal to $\bar{a}'Z(0)$. Define

$$\zeta^* = \sum_{k=0}^{\infty} (A^k - \bar{A})'\zeta ; \tag{3.16}$$

then (3.15) and (3.16) imply that

$$z_i(t) \sim e^{\bar{a}'Z(0) + t\bar{a}'\zeta + \zeta_i^*} . \tag{3.17}$$

In other words, output $z(t)$ grows asymptotically with the growth factor $\exp(\bar{a}'\zeta)$, and with asymptotic relative proportions $\exp(\zeta_i^*)$. The long-run path depends on the initial stock vector $z(0)$ through multiplication by the scalar factor $\exp[\bar{a}'Z(0)]$.

If allocation of inputs to the production of productive commodities is interpreted as investment, then the asymptotic growth factor $\exp(\bar{a}'\zeta)$ can be related to an average investment coefficient for the economy. Let σ_i denote the fraction of commodity i that is allocated to the production of productive commodities, i.e.,

$$\sigma_i = \sum_j f_{ij} , \qquad j \text{ productive.} \tag{3.18}$$

Now imagine that the fraction $(1 - \sigma_i)$ of commodity i that is allocated to the production of non-productive commodities is *re*-allocated to the production of productive commodities, in the same proportions as the original allocations, i.e., define new allocation coefficients $\dot{f}_{ij}$ by

$$\dot{f}_{ij} = \begin{cases} \dfrac{f_{ij}}{\sigma_i} , & \text{for } j \text{ productive ,} \\ 0 , & \text{for } j \text{ non-productive .} \end{cases} \tag{3.19}$$

Use of these allocation coefficients $\dot{f}_{ij}$ would yield an asymptotic growth factor $\exp(\bar{a}'\dot{\zeta})$, where

$$\dot{\zeta} = \beta + \eta(\dot{f}) . \tag{3.20}$$

It can be shown that

$$\exp(\bar{\alpha}'\zeta) = \exp(\sum_i \bar{\alpha}_i \log \sigma_i) \cdot \exp(\bar{\alpha}'\dot{\zeta}) . \tag{3.21}$$

The first factor on the right side of (3.21) is a geometric mean of the investment coefficients for the several commodities. The second factor, the asymptotic growth rate induced by the allocation coefficients $\dot{f}_{ij}$ (all resources devoted to the production of productive commodities), is greater than the growth factor $\exp(\bar{\alpha}'\zeta)$, but it is not the maximum growth factor possible for the economy; for more on this, see Section 5.

Discount Factor Close to Unity

Although the expression for welfare (with an infinite horizon) typically diverges as the discount factor δ approaches 1, it is of interest to examine the limiting characteristics of the optimal path in this case. It will be seen that as $\delta \to 1$, the optimal allocation coefficients and the optimal path do approach limits, such that there is no production of non-productive commodities. Furthermore, it will be seen in Section 6 that this limit path is asymptotically the path of fastest proportional growth.

I maintain here the assumption that A is fully regular. First note from (3.6) that $\tilde{A}$ and $\tilde{\omega}$ diverge as $\delta \to 1$. However,

$$\begin{aligned} \lim_{\delta\to 1} (1-\delta)\tilde{A} &= \bar{A} , \\ \lim_{\delta\to 1} (1-\delta)\tilde{\omega} &= \bar{A}\omega = \bar{\alpha} . \end{aligned} \tag{3.22}$$

Hence, the optimal allocation coefficients (3.8) approach limits

$$\lim_{\delta\to 1} \phi_{ij} = \frac{\alpha_{ij}\bar{\alpha}_j}{\bar{\alpha}_i} \qquad \begin{array}{l} i \text{ productive} , \\ j = 1, \cdots, N . \end{array} \tag{3.23}$$

(I use the fact that $\sum_k \alpha_{ik}\bar{\alpha}_k = \bar{\alpha}_i$.)

But if commodity j is non-productive, then $\bar{\alpha}_j = 0$ (see [2, p. 92]).[6] Hence, for the limit path, there is no production of non-productive commodities.

In particular, in the consumption-saving formulation of the problem outlined in Section 2, consumption falls to zero as δ increases to unity.

4. PROBLEM I: PRIMARY RESOURCES

In this section I expand the model of the previous section to include a third group of commodities, *primary resources*. These are com-

[6] The classification of commodities into the categories productive and non-productive corresponds to the classification of states in a finite Markov chain into the categories essential and non-essential (or ergodic and transient).

modities that are not produced, but whose stocks are determined exogenously in each period, this sequence of stocks being independent of the program chosen. Whether or not a particular commodity should be classified as a primary resource will typically depend upon the circumstances of the problem. For example, in a very poor country the population growth (or decline) may depend upon which economic program is chosen, whereas in a rich country the population might well be taken to be a primary resource, at least as a good approximation. Land should typically be treated as a primary resource, unless the economic programs considered involve possible long-run changes in the fertility of the soil, etc.

The formulas describing the optimal programs for this case are similar to those for the case of no primary resources. However, the evolution of the output of the produced (i.e., non-primary) resources will depend upon the availability of primary resources. For example, if the supply of primary resources is constant, then in an optimal program all outputs, etc., will approach constant levels in the long run. On the other hand, if the various primary resources are growing exponentially, then output in the various sectors will also be asymptotically exponential, with possibly different rates for different sectors.

I shall also show that in this case, unlike the case of no primary resources, the output of desired non-productive commodities does not typically fall to zero as the discount factor δ approaches unity.

Suppose now that to the list of M newly produced and second-hand commodities we add P commodities, called *primary resources*, which enter into the production of the new commodities, but which are themselves exogenously supplied. Let $z_{(1)}(t)$ denote the vector of stocks of the produced commodities (1 to M), and $z_{(2)}(t)$ denote the vector of stocks of primary resources $(M+1)$ to $(M+P)$, at the beginning of period t. Let $Z_{(1)}(t)$ and $Z_{(2)}(t)$ denote the corresponding vectors of logarithms. Using a notation similar to that of (2.13) and (2.14), the production function for produced commodities is assumed to be

$$Z_{(1)}(t) = \beta + \eta(f[t]) + A'Z(t-1) \, , \tag{4.1}$$

where $Z_{(1)}$ is the vector of the first M coordinates of Z. Here the allocation coefficients to be determined are $f_{ij}(t)$ with $i = 1, \cdots, M+P$; $j = 1, \cdots, N$. The matrix A has $(M+P)$ rows and M columns. Condition (2.3) on A becomes

$$\begin{aligned} &\alpha_{ij} \geqq 0 \, , && i = 1, \cdots, M+P, \\ &\sum_{i=1}^{M+P} \alpha_{ij} \leqq 1 \, . && j = 1, \cdots, M \, . \end{aligned} \tag{4.2}$$

The logarithms of the stocks of primary resources are given by

$$Z_{(2)}(t) = Q(t) , \qquad t \geqq 0 , \tag{4.3}$$

where

$$Q_i(t) = \log q_i(t) ,$$

and $q(t)$ is an exogenously given sequence.

If we write

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} , \tag{4.4}$$

where A_1 is made up of the first M rows of A, then the difference equation (4.1) can be solved, together with (4.3), to give the output of produced commodities in every period,

$$\begin{aligned} Z_{(1)}(t) = (A_1')^t Z_{(1)}(0) &+ \sum_{k=0}^{t-1} (A_1')^k \beta \\ &+ \sum_{k=0}^{t-1} (A_1')^k \eta[f(t-k)] \\ &+ \sum_{k=0}^{t-1} (A_1')^k A_2' Q(t-1-k) . \end{aligned} \tag{4.5}$$

Compare this with the solution (3.1) for the case of no primary resources.

Welfare over T periods can be expressed by a formula similar to (3.3). Define

$$\omega_{(1)}(t) = \sum_{k=0}^{t} (\delta A_1)^k \omega_{(1)} , \tag{4.6}$$

where

$$\omega = \begin{pmatrix} \omega_{(1)} \\ \omega_{(2)} \end{pmatrix} ;$$

then welfare over T periods is

$$\begin{aligned} v = \omega_{(1)}(T-1)' A' Z_{(1)}(0) &+ \sum_{1}^{T} \delta^{t-1} \omega_{(1)}(T-t)' \beta \\ &+ \sum_{1}^{T} \delta^{t-1} \omega_{(1)}(T-t)' \eta[f(t)] \\ &+ \sum_{1}^{T} \delta^{t-1} \omega_{(1)}(T-t)' A_2' Q(t-1) \\ &+ \omega_{(2)}' \sum_{1}^{T} \delta^{t-1} Q(t) . \end{aligned} \tag{4.7}$$

The formulas (3.5) for the optimal allocation coefficients derived for the case of no primary resources apply here as well, with $\omega(t)$ replaced by $\omega_{(1)}(t)$:

$$f_{ij}(t) = \frac{\alpha_{ij}\omega_{(1)j}(T-t)}{\sum_{k=1}^{N}\alpha_{ik}\omega_{(1)k}(T-t)}, \qquad \begin{array}{l} i \text{ productive}, \\ j = 1, \cdots, N, \\ t = 1, \cdots, T. \end{array} \tag{4.8}$$

Note that the productive commodities will typically include some (but not necessarily all) of the primary resources. Again, for i non-productive, $f_{ij}(t) = 0$.

Infinite Horizon

The solution here is similar to that of the previous section. Define

$$\tilde{A}_1 = \sum_0^\infty \delta^t A_1^t, \qquad \tilde{\omega}_{(1)} = \tilde{A}_1\omega_{(1)}. \tag{4.9}$$

The optimal allocation coefficients are again independent of t, and are given by

$$\phi_{ij} = \frac{\alpha_{ij}\tilde{\omega}_{(1)j}}{\sum_{k=1}^{N}\alpha_{ik}\tilde{\omega}_{(1)k}}, \qquad \begin{array}{l} i \text{ productive}, \\ j = 1, \cdots, N. \end{array} \tag{4.10}$$

For i non-productive, $\phi_{ij} = 0$.

From (4.7) and (4.8), maximum welfare is

$$\begin{aligned} v = {} & (A_1\tilde{\omega}_{(1)})'Z_{(1)}(0) + \left(\frac{1}{1-\delta}\right)\tilde{\omega}'_{(1)}\zeta \\ & + (A_2\tilde{\omega}_{(1)})'\sum_0^\infty \delta^t Q(t) + \omega'_{(2)}\sum_1^\infty \delta^{t-1}Q(t), \end{aligned} \tag{4.11}$$

where, as before,

$$\zeta = \beta + \eta(\phi). \tag{4.12}$$

(I assume, of course, that the series $\sum_0^\infty \delta^t Q(t)$ converges.)

Asymptotic Properties of Optimal Paths

Continuing the discussion of the case of an infinite horizon, the output of produced commodities along the optimal path is determined by

$$\begin{aligned} Z_{(1)}(t) = {} & (A_1')^t Z_{(1)}(0) + \sum_0^{t-1}(A_1')^k\zeta \\ & + \sum_0^{t-1}(A_1')^k A_2' Q(t-1-k). \end{aligned} \tag{4.13}$$

The long-run behavior of $Z_{(1)}(t)$ thus depends upon the long-run behavior of $Q(t)$.

Suppose that the supply of each primary resource grows (or declines) *geometrically, i.e., at some constant rate.* In particular, this includes the case of a constant supply of primary resources. Thus, suppose

$$(4.14)\qquad \begin{aligned} q_j(t) &= q_j(0)q_j^t\,, \quad \text{or} \\ Q_j(t) &= Q_j(0) + tQ_j\,, \quad \text{or} \\ Q(t) &= Q(0) + tQ\,. \end{aligned}$$

If all of the primary resources are non-productive, then $A_2 = 0$, and one is essentially back in the case of Section 3. Indeed, if any primary resource is unproductive, it can be ignored, as far as the problem of choosing an optimal path is concerned. Therefore, without loss of generality, I can assume that all primary resources are productive.

It follows that the characteristic roots of A_1 are all not greater in absolute value than some number $\rho_1 < 1$, since the column sums of A_1 are all less than 1. (We can take ρ_1 to be the largest non-negative characteristic root of A_1.) Hence

$$(4.15)\qquad \begin{aligned} &\lim_{t\to\infty} A_1^t = 0\,, \\ &\sum_{t=0}^{\infty} A_1^t = (I - A_1)^{-1}\,. \end{aligned}$$

With the assumption (4.14) of geometric growth of the primary resources, the third term on the right side of expression (4.13) for the output path can be calculated to give

$$(4.16)\qquad \sum_{k=0}^{t-1} (A_1')^k [A_2'Q(0) - (I - A_1')^{-1}A_2'Q] + t[A_2(I - A_1)^{-1}]'Q\,.$$

As t gets large, the first term in (4.16) approaches a constant. The second term is linear in t.

To summarize the asymptotic behavior of output, define

$$(4.17)\qquad \begin{aligned} H &= (I - A_1')^{-1}[\zeta + A_2'Q(0) - S]\,, \\ S &= [A_2(I - A_1)^{-1}]'Q\,, \\ h_i &= \exp(H_i)\,, \qquad s_i = \exp(S_i)\,, \qquad i = 1, \cdots, M\,. \end{aligned}$$

Then

$$(4.18)\qquad \lim_{t\to\infty} [Z_{(1)}(t) - tS] = H\,,$$

and

$$(4.19)\qquad z_i(t) \sim h_i s_i^t\,, \qquad i = 1, \cdots, M\,.$$

In other words, *each produced commodity grows asymptotically at a constant rate, which rates may differ from one produced commodity to the other, but do not depend upon the discount factor* δ.

One can show further that no growth factor s_i of a produced commodity can exceed the largest of the growth factors of the primary resources; indeed, if there are constant returns to scale, then each growth factor s_i is a (weighted) geometric mean of the growth factors q_j of the primary resources. To see this, consider the matrix

$$B = \begin{pmatrix} A_1 & 0 \\ A_2 & I \end{pmatrix}, \tag{4.20}$$

and observe that

$$B^t = \begin{pmatrix} A_1^t & 0 \\ A_2 \sum_{k=0}^{t-1} A_1^k & I \end{pmatrix}, \tag{4.21}$$

$$\lim_{t\to\infty} B^t = \begin{pmatrix} 0 & 0 \\ A_2(I - A_1)^{-1} & I \end{pmatrix}. \tag{4.22}$$

The column sums of B are not greater than 1 (recall the assumption of non-increasing returns to scale); hence the column sums of $A_2(I-A_1)^{-1}$ are not greater than 1. It follows from (4.17) that

$$\begin{aligned} S_i &\leqq \max_j Q_j\,, \\ s_i &\leqq \max_j q_j\,. \end{aligned} \tag{4.23}$$

If constant returns to scale prevail, then the column sums of B are all equal to 1; hence so are those of $A_2(I - A_1)^{-1}$. Therefore each S_i is a weighted arithmetic mean of the Q_j, and s_i is a weighted geometric mean of the q_j.[7]

In the case of constant returns to scale, *if all of the primary resources grow at the same rate* (i.e., $q_j = \bar{q}$) then for every i, $s_i = \bar{q}$ also, so that one has *long-run proportional growth.* In particular, if the supplies of primary resources are *constant,* then the outputs of produced resources approach constants.

Discount Factor Close to Unity

As the discount factor δ approaches unity, the matrix $\tilde{A}_1 = \sum_0^\infty \delta^t A_1^t$ and the vector $\tilde{\omega}_{(1)} = \tilde{A}_1\omega_{(1)}$ approach limits (contrast this with the case of no primary resources)

$$\begin{aligned} \lim_{\delta\to1} \tilde{A}_1 &= \sum_0^\infty A_1^k = (I - A_1)^{-1}\,, \\ \lim_{\delta\to1} \tilde{\omega}_{(1)} &= (I - A_1)^{-1}\omega\,. \end{aligned} \tag{4.24}$$

The allocation coefficients (4.10) approach corresponding limits.

[7] Note that in the matrix B, the primary resources correspond to *absorbing states* for the corresponding finite Markov chain.

But in this case, unlike the case of no primary resources, a coordinate of $\lim \tilde{\omega}_{(1)}$ that corresponds to a non-productive commodity will *not* be zero, unless the corresponding coordinate of ω is zero; i.e., unless that non-productive commodity is not desired. Hence, *as δ approaches* 1, *the output of desired non-productive commodities does not fall to zero.* In particular, in the consumption-saving formulation of the problem, consumption does *not* fall to zero as δ increases to unity.

5. MAXIMAL ASYMPTOTIC GROWTH: PROBLEMS III AND IV

For the case of no primary resources, the optimal paths derived in Section 3 have the property that all outputs grow asymptotically at the same constant rate (provided the economy is "fully regular"). The asymptotic growth rate depends, among other things, on the discount factor δ. In this section I analyze an alternative problem: Among all proportional growth paths, find the one with the largest growth rate. The solution will show that this maximum growth rate for proportional growth paths equals the limit of the asymptotic growth rate for an optimal path of Section 3 as the discount factor δ increases towards 1.

In the case of primary resources growing at constant rates (Section 4), optimal output of each produced commodity also grows asymptotically at a constant rate, but these rates are independent of the discount factor δ, thus

$$z_i(t) \sim h_i s_i^t \tag{5.1}$$

(see (4.19)). The "one-period-welfare" approaches a linear function of t in this case (see (4.18)), in the sense that

$$\lim_{t\to\infty} [\sum_i \omega_i (\log z_i(t) - t \log s_i)] = \sum_i \omega_i \log h_i \,. \tag{5.2}$$

Although the growth factors s_i do not depend upon the discount factor δ, the coefficients h_i do. I shall show in this section that the limit, $\sum_i \omega_i \log h_i$, in (5.2) is maximum when the discount factor δ equals unity.

The following heuristic remarks may make these results plausible. Suppose that output $z(t)$ grows at a constant rate, i.e.,

$$z(t) = g^t \bar{z} \; ; \tag{5.3}$$

then total welfare v, as defined by (2.4), is

$$\begin{aligned} v &= \sum_{t=1}^{\infty} \delta^{t-1}[t \log g + \omega' \bar{Z}] \\ &= \Big(\sum_{t=1}^{\infty} \delta^{t-1}\Big) \log g + \Big(\sum_{t=1}^{\infty} \delta^{t-1}\Big)\omega' \bar{Z} \\ &= \frac{1}{(1-\delta)^2} \log g + \frac{1}{(1-\delta)} \omega' \bar{Z} \,, \end{aligned} \tag{5.4}$$

where $\bar{Z}_i = \log \bar{z}_i$. Although v is unbounded as $\delta \to 1$ (unless $v = 0$),

$$\lim_{\delta \to 1} (1 - \delta)^2 v = \log g \ . \tag{5.5}$$

Hence for δ close to 1, maximizing v will be approximately equivalent to maximizing the growth factor g.

However, in the case of primary resources growing at constant rates, the asymptotic growth rates of output for optimal paths do not depend upon δ. Suppose all primary resources grow at the same rate, say with a growth factor g; then for *any* path with an asymptotic growth factor g,

$$Z(t) = t \log g + H + \varepsilon^t U(t) \ , \tag{5.6}$$

where $0 \leqq \varepsilon < 1$, and $U(t)$ is bounded in t. Hence

$$\begin{aligned} v &= \sum_{t=1}^{\infty} \delta^{t-1} \omega' Z(t) \\ &= \frac{\log g}{(1 - \delta)^2} + \frac{\omega' H}{1 - \delta} + \sum_{t=1}^{\infty} \delta^{t-1} \varepsilon^t \omega' U(t) \ . \end{aligned} \tag{5.7}$$

Since the growth factor g is fixed, maximizing v is equivalent to maximizing

$$v^* \equiv \frac{\omega' H}{1 - \delta} + \sum_{t=1}^{\infty} \delta^{t-1} \varepsilon^t \omega' U(t) \ . \tag{5.8}$$

But

$$\lim_{\delta \to 1} (1 - \delta) v^* = \omega' H \ , \tag{5.8}$$

so that for δ close to 1, maximizing v is approximately equivalent to maximizing $\omega' H$.

Case 1. *No Primary Resources*

I now take up the problem of determining the proportional growth path with the largest rate of growth, for the case of no primary resources. I assume that there are constant returns to scale and that the matrix A is fully regular.

Consider an arbitrary proportional growth path $z(t)$; by definition, it satisfies

$$z(t) = g^t z(0) \tag{5.9}$$

or

$$Z(t) = tG + Z(0) \ , \tag{5.10}$$

where G is a vector with identical coordinates equal to $\log g$, and, as usual, $Z_i = \log z_i$.

For a proportional growth path, the allocation coefficients f_{ij} are constant, so that using the production function (2.14) one has

$$Z(t) = \zeta + A'Z(t-1) , \quad \zeta = \beta + \eta(f) . \tag{5.11}$$

The solution of this difference equation is

$$Z(t) = \sum_{k=0}^{t-1} (A')^k \zeta + (A')^t Z(0) . \tag{5.12}$$

Combining (5.10) and (5.12) gives

$$\sum_{k=0}^{t-1} (A')^k \zeta + (A')^t Z(0) = tG + Z(0) . \tag{5.13}$$

Dividing both sides of (5.13) by t, and letting $t \to \infty$, one gets in the limit

$$\bar{A}'\zeta = G , \tag{5.14}$$

where

$$\bar{A} = \lim_{t\to\infty} A^t .$$

Since $\bar{A}$ is composed of identical column vectors $\bar{a}$, (5.14) is equivalent to

$$\bar{a}'\zeta = \log g . \tag{5.15}$$

Hence maximizing the growth factor g is equivalent to maximizing $\bar{a}'\zeta$. From this it is easy to verify that the allocation coefficients f_{ij} that maximize g are given by formula (3.20) for the limits of the optimal allocation coefficients ϕ_{ij} as $\delta \to 1$, and that the corresponding proportions are given by (3.17).

Case 2. *Primary Resources*

Suppose each primary resource grows at a constant rate, and consider again the class of paths for which the allocation coefficients f_{ij} are constant. A calculation like that leading to (4.17) and (4.18) shows that

$$z_i(t) \sim h_i s_i^t , \qquad i = 1, \cdots, M \tag{5.16}$$

and that

$$\lim_{t\to\infty} \omega'_{(1)}[Z_{(1)}(t) - tS] = \omega'_{(1)}H . \tag{5.17}$$

Furthermore S is independent of the allocation coefficients f_{ij}, whereas H depends on them through the term

$$(I - A'_1)^{-1}\zeta \tag{5.18}$$

as is seen from (4.17). The term (5.18) can be rewritten as

$$\sum_{k=0}^{\infty} (A'_1)^k[\beta + \eta(f)] , \tag{5.19}$$

and from this it is routine to verify that the allocation coefficients f_{ij} that maximize $\omega'_{(1)}H$ are given by (4.10) with $\delta = 1$.

6. MAXIMIZING WELFARE IN THE FINAL PERIOD ONLY: PROBLEM II

As an alternative to a welfare function defined on the entire path of growth, one may consider a welfare function defined only on the final state of the economy, at the end of a fixed horizon T. Such a formulation might be thought of as expressing the idea of a "crash program," e.g., achieve the highest standard of living that can be reached in 10 years.

I shall call a problem of this type a *final state problem*, to distinguish it from a problem of the type considered in Sections 2–4, which might be called a *path problem*.

To be precise, in this section I consider the problem of finding a feasible sequence of outputs $z(1), \cdots, z(T)$ that maximizes welfare defined as

$$v = \sum_j \omega_j \log z_j(T) , \tag{6.1}$$

given an initial stock vector $z(0)$, a fixed horizon T, and a technology of the type discussed in the previous sections.

Again, it will be important to distinguish two cases, according to whether primary resources are or are not present. In the case of no primary resources, if the economy is "fully regular" (see Section 3), then for large T the solution of the final state problem is approximately the same as the limit of the solutions of the path problem as the discount factor δ approaches 1. It is therefore also close, for large T, to the path of fastest proportional growth. This last proposition is a special case of the so-called "turnpike theorem" (see [5, 6, 8]); it is special because of the special welfare function and technology being considered here.

On the other hand, in the case of primary resources growing at constant rates, the solution of the final state problem does *not* correspond to a solution of a path problem for any δ less than 1, nor to the limit solution as δ approaches 1. It is therefore not the optimal path, either, for the problem of maximal asymptotic growth analyzed in Section 5.

Case 1. *No Primary Resources*

I consider now the problem of maximizing the welfare (6.1), with the technology of Section 2. From (3.1) the coefficient of $\eta[f(t)]$ in the expression for welfare at the final state is $\omega'(A')^{T-t}$. Define

$$\upsilon(t) = A^t\omega ; \tag{6.2}$$

then the optimal allocation coefficients are

$$f_{ij}(t) = \frac{\alpha_{ij}\upsilon_j(T-t)}{\sum_{k=1}^{N}\alpha_{ik}\upsilon_k(T-t)} \qquad \begin{array}{l} i \text{ productive}, \\ j = 1, \cdots, N, \\ t = 1, \cdots, T. \end{array} \tag{6.3}$$

Assume that A is fully regular. Then

$$\lim_{t\to\infty} \upsilon(t) = \bar{A}\omega = \bar{a} , \tag{6.4}$$

where $\bar{A} = \lim_{t\to\infty} A^t$, and $\bar{A}$ has identical columns $\bar{a}$. For fixed t

$$\lim_{T\to\infty} f_{ij}(t) = \frac{\alpha_{ij}\bar{a}_j}{\bar{a}_i} = \bar{\phi}_{ij} .$$

These are the same allocation coefficients (3.20) as in the limit of solutions for the path problem as $\delta \to 1$. Hence, when T is large, the optimal path for the final state problem is close to the path of fastest proportional growth for most of the T periods, in the following sense: Let proportional growth be fastest along the ray through the vector $\bar{z}$. For every $z(0)$ and every $\varepsilon > 0$ there exist T_0 and T_1 such that if $T \geqq T_0 + T_1$, then for the optimal path

$$| Z_i(t) - \bar{a}'Z(0) - t\bar{a}'\zeta - \log \bar{z}_j | < \varepsilon \tag{6.5}$$

for $T_0 \leqq t \leqq T - T_1$. The details of the proof of (6.5) are routine, and are omitted.

Case 2. *Primary Resources*

Consider now the final state problem in the case with primary resources. From (6.1) and (4.5), the coefficient of $\eta[f(t)]$ in the expression for welfare at the final state is $\omega'_{(1)}(A'_1)^{T-t}$. Define

$$\upsilon_{(1)}(t) = A_1^t\omega_{(1)} ; \tag{6.6}$$

then the optimal allocation coefficients are

$$f_{ij}(t) = \frac{\alpha_{ij}\upsilon_{(1)j}(T-t)}{\sum_{k=1}^{N}\alpha_{ik}\upsilon_{(1)k}(T-t)} \qquad \begin{array}{l} i \text{ productive}, \\ j = 1, \cdots, N, \\ t = 1, \cdots, T. \end{array} \tag{6.7}$$

Let ρ_1 be the largest non-negative characteristic root of A_1. If

$\rho_1 = 0$, then for some n, $A^t = 0$ for all $t \geqq n$, so that $v_{(1)}(t) = 0$ for all $t \geqq n$. Hence welfare v is independent of the allocation coefficients $f_{ij}(t)$ for $t \leqq T - n$. Note that $\rho_1 = 0$ if and only if there is some renumbering of the first M commodities such that $\alpha_{ij} = 0$ for $i \leqq j$; in other words, if and only if any commodity i can enter only into the production of lower numbered commodities.

If $\rho_1 > 0$, then

$$\lim_{T\to\infty} \rho_1^{-T} A^{T-t} = \bar{A}_1 = (\bar{a}_{(1)}, \cdots, \bar{a}_{(1)}) , \tag{6.8}$$

provided that (A_1/ρ_1) is fully regular. Hence, for fixed t, letting T get large, one obtains

$$\lim_{T\to\infty} \rho_1^{-T} v_{(1)}(T - t) = \bar{a}_{(1)} \tag{6.9}$$

$$\lim_{T\to\infty} f_{ij}(t) = \frac{\alpha_{ij}\bar{a}_{(1)j}}{\rho_1 \bar{a}_{(1)i}} , \qquad \begin{matrix} i \text{ productive,} \\ j = 1, \cdots, N . \end{matrix} \tag{6.10}$$

Comparing these results (for $\rho_1 > 0$) with the solution (4.8) of the path problem with an infinite horizon, one sees that if one lets the discount factor δ approach $(1/\rho_1)$ in (4.8), one obtains in the limit formula (6.10). But the welfare for the path problem does not converge for $\delta \geqq 1$, and it is not possible to obtain formula (6.10) as a solution or limit of solutions of the path problem.

7. SHADOW PRICES

To conclude this paper, I discuss the *shadow prices* corresponding to optimal programs for programs for Problem I (Sections 2–4), that is, the prices of inputs and outputs in each period such that the production plan in that period under the optimal program would be profit maximizing. I limit myself to an analysis of the case of constant returns to scale.

The vectors r and p are said to be *shadow prices* for a feasible input-output pair $\hat{x}, \hat{y}$ if the *profit*

$$p'\hat{y} - r'\hat{x}$$

for the input-output pair $\hat{x}, \hat{y}$ is the maximum possible profit for all feasible input-output pairs, i.e.,

$$p'\hat{y} - r'\hat{x} = \max_{y \in \mathscr{P}(x)} p'y - r'x . \tag{7.1}$$

A sequence $\{r(t), p(t)\}$ of vectors is a sequence of shadow prices for a feasible program $\{z(t)\}$ if, for each t, $r(t)$ and $p(t)$ are shadow prices for the input-output pair $[z(t-1), z(t)]$.

In an optimal program, the output $z(t)$ in any given period t is the optimal output in a *one-period* program with initial stocks equal to $z(t-1)$, and with welfare function

$$u(z) = \sum_j w_j(t) \log z_j ,$$

for suitably defined coefficients $w_j(t)$, as will be seen. It will be shown below that for such a one-period problem the shadow prices can be taken to be

$$\text{(7.2)} \quad \begin{aligned} p_j(t) &= \frac{w_j(t)}{z_j(t)} , && \text{for } j \text{ newly produced;} \\ r_i(t) &= \frac{\sum_k \alpha_{ik} w_k(t)}{z_i(t-1)} , && \text{for } i \text{ productive.} \end{aligned}$$

This permits one to calculate the shadow prices for the entire optimal program in question.

A special case of interest is that of separability of consumption and production; i.e., $\omega_i = 0$ for i productive (see Section 1). It will be shown that in this case one can choose the shadow prices for an optimal program so that

$$\text{(7.3)} \quad r(t) = p(t-1) ,$$

and the prices $p(t)$ can be thought of as *discounted* prices in the usual way (this illustrates the theorem of Malinvaud, [4]). Furthermore, if commodity i is both productive and (newly) produced, then (as will be shown)

$$\text{(7.4)} \quad \frac{p_i(t-1)}{p_i(t)} = \left(\frac{1}{\delta}\right) \frac{z_i(t)}{z_i(t-1)} .$$

where, as before, δ is the discount factor in the overall welfare function (2.4). Hence, in this case, the own-rate of interest of commodity i exceeds its growth rate. In particular, if the optimal program has the property of proportional growth at a rate g, then there will be a naturally defined rate of interest d given by

$$\text{(7.5)} \quad 1 + d = \frac{1}{\delta}(1 + g) ,$$

with, of course,

$$\text{(7.6)} \quad d > g .$$

Profit Maximizing for One-Period Programs

Given prices r_i of inputs, and p_j of outputs, the profit for an input-output pair (x, y) is defined as

$$\sum_j p_j y_j - \sum_i r_i x_i \, . \tag{7.7}$$

If a quantity x_{ij} of commodity i is allocated to the production of commodity j, then profit in the production of j is

$$p_j y_j - \sum_i r_i x_{ij} \, . \tag{7.8}$$

Since constant returns to scale prevail, a profit maximizing program will have non-zero output of j only if maximum profit in the production of j is zero; furthermore, with non-zero output giving maximum profit of zero, output will be indeterminate.

It is therefore convenient to consider first the problem of maximizing profit in the production of j, subject to the constraint that the cost $\sum_i r_i x_{ij}$ be equal to a given value c_j. Recalling that output y_j is given by

$$y_j = e^{\beta_j} \prod_i (x_{ij})^{\alpha_{ij}} \, , \qquad \alpha_{ij} \geqq 0 \, , \qquad \sum_i \alpha_{ij} = 1 \, ,$$

(where the product is taken over all i that are productive for j), it is easily verified that the solution to this problem is

$$x_{ij} = \frac{\alpha_{ij} c_j}{r_i} \, . \tag{7.9}$$

Maximum profit is

$$c_j \left[p_j e^{\beta_j} \prod_i \left(\frac{\alpha_{ij}}{r_i} \right)^{\alpha_{ij}} - 1 \right] . \tag{7.10}$$

Now consider the problem of maximizing profit in the production of commodity j, *without* any constraint on cost. For this to have a solution with (finite) positive output, the coefficient of c_j in (7.10) must be zero; this condition can be written

$$p_j e^{\beta_j} \prod_i \alpha_{ij}^{\alpha_{ij}} = \prod_i r_i^{\alpha_{ij}}$$

or

$$\log p_j + \beta_j + \sum_i \alpha_{ij} \log \alpha_{ij} = \sum_i \alpha_{ij} \log r_i \, . \tag{7.11}$$

Any inputs x_{ij} that satisfy (7.9) for some c_j will be profit-maximizing.

Finally, consider the problem of maximizing total profit in the production of all new commodities, as given by (7.7). A solution to this will be provided by the profit maximizing programs for the production of each commodity, so that for a solution with positive outputs, the inputs must satisfy (7.9) for some positive numbers c_j, and the prices must satisfy (7.11).

Notice that if we write

$$x_i = \sum_j x_{ij} \,, \qquad (7.12)$$
$$x_{ij} = f_{ij} x_i \,,$$

then the allocation coefficients f_{ij} are given by

$$f_{ij} = \frac{\alpha_{ij} c_j}{\sum_k \alpha_{ik} c_k} \,, \qquad i \text{ productive} \,. \qquad (7.13)$$

In summary, a one-period profit maximizing program with positive outputs is characterized by (7.9), (7.11), and

$$c_j = \sum_i r_i x_{ij} \,. \qquad (7.14)$$

As an alternative to (7.9), one may take (7.12) and (7.13).

Shadow Prices for Optimal One-Period Programs

Consider now the problem of maximizing the following function of outputs

$$\sum_j w_j \log y_j \,, \qquad w_j \geqq 0 \,, \quad \sum_j w_j > 0 \,, \qquad (7.15)$$

given the total inputs x_i (where the w_j are given coefficients). In terms of the allocation coefficients f_{ij}, the solution is easily verified to be

$$f_{ij} = \frac{\alpha_{ik} w_k}{\sum_k \alpha_{ik} w_k} \,, \qquad i \text{ productive} \,. \qquad (7.16)$$

It is easily verified, using (7.9) and (7.11)–(7.14), that the following are shadow prices for such an optimal one-period program:

$$r_i = \frac{\sum_k \alpha_{ik} w_k}{x_i} \,, \qquad i \text{ productive} \,, \qquad (7.17)$$

$$p_j = \frac{w_j}{y_j} \,, \qquad j \text{ produced} \,. \qquad (7.18)$$

Shadow Prices for Optimal Programs of Problem I

In calculating the shadow prices for Problem I, there is no loss of simplicity in going directly to the case in which primary resources may be present.

One sees immediately from (4.8) that for an optimal program the production in period t is the same as the production for an optimal one-period program with weights

$$w_j = \delta^t \omega_{(1)j}(T - t) \,. \qquad (7.19)$$

Hence the shadow prices may be taken to be

$$r_i(t) = \frac{\delta^t \sum_k \alpha_{ik}\omega_{(1)k}(T-t)}{z_i(t-1)}, \qquad i \text{ productive}, \tag{7.20}$$

$$p_j(t) = \frac{\delta^t \omega_{(1)j}(T-t)}{z_j(t)}, \qquad j \text{ produced}. \tag{7.21}$$

For programs with an infinite horizon this reduces to

$$r_i(t) = \frac{\delta^t \sum_k \alpha_{ik}\tilde{\omega}_{(1)k}}{z_i(t-1)}, \tag{7.22}$$

$$p_j(t) = \frac{\delta^t \tilde{\omega}_{(1)j}}{z_j(t)}. \tag{7.23}$$

From (4.9)

$$\begin{aligned} \tilde{\omega}_{(1)} &= (I - \delta A_1)^{-1}\omega_{(1)}, \\ A_1\tilde{\omega}_{(1)} &= \left(\frac{1}{\delta}\right)(\tilde{\omega}_{(1)} - \omega_{(1)}). \end{aligned} \tag{7.24}$$

Hence one finally arrives at the formulas for the shadow prices

$$r_i(t) = \frac{\delta^{t-1}[\tilde{\omega}_{(1)i} - \omega_{(1)i}]}{z_i(t-1)}, \qquad i \text{ productive}, \tag{7.25}$$

$$p_j(t) = \frac{\delta^t \tilde{\omega}_{(1)j}}{z_j(t)}, \qquad j \text{ produced}. \tag{7.26}$$

For i both productive and produced,

$$\frac{r_i(t)}{p_i(t)} = \left[1 - \frac{\omega_{(1)i}}{\tilde{\omega}_{(1)i}}\right]\left[\frac{1}{\delta}\right]\left[\frac{z_i(t)}{z_i(t-1)}\right]. \tag{7.27}$$

If one defines the own-interest rate $d_i(t)$, the growth rate $g_i(t)$, and the parameter μ_i by

$$\begin{aligned} 1 + d_i(t) &= \frac{r_i(t)}{p_i(t)}, \\ 1 + g_i(t) &= \frac{z_i(t)}{z_i(t-1)}, \\ \mu_i &= \frac{\omega_{(1)i}}{\tilde{\omega}_{(1)i}}, \end{aligned} \tag{7.28}$$

then the own-rate of interest of commodity i is given in terms of i's growth rate by

$$d_i(t) = g_i(t) + (1 - \mu_i - \delta)\left[\frac{1 + g_i(t)}{\delta}\right]. \tag{7.29}$$

The quantity $(1 - \mu_i - \delta)$ may be positive or negative, so that a commodity's own-rate of interest may be greater or less than its growth rate, depending upon the numerical values of the various coefficients in the production and welfare functions.

However, if $\omega_i = 0$ (recall that i is both produced and productive), then $\mu_i = 0$, and we get the conclusions

$$(7.30) \qquad \begin{aligned} 1 + d_i(t) &= \left(\frac{1}{\delta}\right)[1 + g_i(t)] \\ d_i(t) &> g_i(t) \\ p_i(t) &= r_i(t+1) . \end{aligned}$$

In particular, note that in this case (a) there is in effect only a single sequence of shadow prices, with, say, $p(t)$ used to value both the outputs in period t and the inputs in period $t + 1$; and (b) the own-rate of interest for any commodity that is produced and productive is greater than its growth rate.

Continuing the case of $\mu_i = 0$, if growth is proportional (as it will tend to be for large t, under the condition of full regularity), then $g_i(t)$ is a constant, g, independent of i and t, and one has

$$(7.31) \qquad p_j(t) = \left(\frac{\delta}{1+g}\right)^t \frac{\tilde{\omega}_{(1)j}}{z_j(0)} .$$

A common interest rate d is given by

$$(7.32) \qquad (1 + d) = \left(\frac{1}{\delta}\right)(1 + g) .$$

Discount Factor δ Close to Unity

As the discount factor δ approaches 1, one may get different results according to whether primary resources are, or are not, present. If primary resources *are* present (and productive), then $\tilde{\omega}_{(1)}$ remains bounded as δ approaches 1, and (7.27) simply reduces to

$$(7.33) \qquad 1 + d_i(t) = (1 - \mu_i)[1 + g_i(t)] .$$

On the other hand, if primary resources are not present, then

$$\lim_{\delta \to 1} \frac{\omega_i}{\tilde{\omega}_i} = \lim_{\delta \to 1} \mu_i = 0 ,$$

so that in the limit one has

$$(7.34) \qquad d_i(t) = g_i(t)$$

and

$$(7.35) \qquad \lim_{t \to \infty} d_i(t) = \bar{g} ,$$

where $\bar{g}$ is the maximum growth rate that is technologically feasible. In other words, in the limiting case of $\delta = 1$, the own-rates of interest equal the growth rates all along the path, and tend, as time increases, to the maximum growth rate $\bar{g}$.

Notice, too, that in the case of no primary resources, for fixed j and t,

$$\lim_{\delta \to 1} (1 - \delta) p_j(t) = \frac{\bar{a}_j}{z_j(t)}, \tag{7.36}$$

provided A is fully regular (see (3.13) and (3.19)).

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