EFFICIENCY PRICES FOR INFINITE HORIZON PRODUCTION PROGRAMS

By

Roy Radner

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CENTER FOR RESEARCH IN MANAGEMENT SCIENCE
University of California
Berkeley, California
94720

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1. INTRODUCTION

It is a well-known proposition of economic analysis that, under " classical " assumptions of non-increasing returns to scale, non-increasing marginal productivity, continuity, etc., an efficient production programme also maximizes the value of net output if value is calculated using suitable prices. In a dynamic context, in which commodities are distinguished according to the date at which they are used or made available, the value of the production plan is " present value ", and the price system includes discounted future, as well as present, prices.

The extension of this theory to the case of an infinite planning horizon poses certain mathematical difficulties, which in turn raise conceptual problems concerning the proper definition of " price " and " present value ". In the case of a finite horizon (with discrete time and a finite number of commodities at each date), the net outputs resulting from a particular plan can be listed as the coordinates of a vector in a finite dimensional vector space. Thus, if y_{ti} is the net output of commodity i at date t, and if there are T dates and M commodities at each date, the list of net outputs y_{ti} can be thought of as a vector y in TM dimensional space, R^{TM}. If p_{ti} is the (discounted) price of commodity i at date t, then the price system can also be represented as a vector, p, in R^{TM}, and the present value of the net output programme y is

$$p \cdot y \equiv \sum_{i,t} p_{ti} y_{ti},$$

the " scalar product " of p and y. Writing y_t for the vector of net outputs at date t, p_t for the vector of (discounted) prices at date t, and $p_t y_t$ for the scalar product $\sum_i p_{ti} y_{ti}$, the above expression for present value can be written as

$$p \cdot y = \sum_t p_t y_t. \qquad \ldots(1.1)$$

Expression (1.1) has several mathematical properties that are important to its usefulness as a tool of economic theory: (a) it is in the form of a scalar product of the vector p of prices and the vector y of net outputs; (b) it is well defined for every p and y in R^{TM}; (c) for fixed p it is a homogeneous linear function of y, and vice versa; (d) it is a continuous function of p and y. These properties are so taken for granted that the reader may wonder why I have bothered to mention them explicitly. Unfortunately, in the case of an *infinite horizon T*, these properties may not all be mutually compatible. On the one hand, the scalar product (1.1) is not well defined for all sequences (p_t) and (y_t), because the series may fail to converge. On the other hand, depending on the richness of the set of available net output programmes, there may be continuous linear functions of y that cannot be expressed as a scalar product. For example, on the space of all convergent sequences

[1] This research was supported in part by the Office of Naval Research under Contract ONR 222(77) with the University of California and in part by a grant from the National Science Foundation. Reproduction in whole or in part is permitted for any purpose of the United States Government. The essential results of this paper were presented to the Summer Conference on Optimal Economic Growth at the Centre for Advanced Study in the Behavioural Sciences, Stanford, California, during July 1965, and I would like to express my appreciation to the conference participants, and especially to Professor Daniel McFadden, for their comments. I have also had the benefit of discussions on this topic with Professor Gerard Debreu.

(x_t), the limit, $\lim_{t \to \infty} x_t$, is a continuous linear function (provided one chooses a suitable definition of distance in the space).

An important consequence of the present value maximizing property of efficient programmes in the finite horizon case is their property of "intertemporal profit maximization". Suppose that at each date t, for each gross input vector a there is a set, $P_t(a)$, of feasible gross output vectors b, the output being available at date $(t+1)$ either for consumption or as input into further production. For given sequences $a = (a_1, ..., a_{T-1})$ of inputs and $b = (b_2, ..., b_T)$ of outputs, the net output sequence $y = (y_1, ..., y_T)$ is

$$\begin{aligned} y_1 &= a_1, \\ y_t &= b_t - a_t, \quad 1 < t < T, \\ y_T &= b_T. \end{aligned} \qquad \text{...(1.2)}$$

To (1.2) must be added the condition of technological feasibility already mentioned,

$$b_{t+1} \text{ in } P_t(a_t), \quad 1 \leqq t \leqq T-1. \qquad \text{...(1.3)}$$

Conditions (1.2) and (1.3) define the set Y of net output programmes.

For a given sequence $p = (p_1, ..., p_T)$ of price vectors, the present value of y is equal to

$$p \cdot y = \sum_{t=1}^{T-1} (p_{t+1} b_{t+1} - p_t a_t). \qquad \text{...(1.4)}$$

Notice that there is a one-to-one correspondence between input-output pairs (a_t, b_{t+1}), constraints in (1.3), and terms in the sum (1.4). Hence $\hat{y}$ maximizes $p.y$ on the set Y if and only if, for each t, the corresponding $(\hat{a}_t, \hat{b}_{t+1})$ maximizes $(p_{t+1}b_{t+1} - p_t a_t)$ on the set G_t of pairs (a, b) for which b is in $P_t(a)$, i.e.

$$p_{t+1}\hat{b}_{t+1} - p_t \hat{a}_t \geqq p_{t+1} b - p_t a, \quad \text{for all } (a, b) \text{ in } G_t. \qquad \text{...(1.5)}$$

Thus for given prices, the problem of maximizing present value over the whole set of net output programmes can be reduced to a sequence of $T-1$ problems of maximizing "profit", $p_{t+1}b - p_t a$, on the set G_t of input-output pairs that are feasible at date t. I shall call (1.5) the condition of *intertemporal profit maximization.*

The condition of present value maximization requires that the present value be defined for the entire programme, and in the infinite horizon case leads one to adopt the linear function concept as a formalization of "price system". This was the approach taken by Debreu [2].[1] On the other hand, the condition of intertemporal profit maximization requires a sequence of price vectors; this was the approach of Malinvaud [6], [7]. These two approaches are not *a priori* consistent in the infinite horizon case, as was noted above, since there may be linear functions that do not have any associated sequence of price vectors, and there may be sequences of price vectors that do not define a linear function.

A simple example illustrates this phenomenon. Suppose that there is only one commodity at each date, and that for any input $a > 0$, the output is $a^{\frac{1}{2}}$. Suppose further that $a_1 = \frac{1}{4}$. One can show that for any number s between 0 and $\frac{1}{2}$, the policy

$$a_t = s b_t \qquad \text{...(1.6)}$$

yields an efficient net output programme (see, e.g. Radner [9]). The parameter s may be thought of as the savings ratio. It is easily verified that

$$\log b_t = \left(\sum_{k=1}^{t-1} \frac{1}{2^k} \right) \log s + \frac{1}{2^{t-1}} \log a_1,$$

$$y_t = (1-s) b_t.$$

[1] See also Hurwicz [4] on programming in linear spaces.

In particular,

$$\lim_{t\to\infty} b_t = s,$$
$$\lim_{t\to\infty} a_t = s^2, \qquad \text{...(1.7)}$$
$$\lim_{t\to\infty} y_t = (1-s)s.$$

It is also easily verified that for prices p_t, p_{t+1}, the intertemporal profit maximizing input is

$$a_t = \left(\frac{p_{t+1}}{2p_t}\right)^2. \qquad \text{...(1.8)}$$

If for example, $s = \frac{1}{2}$, then for every $t \geqq 2$

$$a_t = y_t = \tfrac{1}{4},$$
$$b_t = \tfrac{1}{2},$$

and the " efficiency prices " must be all equal, e.g. $p_t = 1$. More generally, from (1.8) it follows that the efficiency prices corresponding to a given s must satisfy

$$\lim_{t\to\infty}\left(\frac{p_{t+1}}{p_t}\right) = 2s. \qquad \text{...(1.9)}$$

Since, for all the admissible values of s $(0 \leqq s \leqq 1)$, y_t converges, it is clear from (1.9) that $\Sigma p_t y_t$ will converge if $0 \leqq s < \frac{1}{2}$, and diverge if $s = \frac{1}{2}$.

Hence for the case $s = \frac{1}{2}$ the prices that conserve the principle of intertemporal profit maximization do not define a present value for the entire sequence of net outputs. Nevertheless, one can demonstrate the existence of a continuous linear function $p(y)$ on the set Y of feasible net output sequences such that the sequence $(-\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, ...,$ etc.) has maximum present value in Y. It is difficult to describe in a simple way this present value function, but it has the property that if the long-run-average of the y_t exists, i.e.

$$\lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} y_t$$

exists, then $p(y)$ is equal to this long-run-average. In particular, if (y_t) converges, then $p(y)$ is equal to the limit. Roughly speaking, then, we can say that the programme generated by $s = \frac{1}{2}$ maximizes (generalized) long-run-average net output, and that this (generalized) long-run-average should be considered as the appropriate definition of present value in this case. But the long-run-average of a sequence is not changed by changing a finite number of terms of the sequence; in particular, if all but a finite number of terms of a sequence are zero, then the long-run-average of the sequence is zero. Hence the long-run-average is useless for applying the criterion of intertemporal profit maximization.

The example is useful in illustrating one more aspect of the general problem. First, one notices that in the example the case $s = \frac{1}{2}$ is a boundary case, in the sense that the sequence of net outputs generated by $s = \frac{1}{2}$ can be approximated by taking s close to $\frac{1}{2}$, and for any $s < \frac{1}{2}$, no matter how close, the corresponding efficiency price sequence does define a present value for any bounded sequence of net outputs. Furthermore, from (1.9) one sees that for large t the prices p_t behave asymptotically like $(2s)^t p_0$. Now it is known that

$$\lim_{c\to 1}(1-c)\sum_{t=1}^{\infty} c^t y_t = \lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T} y_t,$$

provided these limits exist. Hence, by *suitable normalization*, as a function of s, the present value determined by the price sequence (p_t) for $s < \frac{1}{2}$ approaches the present value for the case $s = \frac{1}{2}$. This is so even though for each fixed t the normalized price p_t approaches 0 as s approaches $\frac{1}{2}$. That these approximation phenomena are fairly general is shown by Theorem 4.3.

Indeed Theorem 4.3 is the central result of this paper, since it shows that the cases in which the price sequence approach gives different results from the linear function approach are " boundary cases ", in the sense that they can be approximated by the " typical " case in which both approaches give the same result.

Section 2 sets up the formal context of the problem, and Section 3 explores the continuous linear function approach, characterizing efficient programmes in terms of maximization of present value. Theorem 3.3 shows that efficient programmes can be approximated by efficient programmes for which the corresponding efficiency prices are positive.

Section 4 relates the linear function approach to the price sequence approach of Malinvaud, and shows in particular that the sequence of prices derived by Malinvaud can also be derived as limits of price ratios, as one approximates the given efficient programme by programmes whose efficiency prices are all positive.

The application of the approximation theorem of Section 3 is extended if there are certain bounds on production; these bounds are derived in Section 5. Essentially, it is shown that, under " classical " assumptions about production, for any fixed sequence of exogenous supplies of resources one can choose the units of measurement of commodities at each date so that all feasible programmes converge to zero uniformly at any desired rate.

Finally, I should mention another connection between the linear function and price sequence approaches. Even though the Malinvaud prices do not in general determine a linear function on the set of feasible programmes, they may determine the present value of *changes*, Δy_t, from the given efficient programme, provided such changes converge to zero sufficiently rapidly. This idea has been explored by McFadden [8].[1]

2. PROGRAMMES, PRICES AND PRESENT VALUE

Suppose that there is a fixed list of commodities (goods and services), say M in number, each of which is capable of quantitative measurement. At any given date, inputs, outputs, resources, etc. will be represented by vectors in R^M, M-dimensional Euclidean space. Take time as discrete, and assume that at each date $t(= 1, 2, \text{etc.} \ldots)$, for every non-negative *input* vector, a, in R^M there is a set, $P_t(a)$, of possible non-negative *output* vectors. Such an output vector is available at date $(t+1)$, for consumption, trade, and/or input into further production. The correspondence P_t will be called the *production correspondence at date t*.

A production programme is a sequence, $\{(a_t, b_{t+1}): t = 1, 2, \text{etc.} \ldots\}$, such that for each t, a_t and b_{t+1} are non-negative vectors in R^M, and b_{t+1} is in $P_t(a_t)$. For any production programme $\{(a_t, b_{t+1})\}$, the *net output programme*, $y = (y_1, y_2, \text{etc.} \ldots)$, is defined by

$$y_1 = -a_1$$
$$y_t = b_t - a_t, \quad t \geqq 2. \qquad \ldots(2.1)$$

The set of net output programmes y corresponding to the set of all possible production programmes will be denoted by Y. I shall sometimes refer to the *programme* (y, a, b), it being understood that $a = (a_t)$, $b = (b_t)$, $y = (y_t)$, (a, b) is a production programme, and y is the corresponding net output programme.

Let E denote the set of all infinite sequences of vectors in R^M. The set E may be considered as a linear space with the natural definition of addition and scalar multiplication of sequences: for $x = (x_t)$ and $z = (z_t)$ in E, and c real,

$$x+z \equiv (x_t+z_t),$$
$$cx \equiv (cx_t).$$

[1] The results of the present paper were obtained before I had an opportunity to study McFadden's results, and the relation between the two papers is only incompletely sketched at the end of Section 4.

In order to construct price systems with the desired economic interpretations it is necessary to have some notion of continuity of present value as a function of the programme (at least to be able to apply the mathematical techniques presently available); hence E must also be considered as a topological space. For reasons that will become clearer later, I have chosen to topologize E, or rather part of E, as follows. Let E_B denote the set of all *bounded* sequences of vectors in R^M. For a vector $x = (\xi_m)$ in R^M define the norm

$$\| x \| = \max_m | \xi_m |, \qquad ...(2.4)$$

and for a sequence $x = (x_t)$ in E_B define the norm

$$\| x \| = \sup_t \| x_t \|. \qquad ...(2.5)$$

The space E_B with the linear operations (2.3) and the norm (2.5) is a Banach space (see Dunford and Schwartz[1] [3], Chapter IV, Sections 2 and 5). Note that E_B is isomorphic to the space of bounded sequences of real numbers, commonly called l_∞; the division into blocks of cooordinates of length M is retained here to facilitate the economic interpretation.

For any vector $x = (\xi_m)$ in R^M define a second norm

$$\| x \|_1 = \sum_{m=1}^{M} | \xi_m |.$$

One can show that a real-valued function p on E_B is linear and continuous if and only if it can be represented in the form

$$p(x) = \sum_{t=1}^{\infty} p_t x_t + p_\infty(x), \qquad ...(2.6)$$

where

(*a*) $(p_t: t < \infty)$ is a sequence of vectors in R^M such that $\sum_{t<\infty} \| p_t \|_1 < \infty$,

(*b*) p_∞ is an integral with respect to a bounded (finitely) additive measure on the integers such that $p_\infty(x) = 0$ for every $x = (x_t)$ for which $x_t = 0$ except for a finite number of dates t.

The norm of p is defined as

$$\| p \|_* = \sup \{ | p(x) | : x \text{ in } E_B . \| x \| = 1 \}.$$

The set of all continuous linear functions on E_B will be denoted by E_B^* (see DS, Ch. IV, Sec. 5).

The infinite series in (2.6) corresponds to our usual "scalar product" concept of present value; it converges absolutely, since

$$\sum_{t<\infty} | p_t x_t | \leq [\sup_t \| x_t \|] \sum_{t<\infty} \| p_t \|_1 = \| x \| \sum_{t<\infty} \| p_t \|_1. \qquad ...(2.7)$$

The term $p_\infty(x)$ depends only on the asymptotic behaviour of the sequence (x_t). Indeed, it is easy to show, from (*b*) above, that if $\lim_{t\to\infty} x_t = 0$, then $p_\infty(x) = 0$. I shall call $\sum_t p_t x_t$ the *series part* of p, and p_∞ its *asymptotic part*.

The presence of the asymptotic part of p is perhaps an unexpected, and annoying, aspect of the linear function approach to prices in the infinite horizon case. However, the asymptotic part is not without intuitive meaning, as already mentioned in Section 1. For example, consider those sequences x in E_B for which the "long-run-average",

$$A(x) \equiv \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} x_t$$

[1] Since I shall refer frequently to this treatise, I shall use the abbreviation "DS".

exists. It can be shown that $A(x)$ can be extended to the whole of E_B in such a way that it satisfies the properties of a p_∞ as in (*b*) above (see DS [3], p. 73, Exercise 22 on " Banach limits ").

3. EFFICIENT PROGRAMMES AND EFFICIENCY PRICES

In this section I consider some generalizations to the infinite horizon case of propositions about efficient programmes, and corresponding efficiency prices, that do not depend directly on the period-by-period description of the production correspondences. Thus these generalizations would apply, in principle, to any case of a denumerable infinity of commodities.

Since I shall be dealing with efficiency, I shall first recall the definition of the usual partial ordering of vectors. For two vectors (ξ_m) and (ξ'_m) in R^M define

$$(\xi_m) \geqq (\xi'_m) \text{ means } \xi_m \geqq \xi'_m \text{ for } m = 1, \ldots, M,$$

$$(\xi_m) > (\xi'_m) \text{ means } (\xi_m) \geqq (\xi'_m) \text{ but } (\xi_m) \neq (\xi'_m),$$

$$(\xi_m) \gg (\xi'_m) \text{ means } \xi_m > \xi'_m \text{ for } m = 1, \ldots, M.$$

Similar definitions apply to the ordering of sequences (points in E).

Let C be a set in E, and x a point in C; x is said to be *maximal in C* if there is no z in C such that $z > x$. If C is a set of net output programmes, then a maximal programme in C is usually also said to be *efficient* in C; the term " efficient " may also be applied to the corresponding production programme.

For the remainder of this section I shall consider only bounded net output programmes. The first proposition asserts that if a programme is maximal in a convex set C of net output programmes, then, for a suitable non-zero, non-negative price system, it has maximum present value in C. (The restriction to bounded programmes will be justified in Sections 4 and 5).

For any two linear functions, p and q, on E_B, define:

$$p \geqq q \text{ means } p(x) \geqq q(x) \text{ for all } x \geqq 0 \text{ in } E_B,$$

$$p \gg q \text{ means } p(x) > q(x) \text{ for all } x > 0 \text{ in } E_B.$$

Theorem 3.1. *If C is a convex subset of E_B, and $\hat{y}$ is maximal in C, then there is a non-zero, non-negative continuous linear function p on E_B such that*

$$p(y) \leqq p(\hat{y})$$

for all y in C.

Proof. Let D denote the set of all points $z \geqq \hat{y}$ in E_B. D is convex and has a non-empty interior. (To see the latter, let z be the point in E_B obtained by increasing every coordinate of every $\hat{y}_t$ by 1; then the open " sphere ", $\{z\colon \| z - \hat{z} \| < \frac{1}{2}\}$ is entirely contained in D.) No point in the interior of D is in C, otherwise $\hat{y}$ would not be maximal in C. Therefore, by a separation theorem for convex sets (see, e.g. DS, Theorem V.2.8), there is a non-zero continuous linear function p on E_B such that $p(y) \leqq p(z)$ for all y in C and z in D. Since $\hat{y}$ is both in C and in D it follows in particular that

$$p(y) \leqq p(\hat{y}) \text{ for all } y \text{ in } C, \qquad \ldots(3.1)$$

$$p(z) \geqq p(\hat{y}) \text{ for all } z \text{ in } D. \qquad \ldots(3.2)$$

Every point in D can be written as $(\hat{y} + x)$, with $x \geqq 0$; hence (3.2) implies that $p(x) \geqq 0$ for every $x \geqq 0$, i.e. $p \geqq 0$, which completes the proof.

I shall say that a net output programme $\hat{y}$ in a set C is *value maximizing* in C with respect to p if $p(y)$ attains a maximum in C at $\hat{y}$. Theorem 3.1 says that every maximal net output programme in a convex set C (in E_B) is value maximizing with respect to some non-zero, non-negative continuous linear function p. Even in the finite-dimensional case the converse is not true; however, the following partial converse generalizes the corresponding finite-dimensional result.

Theorem 3.2. *If a net output programme is value maximizing in a set C with respect to a strictly positive ($\gg 0$) linear function, then it is maximal in C. (The proof is left to the reader.)*

It is of interest to note that a non-negative continuous linear function p on E_B is strictly positive if and only if its series part is strictly positive, i.e. in the representation (2.6), $p_t \gg 0$ for every $t<\infty$. However, in Theorem 3.2 it is not required that p be continuous, nor that C be convex.

Although a net output programme that is value maximizing with respect to a non-negative linear function need not be maximal, the following theorem shows that every maximal net output programme, $\hat{y}$, can be approximated by maximal programmes, y^m, that are value maximizing with respect to strictly positive continuous linear functions, say p^m corresponding to y^m, provided the set C is convex and compact; and further that the functions p^m can be taken to converge to a non-zero, non-negative linear function, $\hat{p}$, such that $\hat{y}$ is value maximizing with respect to $\hat{p}$.

Theorem 3.3. *If $\hat{y}$ is maximal in a compact convex subset C of E_B, then there is a $\hat{p}$ in E_B^* such that*

(a) $\hat{p} \geqq 0$, $\| \hat{p} \|_* = 1$;

(b) $\hat{y}$ maximizes $\hat{p}(y)$ on C;

(c) for every number $\varepsilon>0$, and every finite subset F of E_B, there is a y' in C and a $p' \gg 0$ in E_B^* such that $\| y'-\hat{y} \| <\varepsilon$, $| p'(x)-\hat{p}(x)| <\varepsilon$ for every x in F, $\| p' \|_* = 1$, and y' maximizes $p'(y)$ on C (and therefore y' is maximal in C).

For a proof of this theorem,[1] the reader is referred to Radner [10].

I shall show later (Section 5) that the condition of compactness on C is not really a restriction under " neoclassical " assumptions about production. It should be borne in mind, however, that compact sets in E_B are very " thin "; for example, the interior of a compact set in E_B is empty. The following class of compact sets will be useful. For any sequence $\lambda = (\lambda_t)$ of numbers, let $E(\lambda)$ denote the set of points $x = (x_t)$ in E for which $\| x_t \| \leqq \lambda_t$ for every t.

Lemma. *If $\lambda = (\lambda_t)$ is a sequence of numbers converging to zero, then $E(\lambda)$ is compact in E_B. (The lemma follows immediately from* DS [3], Theorem IV.5.6.)

[Technical note on Theorem 3.3: Let E_B^* have the weak* topology, and $E_B \times E_B^*$ the product topology. The conclusion of Theorem 3.3 can be rephrased as follows: There is a $\hat{p}$ in E_B^* such that (a) and (b) hold, and such that (c') there is a net (generalized sequence) of points in $C \times E_B^*$, $\{(y^d, p^d)\colon d \in D\}$, converging to $(\hat{y}, \hat{p})$, such that, for each d in D, $p^d \gg 0$, $\| p^d \|_* = 1$, and y^d maximizes $p^d(y)$ on C.

Let S denote the set of all p in E_B^* for which $\| p \|_* = 1$. One can show that S is compact (in the weak* topology), and that the bilinear function $f(x, p) \equiv p(x)$ is continuous on $E_B \times S$. Hence $\lim_{d \in D} p^d(y^d) = \hat{p}(\hat{y})$.]

4. INTERTEMPORAL PROFIT MAXIMIZATION AND TERMINAL COST MINIMIZATION

In this section I relate the price systems provided by the linear function approach to sequences of price vectors of the type derived by Malinvaud [6], [7].

To begin with, I assume only that the set of production possibilities is convex, and

[1] This proposition generalizes a result of Arrow, Barankin, and Blackwell [1] for finite-dimensional spaces.

that zero production is possible. For each production correspondence P_t, define its *graph*, G_t, as the set of all input-output pairs (a, b) such that b is in $P_t(a)$.

Assumption 1. For each t, G_t is convex, and contains (0, 0).

In order to apply the results of the previous section, one wants the maximal programme $(\hat{y}, \hat{a}, \hat{b})$ to be such that the sequences $(\hat{y}_t)$, $(\hat{a}_t)$ and $(\hat{b}_t)$ are all bounded, or even converging to zero as fast as a given sequence of numbers. This can be accomplished by a suitable change of units at each date. Let $\lambda = (\lambda_t)$ be any given sequence of positive numbers. Define for each t

$$k_t \equiv \max(\| \hat{y}_t \|, \| \hat{a}_t \|, \| \hat{b}_t \|, 1),$$
$$c_t \equiv \lambda_t / k_t. \quad \ldots(4.1)$$

Consider the linear transformation of E into itself by

$$x_t \to c_t x_t. \quad \ldots(4.2)$$

This transformation carries the correspondence P_t into the correspondence P'_t defined by:

$$b \text{ in } P'_t(a) \text{ if and only if } \left(\frac{b}{c_{t+1}}\right) \text{ in } P_t\left(\frac{a}{c_t}\right).$$

It is easily verified that the new production correspondences, P'_t, also satisfy Assumption 1, and that the image $(\hat{y}', \hat{a}', \hat{b}')$ of $(\hat{y}, \hat{a}, \hat{b})$ satisfies

$$\| \hat{y}'_t \| \leqq \lambda_t, \ \| \hat{a}'_t \| \leqq \lambda_t, \ \| \hat{b}'_t \| \leqq \lambda_t, \text{ for every } t. \quad \ldots(4.3)$$

Recalling that $E(\lambda)$ is the set of all points $x = (x_t)$ in E for which $\| x_t \| \leqq \lambda_t$ for every t, one may abbreviate (4.3) by saying that $(\hat{y}', \hat{a}', \hat{b}')$ is in $E(\lambda)^3$.

Consider now a given programme $(\hat{y}, \hat{a}, \hat{b})$. In view of the foregoing remarks, there is no loss of generality in assuming that $(\hat{y}, \hat{a}, \hat{b})$ is in E_B^3, i.e. that $(\hat{y}_t)$, $(\hat{a}_t)$ and $(\hat{b}_t)$ are bounded. With the conclusion of Theorem 3.1 in mind, suppose further that p is a non-zero, non-negative continuous linear function on E_B such that $\hat{y}$ is value-maximizing with respect to p on the set $Y \cap E_B$ (recall that Y is the set of net output programmes), i.e.

$$p(y) \leqq p(\hat{y}) \text{ for all } y \text{ in } Y \cap E_B. \quad \ldots(4.4)$$

I shall show that the net output programme $\hat{y}$ satisfies the condition of intertemporal profit maximization with respect to the sequence of price vectors in the series part of p, and also satisfies a certain cost minimizing condition.

As in (2.6), let p be represented in the form

$$p(x) = \sum_{t<\infty} p_t x_t + p_\infty(x), \quad \sum_{t<\infty} \| p_t \|_1 < \infty. \quad \ldots(4.5)$$

A programme $(\hat{y}, \hat{a}, \hat{b})$ will be called *intertemporally profit maximizing* with respect to a sequence (p_t) of vectors in R^M if, for every $t \geqq 1$,

$$p_{t+1} b_{t+1} - p_t a_t \leqq p_{t+1} \hat{b}_{t+1} - p_t \hat{a}_t, \text{ for every } (a, b) \text{ in } G_t. \quad \ldots(4.6)$$

For any $T \geqq 1$, let A_T denote the set of non-negative vectors a in R^M for which there exists a y in Y such that

$$y_t = \begin{cases} 0, & t < T, \\ -a, & t = T, \\ \hat{y}_t, & t > T. \end{cases} \quad \ldots(4.7)$$

The set A_T is the set of possible input vectors at date T that would permit the net outputs $\hat{y}_t$ at dates subsequent to T. Note that at least $\hat{a}_T$ is in A_T. The programme $(\hat{y}, \hat{a}, \hat{b})$ will be called *terminally cost minimizing* with respect to a sequence (p_t) if, for every $t \geqq 1$,

$$p_t \hat{a}_t \leqq p_t a, \text{ for every } a \text{ in } A_t. \quad \ldots(4.8)$$

Theorem 4.1. *If $(\hat{y}, \hat{a}, \hat{b})$ is a programme in E_B^3, p is a non-zero, non-negative continuous linear function on E_B, $\hat{y}$ is value maximizing with respect to p in $Y \cap E_B$, and (p_t) is the series part of p, then $(\hat{y}, \hat{a}, \hat{b})$ is both intertemporally profit maximizing and terminally cost minimizing with respect to (p_t).*

Proof. Suppose that (4.6) is contradicted for some t and some (a, b) in G_t. Define the net output programme y' by

$$y'_s = \begin{cases} \hat{y}_s, & \text{for } s \neq t, t+1, \\ \hat{b}_t - a, & \text{for } s = t, \\ b - \hat{a}_{t+1}, & s = t+1; \end{cases} \qquad \ldots(4.9)$$

then y' is in $Y \cap E_B$, and

$$p(y') - p(\hat{y}) = (p_{t+1}b - p_t a) - (p_{t+1}\hat{b}_{t+1} - p_t \hat{a}_t) > 0,$$

which contradicts (4.4).

Similarly, suppose that (4.8) is contradicted for some t and some a in A_t. Define y'' by

$$y''_s = \begin{cases} \hat{y}_s, & \text{for } s \neq t, \\ \hat{b}_t - a, & \text{for } s = t; \end{cases} \qquad \ldots(4.10)$$

then y'' is in $Y \cap E_B$ and

$$p(y'') - p(\hat{y}) = -p_t a + p_t \hat{a}_t > 0,$$

which also contradicts (4.4), completing the proof of the theorem.

Corresponding to Theorem 4.1, there is a proposition that applies to the case in which $(\hat{y}_t)$, $(\hat{a}_t)$ and $(\hat{b}_t)$ converge to zero. The importance of this case is suggested by the lemma at the end of Section 3 (on a class of compact sets in E_B).

Theorem 4.2. *If*

(i) *$\lambda = (\lambda_t)$ is a given sequence of positive numbers,*
(ii) *$(\hat{y}, \hat{a}, \hat{b})$ is a programme in $E(\lambda)^3$,*
(iii) *p is a non-zero, non-negative continuous linear function on E_B such that $p(y) \leqq p(\hat{y})$ for all programmes (y, a, b) in $E(\lambda)^3$,*

then $(\hat{y}, \hat{a}, \hat{b})$ is both intertemporally profit maximizing and terminally cost minimizing with respect to the series part, (p_t), of p, at all t where $\| \hat{y}_t \|, \| \hat{a}_t \|, \| \hat{b}_t \| < \lambda_t$ and $\| \hat{y}_{t+1} \| < \lambda_{t+1}$.

Proof. The proof is a variation on that of Theorem 4.1. Suppose that for some t and some (a, b) in G_t, condition (4.6) is contradicted, and define the programme (y', a', b') by replacing $(\hat{a}_t, \hat{b}_{t+1})$ by (a, b) as in (4.9). For any $\varepsilon > 0$

$$p(\varepsilon y' + [1-\varepsilon]\hat{y}) - p(\hat{y}) = \varepsilon[(p_{t+1}b - p_t a) - (p_{t+1}\hat{b}_{t+1} - p_t \hat{a}_t)] > 0;$$

but for ε sufficiently small, $\varepsilon(y', a', b') + (1-\varepsilon)(\hat{y}, \hat{a}, \hat{b})$ is a programme in $E(\lambda)^3$, which contradicts hypothesis (iii) of the theorem. Condition (4.8) is established in a similar manner, completing the proof.

Theorems 3.1 and 4.1 enable one to associate with every maximal programme a sequence of non-negative price vectors with respect to which the given programme is intertemporally profit maximizing, and terminally cost minimizing. However, if the series part of the corresponding linear function is zero, then the conclusion is of no interest. Nevertheless, with two additional assumptions, a suitable non-zero sequence can be found; roughly speaking, this is done by an appropriate normalization of the approximating prices of Theorem 3.3.

The first additional assumption is a standard " continuity " condition on production.

Assumption 2. For every t, G_t is closed.

To express the second additional assumption I need to distinguish between producible and non-producible commodities. A commodity is *non-producible* at date t if the corresponding coordinate of the output vector b is zero for every (a, b) in G_t; otherwise it is producible. Suppose that the (possibly empty) set of non-producible commodities is the same at every date; inputs a and outputs b can then be represented in the form

$$a = \begin{pmatrix} c \\ e \end{pmatrix}, \quad b = \begin{pmatrix} f \\ 0 \end{pmatrix}, \qquad \ldots(4.11)$$

where c and e are the vectors of inputs of producible and non-producible commodities, respectively, and f is the vector of outputs of producible commodities. A programme (y, a, b) will be called *tight at date t* if there is no (a', b') in G_t such that

$$\begin{aligned} e'_t &\ll e_t, \\ f'_{t+1} &\gg f_{t+1}, \end{aligned} \qquad \ldots(4.12)$$

where the first line of (4.12) is understood to apply only if there are non-producible commodities. The second additional assumption, incorporated into the hypothesis of the following theorem, is that the programme not be tight at any date. (This condition is due to Malinvaud [7].)

The following theorem was proved by Malinvaud [7], using a different proof. The contribution of the present proof is to relate the Malinvaud prices to the linear function approach. Roughly speaking, one can summarize the proof as follows. Let $\hat{y}$ be an efficient programme, and let $\{y^m\}$ be the " generalized sequence " of efficient programmes converging to $\hat{y}$ whose existence is asserted by Theorem 3.3; to each y^m corresponds a strictly positive linear function, p^m, such that y^m is value maximizing with respect to p^m. Let (p^m_t) denote the series part of p^m. Using the hypothesis of non-tightness one can demonstrate the existence of a sequence, (H_t), of positive numbers such that for each t one has eventually (in m),

$$\frac{\| p^m_t \|_1}{\| p^m_1 \|_1} \leqq H_t.$$

Hence for each t there is a " generalized subsequence " (p^n_t) such that $p^n_t/\| p^n_1 \|_1$ converges. By the usual method of considering successive subsequences (extended to generalized sequences) one constructs a sequence of limits, p_t, which are the Malinvaud prices.

Theorem 4.3. (Malinvaud). *If*

(i) *the production correspondences, P_t, satisfy Assumptions* 1 *and* 2,
(ii) $\lambda = (\lambda_t)$ *is a given sequence of positive numbers converging to zero,*
(iii) $(\hat{y}, \hat{a}, \hat{b})$ *is a programme in* $E(\lambda)^3$ *and is efficient in* $Y \cap E^3_B$,
(iv) $(\hat{y}, \hat{a}, \hat{b})$ *is not tight at any date,*[1]

then *there exists a sequence* $(\bar{p}_t)$ *of non-negative vectors in* R^M, *not all zero, such that* $(\hat{y}, \hat{a}, \hat{b})$ *is intertemporally profit maximizing with respect to* $(\bar{p}_t)$.

Proof. Let C be the set of all net output programmes y such that for some a and b, (y, a, b) is a programme in $E(2\lambda)^3$. In order to apply Theorem 3.3, I shall first show that C is compact. By the lemma at the end of Section 3 it suffices to show that C is closed. To this end, let $\{y^d\colon d \in D\}$ be a net[2] in C converging to a point $\bar{y}$ (in E_B), and let a^d and b^d be the inputs and outputs corresponding to y^d. Since $E(2\lambda)^3$ is compact, there is a subnet $\{y^d\colon d \in D'\}$ of the original net such that

$$\bar{a} \equiv \lim_{d \in D'} a^d, \quad \bar{b} \equiv \lim_{d \in D'} b^d$$

[1] Notice that if there are constant returns to scale, then in the case in which all commodities are producible, no programme can be tight at any date.
[2] For a discussion of nets and subnets, generalisations of sequences and subsequences, see e.g. Kelly [5].

exist; clearly $(\bar{y}, \bar{a}, \bar{b})$ is in $E(2\lambda)^3$. By Assumption 2, $(\bar{a}_t, \bar{b}_{t+1})$ is in G_t, for every t. Furthermore

$$\bar{y}_t = \lim_{d \in D'} y_t^d = \bar{b}_t - \bar{a}_t,$$

so that $\bar{y}$ is in C, thus showing that C is closed. Note $\hat{y}$ is efficient in C.

One can now apply Theorem 3.3. According to the remarks at the end of Section 3, there is a non-negative continuous linear function $\hat{p}$ on E_B such that $\| \hat{p} \|_* = 1$, and there is a net $\{(y^d, p^d)\colon d \in D\}$ in $C \times E_B^*$ converging to $(\hat{y}, \hat{p})$ such that for every d in D, $p^d \gg 0$, $\| p^d \|_* = 1$, and y^d maximizes $p^d(y)$ on C; furthermore, $\hat{y}$ maximizes $\hat{p}(y)$ on C. By the same argument used to show that C is compact, one can associate with each y^d inputs a^d and outputs b^d such that (y^d, a^d, b^d) is a programme in $E(2\lambda)^3$, and such that some subnet of the net

$$\mathcal{N} \equiv \{(y^d, a^d, b^d, p^d)\colon d \in D\} \qquad \text{...(4.13)}$$

converges to $(\hat{y}, \hat{a}, \hat{b}, \hat{p})$. Without loss of generality, then, we can take $\mathcal{N}$ to be such that

$$\lim_{\mathcal{N}} (y^d, a^d, b^d, p^d) = (\hat{y}, \hat{a}, \hat{b}, \hat{p}). \qquad \text{...(4.14)}$$

(It is to be understood here that E_B^* has the weak* topology, and that $E(2\lambda)^3 \times E_B^*$ has the corresponding product topology.)

Consider first the case in which there are non-producible commodities; let the corresponding representation of inputs and outputs be as in (4.11), and the corresponding representation of price vectors be

$$p_t^d = \begin{pmatrix} q_t^d \\ s_t^d \end{pmatrix}. \qquad \text{...(4.15)}$$

Since $(\hat{y}, \hat{a}, \hat{b})$ is not tight at any date t, there exist, for every t, an input-output pair (a_t', b_{t+1}') in G_t and a positive number ϕ_t such that

$$\left.\begin{aligned} e_{tj}' &\leqq \hat{e}_{tj} - 2\phi_t, \quad \text{for every non-producible commodity } j, \\ f_{ti}' &\geqq \hat{f}_{ti} + 2\phi_t, \quad \text{for every producible commodity } i. \end{aligned}\right\} \qquad \text{...(4.16)}$$

Hence for any fixed $t \geqq 1$ one has eventually in the net $\mathcal{N}$

$$\left.\begin{aligned} e_{tj}' &\leqq e_{tj}^d - \phi_t, \quad \text{for every non-producible } j, \\ f_{ti}' &\geqq f_{ti}^d + \phi_t, \quad \text{for every producible } i. \end{aligned}\right\} \qquad \text{...(4.17)}$$

In addition, for every t

$$\| c_t' - c_t^d \| \leqq \| c_t' - \hat{c}_t \| + \| \hat{c}_t - c_t^d \|,$$

so that eventually in the net $\mathcal{N}$

$$\| c_t' - c_t^d \| \leqq \| c_t' - \hat{c}_t \| + 1 \equiv \delta_t. \qquad \text{...(4.18)}$$

Hence, by (4.17) and (4.18), for each t one has eventually in the net $\mathcal{N}$

$$\left.\begin{aligned} q_{t+1}^d(f_{t+1}' - f_{t+1}^d) &\geqq \phi_{t+1} \| q_{t+1}^d \|_1, \\ s_t^d(e_t^d - e_t') &\geqq \phi_t \| s_t^d \|_1, \\ q_t^d(c_t' - c_t^d) &\leqq \delta_t \| q_t^d \|_1 \end{aligned}\right\} \qquad \text{...(4.19)}$$

(recall that for a vector $x = (\xi_i)$ in R^M, $\| x \|_1 \equiv \Sigma \mid \xi_i \mid$).

By Theorem 4.2, for every t the programmes (y^d, a^d, b^d) are eventually intertemporally profit maximizing with respect to the series part of p^d at t; hence

$$p_{t+1}^d b_{t+1}' - p_t^d a_t' \leqq p_{t+1}^d b_{t+1}^d - p_t^d a_t^d$$

which can be rearranged to give

$$q^d_{t+1}(f'_{t+1}-f^d_{t+1})+s^d_t(e^d_t-e'_t) \leqq q^d_t(c'_t-c^d_t). \qquad \ldots(4.20)$$

Hence, by (4.19), for every t eventually one has in the net $\mathcal{N}$

$$\phi_{t+1} \| q^d_{t+1} \|_1 + \phi_t \| s^d_t \|_1 \leqq \delta_{t+1} \| q^d_t \|_1,$$

and therefore

$$\| q^d_{t+1} \|_1 \leqq \left(\frac{\delta_t}{\phi_{t+1}}\right) \| q^d_t \|_1,$$

$$\| s^d_t \|_1 \leqq \left(\frac{\delta_t}{\phi_t}\right) \| q^d_t \|_1. \qquad \ldots(4.21)$$

From (4.21) one easily constructs a sequence (β_t) of positive numbers such that for every $t \geqq 1$, eventually in $\mathcal{N}$

$$\frac{\| q^d_t \|_1}{\| q^d_1 \|_1} \leqq \beta_t, \quad \frac{\| s^d_t \|_1}{\| q^d_1 \|_1} \leqq \beta_t. \qquad \ldots(4.22)$$

Recall that for every d in D, $p^d \gg 0$, and hence in particular $\| q^d_1 \|_1 > 0$.

By (4.22) there is a subnet $\mathcal{N}_1$ of $\mathcal{N}$ such that

$$\bar{q}_1 \equiv \lim_{\mathcal{N}_1} \frac{q^d_1}{\| q^d_1 \|_1},$$

and

$$\bar{s}_1 \equiv \lim_{\mathcal{N}_1} \frac{s^d_1}{\| q^d_1 \|_1}$$

exist.[1] Repeating this process, one can construct a sequence $\mathcal{N}_t$ of nets

$$\{(y^d, a^d, b^d, p^d): \ d \in D_t\},$$

with $\mathcal{N}_0 \equiv \mathcal{N}$, such that for every $t \geqq 1$, $\mathcal{N}_t$ is a subnet of $\mathcal{N}_{t-1}$, and the limits

$$\bar{q}_t \equiv \lim_{\mathcal{N}_t} \frac{q^d_t}{\| q^d_1 \|_1}, \quad \bar{s}_t \equiv \lim_{\mathcal{N}_t} \frac{s^d_t}{\| q^d_1 \|_1} \qquad \ldots(4.23)$$

exist. Define

$$\bar{p}_t \equiv \begin{pmatrix} \bar{q}_t \\ \bar{s}_t \end{pmatrix}. \qquad \ldots(4.24)$$

I shall show that $(\bar{p}_t)$ is the required sequence of price vectors. Note that $\| \bar{p}_1 \|_1 \geqq 1$.

By Theorem 4.2, for every $t \geqq 1$, every (a, b) in G_t, and every d in D_t,

$$p^d_{t+1}b - p^d_t a \leqq p^d_{t+1}b^d_{t+1} - p^d_t a^d_t$$

or

$$\frac{p^d_{t+1}}{\| q^d_1 \|_1}(b-b^d_{t+1}) - \frac{p^d_t}{\| q^d_1 \|}(a-a^d_t) \leqq 0. \qquad \ldots(4.25)$$

Fixing t and (a, b) in (4.25) and taking the limit for the net $\mathcal{N}_t$, yields

$$\bar{p}_{t+1}(b-\hat{b}_{t+1}) - \bar{p}_t(a-\hat{a}_t) \leqq 0,$$

which completes the proof of the theorem for the case in which there are non-producible commodities.

[1] If D_1 denotes the directed set corresponding to the net $\mathcal{N}_1$, there is no assurance that $D_1 \subset D$; with this point in mind there should be no confusion caused by using the symbol d to indicate the generic element of D_1 as well as of D (see Kelly [5], p. 70).

The case in which all commodities are producible can clearly be treated by the same method, eliminating all reference to the components e_t and s_t.

With an additional continuity assumption about production possibilities, one can obtain the condition of terminal cost minimization in the context of Theorem 4.3. For any programme (y, a, b) in $E(\lambda)^3$ and any $T \geqq 1$, let $A_T(y)$ denote the set of inputs at date T that would permit the net outputs y_t at date $t > T$ [i.e. define $A_T(y)$ with respect to y just as A_T was defined with respect to $\hat{y}$ in (4.7)].

Assumption 3. If a net $\{(y^d, a^d, b^d): d \in D\}$ of programmes in $E(\lambda)^3$ converges to a programme (y, a, b) in $E(\lambda)^3$ then for every T, every $\tilde{a}$ in $A_T(y)$, and every d in D, there is an input vector $\tilde{a}_T^d$ in $A_T(y^d)$ such that the net $\tilde{a}^d$ converges to $\tilde{a}$.

Corollary to Theorem 4.3. If Assumption 3 is added to the hypothesis of Theorem 4.3, then $(\hat{y}, \hat{a}, \hat{b})$ is terminally cost minimizing with respect to $(\bar{p}_t)$.

Proof. Suppose $\tilde{a}$ is in $A_T(\hat{y})$; then by Assumption 3, applied to the net $\mathcal{N}_T$, there is for every d in D_t an $\tilde{a}_T^d$ in $A_T(y^d)$ such that

$$\lim_{\mathcal{N}_T} \tilde{a}_T^d = \tilde{a}.$$

by Theorem 4.2, for every d in D_T

$$p_T^d(a_T^d - \tilde{a}_T^d) \leqq 0;$$

hence

$$\frac{p_T^d}{\| q_1^d \|_1} (a_T^d - \tilde{a}_T^d) \leqq 0. \qquad \ldots(4.26)$$

Taking the limit of (4.26) in the net $\mathcal{N}_T$ completes the proof of the corollary.

I now consider some consequences of constant returns to scale in production.

Assumption 4. For every t, G_t is a cone.

If there are constant returns to scale at a date t, then, as is well known, the condition (4.6) of intertemporal profit maximization at t implies, with respect to the price vector $(\bar{p}_t)$,

$$\begin{aligned} &\bar{p}_{t+1}\hat{b}_{t+1} - \bar{p}_t\hat{a}_t = 0, \\ &\bar{p}_{t+1}b - \bar{p}_t a \leqq 0, \text{ for all } (a, b) \text{ in } G_t. \end{aligned} \qquad \ldots(4.27)$$

If there are constant returns to scale at every date then (4.27) in turn implies that for every programme (y, a, b) and every date $T \geqq 1$,

$$\sum_{t=1}^{T} \bar{p}_t y_t = \sum_{t=1}^{T-1} (\bar{p}_{t+1}b_{t+1} - \bar{p}_t a_t) - \bar{p}_T a_T \leqq 0, \qquad \ldots(4.28)$$

so that

$$\limsup_{T} \sum_{t=1}^{T} \bar{p}_t y_t \leqq 0. \qquad \ldots(4.29)$$

I close this section with some remarks on the particular way that I have chosen to implement the linear function approach to price systems in the infinite horizon case. The set of continuous linear functions on a subset of the set of all programmes depends upon the subset and upon the topology with which that subset is provided. The goal of extending the analysis to as wide as possible a class of programmes may conflict with the goal of conserving certain useful mathematical properties. In this case the choice of the set of bounded programmes with the sup norm topology was influenced by the facts that (1) the non-negative orthant in E_B has a non-empty interior, and (2) the set of non-negative p in E_B^* with $\| p \|_* = 1$ is closed in the weak* topology, and therefore compact. This latter fact was used to prove Theorem 3.3 (see Radner, [10]).

Instead of attempting to define the present value of individual net output programmes, one may be content to define the present value of differences $(\hat{y}-y)$ between a given net output programme $\hat{y}$ and net output programmes y for which the sequence of differences, $(\hat{y}_t-y)$, is bounded, converges to zero sufficiently rapidly, or has some other prescribed asymptotic behaviour. For example, the sequence $(\bar{p}_t)$ of price vectors in Theorem 4.3 may not define a present value for the net output programme $\hat{y}$, but since $\| \bar{p}_t \| \leqq \beta_t$, the series

$$\sum_{t=1}^{\infty} \bar{p}_t(\hat{y}_t - y_t)$$

will converge for any net output programme for which

$$\sum_{t=1}^{\infty} \beta_t \| \hat{y}_t - y_t \|_1 < \infty.$$

Given a sequence (β_t) of positive numbers, the set of sequences $x = (x_t)$ of vectors in R^M for which

$$\sum_{t=1}^{\infty} \beta_t \| x_t \|_1 < \infty \qquad \ldots(4.30)$$

is indeed a Banach space with the norm of x defined by (4.30), and every continuous linear function p on this space can be represented in the form

$$p(x) = \sum_{t=1}^{\infty} p_t x_t,$$

where (p_t) is a sequence of vectors in R^M such that the sequence $(\| p_t \|/\beta_t)$ is bounded. An approach along these lines has been taken by McFadden [8].

5. BOUNDS ON PRODUCTION

In this section I establish certain bounds on production, under " neo-classical " assumptions. Given the exogenous supplies of resources at each date these bounds enable one to choose the units of measurement in such a way that the set of feasible programmes is compact. In this case Theorems 3.3 and 4.3 can be applied to the whole set of feasible programmes.

A production correspondence P_t will be called neo-classical if, in addition to Assumptions 1, 2 and 4 (convexity, " continuity ", and constant returns to scale), zero input implies zero output. These assumptions may be summarized by the statement: the graph, G_t, of P_t is a closed convex cone in R^{2M}, and $P_t(0) = \{0\}$.

Lemma. *If the production correspondence P_t is neo-classical, then there is a number, M_t,* such that $\| b \| \leqq M_t \| a \|$ for all (a, b) in G_t.

Proof. The proof uses a by now standard argument. Suppose to the contrary that there is an infinite sequence (a_n, b_n) such that

(i) (a_n, b_n) is in G_t for every n,

(ii) $\dfrac{\| a_n \|}{\| b_n \|}$ converges to zero.

Let $a'_n \equiv a_n/\| b_n \|$, $b'_n \equiv b_n/\| b_n \|$. Since G_t is a cone, (a'_n, b'_n) is in G_t for every n. The sequence (a'_n, b'_n) is bounded, and hence has a limit point, say (a', b'), By (ii) above, and the construction of (a'_n, b'_n), it follows that $a' = 0$ and $\| b' \| = 1$; but since G_t is closed, (a', b') is in G_t, which contradicts the assumption that $P_t(0) = \{0\}$. Hence $\| b \|/\| a \|$ is bounded for the set of (a, b) in G_t with $a \neq 0$, which completes the proof of the lemma.

For the rest of this section I assume:

Assumption 5. For every t, P_t is neo-classical.

For every t, let w_t be the vector of quantities of commodities exogenously supplied to the economy in question, i.e., its exogenous resources. For a programme (y, a, b) to be *feasible* not only must one have, as in (2.1)

$$\begin{aligned} y_1 &= -a_1, \\ y_t &= b_t - a_t, \quad t \geqq 2, \\ b_{t+1} &\text{ in } P_t(a_t), \quad t \geqq 1, \end{aligned} \qquad \text{...(5.1)}$$

but also

$$y_t + w_t \geqq 0, \quad t \geqq 1. \qquad \text{...(5.2)}$$

Combining (5.1) and (5.2) one gets

$$\begin{aligned} a_1 &\leqq w_1 \\ a_t &\leqq b_t + w_t, \quad t \geqq 2, \\ b_{t+1} &\text{ in } P_t(a_t), \quad t \geqq 1. \end{aligned} \qquad \text{...(5.3)}$$

For every t, let

$$M_t \equiv \sup \{\| b \| / \| a \| : (a, b) \in G_t . a \neq 0\}; \qquad \text{...(5.4)}$$

the finiteness of M_t is assured by the lemma. Define

$$\begin{aligned} K_1 &\equiv \bar{K}_1 \equiv \| w_1 \|, \\ \bar{K}_1 &\equiv M_{t-1} \bar{K}_{t-1} + \| w_t \|, \quad t \geqq 2, \\ K_t &\equiv M_{t-1} \bar{K}_{t-1} + \bar{K}_t. \end{aligned} \qquad \text{...(5.5)}$$

It is easy to verify that, for every t,

$$\begin{aligned} \| a_t \| &\leqq \bar{K}_t, \\ \| b_{t-1} \| &\leqq M_t \bar{K}_t, \\ \| y_t \| &\leqq K_t. \end{aligned} \qquad \text{...(5.6)}$$

In view of the bounds (5.5), the units of measurement of commodities at different dates can be chosen so as to force $\| a_t \|$, $\| b_t \|$, and $\| y_t \|$ to decrease to zero arbitrarily fast, uniformly for *all feasible programmes* (following the procedure described at the beginning of Section 4).

These remarks are summarized in the following theorem. Recall that for any sequence $\lambda = (\lambda_t)$ of positive numbers, $E(\lambda)$ denotes the set of points $x = (x_t)$ in E such that $\| x_t \| \leqq \lambda_t$, for every t.

Theorem 5.1. *If production is neoclassical, then for every sequence $\lambda = (\lambda_t)$ of positive numbers and every sequence (w_t) of exogenous resources, there is a choice of units at each date such that every feasible programme is in $E(\lambda)^3$.*

Corollary. If production is neoclassical, then for every sequence of exogenous resources there is a choice of units at each date such that the set of *all* feasible programmes is compact.

Proof of Corollary. Take (λ_t) converging to zero, and apply the argument in the proof of Theorem 4.3 demonstrating the compactness of C.

University of California, Berkeley. R. Radner.

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