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COMPETITIVE EQUILIBRIUM UNDER UNCERTAINTY[1]

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This paper explores how far one can go in applying the modern theory of competitive equilibrium to the case of uncertainty. In the first part, the analyses of Arrow and Debreu are extended to the case in which different economic agents may have different information about the environment. The second part deals with the limitations of the Arrow-Debreu type of model, and discusses the difficulties associated with nonconvexities in the production of information, with information generated by spot markets, and with limitations on the computational capacities of economic agents. It is argued that the demand for liquidity arises from, among other things, the last two phenomena, and thus does not appear to be amenable to analysis by means of the "neoclassical" theory of competitive equilibrium.

1. INTRODUCTION[2]

The purpose of this paper is to explore how far one can go in applying the modern theory of competitive equilibrium (as exemplified, say, by Debreu's *Theory of Value*)[3] to the case of uncertainty. In Sections 2–9, I extend the analysis of Arrow [1] and Debreu [2] to the case in which different economic agents may have different information about the environment. The treatment of information used here derives from statistical decision theory generally (e.g., Savage [7]), and more particularly from the theory of teams (Marschak and Radner [4], Radner [5]).

I conclude that if economic decision makers have unlimited computational capacity for choice among strategies, then even if there is uncertainty about the environment, and different agents have different information and different beliefs about the environment, then one can apply the standard theorems on the existence and optimality of competitive equilibrium. In such a theory there is no role for money and liquidity. All contracts are negotiated at the beginning of the history of the economy, and from then on all actions are determined by already chosen strategies. Such strategies may, of course, take account of new information as it becomes available.

On the other hand, I argue that a demand for liquidity arises from *computational limitations*, and would be present even in a world of certainty about the environment if that world were sufficiently complicated. I suggest further that there is a basic difficulty in incorporating computational limitations in a "classical" equilibrium theory based on optimizing behavior, and that this presents an obstacle to

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[2] I am grateful to G. Debreu and K. J. Arrow for helpful comments on an earlier draft of this paper. In particular, I feel they helped to clarify my thoughts on the material of Sections 10 and 11, although they might well disagree after reading this material.

[3] See Debreu [2] in the list of references at the end of the paper.

an extension of the classical theorems of welfare economics to cover the case of a monetary economy.

I also argue that if decision makers receive information about each other's behavior as well as about the environment, then this introduces a type of externality (interdependence) among their decision rules. This type of externality has the result that decision makers must take account of uncertainty about each other's behavior as well as about the environment. It, too, may give rise to a demand for liquidity. In particular, the introduction of "spot" markets as well as futures markets results in this type of externality, and thus does not appear to be amenable to analysis by means of the "classical" theory of competitive equilibrium.

The distinction between (1) uncertainty and information about the environment, and (2) uncertainty and information about others' behavior or the outcome of as yet unperformed computations appears to be fundamental. The analyses of Arrow [**1**] and Debreu [**2**] deal with uncertainty about the environment.[4] The "world" is divided into two sets of variables: decision variables, which are controlled by economic agents, and environmental variables, which are not controlled by any economic agent. Following the terminology of statistical decision theory, I shall call a complete specification of the environment a "state of nature." A state of nature is a complete description of the environment from the beginning to the end of the economic system in question. In Sections 2–8 of this paper I assume that each economic agent (consumer or producer) obtains or has information about the environment at every date, and that he knows in advance what kind of information he will have (even though he typically doesn't know in advance exactly what that information will be). At the beginning of time each agent chooses a strategy, subject to some constraints; this strategy determines his receipts and deliveries of goods and services at all dates. The actual receipts and deliveries at a given date will depend upon the information that the agent has at that date. This information, in turn, depends upon the state of nature, so that, in effect, a strategy determines how receipts and deliveries at each date will depend upon the state of nature. Different agents may, of course, have different information.

Since receipts and deliveries at each date depend, directly or indirectly, on the state of nature, it is natural to assume that a market exists for delivery of each commodity at each date conditional on each state of nature, at least in principle. I hasten to add that this assumption is natural only if each agent has unlimited computational ability. (As a matter of fact, depending upon the structure of information available to the various agents, some of these markets may be unnecessary or inactive; this topic is discussed in Section 8.) I assume, therefore, that there is a price for delivery of each commodity at each date conditional on each state of nature. With such a system of prices there is no uncertainty about the cost or value of a given strategy. This last point may at first surprise the reader, and needs some

[4] In general, so does statistical decision theory; see Savage [**7**].

emphasis. Since the consequences of any strategy are completely described by specifying the receipts or deliveries of each commodity at each date in each state, and each such receipt or delivery can be valued according to its corresponding (conditional) price, the cost of any strategy is simply the sum of products of such conditional prices times the quantities. Note that the "price" of *unconditional* delivery of a given commodity at a given date is the sum, over all states of nature, of the corresponding conditional prices.

In outlining the assumptions about the behavior of producers and consumers, I shall first limit myself to the case of a fixed structure of information for each economic agent in the economy. For certain purposes it is useful to make a distinction between a *strategy* and an *act*. A *strategy* determines inputs and outputs at each date as a function of incoming information. The incoming information is determined by some information process (e.g., observation) as a function of the state of nature. Combining an information process with a strategy yields an *act*, namely a function that determines inputs and outputs at each date as a function of the state of nature. (The distinction between strategies and acts is of importance in the case of communication among decision makers; see discussion below, and Sections 10 and 11.)

Each producer is characterized by a *production possibility set*, i.e., a set of feasible production acts. This set expresses all of the constraints on his production, including the constraints imposed on his acts by the structure of information available to him at each date. Any one act of a producer determines his inputs and/or outputs of each commodity at each date in each state of nature. For a given set of prices, the *profit* corresponding to a production act is the net value of the act, computed in the manner described above.

A consumer is characterized by a consumption possibility set, preferences, resources, and shares in productive enterprises. His *consumption possibility* set is a set of feasible consumption acts, expressing, among other things, the constraints imposed by the structure of his information. His *preferences* among acts reflect his tastes for consumption, his beliefs about the relative likelihood of the several states of nature, and his attitude towards risk. Following Debreu, I do not find it necessary to assume that a consumer's preferences are sufficiently regular so as to permit scaling in terms of subjective probabilities and von Neumann-Morgenstern utilities. I do assume, however, that consumers are risk-averse or risk-neutral (convexity of preferences). The *resources* of each consumer consist of specified quantities (possibly zero) of each commodity at each date in each state of nature. His *shares* consist of some fraction (possibly zero) of the profit of each producer (not necessarily the same fraction for each). For given prices, his *wealth* is the sum of the value of his resources and his profit shares.

An *equilibrium* is a set of prices, together with acts of consumers and producers, such that (a) each consumer maximizes his preferences within his consumption possibility set, subject to his wealth constraint; (b) each producer maximizes profits within his production possibility set; and (c) total demand equals total supply, at every date and in every state of nature. Note that it is assumed that producers and consumers are "price-takers." Note, too, that in maximizing profits the beliefs concerning likelihoods of states and the attitudes towards risk of the producers are irrelevant and play no role in their behavior, since for given prices there is no uncertainty about the value of a production act. On the other hand, beliefs and attitudes towards risk do play a role in consumer behavior, although for given prices and given production plans there is no uncertainty about a consumer's wealth.

In such an economy, which I might call the "Arrow-Debreu world,"[5] all contracts are negotiated at the beginning of the history of the economy, as I mentioned above, and from then on all actions are determined by the already chosen acts. There is no need to revise any strategies, because the choice of a strategy has already taken account of the structure of information in the future, i.e., what information will be available at each date. Since (1) all accounts are settled at the beginning, (2) there is no revision of strategies, and (3) the present value of each producer and each consumer is known with certainty at the beginning, it follows that there is no need for money or liquidity and no incentive to trade shares. Using the techniques of Debreu [**2, 3**], one can demonstrate, under "classical conditions" (convexity and continuity of production and consumption sets, and of preferences, etc.), the existence of an equilibrium, the Pareto optimality of an equilibrium, and that, roughly speaking, every Pareto-optimal configuration of strategies is an equilibrium relative to some price system for some distribution of resources (Sections 5 and 6).

The Arrow-Debreu world has been criticized as a model of reality, or even as a normative model of planning, for requiring the existence of too many markets. The limits on the information available to the agents, however, and in particular the differences among the information structures of the several agents, will typically have the effect of drastically reducing the number of required markets (Section 8). First, there is no need for contracts that depend upon information that is not available in the economy. Second, the net trade between any group of agents and the group of all other agents in the economy can at most depend upon information that is common to both groups; "common to both groups" means here that at least one agent in each group must have the information in question. These considerations somewhat diminish the force of the above criticism, but are far from eliminating it altogether.

Up to this point, I have assumed that the structure of information available to each agent at each date is fixed in advance. The choice of information structure can be included in the model simply by taking account of the real costs of obtaining information in the formal description of the consumption and production possibil-

[5] The model just described, with the exception of the possibility of different agents having different information, is due to Debreu [**2**, Ch. 7], and represents an extension and generalization of Arrow [**1**].

ity sets. With this formulation, however, the convexity assumption may be seriously criticized. This problem, with an example, is discussed in Section 9.

In the Arrow-Debreu world a competitive equilibrium achieves a Pareto-optimum relative to a given structure of information by making available to all agents in the economy some additional information, namely the equilibrium prices. As I have already emphasized, these are prices in a futures market; all strategies are determined at the beginning and no revision of strategies is contemplated. Suppose, however, that new markets were introduced at later dates; would there be any incentive to trade in these new markets? In general there would, because the equilibrium prices in such markets would convey additional information beyond that contemplated in the original structure of information. These prices would depend, at a given date, on the evolution of the economy up to that date, including the evolution of the environment, both through direct observations of the environment made by economic agents, and indirectly through the decisions made up to that date, which determine the stocks at the beginning of that date. Unfortunately, in order correctly to infer something about the state of nature from the value of the new prices, an agent must in principle know the strategies used by other agents up to that date.

An agent may choose a strategy determining how his inputs and outputs will depend upon his information, including the "spot" prices in the later markets, but such a strategy will not independently determine an act! The acts of all the agents will be determined jointly by their choice of strategies. In this sense, the introduction of spot markets introduces "external effects" among the acts of the several agents in the economy. Thus, although the introduction of spot markets makes available to the economy acts that were not available with the initial structure of information, it also destroys one of the important conditions for the "classical" analysis of competitive equilibrium. In particular, an agent will no longer be able to assign a definite value to a strategy for given prices in the futures market. There would typically arise a demand for liquidity, but unless agents could correctly predict the strategies of others, and calculate the consequences, he could not determine his optimal demand for liquidity!

The structure of information generated by spot markets is a special case of what has elsewhere been called *network information* (Marschak and Radner, [**4**]). The model of network information is described in Section 10, with a discussion of the resulting externalities, and the particular case of market information is discussed in Section 11.

The Arrow-Debreu world is strained to the limit by the problem of choice of information. It breaks down completely in the face of limits on the ability of agents to compute optimal strategies. I have already hinted at the nature of the difficulty at the beginning of this introduction. This problem is explored a little in Section 12. The discussion there points to a model in which there is a succession of temporary equilibria, with the existence at each date of a limited number of markets for current and future delivery. In such a model each agent faces not only uncertainty about the environment, but also about the outcome of his own future computations, about future prices (as distinct from "futures prices") and about the existence of various markets in the future. This additional type of uncertainty leads to the constant revision of strategies, and therefore to trading at every date (not just in the beginning of time, as in the Arrow-Debreu world), to the importance of probability judgments and risk-taking by producers, to the demand for money and liquidity, and to the trading of shares, to mention only a few additional phenomena. This computational uncertainty also seems to be an obstacle to the definition of concepts of individual and social optimum, except possibly in a long-run statistical sense.

I conclude this Introduction with some suggestions on strategies for reading this paper. The reader armed only with arithmetic and a little algebra, and without much patience for abstract formulation and symbolism, should proceed immediately to Section 7, in which I present an extended example, and then read the example of Section 9, and Sections 10 through 13. The rest of the paper, with the exception of Theorems 1 and 2 of Section 5 on the existence of equilibrium, is abstract but mathematically elementary. The new techniques of Debreu [3] are helpful in analyzing the question of existence of equilibrium, as is shown in Theorems 1 and 2 of Section 5. Here the difficulty is that, because of the limits on information, each agent's set of feasible strategies lies in a linear subspace of the space of all possible strategies, and these subspaces are typically different for different agents. Thus, for example, the assumption of "free disposal" is not appropriate here.

2. STATES AND ACTS

Consider an economy with T dates (elementary time periods), and C different "commodities" at each date, where "commodities" are to be interpreted broadly, including goods and services, distinguished possibly according to age, location, etc.

Let S denote a finite set, to be interpreted as the set of alternative "states of nature." The states in S are mutually exclusive, and the set S is exhaustive. Each state in S determines the entire history of all aspects of the economy that are beyond the control of any of the agents. (See Savage [7, Chapter 2] for a detailed discussion of this concept.)

An *act* of an economic agent (producer or consumer) is a T-tuple $\alpha=(\alpha_1, \alpha_2, \ldots, \alpha_T)$, where for each t, α_t is a function from S to the C-dimensional Euclidean vector space R^C. For a producer, the cth coordinate of $\alpha_t(s)$ is to be interpreted as his output of commodity c at date t, if s is the true state of nature ("outputs" are positive, "inputs" are negative). For a consumer the cth coordinate of $\alpha_t(s)$ is his input of commodity c at date t, if s is the true state (but here "inputs" (consumption) are positive, "outputs" are negative).[6]

A *price system* is a nonzero T-tuple $p=(p_1, \ldots, p_T)$, where for each t, p_t is a function from S to R^C. The cth coordinate of $p_t(s)$ is to be interpreted as the price of commodity c at date t if state s obtains.

For two vectors, $u=(u^c)$ and $v=(v^c)$, in R^C denote by $u \cdot v$ the inner product $u \cdot v \equiv \Sigma_c u^v v^c$. The same "dot product" notation will be used to denote the *value*, $p \cdot \alpha$, *of an act* α *relative to a price system* p: $p \cdot \alpha \equiv \Sigma_s \Sigma_t p_t(s) \cdot \alpha_t(s)$.

If S has $|S|$ elements (finite), then every act and every price system may be regarded as a point in a Euclidean space of $T \cdot C \cdot |S|$ dimensions. This space will be denoted by A.

3. FIXED INFORMATION ABOUT THE ENVIRONMENTS; STRATEGIES

In Sections 4–9 of this paper the information on which a given economic agent bases his decisions at a given date t will be characterized by a partition, say $\mathscr{S}_t$, of the set S of states. The T-tuple $\mathscr{S}=(\mathscr{S}_1, \ldots, \mathscr{S}_T)$ of these partitions will be called the *information structure* for the agent in question. The information structure $\mathscr{S}$ is assumed to constrain the acts of the agent as follows: for every t, every set M in $\mathscr{S}_t$,

$$\text{(3.1)} \qquad \text{if } s \text{ and } s' \text{ are both in } M, \text{ then } \alpha_t(s) = \alpha_t(s').$$

To say that M is a set in $\mathscr{S}_t$ expresses the idea that if the true state is in M, then the agent's information at date t enables him to determine that the true state is in M, but not to determine which of the states in M is the true state. (See Savage [7, Ch. 6].) If the agent cannot distinguish, at date t, between the different states in M, then he cannot make his decision at t depend upon s within M; hence the constraint (3.1).

For any information structure $\mathscr{S}$ let $\mathscr{A}(\mathscr{S})$ denote the set of all acts that are compatible with $\mathscr{S}$, i.e., satisfy (3.1). The set $\mathscr{A}(\mathscr{S})$ is a linear subspace of A, the linear space of all acts, with the usual definitions of addition and scalar multiplication of functions.[7]

Relative to the given set of states, at a given date, the partition in which every set consists of a single state represents "complete information;" whereas, the partition consisting of the set S alone represents "no information."

For two partitions, $\mathscr{U}$ and $\mathscr{V}$, of S, I shall say that $\mathscr{U}$ *is as fine as* $\mathscr{V}$ if for every $U \in \mathscr{U}$ and $V \in \mathscr{V}$ either $U \subseteq V$ or $U \cap V = \varnothing$.

[6] I have followed closely the terminology of Debreu [2]. However, Debreu calls an "action" what I have called an "act"; the latter term is used in Savage [7].

[7] For each t let $\mathscr{S}_t^*$ be the smallest algebra of sets containing the partition $\mathscr{S}_t$. Then condition (3.1) is equivalent to the requirement that α_t be $\mathscr{S}_t^*$-measurable. In any extension of this treatment to infinite sets of states, it would be natural to describe information in terms of σ-algebras $\mathscr{S}_t^*$, and acts α such that α_t is $\mathscr{S}_t^*$-measurable. In the finite case, $\mathscr{S}_t^*$ is the family of all unions of sets in $\mathscr{S}_t$.

If an agent did not forget any information from one date to the next, then one would represent this by a structure of information $\mathscr{S}$ in which each partition $\mathscr{S}_t$ was at least as fine as the preceding one; one might call this an *expanding* information structure. Debreu [**2**, Ch. 7] treats the case in which all the economic agents have the same expanding information structure. Arrow [**1**] does not consider dated information and action; he also assumes implicitly that all agents have the same information.

It may help the reader to relate the present abstract representation of information about the environment to an equivalent one that is perhaps more familiar. One may think of the information about the environment as deriving from an observation, expressed in the form of some measurement, signal, etc. For a given economic agent at a given date t, let the set of possible alternative observations be denoted by $\tilde{B}_t$. A decision rule at that date would assign to every possible observation $\tilde{b}_t$ in $\tilde{B}_t$ a vector of inputs and outputs, i.e., a point in the commodity space, R^C. A T-tuple of such rules will be called a *strategy*. Thus a strategy is a T-tuple $\beta=(\beta_1, \ldots, \beta_T)$ such that for each t, β_t is a function from $\tilde{B}_t$ to R^C.

The particular method of observation used will determine a relation between the state of nature and the observation. Assuming that the state of nature includes a description of all environmental factors relevant to the method of observation (including disturbances, defects of the observational instruments, "noise," etc.), the method of observation will be characterized by a function from S to $\tilde{B}_t$, say $\tilde{\beta}_t$. Every such function $\tilde{\beta}$ determines a partition $\mathscr{S}_t$ of S, as follows: a set is in $\mathscr{S}_t$ if and only if it is the set of all states mapped into a given observation in $\tilde{\beta}_t$. Conversely, every partition $\mathscr{S}_t$ of S may be represented as a "method of observation" by taking $\tilde{B}_t$ to be identical with $\mathscr{S}_t$, and $\tilde{\beta}_t$ to be the function that assigns to every state in S the set in $\mathscr{S}_t$ that contains it. Of course, more than one "method of observation" may lead to the same partition, but (aside from the question of cost) they will be equivalent from the point of view of the decision maker.

If a method of observation $(\tilde{B}_t, \tilde{\beta}_t)$ is followed by the use of a decision rule β_t, the result will be a component of an act, namely $\alpha_t(s)=\beta_t[\tilde{\beta}_t(s)]$. Indeed, given a T-tuple of methods of observation, with corresponding information structure $\mathscr{S}=(\mathscr{S}_1,\ldots, \mathscr{S}_T)$, the set of acts generated by the set of all possible strategies is exactly the set $\mathscr{A}(\mathscr{S})$ of all acts that are compatible with $\mathscr{S}$.

4. CONSUMERS AND PRODUCERS

Following Debreu [**2**], the agents of an economy are divided into *consumers* and *producers*. Consumer i is characterized by:

(a) an information structure $\mathscr{S}_i=(\mathscr{S}_{i1}, \ldots, \mathscr{S}_{iT})$;

(b) a consumption set, X_i, assumed to be a subset of $\mathscr{A}(\mathscr{S}_i)$, representing the set of feasible acts for i;

(c) a complete preordering, $\lesssim_i$, on X_i, describing the preferences of i;

(d) a T-tuple, $\omega_i = (\omega_{i1}, \ldots, \omega_{iT})$, in $\mathscr{A}(\mathscr{S}_i)$, representing the resources of i (thus $\omega_{it}(s)$ represents the vector of quantities of commodities available to consumer i at date t if s is the true state, not as a consequence of any economic action; for example, these might include stocks inherited from the past at the first date, and hours of available labor at future dates);

(e) a set of nonnegative numbers θ_{ij}, where j runs over the set of producers; the number θ_{ij} represents the share in producer j held by consumer i. The numbers θ_{ij} must satisfy, of course, the accounting constraint $\Sigma_i \theta_{ij} = 1$, each j.

Producer j is characterized by:

(a′) an information structure $\mathscr{T}_j = (\mathscr{T}_{j1}, \ldots, \mathscr{T}_{jT})$;

(b′) a production set Y_j, assumed to be a subset of $\mathscr{A}(\mathscr{T}_j)$, representing the set of feasible acts for j.

Given a price system p, the *profit* for producer j is the value of his act, i.e., $p \cdot y_j$. The *wealth* of consumer i is the sum of the value of his resources plus his share of the profits of the producers, i.e., $w_i = p \cdot \omega_i + \Sigma_j \theta_{ij} p \cdot y_j$.

In the descriptions of consumers and producers just given, the information structures are behind the scenes to the extent that every agent's acts are constrained to be compatible with his information structure; recall that $X_i \subseteq \mathscr{A}(\mathscr{S}_i)$ and $Y_j \subseteq \mathscr{A}(\mathscr{T}_j)$. One may bring these constraints more clearly to the fore by supposing that for each agent there is a basic set of "potentially feasible" acts, and that his actual feasible set consists of all potentially feasible acts that are compatible with a given information structure.

In the case of a consumer with a given sequence ω_i of resources, it is not meaningful to consider an information structure $\mathscr{S}_i$ that is so "coarse" that $\omega_i \notin \mathscr{A}(\mathscr{S}_i)$. Since producers are not assumed to have any initial resources, no such limitation need be placed on their information structures.

For each consumer i let X_i^1 denote his set of potentially feasible acts, and let Y_j^1 denote the corresponding set for producer j. Further, for each consumer i let ω_i denote his (given) resources.

For two information structures $\mathscr{S} = (\mathscr{S}_t)$, $\mathscr{S}' = (\mathscr{S}'_t)$ I shall say that $\mathscr{S}$ is *as fine as* $\mathscr{S}'$ if, for every t, $\mathscr{S}_t$ is as fine a partition as $\mathscr{S}'_t$ (see Section 3). For each consumer i let $\mathscr{S}_i^0$ denote the least fine information structure with which ω_i is compatible.

By an *information structure for the economy* I shall mean an $(m+n)$-tuple $\mathscr{I} = ((\mathscr{S}_i), (\mathscr{T}_j))$ of information structures, with $\mathscr{S}_i$ and $\mathscr{T}_j$ corresponding to consumer i and producer j, respectively. $\mathscr{I}$ is *admissible* if, for every consumer i, $\mathscr{S}_i$ is as fine as $\mathscr{S}_i^0$. As noted above, for any given resources of consumers, it makes sense to consider only admissible information structures for the economy. The minimal admissible information structure, $\mathscr{I}^0$, is defined by

$$\mathscr{S}_i = \mathscr{S}_i^0 \qquad (i = 1, \ldots, m),$$

$$\mathscr{T}_{jt} = \{R^C\} \equiv \mathscr{T}_{jt}^0 \qquad (j = 1, \ldots, n;\ t = 1, \ldots, T).$$

The maximal information structure, $\mathscr{I}^1$, is characterized by taking every $\mathscr{S}_{it}$ and every $\mathscr{T}_{jt}$ to be the partition of S into one-element sets (i.e., complete information; see Section 3).

For any admissible information structure $\mathscr{I} = ((\mathscr{S}_i), (\mathscr{T}_j))$ for the economy, define

$$X_i(\mathscr{I}) = X_i^1 \cap \mathscr{A}(\mathscr{S}_i),$$

$$Y_j(\mathscr{I}) = Y_j^1 \cap \mathscr{A}(\mathscr{T}_j).$$

An economy is generated by an information structure $\mathscr{I}$ by taking $X_i(\mathscr{I})$ to be the consumption set for consumer i, with resources ω_i, and with the restriction to $X_i(\mathscr{I})$ of the original preference preordering $\lesssim_i$ on X_i^1; and by taking $Y_j(\mathscr{I})$ to be the production set for producer j.

As pointed out in Section 3, for any information structure $\mathscr{S}$ for an agent, $\mathscr{A}(\mathscr{S})$ is a linear subspace of the set A of all possible acts. If consumer i has information structure $\mathscr{S}_i$, then his set X_i of consumption acts is obtained from the potentially feasible set, X_i^1, by adding a number of linear constraints, namely the constraints (3.1). A corresponding remark applies to producers.

Economic agents may come to the market with prior information about the environment. Prior information enters the model in two ways: (i) if an agent knows at the beginning that the state s is in a certain state E (i.e., that the event E has occurred), then his information structure at each date should reflect the fact that at each date he will know whether or not the true state is in E; (ii) a consumer who knows that s is in E will reflect this fact in his preference ordering; thus for him the marginal utility of consumption contingent on any state not in E will be zero; if his preferences are representable in terms of expected utility, then his personal prior probability of E will be unity.[8] Some comments are perhaps called for here. Although formally every agent is assumed to plan what he would do in each state, it would not in fact be necessary for a consumer to plan what he would consume in states he considers impossible *a priori*. Presumably he would plan to consume nothing in such states, or at least he would not pay anything for consumption in such states. If an agent knew that an event E obtained, but not all agents knew this, the agent in question might find himself in a position in which he could sell, at a positive price, a contract for delivery contingent on an event that he already knew could not occur. *Caveat emptor*! Whether or not this raises any moral questions, it does raise the question of whether or not an agent's information includes knowledge of other agents' information structures. In the real world of contracts between individuals this question does arise. For example, it is not considered correct to make a bet on the outcome of a race whose results you already know.

[8] More generally, his prior information will be summarized in his own personal probability distribution.

But in the Arrow-Debreu world, individuals make contracts with an impersonal "market," so this issue does not arise, and individuals are free to make such contracts.

For each fixed information structure, each consumer is required only to have a preference preordering on the set of acts available to him, where this set already reflects his information structure. On the other hand, if we wish to consider the whole range of economies generated by varying the information structures, then we must assume that each consumer's preferences are defined on the entire set of his potentially feasible acts.

In the case of a fixed information structure, a consumer can, in his preferences, "aggregate" events that his information will not permit him to distinguish. This is brought out clearly in the case in which his preferences are representable by an expected utility function.

In a "static" situation, let $x(s)$ denote his consumption in state s, $P(s)$ his personal prior probability of state s, and $u(c)$ his utility for the sure consumption c. His expected utility of a consumption plan $x(\cdot)$ is $\Sigma_s P(s)u[x(s)]$. If $E_1, \ldots, E_K$ is a partition of S, and $x(s)$ is constant on each event E_k, say equal to $x(E_k)$, then the expected utility of $x(\cdot)$ is

$$\sum_k \left[\sum_{s \in E_k} P(s) \right] u[x(E_k)] = \sum_k \mathrm{Prob}(E_k) u[x(E_k)] .$$

Thus the individual need only make precise his personal probabilities of the events E_k. Warning: this does not imply that if he were to make precise the individual probabilities $P(s)$ they then would be equal for all states in a given event E_k.

5. EQUILIBRIUM

An equilibrium is a price system together with acts of consumers and producers such that, for the given price system, (a) each consumer maximizes his preferences subject to his wealth constraint, (b) each producer maximizes his profits, and (c) total demand equals total supply. Formally, if there are m consumers and n producers, an equilibrium is an $(m+n+1)$-tuple $(x_1^*, \ldots, x_m^*, y_1^*, \ldots, y_n^*, p^*)$ such that p^* is a price system and such that

(a.1) for every i, $x_i^* \in X_i$ and $p^* \cdot x_i^* \leqslant w_i^* \equiv p^* \cdot \omega_i + \Sigma_j \theta_{ij} p^* \cdot y_j^*$;
(a.2) for every i, if $x_i \in X_i$ and if $p^* \cdot x_i \leqslant w_i^*$, then $x_i \lesssim_i x_i^*$;
(b) for every j, $y_j^* \in Y_j$, and if $y_j \in Y_j$ then $p^* \cdot y_j \leqslant p^* \cdot y_j^*$;
(c) $\Sigma_i (x_i^* - \omega_i) = \Sigma_j y_j^*$.

In this section I shall first give sufficient conditions for the existence of an equilibrium for a given structure of information in the economy. I shall then give sufficient conditions on the potentially feasible sets for there to exist an equilibrium for any structure of information in the economy that is compatible with the resources of consumers.[9]

First, then, I consider the case of a given structure of information for each agent in the economy, as described in the previous sections. The main difficulty that arises here in the question of existence of equilibrium is that, with the restrictions imposed on acts by a given information structure, the simpler theorems on existence will typically not be applicable. For example, the assumption of free disposal is inappropriate here, at least in its usual form; even though a given act α is compatible with an information structure $\mathscr{S}$, the set of acts $\leqslant \alpha$ (in the vector sense) will typically contain acts that are not compatible with $\mathscr{S}$. However, the analysis of Debreu [3] does provide appropriate conditions. Before stating his result, I need to introduce some additional notation. An $(m+n)$-tuple $((x_i), (y_j))$ of acts is attainable if (a) for each i, $x_i \in X_i$; (b) for each j, $y_j \in Y_j$; and (c) $\Sigma_i(x_i - \omega_i) = \Sigma_j y_j$. An act x_i is attainable for consumer i if it is part of some attainable $(m+n)$-tuple of acts. Let $\hat{X}_i$ denote the set of attainable acts for consumer i. Define $x_i >_i \hat{X}_i$ to mean $x_i >_i x_i'$ for all $x_i' \in \hat{X}_i$. Further define:

$$D_i \equiv \{x_i : x_i \in X_i \cdot x_i >_i \hat{X}_i\} ,$$

$$D' \equiv \Sigma_i(D_i - \{\omega_i\}) ,$$

and define D as the smallest cone with vertex at 0 that contains D'. The set $D_i - \{\omega_i\}$ may be interpreted as the set of net trades by consumer i that would make him better off than in any attainable state of the economy.

Also define $X \equiv \Sigma_i X_i$, $Y \equiv \Sigma_j Y_j$, and $\omega \equiv \Sigma_i \omega_i$. Finally, for any set W, its *asymptotic cone*[10] will be denoted by $\underset{\sim}{A}W$.

The following theorem on the existence of an equilibrium is drawn from Debreu [3, pp. 259–260, 269–270].

THEOREM 1: *An equilibrium exists if:*

(a.1) $\underset{\sim}{A}X \cap (-\underset{\sim}{A}X) = \{0\}$;

for every i;

(a.2) X_i *is closed and convex*;

(b.1) (*nonsatiation in attainable set*) *for every* x_i *in* $\hat{X}_i$ *there is an* x_i' *in* X_i *such that* $x_i' >_i x_i$;

(b.2) (*continuity of preference*) *for every* x_i' *in* X_i *the sets* $\{x_i : x_i \in X_i \cdot x_i >_i x_i'\}$ *and* $\{x_i : x_i \in X_i \cdot x_i \lesssim_i x_i'\}$ *are closed*;

(b.3) (*convexity of preference*) *for every* x_i' *in* X_i *the set* $\{x_i : x_i \in X_i \cdot x_i \gtrsim_i x_i'\}$ *is convex*;

[9] This section is substantially more technical than the rest of the paper. The reader may skip here to the next section without loss of continuity.

[10] Given W, for any number $k \geqslant 0$ let W_k denote the smallest closed cone with vertex 0 containing all points in W with norm (length) at least k; then $\underset{\sim}{A}W \equiv \cap_{k \geqslant 0} W_k$. See Debreu [2, pp. 22, 23].

(c.1) *Y is closed and convex and the relative interiors*[11] *of* $(X-\{\omega\})$ *and of* Y *have a nonempty intersection*;

(c.2) *for every* i, $[\underset{\sim}{A}Y - \mathrm{Int}_C D] \cap [X_i - \{\omega_i\}] \neq \emptyset$, *where*[12] $C \equiv \Sigma_i \mathscr{A}(\mathscr{S}_i)$;

(d.1) 0 *is in* Y_j, *for every* j; *and*

(d.2) $(\underset{\sim}{A}X) \cap (\underset{\sim}{A}Y) = \{0\}$.

PROOF:[13] By the theorem and footnote 7 of Debreu [3, p. 259], there exists a quasi-equilibrium, say $((x_i^*), (y_j^*), p^*)$, such that[14] $p^* \cdot (\underset{\sim}{A}Y - D) \leqslant 0$, and hence $p^* \cdot D \geqslant 0$ and $p^* \cdot \underset{\sim}{A}Y \leqslant 0$. It follows that either p^* is strictly positive on $\mathrm{Int}_C D$, or $p^* \cdot C = 0$. Suppose that $p^* \cdot C = 0$; then for every i, $p^* \cdot X_i = 0$, by (b) of Section 2.3 and the definition of C. Hence for every consumer i, $p^* \cdot x_i^* = \min\{p^* \cdot x_i : x_i \in X_i\}$. By the same argument used to prove the proposition on p. 270 of Debreu [3] one can show that this contradicts hypothesis (c.1) of the present theorem. Hence p^* is strictly positive on $\mathrm{Int}_C D$, and therefore strictly negative on $(\underset{\sim}{A}Y - \mathrm{Int}_C D)$. By (c.2) and the fact just proved, for every consumer i there exists $x_i \in X_i$ such that $p^* \cdot (x_i - \omega_i) < 0$. Since $0 \in Y_j$ for every producer j, $p^* \cdot y_j^* \geqslant 0$, and hence $p^* \cdot \omega_i + \Sigma_j \theta_{ij} p^* \cdot y_j^* \geqslant p^* \cdot \omega_i > p^* \cdot x_i$. Hence the quasi-equilibrium is an equilibrium, which completes the proof of the theorem.

I now turn to the case in which each agent has a basic set of "potentially feasible" acts, but his actual feasible set consists of all those potentially feasible ones that are compatible with a given information structure (see Section 4). Let A_0 denote the set of all acts that are constant on S for each date t.

THEOREM 2: *If conditions* (a.1)–(b.3), (d.1), (d.2) *of Theorem* 1 *are satisfied for the potentially feasible economy generated by complete information,* $\mathscr{I}^1$, *and if* (c.1)* *the potentially feasible production sets* $Y_1^1, \ldots, Y_n^1$ *are each closed and convex, and their asymptotic cones are positively semi-independent*;[15] *furthermore, there is a* z *such that*

$$z = \sum_i z_i = \sum_j y_j,$$

$$z_i + \omega_i \in \mathscr{A}(\mathscr{S}_i^0) \cap \mathit{Int}\, X_i^1, \qquad \text{every } i,$$

$$y_j \in \mathscr{A}(\mathscr{T}_j^0) \cap \mathit{Int}\, Y_j^1, \qquad \text{every } j;$$

[11] The relative interior of a set W in R^N is the interior of W relative to the smallest linear variety containing W. A linear variety is a set of the form $\{x\} + V$, where V is any linear subspace of R^N and x is any point of R^N (not necessarily in V). Let w_0 be any point of W; then the smallest linear variety containing W is the set of all points of the form $w_0 + \Sigma_{i=1}^I \alpha_i(w_i - w_0)$ where I is any positive integer, $\alpha_1, \ldots, \alpha_I$ are any real numbers, and $w_1, \ldots, w_I$ are any points in W.

[12] Int_C here denotes "interior relative to C."

[13] This proof assumes knowledge of the terminology of Debreu [3].

[14] For any set $W \subseteq A$, write $p^* \cdot w \leqslant 0$ if $p^* \cdot w \leqslant 0$ for all $w \in W$ (and similarly for $\geqslant$ and $=$).

[15] See Debreu [2, p. 22].

and (c.2)* *for each i there exist* z_i, v_i, d_i *such that* $z_i = v_i - d_i$, $z_i \in X_i^0 - \{\omega_i\}$, $z_i \in A_0$, $v_i \in \underset{\sim}{A} Y^0$, *and* $d_i = h\Sigma_{k=1}^m (x_{ik} - \omega_k)$ *for some* $h > 0$ *and some* $x_{il}, \ldots, x_{im}$ *for which* $x_{ik} \in \mathit{Int}\, D_k^1$ *and* $(x_{ik} - \omega_k) \in A_0$ *for each* $k = 1, \ldots, m$; *then for every admissible information structure* $\mathscr{I}$ *for the economy there is an equilibrium (where the sets* Y^0, D^1, C^1 *are the sets* Y, D, C *corresponding to* $\mathscr{I}^0$, $\mathscr{I}^1$ *and* $\mathscr{I}^1$, *respectively).*

PROOF: For any set W, let $\mathscr{L}(W)$ denote the smallest linear variety containing W. I preface the proof with two lemmas about linear varieties and relative interiors.

It is to be understood that everything takes place in a fixed (finite dimensional) Euclidean space.

LEMMA 1: *Let* U *be an open set and* L *a subspace such that* $U \cap L \neq \varnothing$; *then* $L = \mathscr{L}(U \cap L)$.

PROOF: Let u be a point in $U \cap L$, and let V be the unit sphere in L. For $\varepsilon > 0$ sufficiently small, $(\{u\} + \varepsilon V) \subset U \cap L$. Hence

$$L = \{u\} + L = \mathscr{L}(\{u\} + \varepsilon V) \subset \mathscr{L}(U \cap L) \subset L,$$

which proves Lemma 1.

LEMMA 2: *Let* $W_1, \ldots, W_K$ *be sets with nonempty interiors; for each* $k = 1, \ldots, K$ *let* L_k *be a subspace that intersects* $\mathit{Int}(W_k)$; *and let* $L \equiv \Sigma_{k=1}^K L_K$ *and* $W^* \equiv \Sigma_k (W_k \cap L_k)$; *then*

$$\sum_k (\mathit{Int}_{L_k}[W_k \cap L_k]) \subset \mathit{Int}_L(W^*), \tag{5.1}$$

$$\mathscr{L} \sum_k [W_k \cap L_k] = L. \tag{5.2}$$

PROOF: For every k let w_k be a point of $\mathrm{Int}_{L_k}[W_k \cap L_k]$, and let $w = \Sigma_k w_k$. There exists an $\varepsilon > 0$ such that for every k the set U_k defined by

$$U_k \equiv \{u_k : u_k \in L_k \cdot \|u_k - w_k\| < \varepsilon\}$$

is contained in $W_k \cap L_k$. To demonstrate that $w \in \mathrm{Int}_L(W^*)$, I shall show that for some $\sigma > 0$ the set U defined by

$$U \equiv \{u : u \in L \cdot \|u - w\| < \sigma\} \tag{5.3}$$

is contained in W^*.

There exist $P_1, \ldots, P_K$ such that for each k, P_k is a linear transformation from L into L_k, $\Sigma_{j=1}^k P_j$ is a projection onto $\Sigma_{j=1}^k L_j$. For each k, $P_k' P_k$ is nonnegative semi-definite (P' denotes the adjoint of P), and its largest characteristic root, say r_k, is nonnegative. Let $r^2 \equiv \max(r_1, \ldots, r_K)$. For any $v \in L$, $v = \Sigma_{k=1}^K P_k v$, $\|P_k v\| \leqslant r\|v\|$. Hence $\|v\| < (\varepsilon/r)$ implies that for every k, $\|P_k v\| < \varepsilon$. Let U be defined as in (5.3) with $\sigma=(\varepsilon/r)$; then for any $u\in U$, $u-w=\Sigma_k P_k(u-w)$, $\|P_k(u-w)\|<\varepsilon$, $P_k(u-w)\in L_k$. Let $u_k\equiv w_k+P_k(u-w)$; then $u_k\in L_k$, $\|u_k-w_k\|<\varepsilon$, $\Sigma_k u_k=w+\Sigma_k P_k(u-w)=u$. Hence $u\in\Sigma_k U_k$. Therefore

$$U\subset\sum_k U_k\subset\sum_k(W_k\cap L_k)=W^*\,.$$

To prove (5.2), it follows from Lemma 1 that

$$L=\mathscr{L}(U)\subset\mathscr{L}(\sum_k[W_k\cap L_k])\subset\sum_k L_k=L\,.$$

I turn now to the proof of the theorem. Consider a fixed admissible information structure $\mathscr{I}$, and the corresponding economy with consumption sets $X_i\equiv X_i(\mathscr{I})$, preferences $\precsim_i$, resources ω_i, and production sets $Y_j\equiv Y_j(\mathscr{I})$. It is straightforward to verify that (a.1)–(b.3), (d.1), and (d.2) of Theorem 1 are satisfied, and also that $Y\equiv\Sigma_j Y_j$ is convex. Furthermore, every Y_j is closed, and the asymptotic cones $\underset{\sim}{A}Y_1,\ldots,\underset{\sim}{A}Y_n$ are positively semi-independent; hence Y is closed (see Debreu [2, p. 23]).

To verify the rest of (c.1) of Theorem 1, I first show that, in (c.1)* of the present theorem, $z=\Sigma_j y_j$ is in the relative interior of Y. Let $L_j\equiv\mathscr{A}(\mathscr{T}_j)$, $L\equiv\Sigma_j L_j$, and note that $y_j\in L_j$, since $L_j\supset\mathscr{A}(\mathscr{T}_j^0)$. Hence, for every j, $y_j\in\mathrm{Int}_{Lj}\,Y_j$. Since $Y_j=Y_j^1\cap L_j$, it follows from Lemma 2 that $z\in\mathrm{Int}_L\,Y$, which equals the relative interior of Y. Similarly, one can verify that z is in the relative interior of $\Sigma_i(X_i-\omega_i)$.

I turn now to the verification of (c.2) of Theorem 1. It follows from (c.2)* of the present theorem that for every i and k, $x_{ik}\in\mathscr{A}(\mathscr{S}_k)$, $x_{ik}\in\mathrm{Int}\ W_k$, where $W\equiv\{x_k:x_k\in Y_k^1\cdot x_k>_k X_k(\mathscr{I})\}$, since $\hat{X}_k(\mathscr{I})\subset\hat{X}_k^1$. Note that $D_k(\mathscr{I})=W_k\cap\mathscr{A}(\mathscr{S}_k)$; hence $x_{ik}\in\mathrm{Int}_{\mathscr{A}(\mathscr{S}k)}D_k(\mathscr{I})$, $x_{ik}-\omega_k\in\mathrm{Int}_{\mathscr{A}(\mathscr{S}k)}[D_k(\mathscr{I})-\{\omega_k\}]$. Hence by Lemma 2

$$\sum_{k=1}^{m}(x_{ik}-\omega_k)\in\mathrm{Int}_C\sum_{k=1}^{m}[D_k(\mathscr{I})-\{\omega_k\}],$$

where $C\equiv C(\mathscr{I})=\Sigma_{i=1}^m\mathscr{A}(\mathscr{S}_i)$. Hence $d_i\in\mathrm{Int}_C\,D(\mathscr{I})$. Also $v_i\in\underset{\sim}{A}Y$, since $Y^0\subset Y$. Therefore $z_i\in\underset{\sim}{A}Y-\mathrm{Int}_C D(\mathscr{I})$. Finally, $z_i\in X_i-\{\omega_i\}$, since $X_i^0\subset X_i$. This completes the verification of (c.2), and therefore the proof of Theorem 2.

6. OPTIMUM

The concepts of *optimum*[16] and of *equilibrium relative to a price system* (see Debreu [2, Ch. 6]) can be applied directly to the present model of an economy under uncertainty. It should perhaps be emphasized that "optimum" here must be interpreted as "optimum relative to a given structure of information in the economy." In particular, in the context of the end of Section 4, in which an economy

[16] Sometimes called Pareto optimum.

is defined in terms of potentially feasible acts constrained by a given structure of information, a refinement of information structure results in an enlargement of the set of actually feasible acts, and in principle a new set of optima.

Debreu [**2**, Ch. 6, Sections 3 and 4] gives conditions under which an equilibrium relative to a price system is an optimum, and under which the converse is true. The reader is referred to this source for the details. I might mention that again convexity of preferences, and therefore in this context, risk aversion, plays a role.

7. AN EXAMPLE

An example may help the reader to understand the formal description of the theory. Furthermore, this example will illustrate the constraints on trade that may result from differences in information among agents. To keep the example simple, time (i.e., dating of commodities and information) is left out.

Consider an economy with two consumers, one producer, and two "commodities." To aid the imagination, the two commodities may be thought of as "labor" and "food." The producer can transform any nonnegative number of units of labor into an equal number of units of food (the units of measurement are fixed). The resources of the economy consist entirely of labor, held by the two consumers; however, there is uncertainty about the actual quantities of these resources. The uncertainty is described in terms of three states of nature, as in Table I. Thus, in states 1 and 2 consumer 1 will have at his disposal one unit of labor, whereas in state 3 he will have two units, etc.

TABLE I

RESOURCES

States of Nature	Labor Resources of: Consumer 1	Consumer 2
1	1	1
2	1	2
3	2	2

To complete the description of the uncertainty, we must say what each agent knows about the state of nature when he makes his economic decisions. Assume that the producer knows nothing beyond the facts in Table I, whereas, each consumer knows (in addition) the quantity of labor that he himself has at his disposal. More precisely, one may say that (i) all three agents know Table I; (ii) if the true state of nature is 1 or 2, then consumer 1 knows only that it is one of these two (but not which one), whereas, if the true state is 3, then he knows that it is 3; (iii) if the true state is 1, then consumer 2 knows it, whereas, if the true state is 2 or 3, then he knows only that it is one of these two; (iv) the producer does not know which of the three is the true state.

A decision of a consumer is a trade, and any trade results in a net consumption vector for him. A decision of a producer is a production plan, which results in a net production vector. Since consumption and production may depend upon which state of nature is realized, I shall distinguish six commodities, two for each state, as in Table II. Using this notation, the resource vectors, ω_1 and ω_2, of consumers 1 and 2, respectively, are

$$\omega_1 = (1, 0, 1, 0, 2, 0)\,, \qquad \omega_2 = (1, 0, 2, 0, 2, 0)\,. \tag{7.1}$$

Let $x_i = (x_i^1, \ldots, x_i^6)$ denote a (net) consumption vector for consumer i ($i = 1, 2$). Consumption of labor may be interpreted as leisure, and equals the difference between resources and what is traded. To describe the constraints on the decisions of consumer 1 that are imposed by the structure of his information (see (ii) above), I assume

$$x_1^1 = x_1^3, \quad x_1^2 = x_1^4\,. \tag{7.2}$$

Similarly, the information constraints for consumer 2 are (see (iii) above)

$$x_2^3 = x_2^5, \quad x_2^4 = x_2^6\,. \tag{7.3}$$

If $y = (y^1, \ldots, y^6)$ denotes a (net) production vector (here outputs are positive and

TABLE II

LIST OF COMMODITIES

Commodity	Interpretation
1	labor in state 1
2	food in state 1
3	labor in state 2
4	food in state 2
5	labor in state 3
6	food in state 3

inputs are negative), the producer's information constraints are (see (iv) above)

$$y^1 = y^3 = y^5, \quad y^2 = y^4 = y^6\,. \tag{7.4}$$

To further specify the set of feasible consumption vectors, suppose, for the purposes of this example, that whatever the state of nature, (1) each consumer must consume at least 0.1 units of food, (2) consumption of labor is nonnegative, and (3) quantity of labor sold is nonnegative. These conditions are described by the constraints

$$
\begin{aligned}
&x_i^2, x_i^4, x_i^6 \geqslant 0.1, \qquad i=1,2;\\
(7.5)\qquad &x_i^1, x_i^3, x_i^5 \geqslant 0;\\
&\quad x_1^1, x_1^3 \leqslant 1; \quad x_1^5 \leqslant 2\ ;\\
&\qquad x_2^1 \leqslant 1; \quad x_2^3, x_2^5 \leqslant 2\,.
\end{aligned}
$$

The constraints on the producer's decisions are further specified by

$$(7.6)\qquad y^1 = -y^2, \quad y^3 = -y^4, \quad y^5 = -y^6; \quad y^2, y^4, y^6 \geqslant 0\,.$$

The constraints (7.2) and (7.5) define consumer 1's *consumption set*, X_1. Similarly, (7.3) and (7.5) define X_2. Constraints (7.4) and (7.6) define the *production set*, Y.

Suppose that there is a "market" for every commodity, i.e., for labor and food in each of the states. Let p^i denote the price of commodity i ($i=1, \ldots, 6$). Suppose further that the two consumers own between them all the shares of the production enterprise, in equal numbers. The profit of a production vector y is

$$(7.7)\qquad p \cdot y \equiv \sum_{k=1}^{6} p^k y^k\,.$$

The wealth w_i of consumer i is the value of his resources plus his share of the profits of production:

$$(7.8)\qquad w_i \equiv p \cdot \omega_i + \tfrac{1}{2} p \cdot y\,.$$

Finally, assume that the preferences of consumer i are represented by the function

$$(7.9)\qquad U(x_i) = 3(x_i^2 + x_i^4 + x_i^6) + (x_i^1 + x_i^3 + x_i^5)\,.$$

For simplicity, I have assumed that the two consumers have identical preferences.

If, given a price vector p, the producer chooses a production vector y to maximize profit $p \cdot y$, and given p and w_i, each consumer i chooses a consumption vector in X_i to maximize $U(x_i)$, subject to the wealth constraint $p \cdot x_i \leqslant w_i$, then it is easily verified that an equilibrium of this economy is

$$
\begin{aligned}
&\hat{p} = (7, 3, 1, 3, 1, 3)\,,\\
(7.10)\qquad &\hat{y} = (-2, 2, -2, 2, -2, 2)\,,\\
&\hat{x}_1 = (0, 1, 0, 1, 1, 1)\,,\\
&\hat{x}_2 = (0, 1, 1, 1, 1, 1)\,.
\end{aligned}
$$

Note that for this equilibrium, profit from production is 0, and the wealths are

$$
\begin{aligned}
(7.11)\qquad &w_1 = (7+1+2) + \tfrac{1}{2}(0) = 10\,,\\
&w_2 = (7+2+2) + \tfrac{1}{2}(0) = 11\,.
\end{aligned}
$$

A few remarks on the interpretation of this example are in order. First, although markets for all six commodities have been assumed, the clearing of these markets, together with the constraints (7.1)–(7.4) imposed by the structure of information, imply that only contracts for delivery independent of the state of nature are in fact possible. The market clearing condition is

$$(7.12) \qquad (x_1-\omega_1)+(x_2-\omega_2)=y\,.$$

Define $\xi_i \equiv x_i - \omega_i$; then (7.12) can be rewritten as

$$(7.13) \qquad \xi_1+\xi_2=y.$$

It is easy, if slightly tedious, to verify that (7.1)–(7.4) and (7.13) imply

$$(7.14) \qquad \begin{array}{ll} \xi_1^1=\xi_1^3=\xi_1^5, & \xi_1^2=\xi_1^4=\xi_1^6\,, \\ \xi_2^1=\xi_2^3=\xi_2^5, & \xi_2^2=\xi_2^4=\xi_2^6\,. \end{array}$$

But ξ_i^k is the net amount of commodity k bought by consumer i (selling is buying a negative amount), so that an examination of (7.14) reveals that consumer 1 must sell the same quantity of labor in each state of nature, and buy the same quantity of food in each state. The same statement applies to consumer 2. The producer, by the structure of his information (constraint (7.4)), must make the same trade in each state.

A corollary of the fact that, in equilibrium, only contracts that are constant over states ("sure" contracts) will be observed (for this example), is the fact that essentially only two prices, instead of six, will be operative. To see this, note that a contract to (say) buy F units of food and sell L units of labor, in each state, has a net cost of $(p^2+p^4+p^6)F-(p^1+p^3+p^5)L$. Hence the two relevant prices for sure contracts are $(p^2+p^4+p^6)$, the price of sure delivery of food, and $(p^1+p^3+p^5)$, the price of sure delivery of labor.

The restriction to sure contracts in this example is an extreme illustration of the general phenomenon that clearing markets together with differences in information among economic agents will lead to a reduction in the number of markets.

8. INFORMATIONAL CONSTRAINTS ON TRADE

For any consumer i, and any consumption act x_i, define his *trade act* z_i by $z_i=\omega_i-x_i$. Thus $z_{itc}(s)$ is the quantity of commodity c at date t in state s that consumer i sells (if $z_{itc}(s)\geqslant 0$) or buys (if $z_{itc}(s)\leqslant 0$). Define the trade act of a producer to be identical to his production act. For the purpose of this section, I shall ignore the distinction between consumers and producers, and simply consider that there is a set K of $(m+n)$ economic agents, that agent k has an information structure $\mathscr{S}_k$, and that he must choose a trade act $z_k \in Z_k$, where Z_k, the set of feasible trades for k, is a subset of $\mathscr{A}(\mathscr{S}_k)$.

If the $(m+n)$-tuple (z_k) of trades satisfies

$$(8.1) \qquad \sum_{k\in K} z_k=0\,,$$

then (z_k) clears the market.

Let $\underset{\sim}{z}=(z_1, \ldots, z_{m+n})$ be an $(m+n)$-tuple of trades, and $I \subset K$ a subset of agents; the *net trade* of I is

$$(8.2) \qquad \zeta(\underset{\sim}{z}, I) \equiv \sum_{k \in I} z_k .$$

If $\underset{\sim}{z}$ clears the market then for every subset I of K

$$(8.3) \qquad \zeta(\underset{\sim}{z}, I) = -\zeta(\underset{\sim}{z}, \sim I),$$

(where $(\sim I)$ denotes the complement of I in K). Now note that, for every $I \subset K$,

$$(8.4) \qquad \zeta(\underset{\sim}{z}, I) \in \sum_{k \in I} \mathscr{A}(\mathscr{S}_k) ;$$

hence (8.3) implies that for every market clearing $\underset{\sim}{z}$ and every subset $I \subset K$ the net trade of I satisfies

$$(8.5) \qquad \zeta(\underset{\sim}{z}, I) \in \Big[\sum_{k \in I} \mathscr{A}(\mathscr{S}_k)\Big] \cap \Big[\sum_{k \notin I} \mathscr{A}(\mathscr{S}_k)\Big] .$$

Condition (8.5) is a formal expression of the fact that *the net trade between any group of agents and the group of all other agents in the economy can at most depend upon information that is common to both groups of agents.*

Since no agent can have a trade act that depends on information not available to him, there need be no markets for contracts that depend upon information that is not available to someone in the economy. Formally, for each k,

$$(8.6) \qquad z_k \in \sum_{k} \mathscr{A}(\mathscr{S}_k) .$$

In particular, prices for delivery at date t will not depend upon events that will occur, if at all, on some date later than t (see Debreu [**2**, Ch. 7, Sections 2, 3]).

However, the set of effective markets implied by (8.5) will typically be even smaller than that implied by (8.6) alone. This phenomenon shows up in the example of Section 7.

9. CHOICE OF INFORMATION

There is no problem in incorporating the choice of information into the present formal framework. There is a problem, however, in determining under what economically meaningful conditions the resulting model satisfies the "classical" conditions for the existence and optimality of competitive equilibrium. My general impression is that it is not typically realistic to suppose that these conditions are satisfied.

For a given act α, let $\mathscr{S}(\alpha)$ denote the least fine information structure with which α is compatible; I shall say that $\mathscr{S}(\alpha)$ is the information structure *required by* α. Thus an act α such that $\alpha_t(s)=\alpha_t(s')$ for all dates t and all states s and s' in S requires the minimal information structure ("no information"), in which $\mathscr{S}_t=\{S\}$ for all t

(see Section 3). On the other hand, an act α such that $\alpha_t(s) \neq \alpha_t(s')$ for all dates t and all *distinct* states s and s' in S ($s \neq s'$), requires complete information. Indeed, one may simply characterize $\mathscr{S}(\alpha)$ as follows: the elements of the partition $\mathscr{S}_t$ are inverse images $\alpha_t^{-1}(v)$ of points v in R^C.

The acquisition and use of information typically requires the use of resources. This may be reflected in the description of the set of feasible acts. Thus, in the example of Section 7, suppose that for the producer to obtain and use complete information about the state of nature requires the input of one unit of labor. To express this, I adjoin to his set of feasible acts the set Y'' of all $y = (y^1, \ldots, y^6)$ such that

$$y^1 = -1 - y^2,\ y^3 = -1 - y^4,\ y^5 = -1 - y^6;\ y^2, y^4, y^6 \geqslant 0\,. \tag{9.1}$$

Recall that in Section 7 it was assumed that the following production acts were also feasible: the set Y' of all y such that

$$y^1 = -y^2,\ y^3 = -y^4,\ y^5 = -y^6; \quad y^2, y^4, y^6 \geqslant 0\ ; \tag{9.2a}$$

$$y^1 = y^3 = y^5,\ y^2 = y^4 = y^6 \tag{9.2b}$$

(see equations (7.6) and (7.4)). Recall that (9.2b) expresses the constraint of "no information"; in the present context (9.2a) expresses the "potentially feasible" production possibilities, but without taking account of the "cost of information."

We may now suppose that the total production possibility set for the producer is

$$Y = Y' \cup Y''\,, \tag{9.3}$$

the union of the possibilities with and without information. From a formal descriptive point of view, this is satisfactory, but it is easy to verify that, whereas Y' and Y'' are each convex, Y is not. For example, $y' = (-1, 1, -1, 1, -1, 1) \in Y'$, $y'' = (-2, 1, -3, 2, -4, 3) \in Y''$, but $(\frac{1}{2}y' + \frac{1}{2}y'')$ is neither in Y' nor in Y''. In this example, the cost of information is a "set-up cost" (a well-known destroyer of convexity!). Notice that the set Y'' includes acts that do not require any information about the state of nature, even though the cost of obtaining that information has, in some sense, been incurred (e.g., the act $(-2, 1, -2, 1, -2, 1)$).

An important class of cases in which convexity may be reasonable is generated by situations in which future information depends in some way upon current action, and all actions can be "scaled down" to any desired size. For example, the model of dynamic production in Radner [**6**] can be extended in a straightforward way to the case of uncertainty, following the usual model of dynamic programming, in a manner that preserves convexity.

10. NETWORK INFORMATION

Thus far it has been assumed that each agent's set of available acts is constrained by a fixed information structure, or by a family of alternative fixed information structures, in the sense of Section 3. In any case, the only joint constraint has been the one that total supply equal total demand; apart from that constraint each agent could choose his act independently of the choices of other agents. In other words, there have been no *external effects* among the agents' acts. To make this quite precise, recall that if for each i consumer i chooses the act x_i from his consumption set X_i, then the total consumption for the economy is

$$x = \sum_i x_i \,. \tag{10.1}$$

In (10.1), x is also, in a sense, an act, i.e., it is T-tuple of functions from the set S of states of nature to the commodity space. On the other hand, any act x that can be represented in the form (10.1), with x_i in X_i for each i, is a possible total consumption for the economy. In other words, the total consumption set, say X, for the economy is the (vector) sum of the individual consumption sets. Similarly, the total production set, say Y, for the economy is the sum of the individual production sets Y_j. Finally, a set of consumption acts x_i and production acts y_j is attainable if

$$x - y = \omega \,, \tag{10.2}$$

where $\omega = \Sigma_i \omega_i$ (total resources). The net excess demand for the economy, $(x - y)$, is also an act; the set of excess demands for the economy, not taking account of the constraint (10.2), is the set

$$\sum_i X_i - \sum_j Y_j \,. \tag{10.3}$$

The representation (10.3) of the set of excess demands for the economy expresses the fact that there are no external effects among the acts of the several economic agents.

In Sections 3 and 4 the assumption of fixed information structure, represented by a partition of the set of states of nature, was interpreted as representing information about the environment. The constraint corresponding to such information was superimposed on a set of "potentially feasible" acts, i.e., a set of acts that would be feasible with complete information about the environment. Such constraints, together with the assumption that there be no external effects among the sets of potentially feasible acts, resulted in the absence of external effects among the sets of acts that were feasible under the given structure of information.

By contrast, I shall argue that if agents receive information about each other's decisions, then even if there are no external effects among the sets of potentially feasible acts, the sets of acts constrained by information will typically exhibit external effects. To give a precise idea of the nature of these externalities I shall describe a formal model of the kind of information structure I have in mind.[17] Since the model represents, in part, a communication network among the decision makers, this will be called *network information*.

[17] This model is based on Marschak and Radner [**4**, Chapter VIII].

Suppose that at each date each agent may receive messages from, and send messages to, other agents in the economy. For the purposes of the present discussion it is not necessary to distinguish between consumers and producers; the $(m+n)$ agents in the economy will simply be numbered from 1 to K. A message sent by agent h to agent k on date t is assumed to be received by k on date $(t+1)$. Let B_{hkt} denote the set of possible alternative messages b_{hkt} that h can send to k on date t. Direct observation of the environment will be similarly represented; let $B_{0k,t-1}$ be the set of possible alternative observations of the environment on date t. Such an observation may be interpreted as a "message from Nature to k;" hence the notation. Decisions about inputs and outputs can also be represented as "messages;" thus for every k and t let B_{k0t} be R^C, the commodity space. Finally, the process of remembering may be represented as the sending of messages to oneself; thus for every k and t, let B_{kkt} denote the set of alternative complete descriptions of what k is capable of remembering from date t to date $t+1$.

The task of each agent at each date is to transform incoming messages into outgoing messages. In particular, this includes decisions about inputs and outputs. For agent k, the set of alternative incoming messages, taking account of all sources including Nature and memory, is the Cartesian product

$$(10.4)\qquad \tilde{B}_{kt} = \mathop{\times}_{h=0}^{K} B_{hk,t-1}\ ;$$

the set of alternative outgoing messages is likewise the Cartesian product

$$(10.5)\qquad B_{kt} = \mathop{\times}_{h=0}^{K} B_{kht}\ .$$

For each k and t, agent k will have available to him some set $\mathscr{B}_{kt}$ of functions β_{kt} from $\tilde{B}_{kt}$ to B_{kt}. One may call such a T-tuple, $\beta_k \equiv (\beta_{k1}, \ldots, \beta_{kT})$, a *strategy*, thus extending the concept of strategy introduced in Section 3. The set of strategies for k will be denoted by $\mathscr{B}_k$.

Notice that the choice of a strategy by an agent is not typically sufficient to determine his act; rather it takes a determination of strategies by all of the agents to jointly determine their acts, at least in principle. There is, however, a recursive feature in the determination of acts. The component α_{kt} of the act of agent k at date t is determined by the components β_{hr} of the strategies of other agents for $r=1, \ldots, t-1$, and by the components β_{kr} of k's strategy for $r=1, \ldots, t$.

Corresponding to this recursive determination of acts is a recursive determination of information structure at each date, in the sense of partitions. Given the components β_{hr} of all strategies for $r=1, \ldots, t-1$, the joint message received by any agent k is a well defined function of the state of nature. This determines a partition, say $\mathscr{S}_{kt}$, of the set S of states, and any strategy component β_{kt} in $\mathscr{B}_{kt}$ determines a component act α_{kt} that is compatible with $\mathscr{S}_{kt}$.

Let $\mathscr{A}$ denote the set of all K-tuples of acts (one for each agent) generated by all K-tuples of strategies $(\beta_1, \ldots, \beta_K)$ such that for each k, β_k is in $\mathscr{B}_k$. For each agent k, let $\mathscr{A}_k$ denote the set of all acts α_k such that α_k is the kth component of a K-tuple in $\mathscr{A}$. It is clear from the above remarks that in general $\mathscr{A}$ is not the Cartesian product of the sets $\mathscr{A}_k$, and hence the set of excess demands for the economy cannot in general be represented in the form (10.3). It is in this sense that network information introduces external effects among the sets of acts of the various agents.

Since the choice of a strategy by an agent does not, in general, determine an act independently of the choice of strategies by others, the agent will be uncertain as to the actual act that follows from a given strategy choice. In particular, he will be uncertain about the value of a given strategy for given prices. It should be emphasized that his uncertainty is not just about what his inputs and outputs will be at various dates, but about what these inputs and outputs will be for a given state of nature. This uncertainty derives from his uncertainty about the strategies of other agents. This feature takes us outside the framework of the standard theory of competitive equilibrium, and I shall not attempt to give a rigorous analysis in the present paper.

11. MARKET INFORMATION AND DEFERRED MARKETS

Return now to the model of Sections 2–6, with fixed information structure. Recall that, under appropriate conditions, an equilibrium relative to a price system is an optimum (with respect to the given information structure), and every optimum is an equilibrium for some distribution of resources (see Section 6). To achieve an equilibrium relative to a price system, those prices have to be communicated or somehow made available to the agents in the economy. Hence the competitive equilibrium achieves an optimum relative to a given structure of information by making available to the agents some additional information, namely an equilibrium price system.

(Thus far, nothing has been said about how equilibrium prices are to be determined. Presumably, this takes place through some process of interaction among economic agents, possibly including agents whose sole function is to aid in the process of price formation. In other words, the process of price formation itself might be described in terms of the model of network information presented in the previous section, with the attendant difficulties alluded to there.)

Once the equilibrium prices have been announced, and acts have been chosen, it is assumed that no further market transactions take place; indeed, there should be no incentive for such further transactions, since the equilibrium is an optimum. But this optimum is relative to a given structure of information; hence to the extent that new market transactions provide new information, there might well be an incentive to enter into them.

Consider the possibility of setting up a new market at the last date, T. Suppose that the given structure of information is such that it does not provide each agent with complete information about the history of the environment up to that date. Nevertheless, since at that date everyone would know his own resources and stocks, and only trade decisions would be required, a market equilibrium would lead to an optimum for the last date, relative to the situation of the economy at the beginning of that period.

Let q denote the price vector for that equilibrium (assuming for the sake of simplicity that it is unique). There is no reason to suppose that q is proportional to p_T, the component of the *ex ante* equilibrium price system. In a sense, p_T is a vector of prices for futures contracts, whereas q is a vector of "spot" prices. As a matter of fact, there is every reason to believe that q would typically be different from p_T. Note that q depends on the evolution of the economy, and therefore the state of nature, up through date $T-1$, both through direct observations of the environment made by the agents, and through the decisions made, which determine the stocks at the beginning of date T. Hence, for given acts of the economic agents (at least specified through date $T-1$), q is a function defined on the set of states of nature. Therefore the announcement of q at the beginning of date T would typically provide each agent with information beyond that contemplated in his original structure of information, to the extent that he could guess the strategies (or acts) of the other agents.

What we have here is an example of network information, with the resulting externalities (see Section 10). The introduction of the spot markets brings with it the need for economic agents to be concerned not only with uncertainty about the environment, but also with uncertainty about other agents' strategies. In particular, producers can no longer assign a definite value to a strategy on the basis of futures contract prices alone. This would typically result in a demand for "liquidity," which was not present in the fixed information structure case. The need for money does not arise except in response to an "imperfection." Unfortunately, the very imperfection that gives rise to the demand for liquidity and money in this case prevents individual agents from determining their optimal demands for these things. In the next section, under the discussion of computation, this phenomenon will appear again.

12. COMPUTATION

Computation plays a role in the behavior of an economic agent in at least two ways. First, the implementation of a given strategy may require computation. Second, the choice of a preferred (or profit-maximizing) strategy from a given set typically requires computation.

The computational costs of implementing a given strategy may be described formally in the same way as was suggested for taking account of the cost of information (Section 9), and with the same attendant difficulties. Indeed, it may be difficult in many cases to meaningfully separate informational and computational costs incurred in the implementation of a given strategy.

Costs incurred in the choice of a strategy present a special problem, which threatens to involve the model builder in an "infinite regression." To focus on this problem, consider a choice by a given economic agent between two information structures. Suppose that the second structure is considerably finer than the first, and that the associated optimization problem is more complicated, requiring more resources for its solution. Until the two optimization problems are solved, or at least some computation is done, the agent cannot decide whether the finer information structure is worth having. In deciding whether or not to perform these computations he is reduced to forming "preferences" or "beliefs" about the outcome of purely logical operations, a phenomenon that has not yet been successfully incorporated into a theory of rational behavior (see Savage [**7**, p. 7, footnote]).

Brushing aside this difficulty, we may suppose that the costs of choice have been separated from the costs of implementation, and that the decision maker has a given capacity for computation associated with choice. He may remain within that capacity by using certain "rules of thumb." Or he may choose to ignore some information, or to ignore the fact that he will have some particular information in the future. To formalize the second device, we may say that in the formulation of his optimization problem he restricts himself to an information structure that is less fine than the one that is actually available to him. In effect, he restricts himself to a smaller set of acts than is actually feasible.

I would like to explore a little this last formulation. In the one-period case it leads immediately to the concept of an equilibrium (or optimum) relative to a given "structure of information and computation." Indeed, the restriction on computation has been reduced to one on information, and no new theory is needed.

In the several-period case, however, a new problem arises. If the capacity for "choice computation" imposes a limit per period, then further choices are possible after the initial period, i.e., after the initial choice of strategy. Thus if in period 1 the decision maker chose a strategy ignoring the fact that he would receive one of two alternative signals (e.g., "rain" or "shine") in period 2, then in period 2, after having actually received one of these signals, he may wish to, and be able to, revise his strategy. I emphasize that I am discussing the revision of an entire strategy, not simply the implementation of a given strategy.

In the model of Sections 2–8 a strategy is embodied in a contract to receive and deliver various commodities on various dates under various circumstances. Revising a strategy requires making a further contract for additional receipts and deliveries (these new decisions may to a certain extent have the effect of cancelling or revising parts of the old contract, as when a loan is refinanced). Such new contracting will take place under new prices. Furthermore, new markets may be opened up, since finer information structures will be taken account of than were in the first period. Note also that in our previous model payment was made only at the beginning of the history of the economic system, when all contracts were made, whereas in the present model the making of new contracts in each period imposes the need for payments in each period. Finally, there will be uncertainty about the new prices and payments, and about which new markets will open up. This uncertainty will involve more than uncertainty about the environment, but also uncertainty about the outcomes of computations by the agents in the economy, and hence extend, through uncertainty about future equilibria, to uncertainty about other aspects of economic behavior. In this respect the situation is similar to that of Section 11.

A host of additional economic phenomena and considerations will be brought into play, which were not present in the original model of uncertainty: deferred payment, trading by consumers in shares of the productive enterprises, liquidity, to name a few. For example, if a new, additional, contract by a consumer has a negative value, then it is possible only if he has not previously exhausted his wealth budget. This suggests the desirability of reserving unspent a part of his wealth budget at any date, i.e., of retaining some liquidity. Unfortunately, the consumer is not in a position to compute his optimal amount of liquidity at any time, since this would require the results of computations that are to be performed only in the future![18]

13. CONCLUSION

This exploration of competitive equilibrium under uncertainty has indicated that if economic decision makers are uncertain about the environment, and if their information is about the environment, then even if they have different information, a once-and-for-all futures market in conditional contracts can achieve an optimum allocation of resources, relative to the given structure of information. In this "Arrow-Debreu world" there is no money and no demand for liquidity. On the other hand, the introduction of information about the behavior of other decision makers introduces externalities among the sets of acts available to them. A particular case of this results from the introduction of deferred, or "spot," markets. The presence of such markets enlarges the set of acts jointly available to the individual decision makers in the economy, but also introduces the above-mentioned externalities. A demand for liquidity is generated, but individual decision makers cannot, in principle, calculate their optimal kind and degree of liquidity without knowing something about the decision rules of other individuals.

The demand for liquidity also arises from computational limitations, and with such limitations would be present even in a world of certainty about the environment, if that world were sufficiently complicated. Of course, uncertainty about the environment vastly complicates a decision problem, and so indirectly contributes to the demand for liquidity.

[18] Of course "liquidity" can also be obtained by holding, or contracting for future delivery of, commodities for which there is sure to be a market. This only complicates the consumer's decision problem.

This exploration suggests that a general equilibrium theory incorporating the most important aspects of money and liquidity cannot be based on a "classical" approach with thoroughgoing optimizing, and therefore that there may be little hope of extending the classical theorems of welfare economics to the more general case. Such an extension, however, does appear possible if one is satisfied with concepts of equilibrium and optimum that are defined in terms of long-run statistical averages; but this represents a program for future research.

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[Footnotes]

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