NEW IDEAS IN PURE THEORY*

PROBLEMS IN THE THEORY OF MARKETS UNDER UNCERTAINTY†

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Introduction

One of the notable intellectual achievements of economic theory during the past twenty years has been the rigorous elaboration of the Walras-Pareto theory of value; that is, the theory of the existence and optimality of competitive equilibrium. Although many economists and mathematicians contributed to this development, the resulting edifice owes so much to the pioneering and influential work of Arrow and Debreu that in this paper I shall refer to it as the "Arrow-Debreu theory." (For a comprehensive treatment, together with references to previous work, see [6].)

The Arrow-Debreu theory was not originally put forward for the case of uncertainty, but an ingenious device introduced by Arrow [1], and further elaborated by Debreu [5], enabled the theory to be reinterpreted to cover the case of uncertainty about the availability of resources and about consumption and production possibilities. (See [6, Chap. 7] for a unified treatment of time and uncertainty.)

In the present paper I take the Arrow-Debreu theory as a starting point and discuss certain extensions, limitations, and possible new departures. In particular, I: (1) show how the theory can be extended to account explicitly for differences in information available to different economic agents, and for the "production" of information; (2) present a critique of the (extended) theory, especially its failure to explain or take account of money, stock markets, and the presence in the real world of active markets at every date; (3) argue for the consideration of a theory of a sequence of markets and suggest several concepts of equilibrium that might be appropriate to such a theory; and (4) present some results on the existence of an equilibrium of plans, prices, and price expectations in a sequence of markets.

The main features of the Arrow-Debreu theory have been available in the literature for more than a decade and were even discussed at a meeting of this Association six years ago [12]. Nevertheless, it seemed to me wise to begin the paper with a brief review of the elements of the theory, although I fear that the review may be too brief to be intelligible to those who are not already familiar with the material!

The consideration of a sequence of markets under conditions of uncertainty is not new in economics but does not seem to have received much attention from value theorists since the publication of Hicks's *Value and Capital* [11]. I would therefore have felt more comfortable presenting this paper in a session entitled, "Old Ideas in Pure Theory," but as far as I know, no such session has been organized for the current meetings.

I. *Review of the Arrow-Debreu Model of a Complete Market for Present and Future Contingent Delivery*

In this section I review the approach of Arrow [1] and Debreu [6] to incorporating uncertainty about the environment into a Walrasian model of competitive equilibrium. The basic idea is that commodities are to be distinguished, not only by their physical characteristics and by the location and dates of their availability and/or use, but also by the environmental event in which they are made available and/or used. For example, ice cream made available (at a particular location on a particular date) if the weather is hot may be considered to be a different commodity from the same kind of ice cream made available (at the same location and date) if the weather is cold. We are thus led to consider a list of "commodities" that is greatly expanded by comparison with the corresponding case of certainty about the environment. The standard arguments of the theory of competitive equilibrium, applied to an economy with this expanded list of commodities, then require that we envisage a "price" for each commodity in the list, or, more precisely, a set of price ratios specifying the rate of exchange between each pair of commodities.

Just what institutions could, or do, effect such

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exchanges is a matter of interpretation that is, strictly speaking, outside the model. I shall present one straightforward interpretation, and then comment briefly on an alternative interpretation.

First, however, it will be useful to give a more precise account of the concepts of environment and event that I shall be employing. The description of the "physical world" is decomposed into three sets of variables: (1) decision variables, which are controlled (chosen) by economic agents; (2) environmental variables, which are not controlled by any economic agent; and (3) all other variables, which are completely determined (possibly jointly) by decisions and environmental variables. A state of the environment is a complete specification (history) of the environmental variables from the beginning to the end of the economic system in question. An event is a set of states; for example, the event "the weather is hot in New York on July 1, 1970" is the set of all possible histories of the environment in which the temperature in New York during the day of July 1, 1970, reaches a high of at least (say) 75°F. Granting that we cannot know the future with certainty, at any given date, there will be a family of elementary observable (knowable) events, which can be represented by a partition of the set of all possible states (histories) into a family of mutually exclusive subsets. It is natural to assume that the partitions corresponding to successive dates are successively finer, which represents the accumulation of information about the environment.

We shall imagine that a "market" is organized before the beginning of the physical history of the economic system. An elementary contract in this market will consist of the purchase (or sale) of some specified number of units of a specified commodity to be delivered at a specified location and date, if and only if a specified elementary event occurs. Payment for this purchase is to be made now (at the beginning), in "units of account," at a specified price quoted for that commodity-location-date-event combination. Delivery of the commodity in more than one elementary event is obtained by combining a suitable set of elementary contracts. For example, if delivery of one quart of ice cream (at a specified location and date) in hot weather costs \$1.50 (now) and delivery of one quart in non-hot weather costs \$1.10, then sure delivery of one quart (i.e., whatever be the weather) costs \$1.50 + \$ 1.10 = \$ 2.60.

There are two groups of economic agents in the economy: producers and consumers. A producer chooses a production plan, which determines his input and/or output of each commodity at each date in each elementary event. (I shall henceforth suppress explicit reference to location, it being understood that the location is specified in the term commodity.) For a given set of prices, the present value of a production plan is the sum of the values of the inputs minus the sum of the values of the outputs. Each producer is characterized by a set of production plans that are (physically) feasible for him: his production possibility set.

A consumer chooses a consumption plan, which specifies his consumption of each commodity at each date in each elementary event. Each consumer is characterized by: (1) a set of consumption plans that are (physically, psychologically, etc.) feasible for him, his consumption possibility set; (2) preferences among the alternative plans that are feasible for him; (3) his endowment of physical resources, i.e., a specification of the quantity of each commodity, e.g., labor, at each date in each event with which he is exogenously endowed; and (4) his shares in producers' profits, i.e., a specification for each producer, of the fraction of the present value of that producer's production plan that will be credited to the consumer's account. (For any one producer, the sum of the consumers' shares is unity.) For given prices and given production plans of all the producers, the present net worth of a consumer is the total value of his resources plus the total value of his shares of the present values of producers' production plans.

An equilibrium of the economy is a set of prices, a set of production plans (one for each producer), and a set of consumption plans (one for each consumer), such that (*a*) each producer's plan has maximum present value in his production possibility set; (*b*) each consumer's plan maximizes his preferences within his consumption possibility set, subject to the additional (budget) constraint that the present cost of his consumption plan not exceed his present net worth; (*c*) for each commodity at each date in each elementary event, the total demand equals the total supply; i.e., the total planned consumption equals the sum of the total resource endowments and the total planned net output (where inputs are counted as negative outputs).

Notice that (1) producers and consumers are "price takers"; (2) for given prices there is no uncertainty about the present value of a production plan or of given resource endowments, nor about the present cost of a consumption plan; (3) therefore, for given prices and given producers' plans, there is no uncertainty about a given consumer's present net worth; (4) since a consumption plan may specify that, for a given commodity at a given date, the quantity consumed is to vary according to the event that actually occurs, a consumer's preferences among plans will reflect not only his "tastes" but also his subjective beliefs about the likelihoods of different events and his attitude towards risk [16].

It follows that beliefs and attitudes towards risk play no role in the assumed behaviour of producers. On the other hand, beliefs and attitudes towards risk do play a role in the assumed behavior of consumers, although for given prices and production plans each consumer knows his (single) budget constraint with certainty.

I shall call the model just described an "Arrow-Debreu" economy. One can demonstrate, under "standard conditions": (1) the existence of an equilibrium, (2) the Pareto optimality of an equilibrium, and (3) that, roughly speaking, every Pareto optimal choice of production and consumption plans is an equilibrium relative to some price system for some distribution of resource endowments and shares [6, Chaps. 5 and 6] [7].

In the above interpretation of the Arrow-Debreu economy, all accounts are settled before the history of the economy begins, and there is no incentive to revise plans, reopen the market or trade in shares. There is an alternative interpretation, which will be of interest in connection with the rest of this paper but which corresponds to exactly the same formal model. In this second interpretation, there is a single commodity at each date—let us call it "gold"—that is taken as a numeraire at that date. A "price system" has two parts: (1) for each date and each elementary event at that date, there is a price, to be paid in gold at date 1, for one unit of gold to be delivered at the specified date and event; (2) for each commodity, date, and event at that date, there is a price, to be paid in gold at that date and event, for one unit of the commodity to be delivered at that same date and event. The first part of the price system can be interpreted as "insurance premiums" and the second part as "spot prices" at the given date and event. The insurance interpretation is to be made with some reservation, however, since there is no real object being insured and no limit to the amount of insurance that an individual may take out against the occurence of a given event. For this reason, the first part of the price system might be better interpreted as reflecting a combination of betting odds and interest rates.

Although the second part of the price system might be interpreted as spot prices it would be a mistake to think of the determination of the equilibrium values of these prices as being deferred in real time to the dates to which they refer. The definition of equilibrium requires that the agents have the access to the complete system of prices when choosing their plans. In effect, this requires that at the beginning of time all agents have available a (common) forecast of the equilibrium spot prices that will prevail at every future date and event.

II. *Extension of the Arrow-Debreu Model to the Case in Which Different Agents Have Different Information*

In an Arrow-Debreu economy, at any one date each agent will have incomplete information about the state of the environment, but all the agents will have the same information. This last assumption is not tenable if we are to take good account of the effects of uncertainty in an economy. I shall now sketch how, by a simple reinterpretation of the concepts of production possibility set and consumption possibility set, we can extend the theory of the Arrow-Debreu economy to allow for differences in information among the economic agents.[1]

For each date, the information that will be available to a given agent at that date may be characterized by a partition of the set of states of the environment. To be consistent with our previous terminology, we should assume that each such information partition must be at least as coarse as the partition that describes the elementary events at that date; i.e., each set in the information partition must contain a set in the elementary event partition for the same date.

For example, each set in the event partition at a given date might specify the high temperature at that date, whereas each set in a given agent's information partition might specify only whether this temperature was higher than 75°F. or not. Or the event partition at a given date might specify the temperature at each date during the past month, whereas the information partition might specify only the mean temperature over the past month.

An agent's information restricts his set of feasible plans in the following manner. Suppose that at a given date the agent knows only that the state of the environment lies in a specified set A (one of the sets in his information partition at that date), and suppose (as would be typical) that the set A contains several of the elementary events that are in principle observable at that date. Then any action that the agent takes at that

[1] This section is based upon [14, Sections 2–6].

date must necessarily be the same for all elementary events in the set A. In particular, if the agent is a consumer, then his consumption of any specified commodity at that date must be the same in all elementary events contained in the information set A; if the agent is a producer, then his input or output of any specified commodity must be the same for all events in A. (I am assuming that consumers know what they consume and producers what they produce at any given date.)

Let us call the sequence of information partitions for a given agent his information structure and let us say that this structure is fixed if it is given independent of the actions of himself or any other agent. Furthermore, in the case of a fixed information structure, let us say that a given plan (consumption or production) is compatible with that structure if it satisfies the conditions described in the previous paragraph, at each date.

Suppose that the consumption and production possibility sets of the Arrow-Debreu economy are interpreted as characterizing, for each agent, those plans that would be feasible if he had "full information" (i.e., if his information partition at each date coincided with the elementary event partition at that date). The set of feasible plans for any agent with a fixed information structure can then be obtained by restricting him to those plans in the full information possibility set that are also compatible with his given information structure.

From this point on, all of the machinery of the Arrow-Debreu economy (with some minor technical modifications) can be brought to bear on the present model. In particular, we get a theory of existence and optimality of competitive equilibrium relative to fixed structures of information for the economic agents. I shall call this the "extended Arrow-Debreu economy."[2]

III. *Choice of Information*

There is no difficulty in principle in incorporating the choice of information structure into the model of the extended Arrow-Debreu economy. I doubt, however, that it is reasonable to assume that the technological conditions for the acquisition and use of information generally satisfy the hypotheses of the standard theorems on the existence and optimality of competitive equilibrium.

The acquisition and use of information about the environment typically require the expenditure of goods and services; i.e., of commodities.

If one production plan requires more information for its implementation than another (i.e., requires a finer information partition at one or more dates), then the list of (commodity) inputs should reflect the increased inputs for information. In this manner a set of feasible production plans can reflect the possibility of choice among alternative information structures.

Unfortunately, the acquisition of information often involves a "set-up cost"; i.e. the resources needed to obtain the information may be independent of the scale of the production process in which the information is used. This set-up cost will introduce a nonconvexity in the production possibility set, and thus one of the standard conditions in the theory of the Arrow-Debreu economy will not be satisfied [14, Sec. 9].

There is another interesting class of cases in which an agent's information structure is not fixed, namely, cases in which the agent's information at one date may depend upon production or consumption decisions taken at previous dates, but all actions can be scaled down to any desired size. Unfortunately space limitations prevent me from discussing this class in the present paper.

IV. *Critique of the Extended Arrow-Debreu Economy*

If the Arrow-Debreu model is given a literal interpretation, then it clearly requires that the economic agents possess capabilities of imagination and calculation that exceed reality by many orders of magnitude. Related to this is the observation that the theory requires in principle a complete system of insurance and futures markets, which appears to be too complex, detailed, and refined to have practical significance. A further obstacle to the achievement of a complete insurance market is the phenomenon of "moral hazard" [2].

A second line of criticism is that the theory does not take account of at least three important institutional features of modern capitalist economies: money, the stock market, and active markets at every date.

These two lines of criticism have an important connection, which suggests how the Arrow-Debreu theory might be improved. If, as in the Arrow-Debreu model, each production plan has a sure unambiguous present value at the beginning of time, then consumers have no interest in trading in shares, and there is no point in a stock market. If all accounts can be settled at the beginning of time, then, there is no need for money

[2] This teminology is not in any way meant to imply that either Arrow or Debreu approve of this way of incorporating information into their model!

during the subsequent life of the economy; in any case, the standard motives for holding money do not apply.

On the other hand, once we recognize explicitly that there is a sequence of markets, one for each date, and no one of them complete (in the Arrow-Debreu sense), then certain phenomena and institutions not accounted for in the Arrow-Debreu model become reasonable. First, there is uncertainty about the prices that will hold in future markets, as well as uncertainty about the environment.

Second, producers do not have a clear-cut natural way of comparing net revenues at different dates and states. Stockholders have an incentive to establish a stock exchange, since it enables them to change the way their future revenues depend on the states of the environment. As an alternative to selling his shares in a particular enterprise, a stockholder may try to influence the management of the enterprise in order to make the production plan conform better to his own subjective probabilities and attitude towards risk.

Third, consumers will typically not be able to discount all of their "wealth" at the beginning of time, because (*a*) their shares of producers' future (uncertain) net revenues cannot be so discounted and (*b*) they cannot discount all of their future resource endowments. Consumers will be subject to a sequence of budget constraints, one for each date (rather than to a single budget constraint relating present cost of his consumption plan to present net worth, as in the Arrow-Debreu economy).

Fourth, economic agents may have an incentive to speculate on the prices in future markets, by storing goods, hedging, etc. Instead of storing goods, an agent may be interested in saving part of one date's income, in units of account, for use on a subsequent date, if there is an institution that makes this possible. There will thus be a demand for "money" in the form of demand deposits.

Fifth, agents will be interested in forecasting the prices in markets at future dates. These prices will be functions of both the state of the environment and the decisions of (in principle, all) economic agents up to the date in question.

V. *Equilibrium of Plans, Prices, and Price Expectations in a Sequence of Markets*

Consider now a sequence of markets at successive dates. Suppose that no market at any one date is complete in the Arrow-Debreu sense; i.e., at every date and for every commodity there will be some future dates and some events at those future dates for which it will not be possible to make current contracts for future delivery contingent on those events. In such a model, several types of "equilibrium" concept suggest themselves. First, we may think of a sequence of "momentary" equilibria in which the current market is cleared at each date. The prices at which the current market is cleared at any one date will depend upon (among other things) the expectations that the agents hold concerning prices in future markets (to be distinguished from future prices on the current market!). We can represent a given agent's expectations in a precise manner as a function (schedule) that indicates what the prices will be at a given future date in each elementary event at that date. This includes, in particular, the representation of future prices as random variables, if we admit that the uncertainty of the agent about future events can be scaled in terms of subjective probabilities [16].

In the evolution of a sequence of momentary equilibria, each agent's expectations will be successively revised in the light of new information about the environment and about current prices. Therefore, the evolution of the economy will depend upon the rules or processes of expectation formation and revision used by the agents. In particular, there might be interesting conditions under which such a sequence of momentary equilibria would converge, in some sense, to a (stochastic) steady state. This steady state, e.g., stationary probability distribution of prices, would constitute a second concept of equilibrium.

I am not aware of any systematic general theory of markets under uncertainty, incorporating one or both of these two concepts of equilibrium, that has appeared since Hicks's *Value and Capital*, and I don't think that we can rest satisfied with Hicks's treatment in terms of "certainty equivalents" and "elasticities of expectation." The desirability of having a better theory and the importance of the role of expectations are well recognized, of course [3]. In the further development of such a theory, we shall no doubt have to face some of the difficult problems that have appeared in recent work on sequences of momentary equilibria under conditions of certainty [10] [17] [18].

A third concept of equilibrium emerges if we investigate the possibility of consistency among the expectations and plans of the various agents. I shall say that the agents have common expectations if they associate the same (future) prices to the same events. (Note that this does not necessarily imply that they agree on the joint probability distribution of future prices, since different agents might well assign different subjective probabilities to the same event.) I shall say that the plans of the agents are consistent if for each commodity, each date, and each event at that date the planned supply of that commodity at that date in that event equals the planned demand and if a corresponding condition holds for the stock markets. An equilibrium of plans, prices, and price expectations is a set of prices on the current market, a set of common expectations for the future, and a consistent set of individual plans, one for each agent, such that, given the current prices and price expectations, each individual agent's plan is optimal for him, subject to an appropriate sequence of budget constraints.

Of the three concepts of optimality, the last is perhaps the closest in spirit to the Arrow-Debreu theory. How far do the conclusions of the Arrow-Debreu theory (existence and optimality of equilibrium) extend to this new situation? Concerning existence, for particular definitions of "individual optimality" and specifications of the agents' "budget constraints," one can prove the following theorem. Before stating the existence theorem I must define what I shall call a pseudo-equilibrium.

The definition of pseudo-equilibrium is obtained from the definition of equilibrium by replacing the requirement of consistency of plans by the condition that at each date and each event the difference between total saving and total investment (by consumers) is smaller at the pseudo-equilibrium prices than at any other prices.[3]

One can prove [15] that under assumptions about technology and consumer preferences similar to those used in the Arrow-Debreu theory: (1) there exists a pseudo-equilibrium; (2) if in a pseudo-equilibrium the current and future prices on the stock market are all strictly positive, then the pseudo-equilibrium is an equilibrium; (3) in the case of a pure exchange economy, there exists an equilibrium.

The crucial difference between this theorem and the corresponding one in the Arrow-Debreu theory seems to be due to the form taken by Walras' law, which in this model can be paraphrased by saying that saving must be at least equal to investment at each date in each event. This form derives from the replacement of a single budget constraint (in terms of present value) by a sequence of budget constraints, one for each date.

With regard to optimality, there is little that can be said at this time. The main difficulty in investigating this question seems to be in characterizing the set of states of the economy that are attainable, given the restrictions on the set of allowable contracts at each date.

[3] This second condition will be automatically satisfied at an equilibrium. It should be noted that at each date the set of current prices is normalized; e.g., by taking the sum to be unity.

VI. *Unsolved Problems*

I can only list here a few unsolved problems that I personally find interesting and promising for further research.

I have already mentioned the question of the optimality properties (if any) of an equilibrium of plans, prices, and price expectations. One possible approach is to consider more explicitly the information that the observation of prices provides for agents in the economy. One might hope to show that an equilibrium is an optimum relative to the set of states of the economy that could be attained with just the same information that is provided by the equilibrium prices (in addition, of course, to the information structures originally available to the individual agents). Notice that since the equilibrium price expectations are self-fulfilling, the observation of the prices in any current market provides information about the true state of the environment (i.e., the specification of the values of particular prices defines an "event" in the set of possible states of the environment). An approach of this kind was tried in a two-period model [13], which was further complicated, however, by allowing agents to make contracts for future delivery contingent on the values taken on by future prices. (An example of such a contract would be a wage contract with a cost-of-living escalation clause.) It was shown that in this model, if the introduction of such contracts enabled all the agents to discount future receipts and costs back to the initial date (i.e., if all uncertainty about the environment could be reflected in some corresponding uncertainty about future prices), then an equilibrium would be an optimum in the above sense. Unfortunately, the existence of an equilibrium in such a model was not demonstrated, and indeed there might be important economic phenomena that would rule out the existence of equilibrium in such a model [13].

I have also already mentioned the unsatisfactory state of the theory of the evolution of momentary equilibria in a sequence of markets and the question of possible convergence of momentary equilibria to a (stochastic) steady state.

In all of these (potential) theories of a sequence of markets we shall need a more detailed theory of the firm than that used in the Arrow-Debreu model. Simple profit maximization is not well defined if future profits are uncertain and cannot fully be discounted back to the present. The model of Section V essentially begged this question by assuming that each producer maximizes a utility function whose arguments are his future net revenues in different events. Such an assumption fails to relate the behavior of the firm to the preferences of the stockholders or potential stockholders. (It is rather an expression of the "divorce of ownership from management"!) An alternative candidate that has been discussed is the assumption that at each date a producer maximizes the current stock market value of his firm. (Note that in the Arrow-Debreu model, profit maximization is equivalent to maximization of the value of the stock.) However, except in the context of a special example considered by Diamond [8], I have not seen a formulation of this hypothesis that enables the producer to act as a price-taker; i.e., that does not imply that the producer is able to calculate the effect of his actions on the equilibrium prices.

We shall also want to incorporate into our theories the process of entry and exit of firms. In particular, the results described in Section V on the relationship between equilibrium and pseudo-equilibrium suggest that the possibility of exit may be important in assuring the existence of such an equilibrium.

Finally, I mention the old problem of incorporating a theory of money and credit in a Walrasian model of general equilibrium [9]. In a sense, the model of Section V allows "secured" loans that are backed either by physical collateral or by contracts for future delivery of commodities. The theory also provides a framework for explaining the holding of "commodity money." The model does not, however, describe any institutions for carrying over "units of account" from one date to the next; the introduction of such institutions seems a natural next step and one for which the model seems to me to be well suited.

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