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Econometrica, Vol. 40, No. 2. (Mar., 1972), pp. 289-303.

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EXISTENCE OF EQUILIBRIUM OF PLANS, PRICES, AND PRICE EXPECTATIONS IN A SEQUENCE OF MARKETS

BY ROY RADNER[1]

Consider a sequence of markets for goods and securities at successive dates, with no market at any date complete in the Arrow-Debreu sense. A concept of *common expectations* is proposed that requires traders to associate the same future prices to the same future exogenous events, but does not require them to agree on the (subjective) probabilities associated with those events. An equilibrium is a set of prices at the first date, a set of common price expectations for the future, and a consistent set of individual plans for consumers and producers such that, given the current prices and price expectations, each individual agent's plan is optimal for him, subject to an appropriate sequence of budget constraints. The existence of such an equilibrium is demonstrated under assumptions about technology and consumer preferences similar to those used in the typical Arrow-Debreu theory of complete markets. However, an equilibrium can fail to exist if some provision is not made for the elimination of "unprofitable" enterprises. The usual assumptions of "rationality" imply, in this model, that agents learn from experience and modify their expectations as Bayesians.

1. INTRODUCTION

CONSIDER A SEQUENCE of markets at successive dates, no one of which is complete in the Arrow-Debreu sense, i.e., at every date and for every commodity there will be some future dates and some events at those dates for which it will not be possible to make current contracts for future delivery contingent on those events. The prices at which the current market is cleared at any one date will depend upon (among other things) the expectations that the traders hold concerning prices in future markets. We can represent a trader's expectations as a function that indicates what the prices will be at a given date in each elementary event at that date. This includes, in particular, the representation of future prices as random variables, if we admit that the uncertainty of the traders about future events can be scaled in terms of subjective probabilities. I shall say that the traders have *common* expectations if they associate the same (future) prices to the same events. This does not necessarily imply that they agree on the joint probability distribution of future prices, since different traders might assign different subjective probabilities to the same event.

I shall say that the plans of the traders are *consistent* if, for each commodity, each date, and each event at that date, the excess planned supply of that commodity at that date in that event is zero. An equilibrium of plans, prices, and price expectations is a set of prices on the first market, a set of common price expectations for the future, and a consistent set of individual plans, one for each trader, such that, given

[1] The research on which this paper is based was supported in part by the National Science Foundation. A preliminary version of this was prepared while I was an Overseas Fellow at Churchill College, Cambridge, 1969–70, and I would like to thank the Master and Fellows of Churchill College and the Faculty of Economics of Cambridge for their facilitation of my research during that period.

the current prices and price expectations, each individual trader's plan is optimal for him, subject to an appropriate sequence of budget constraints.

In the present paper I consider the problem of existence of such an equilibrium for a model that includes production and the trading of shares on a stock market. The analysis of a stock market introduces considerable complication in the model, so I first deal with the case of pure exchange of commodities. This enables the reader to encounter the essential idea of an equilibrium of "self-fulfilling expectations" in as simple a context as possible.

For the case of pure exchange I demonstrate the existence of an equilibrium under hypotheses similar to those used in the now classical Arrow-Debreu theory. For the case of production (with stock markets), I prove first the existence of what I call here a *pseudo-equilibrium*. In a pseudo-equilibrium, (i) the plans of producers and consumers are individually optimal for the given system of commodity and share prices, and (ii) the given prices minimize, at each date and event, the excess of the total value of planned consumer saving over the total value of planned new consumer investment. In an equilibrium, one has the additional condition that, if the plans of producers and consumers are realized, then the commodity and share markets will be cleared at every date in every event. A pseudo-equilibrium is shown to be an equilibrium (under the assumptions of the model) if the initial share market is cleared and the share prices are all positive at every date and event, provided no consumer can be satiated at any date-event pair. This last assumption implies that no consumer attaches subjective probability zero to any observable event.

The gap between pseudo-equilibrium and true equilibrium seems to be related to the problem of how to formulate the phenomena of bankruptcy and exit of producers in a model such as this one. Heuristic comments on this question, and on other aspects of the model, are included in Section 9.

The conceptual framework and the technique of analysis used in this paper rely heavily on the presentation in G. Debreu's *Theory of Value* [**1**]. For a critique of the Arrow-Debreu theory, and hints towards a theory of a sequence of markets, see [**4**, **5**, and **6**]. This last paper also gives references to recent literature on sequences of markets under conditions of certainty.

In the model of this paper, the set of allowable contracts at each date for future (contingent) delivery is given, without any explanation of why some contracts are allowed and others are not (see, however, some general remarks in Radner [**5**]). F. H. Hahn [**3**] has independently studied this question with the aid of a model of transaction technology, under conditions of certainty.

2. COMMODITIES, EVENTS, TRADE CONTRACTS, AND PRICES

Consider an economy extending through a finite sequence of elementary *dates*, $1, \ldots, T$, in an environment with a finite set S of alternative *states*. Each state in S is to be interpreted as a particular history of the environment from date 1 through date T. The set of events observable at date t will be represented by a partition, $\mathscr{S}_t$, of S. It is assumed that the sequence of partitions, $\mathscr{S}_t$, is monotone nondecreasing in fineness, that is, $\mathscr{S}_{t+1}$ is as fine[2] as $\mathscr{S}_t$, for each t. Also, take $\mathscr{S}_1 = \{S\}$. Thus the observable events form a "tree" (see [**1**, pp. 98, 99]).

For each date there is a finite set of commodities, numbered $1, \ldots, H$. An *elementary trade contract at date t in event A*, denoted by $z^h_{tu}(A, B)$, specifies the number of units of commodity h that the trader will deliver "to the market" at date u in event B. (A negative delivery is to be interpreted as a receipt.) For $u = t$ we have a "spot" trade, whereas for $u > t$ we have a "forward" trade.

The allowable trade contracts are described as follows. For each pair of dates t and u, with $t > u$, and each commodity h, there is given a family $\mathscr{A}^h_{tu}$ of events, which is either empty or is a partition of S; in the latter case, $\mathscr{S}_u$ must be as fine as $\mathscr{A}^h_{tu}$. Assume that $\mathscr{A}^h_{tt} = \mathscr{S}_t$, i.e., every spot market is complete. Assume further that if $\mathscr{A}^h_{tu}$ is not empty, and $t \leqslant v \leqslant u$, then $\mathscr{A}^h_{vu}$ is as fine as $\mathscr{A}^h_{tu}$. In other words, if at date t one can contract for delivery at date u contingent on event B, then at a later date, v, one can do the same. A trade contract $z^h_{tu}(A, B)$ is *allowable* if

$$(2.1) \qquad A \text{ is in } \mathscr{S}_t, \qquad B \text{ is in } \mathscr{A}^h_{tu}, \qquad B \subseteq A, \qquad \text{and} \qquad z^h_{tu}(A, B) \leqslant L,$$

where L is a given positive number. An upper bound on allowable contracts is natural; for example, a contract to deliver a quantity vastly greater than the total supply of the commodity would not be credible.

A *trade plan*, z, is an array $\{z^h_{tu}(A, B)\}$ of allowable trade contracts, one for each allowable combination (h, t, u, A, B).

The *price* received at date t in event A (per unit) for delivery of commodity h at date u in event B will be denoted by $p^h_{tu}(A, B)$. An array, p, of such prices will be called a *commodity price system.*

In the situation just described, there is for each date-event pair (t, A), with A in $\mathscr{S}_t$, a "market" in contracts for current and future delivery, with payment to be made currently in units of account. This market need not be complete in the Arrow-Debreu sense, i.e., the partitions $\mathscr{A}^h_{tu}$ need not be identical with $\mathscr{S}_u$.

To simplify the notation, let M denote the set of all pairs (t, A) such that $t = 1, \ldots, T$, and A is in $\mathscr{S}_t$. The set M is thus the index set of the family of "markets" just described. Note that M is partially ordered by the order of t and set inclusion, i.e., for $m = (t, A)$ and $n = (u, B)$ in M:

$$(2.2) \qquad m \leqslant n \text{ if and only if } t \leqslant u \text{ and } A \supseteq B.$$

I shall use the convention that the pair $(1, S)$ will be denoted simply by "1".

For each $m = (t, A)$ in M let Z_m denote the vector space of all arrays of numbers $z^h_u(B)$, one for each combination (h, u, B) such that $h = 1, \ldots, H$; $u = t, \ldots, T$; $B \subseteq A$; and $B \in \mathscr{A}^h_{tu}$. (Note that Z_m is not empty, because $\mathscr{A}^h_{tt} = \mathscr{S}_t$.)

Also, let $Z = \bigtimes_{m \in M} Z_m$. The set of all allowable trade plans is the set of all points in Z whose coordinates do not exceed L.

[2] Partition $\mathscr{S}$ is said to be as fine as partition $\mathscr{S}'$ if, for every A' in $\mathscr{S}'$ and A in $\mathscr{S}$, either $A \subset A'$ or $A \cap A' = \varnothing$.

A commodity price system is also a point in Z. The *revenue* at m for the trade plan z, given the commodity price system p, is the inner product of p_m and z_m, which will simply be denoted by $p_m z_m$. The *vector* of revenues $p_m z_m (m \in M)$, will be denoted by $r(p, z)$.

Consider a trade plan $z = (z_m)$, a commodity h, and two date-event pairs $m = (t, A)$ and $n = (u, B)$ such that $m \leqslant n$. It will be useful to have a symbol for the quantity of commodity h to be delivered at n according to the contract (if any) made at m.

Define z_{mn}^h by:

$$(2.3) \qquad z_{mn}^h = \begin{cases} z_{tu}^h(A, B), & \text{if this contract is allowable,} \\ 0, & \text{otherwise.} \end{cases}$$

I shall write z_{mn} for the vector whose coordinates are z_{mn}^h, $h = 1, \ldots, H$.

3. THE CASE OF PURE EXCHANGE: TRADERS

There is a finite set of traders, denoted by I. Each trader chooses a consumption plan and a trade plan.

A consumption plan for consumer i is a vector, $x_i = (x_{im})_{m \in M}$, where, for each m in M, x_{im} is a vector in H-dimensional space representing i's consumption at the date-event pair m. Let X denote the space of all such vectors, and let X_i denote the set of consumption plans feasible for i.

The *resources* of consumer i are denoted by a vector $w_i = (w_{im})$ in X.

A consumption-trade plan, (x, z), is *feasible* for i, given the price system p, if:

(3.1) $\quad x$ is in X_i;

(3.2) $\quad z$ is allowable (see (2.1));

(3.3) $\quad \sum_{m \leqslant n} z_{mn} \leqslant w_{in} - x_n$, for every n in M; and

(3.4) $\quad p_m z_m \geqslant 0$, for every m in M.

The set of plans feasible for i, given p, will be denoted by $\Gamma_i(p)$.

Condition (3.3) requires that the trader not plan to deliver at any date-event pair more than he would have available from his resources after subtracting his consumption. The inequality in (3.3) expresses "free disposal." Condition (3.4) is a sequence of budget constraints, one for each date-event pair.

The preferences of trader i among the set of consumption-trade plans feasible for him is assumed to be represented by a utility function U_i on the set of consumption plans.

The behavior of a trader is summarized in his *behavior correspondence*, denoted by γ_i, which is defined by:

(3.5) $\quad$ for each price system p, $\gamma_i(p)$ is the set of plans (x, z) in $\Gamma_i(p)$ that maximize $U_i(x)$.

Note that $\gamma_i(p)$ may be empty for some p.

I shall assume that, for each trader i in I:

(3.6) X_i is closed and convex, and there is a vector $\bar{x}_i$ such that $x \geqslant \bar{x}_i$ for all x in X_i;

(3.7) for every x in X_i and every m in M, there is an x' in X_i, differing from x only at m, such that $U_i(x') > U_i(x)$;

(3.8) there is an $\tilde{x}_i$ in X_i such that $\tilde{x}_i \ll w_i$; and

(3.9) U_i is continuous and concave.

The assumption (3.7) of "nonsatiation" at each date-event pair implies that the trader has a positive subjective probability for each event (provided his utility function permits of a scaling in terms of subjective probability and utility). The assumption that U_i is concave implies that the trader does not have a "preference for risk."

The utility function could, of course, be replaced by a suitable preference pre-ordering (see [**1**, pp. 55–59]).

When there is no risk of ambiguity, a reference to "i" is to be understood as a reference to the elements of I; e.g., $\sum_i$ is to be understood as $\sum_{i \in I}$.

4. THE CASE OF PURE EXCHANGE: EQUILIBRIUM

In this section I define an equilibrium of plans, prices, and price expectations, and show that such an equilibrium exists, with nonnegative prices that are not all zero at any date-event pair. The prices will be normalized as follows. For each m in M, let P_m be the set of all nonnegative vectors in Z_m whose coordinates sum to unity; and define $P = \bigtimes_{m \in M} P_m$.

An *equilibrium* of the economy described in Sections 2 and 3 is an array $[(x_i, z_i), p]$ of plans (one for each trader) and a price system such that:

(4.1) p is in P;

(4.2) (x_i, z_i) is in $\gamma_i(p)$, for each i; and

(4.3) $\sum_i z_{im} = 0$, for every m in M.

Condition (4.3) requires that "total excess supply" be zero in every market, i.e., for every type of allowable contract at every date-event pair.

THEOREM: *If the assumptions of Sections* 2 *and* 3 *are satisfied, then the pure exchange economy has an equilibrium.*

PROOF: The technique of proof is very similar to that used in [**1**, pp. 83–88]. I shall therefore only sketch the modifications that would adapt the latter proof to the present situation. These modifications are required by the presence of "separate" markets and budget constraints, one for each date-event pair m in M, and by the constraints (3.3) that "connect" the markets.

First, one can show that the set of attainable plans for the economy is bounded, and replace the original economy by an "equivalent" bounded one. Let $\hat{\gamma}_i$ denote the behavior correspondence of trader i in the bounded economy. For every m in M and every array $(x_i, z_i)_{i \in I}$ of plans, define $\mu_m[(x_i, z_i)_{i \in I}]$ to be the set of p_m in P_m that minimize $p_m \sum_i z_{im}$, and let

$$\mu = \bigtimes_{m \in M} \mu_m,$$

$$\hat{\gamma} = \bigtimes_{i \in I} \hat{\gamma}_i,$$

$$\psi[(x_i, z_i), p] = \hat{\gamma}(p) \times \mu[(x_i, z_i)].$$

The correspondence ψ can be shown to satisfy the hypotheses of the Kakutani fixed point theorem, and therefore to have a fixed point, say $[(x_i^*, z_i^0), p^*]$. This fixed point satisfies:

(4.4) (x_i^*, z_i^0) is in $\hat{\gamma}_i(p^*)$, for every i; and

(4.5) $p_m^* \sum_i z_{im}^0 \leqslant p_m \sum_i z_{im}^0$, for every p_m in P_m, and every m in M.

One can then conclude, in the usual manner, using the budget constraints (3.4) and Assumption (3.7) (nonsatiation at every date-event pair), that, *for every m in M,*

(4.6) $$\sum_i z_{im}^0 \geqslant 0, \qquad \text{and} \qquad p_m^* \sum_i z_{im}^0 = 0.$$

One then makes use of the "free disposal" inequality in (3.3) to define new trade plans, say as follows: let j be any one trader in I, and define

(4.7) $$z_i^* = \begin{cases} z_i^0, & \text{for } i \neq j, \\ z_j^0 - \sum_{i \in I} z_i^0, & \text{for } i = j. \end{cases}$$

It is now straightforward to verify that $[(x_i^*, z_i^*), p^*]$ is an equilibrium.

5. PRODUCERS

I turn now to the case of production. There is a finite set of producers, denoted by J. Each producer chooses a production plan and a trade plan. A production plan is an array of numbers $y_t^h(A)$, one for each $h = 1, \ldots, H$, and (t, A) in M, representing the net output of commodity h at date t in event A. ("Inputs" are represented by negative outputs.) Thus a production plan $y = (y_m)$ is a point in the vector space $E = \bigtimes_{m \in M} E_m$, where, for each m in M, E_m is H-dimensional Euclidean space. The set of production plans that are technologically feasible for j will be denoted by Y_j.

A production-trade plan (y, z) is feasible for j if:

(5.1) y is in Y_j;

(5.2) z is allowable; and

(5.3) $\sum_{m \leqslant n} z_{mn} = y_n$, for every n in M.

Condition (5.3) requires that, at any date-event pair, the total of past and current contracts for current delivery must equal current net output (see Definition (2.3)). The set of production-trade plans feasible for producer j will be denoted by Π_j, a subset of $E \times Z$.

Given the commodity price system p, each producer is assumed to maximize his *utility*, which is a function of his revenue vector $r(p, z_j)$ (recall that $r(p, z)$ is the vector with coordinates $p_m z_m$). Call this utility function V_j.

The behavior of producer j is summarized by his *producer's correspondence*, π_j, defined by the following:

(5.4) For each producer j in J, and each commodity price system p, $\pi_j(p)$ is the set of production-trade plans (y, z) that maximize the utility $V_j[r(p, z)]$ in the set Π_j of feasible plans.

Each producer correspondence π_j is thus a function on Z, whose values are (possibly empty) subsets of Π_j.

I shall assume that for each j in J:

(5.5) Y_j is closed and convex; furthermore, $Y_j \supset (-E^+)$ (free disposal), where E^+ denotes the nonnegative orthant of E;

(5.6) V_j is continuous and strictly concave;

furthermore,

(5.7) $\left(\sum_{j \in J} Y_j\right) \cap \left(-\sum_{j \in J} Y_j\right) = \{0\}$ (irreversibility of total production).

The assumption that V_j is *strictly* concave ensures that all the optimal plans corresponding to any particular commodity price vector p (i.e., all the plans in $\pi_j(p)$) have the same revenue vector $r(p, z_j)$. This last condition in turn will ensure that a consumer's share of a producer's revenue is uniquely determined as a function of the commodity price system. The common revenue vector $r(p, z_j)$ for all z_j in $\pi_j(p)$ will be denoted by $\rho_j(p)$, and the array $(\rho_j[p])_{j \in J}$ by $\rho(p)$, i.e.,

(5.8) $\rho_{jm}(p) = p_m z_{jm}$, for all m in M and z_j in $\pi_j(p)$.

6. THE STOCK MARKETS

Given a commodity price system, each producer's choice of a plan implies a revenue at each date-event pair. These revenues will be distributed among the "shareholders" (the consumers) according to the shares held at the immediately preceding date-event pair. The shares are traded on a stock market at each date except the last. Shareholders retain their shares from one date to the next if they do not sell them.

The beginning of the economy requires some special treatment. Assume that each consumer starts with an initial endowment of shares, and that the first market for shares takes place before date 1, say at date 0, i.e., before the activities of production and consumption begin. The reason for this formulation is discussed in the last section of the paper.

To describe the stock market formally it will be useful to have some additional notation. Let M_0 denote the set of all (t, A) in M such that $t \neq T$, together with the element "$(0, S)$" (to be interpreted as date 0). The element $(0, S)$ will usually be simply denoted as "0". For any $n = (u, B)$ in M, the *predecessor*, n^-, of n is defined to be that date-event pair $(t - 1, A)$ in M_0 such that $A \supseteq B$. For any $m = (t, A)$ in M_0, the set N_m of *successors* of m is defined to be the set of $n = (t + 1, B)$ in M such that $B \subseteq A$.

For every m in M_0, let F_m be the Euclidean vector space of dimension equal to the number of producers, and let $F = \bigtimes_{m \in M_0} F_m$. A *portfolio plan* $f = (f_m)$ is a point in F such that, for each m in M_0,

$$(6.1) \qquad 0 \leqslant f_{mj} \leqslant 1, \qquad j \in J.$$

Here f_{mj} denotes the fraction of producer j's revenue that the shareholder will receive in each successor date-event pair in N_m. The set of portfolio plans will be denoted by F^0. A *share price system* is also a point in F. Given the share price system $v = (v_m)$ in F, the *value* of the portfolio plan $f = (f_m)$ *at* m is the inner product, $v_m f_m$, of v_m and f_m.

Given a (commodity-share) price system, (p, v), and trade plans (z_j) of the producers $(j \in J)$, a particular portfolio plan $f = (f_m)$ will imply that the shareholder's total share of producer revenue at n in M will be

$$(6.2) \qquad \sum_{j \in J} f_{n^- j} p_n z_{jn},$$

and his net revenue from the change in his portfolio will be

$$(6.3) \qquad \sum_{j \in J} v_{nj}(f_{n^- j} - f_{nj}) = v_n(f_{n^-} - f_n) \qquad \text{at every } n \neq 0 \text{ in } M_0.$$

At date 0, the shareholder has no revenue from producers; if $\bar{f} = (\bar{f}_j)$ denotes his initial share holdings, then at date 0 his net revenue from the change in his portfolio is

$$(6.4) \qquad v_0(\bar{f} - f_0).$$

7. CONSUMERS

There is a finite set, I, of consumers. Each consumer chooses a consumption plan, a trade plan, and a portfolio plan.

A *consumption plan* is a point, $x = (x_m)$, in E, where x_m^h is interpreted as the consumption of commodity h at the date-event pair m. The set of consumption plans that are (physiologically, psychologically, culturally, etc.) feasible for consumer i will be denoted by X_i.

Each consumer i has an endowment $(\bar{f}_i, \bar{w}_i)$, where $\bar{f}_i$ is his initial endowment of shares (see Section 6), and $\bar{w}_i$, a point in E, represents endowments of physical resources at each date-event pair in M.

Given a price system, (p, v), and plans (y_j, z_j) of producers (each j in J), a consumption-trade-portfolio plan, (x_i, z_i, f_i), is *feasible* for consumer i if

(7.1) x_i is in X_i;

(7.2) z_i is allowable;

(7.3) $x_m + \sum_{m \leqslant n} z_{imn} = \bar{w}_{in}$, for each $n \in M$;

and

$$(7.4) \quad \begin{cases} v_0(\bar{f}_i - f_{i0}) \geqslant 0, \\ p_m z_{im} + \sum_{j \in J} f_{im^-j} r_{jm} + v_m(f_{im^-} - f_{im}) \geqslant 0, \text{ for } m \in M_0,\ m \neq 0, \\ p_m z_{im} + \sum_{j \in J} f_{im^-j} r_{jm} \geqslant 0, \text{ for } m = (T, A) \in M, \end{cases}$$

where

$$(7.4a) \quad r_{jm} = p_m z_{jm}, \qquad j \in J,\ m \in M.$$

Condition (7.3) is analogous to (5.3), and condition (7.4) represents the consumer's budget constraints. Note that the set of feasible plans for consumer i depends on the producers' plans through the array $\boldsymbol{r} = ((r_{jm}))$ of their revenues. Let $\Gamma_i(p, v, \boldsymbol{r})$ denote the set of plans feasible for i, given p, v, and $\boldsymbol{r}$.

Given the set $\Gamma_i(p, v, \boldsymbol{r})$, each consumer is assumed to maximize his utility, which is a function of his consumption plan alone. Call this utility function U_i.

The behavior of consumer i is summarized by his *consumer's correspondence*, γ_i, defined by the following:

(7.5) For each consumer i in I, each price system (p, v), and each array $\boldsymbol{r}$ of producers' revenues, $\gamma_i(p, v, \boldsymbol{r})$ is the set (possibly empty) of plans (x, z, f) that maximize the utility $U_i(x)$ in the set $\Gamma_i(p, v, \boldsymbol{r})$.

I shall assume that, for every i in I:

(7.6) X_i is closed and convex, and there is a vector $\bar{x}_i$ such that $x \geqslant \bar{x}_i$ for all x in X_i;

(7.7) there is an $\tilde{x}_i$ in X_i such that $\tilde{x}_i \ll \bar{w}_i$;

(7.8) U_i is continuous and concave;

(7.9) for every x in X_i and every m in M, there is an x' in X_i, differing from x only at m, such that $U_i(x') > U_i(x)$;

(7.10) $\bar{f}_{ij} > 0$, every j in J;

and furthermore, for each j in J,

(7.11) $\sum_{i \in I} \bar{f}_{ij} = 1$.

8. WALRAS' LAW

In the present model, Walras' Law takes the form of a set of inequalities, one for each date-event pair, obtained by adding the budget constraints (7.4) over the set of consumers.

For producers' trade plans (z_j), consumers' trade and portfolio plans (z_i, f_i), and consumers' initial share endowments $(\bar{f}_i)$, define D_m^Z and D_m^F by:

$$D_m^Z = \sum_i z_{im} + \sum_{i,j} f_{im^-j} z_{jm}, \qquad m \text{ in } M; \tag{8.1}$$

$$D_m^F = \begin{cases} \sum_i (\bar{f}_i - f_{i0}), & m = 0, \\ \sum_i (f_{im^-} - f_{im}), & m \text{ in } M_0, m \neq 0, \end{cases} \tag{8.2}$$

where the sums over i and j are understood to be for i in I and j in J, respectively.

Adding (7.4) over all i in I yields Walras' Law:

$$\begin{cases} v_0 D_0^F \geqslant 0, & \\ p_m D_m^Z + v_m D_m^F \geqslant 0, & m \text{ in } M_0, m \neq 0, \\ p_m D_m^Z \geqslant 0, & m \text{ in } M - M_0. \end{cases} \tag{8.3}$$

Notice that D_m^Z is identical with the total planned excess supply of commodities at m if and only if

$$\sum_i f_{im^-j} = 1, \quad \text{for each } j \text{ in } J. \tag{8.4}$$

Also, D_m^F is the decrease in the planned demand for shares at m over the demand at the preceding date-event pair, m^-.

In value terms, $p_m D_m^F$ can be interpreted as the total value of planned consumer savings at m, and $(-v_m D_m^F)$ as the total value of planned (new) consumer investment in shares. Since (8.3), for m in M_0, $m \neq 0$, can be written

$$p_m D_m^Z \geqslant -v_m D_m^F, \tag{8.5}$$

Walras' Law can be paraphrased as "the total value of planned consumer saving must be at least as large as the total value of new consumer investment, at each date and event."

9. EQUILIBRIUM AND PSEUDO-EQUILIBRIUM

In this section I define the concepts of equilibrium and pseudo-equilibrium of plans, prices, and price expectations. I state a theorem on the existence of a pseudo-equilibrium with nonnegative prices that are not zero at any date-event pair (the proof is deferred to the next section) and demonstrate that under certain conditions a pseudo-equilibrium is an equilibrium.

The nonnegative price systems will be normalized as follows. For any vector space, the unit simplex is the set of nonnegative vectors whose coordinates sum to unity. Let P_0 be the unit simplex of F_0, P_m the unit simplex of $Z_m \times F_m$ for $m \neq 0$ in M_0, P_m the unit simplex of Z_m for m in $M - M_0$, and

$$P = \bigtimes_{m \in M_0 \cup M} P_m.$$

An *equilibrium* of the economy described in Sections 2 and 5–8 is an array $[(y_j, z_j)_{j \in J}, (x_i, z_i, f_i)_{i \in I}, (p, v)]$ of producers' and consumers' plans and a commodity-share price system such that

(9.1) (p, v) is in P;

(9.2) (y_j, z_j) is in $\pi_j(p)$, for every j in J;

(9.3) (x_i, z_i, f_i) is in $\gamma_i(p, v, \rho[p])$, for every i in I;

(9.4) $\sum_{k \in I \cup J} z_{km} = 0$, for every m in M; and

(9.5) $\sum_{i \in I} f_{imj} = 1$, for every j in J and m in M_0.

Conditions (9.2) and (9.3) express the individual optimality of plans, and (9.4) and (9.5) express the clearance of the commodity and share markets, respectively, at every date-event pair.

A *pseudo-equilibrium* is defined by replacing (9.4) and (9.5) in the definition of equilibrium by the condition

$$(9.6)\quad \begin{cases} v_0 D_0^F \leqslant v_0' D_0^F, \\ p_m D_m^Z + v_m D_m^F \leqslant p_m' D_m^Z + v_m' D_m^F,\ m \neq 0 \text{ in } M_0, \\ p_m D_m^Z \leqslant p_m' D_m^Z,\ m \text{ in } M - M_0, \text{ for all } (p', v') \text{ in } P, \text{ where } D_m^Z \text{ and } D_m^F \text{ are} \\ \text{evaluated at the plans } (z_j), (z_i). \text{ and } (f_i) \text{ according to (8.1) and (8.2).} \end{cases}$$

The following two propositions will be demonstrated, under the assumptions of Sections 2 and 5–8:

I. A pseudo-equilibrium exists.

II. A pseudo-equilibrium in which the share prices are all positive ($v \gg 0$) and the share market is cleared at date 0 ($D_0^F = 0$) is an equilibrium.

A proof of I is given in the next section. To prove II, first note that a pseudo-equilibrium satisfies Walras' Law (8.3), and since $P = \bigtimes_m P_m$, condition (9.6) implies that $D_m^F \geqslant 0$ for every m in M_0 and $D_m^Z \geqslant 0$ for every m in M. Assumption (7.9) (nonsatiation at every date-event pair in M) implies that Walras' Law is in fact satisfied with *equality* at every m in M. Hence, if $v_m^F \gg 0$, then $D_m^F = 0$, for $m \neq 0$ in M_0. By hypothesis, $D_0^F = 0$, too. Therefore, for all m in M_0 and all j in J,

$$\sum_i f_{imj} = \sum_i \bar{f}_{ij} = 1,$$

by (8.2) and (7.11), so that (9.5) is verified. In this case, as in the discussion of (8.4),

$$D_m^Z = \sum_{k \in I \cup J} z_{km}, \qquad \text{all } m \text{ in } M,$$

so that

$$\sum_{k \in I \cup J} z_{km} \geqslant 0,$$
$$(9.7) \qquad p_m \sum_{k \in I \cup J} z_{km} = 0, \qquad \text{all } m \text{ in } M.$$

Consider any arbitrary producer j_0 in J, and let

$$y^*_{j_0 n} = y_{j_0 n} - \sum_{\substack{k \in I \cup J \\ m \leqslant n}} z_{kmn}, \qquad n \in M,$$
$$(9.8) \qquad z^*_{j_0} = z_{j_0} - \sum_{k \in I \cup J} z_k.$$

By the assumption of free disposal in (5.5), $y^*_{j_0}$ is in Y_{j_0}; also, from (9.8), since (y_{j_0}, z_{j_0}) satisfies (5.2) and (5.3), so does $(y^*_{j_0}, z^*_{j_0})$. Finally, by (9.7), $p_m z_{j_0 m} = p_m z^*_{j_0 m}$ for every m in M, so the new plan has the same utility for j_0. Therefore, replacing (y_{j_0}, z_{j_0}) by $(y^*_{j_0}, z^*_{j_0})$ in the pseudo-equilibrium makes it into an equilibrium.

10. PROOF OF EXISTENCE OF A PSEUDO-EQUILIBRIUM

The technique of proof for Proposition I is similar to that used in [**1**, pp. 83–88]. I shall therefore only sketch the proof, indicating the complications peculiar to the present problem.

First, one can show that the set of attainable plans for the economy is bounded, and replace the original economy by a bounded one. Let G_k be the vector space that carries the set of plans of agent k in $K \equiv I \cup J$, and let $G = \bigtimes_{k \in K} G_k$. A plan for the economy is a point in G. A plan in G is *attainable* if conditions (5.1)–(5.3), (7.1)–(7.3), (9.4), and (9.5) are satisfied. Let $\hat{G}$ denote the set of attainable plans; one can show that $\hat{G}$ is bounded (use [**1**, Sec. 5.4], and (5.3), (7.3), and (2.1)).

The following construction shows that $\hat{G}$ is not empty, *provided L is large enough* (see (2.1)). Let

$$(10.1) \qquad x_i = \tilde{x}_i;$$
$$z_{imm} = \bar{w}_{im} - \tilde{x}_{im};$$
$$z_{imn} = 0, \qquad \text{for } m < n, \qquad i \in I;$$
$$y_j = \begin{cases} 0, & j \neq j_0, \\ -\sum_i (\bar{w}_i - \tilde{x}_i), & j = j_0; \end{cases}$$
$$z_{jmm} = y_{jm}; \quad \text{and}$$
$$z_{jmn} = 0, \qquad \text{for } m < n, \qquad j \in J;$$

where j_0 is an arbitrarily chosen j in J.

For each k in K, let $\hat{G}_k$ be the projection of $\hat{G}$ into G_k, $\hat{G}_k$ be a closed cube in G_k containing $\hat{G}_k$ in its interior, and $\hat{G} = \bigtimes_k \hat{G}_k$. Define $\hat{\Pi}_j = \Pi_j \cap \hat{G}_j$, and $\hat{\pi}_j$ and

$\hat{\rho}_j$ as in (5.4) and (5.8), respectively, but with Π_j replaced by $\hat{\Pi}_j$, and consider $\hat{\pi}_j$ and $\hat{\rho}_j$ as defined on P. Further, define

$$\bar{\Gamma}_i(p, v, r) = \Gamma_i(p, v, r) \cap \hat{G}_i, \qquad \hat{\Gamma}_i(p, v) = \bar{\Gamma}_i[p, v, \hat{\boldsymbol{\rho}}(p, v)],$$

and $\hat{\gamma}_i$ as in (7.5) but with Γ_i replaced by $\hat{\Gamma}_i$.

As usual, one shows that $\hat{\pi}_j$ and $\hat{\gamma}_i$ are upper semicontinuous. The main difficulty is in showing that $\hat{\Gamma}_i$ is continuous. First, note that for all (p, v) in P,

$$v_0 > 0,$$
$$(p_m, v_m) > 0, \qquad \text{for } m \neq 0 \text{ in } M_0, \qquad \text{and}$$
$$p_m > 0, \qquad \text{for } m \text{ in } M - M_0.$$

Fix (p, v) in P. Since $\bar{w}_i - \tilde{x}_i \gg 0$, one can divide $(\bar{w}^h_{im} - \tilde{x}^h_{im})$ in *positive* amounts among all the allowable contracts (h, n, m), $n \leqslant m$; this can be done exactly because $\mathscr{A}^h_{tt} = \mathscr{S}_t$ for every t, *provided L is large enough.* Let $\tilde{z}_i$ denote the resulting trade plan. Note that $p_m \tilde{z}_{im} > 0$ for m in $M - M_0$. Also, $\varepsilon > 0$, where

$$\varepsilon \equiv \min \{p_m \tilde{z}_{im} : m \text{ in } M \text{ for which } p_m > 0\}.$$

Let $r = (r_{jm})$ be such that, for every m in M,

$$\text{(10.2)} \qquad r_{jm} = 0 \qquad \text{if} \qquad p_m = 0.$$

By (7.10), $\bar{f}_i \gg 0$; hence, there is a portfolio plan $\tilde{f}_i$ such that, for every m in M,

$$\sum_i \tilde{f}_{imj} r_{jm} > -\varepsilon, \qquad \tilde{f}_{im^-} \gg \tilde{f}_{im}, \qquad \bar{f}_i \gg f_{i0}.$$

Then, for the given (p, v, r), the plan $(\tilde{x}_i, \tilde{z}_i, \tilde{f}_i)$ satisfies *every* budget constraint in (7.4) with a *strict* inequality. Also, this plan is attainable (use a construction similar to (10.1)). Therefore, $\bar{\Gamma}_i$ is continuous at every (p, v, r) that satisfies (10.2), with (p, v) in P (analogue of [**1**, 4.8, (1), p. 63]). But every $r = \boldsymbol{\rho}(p, v)$ does satisfy (10.2). Also, by the strict concavity of V_i (see (5.6)), each $\hat{\rho}_j$ is continuous. Hence, $\hat{\Gamma}_i$ is continuous.

For any plan g in $\hat{G}$, define $\mu_m(g)$ to be the set of points in P_m that satisfy the corresponding line of (9.6), and define:

$$\mu(g) = \bigtimes_m \mu_m(g);$$
$$\hat{\gamma}(p, v) = \bigtimes_i \hat{\gamma}_i(p, v);$$
$$\hat{\pi}(p, v) = \bigtimes_j \hat{\pi}_j(p, v); \quad \text{and}$$
$$\psi(g, p, v) = \hat{\gamma}(p, v) \times \hat{\pi}(p, v) \times \mu(g).$$

The correspondence ψ from $\hat{G} \times P$ to itself satisfies the hypotheses of the Kakutani Fixed Point Theorem [**1**, 1.10, (2), p. 26], and therefore has a fixed point; the proof that this fixed point is a pseudo-equilibrium can be completed using the method of [**1**, p. 87, para. 7].

11. COMMENTS ON THE ASSUMPTIONS AND ON ALTERNATIVE FORMULATIONS

It has already been pointed out that in the present model the "shareholders" have *un*limited liability, and they therefore have a status more like that of partners than of shareholders, as these terms are usually understood. One way to formulate limited liability for shareholders is to impose the constraint on producers that their net revenues be nonnegative at each date-event pair. The reason this formulation has not been adopted in this paper is that I have not yet found reasonable conditions under which the producers' correpondences could be shown to be upper semicontinuous. This is analogous to the problem that arises when, for a given price system, the consumer's budget constraint(s) force him to be on the boundary of his consumption set. In the case of the consumer, this situation is avoided by a condition such as (7.7) (for weaker conditions, see [**1**, Notes to Chapter 5, pp. 88–89; and **2**]. However, for the case of the producer, it is not considered unusual in the standard theory of the firm that, especially in equilibrium, the maximum profit achievable at the given price system be zero (e.g., in the case of constant returns to scale).

It would be interesting to have conditions on the producers and consumers that would directly guarantee the existence of an equilibrium, not just a pseudo-equilibrium. In other words, under what conditions would the share markets be cleared at every date-event pair? Notice that if there is an excess supply of shares of a given producer j at a date-event pair (t, A), then at the successor date-event pairs $(t + 1, B)$ some part of the producer's revenue will not be "distributed." One would expect this situation to arise only if his revenue is to be negative in at least one event B at date $t + 1$; thus at such a date-event pair the producer would have a deficit covered neither by "loans" (i.e., not offset by forward contracts) nor by shareholders' contributions. In other words, the producer would be "bankrupt" at that point.

One approach might be to eliminate from a pseudo-equilibrium all producers for whom the excess supply of shares is not zero at some date-event pair, and then search for an equilibrium with the smaller set of producers, etc., successively reducing the set of producers until an equilibrium is found. This procedure has the trivial consequence that an equilibrium always exists, since it exists for the case of pure exchange[3] (the set of producers is empty!). This may not be the most satisfactory resolution of the problem, but it does point up the desirability of having some formulation of the possibility of "exit" for producers who are not doing well.

Although the present model does not allow for "exit" of producers (except with the modification described in the preceding paragraph), it does allow for "entrance" in the following limited sense. A producer may have zero production up to some date, but plan to produce thereafter; this is not inconsistent with a positive demand for shares at preceding dates.

[3] For this formulation, one would introduce an inequality in condition (7.3), to provide for "free disposal" by consumers, as in condition (3.3).

The creation of new "equity" in an enterprise is also allowed for in a limited sense. A producer may plan for a large investment at a given date-event pair, with a negative revenue. If the total supply of shares at the preceding date-event pair is nevertheless taken up by the market, this investment may be said to have been "financed" by shareholders.

The assumptions of Section 3 describe a mode of producer behavior that is not influenced by the shareholders or (directly) by the prices of shares. A common alternative hypothesis is that a producer tries to maximize the current market value of his enterprise. There seem to me to be at least two difficulties with this hypothesis. First, there are different market values at different date-event pairs, so it is not clear how these can be maximized simultaneously. Second, the market value of an enterprise at any date-event pair is a price, which is supposed to be determined, along with other prices, by an equilibrium of supply and demand. The "market value maximizing" hypothesis would seem to require the producer to predict, in some sense, the effect of a change in his plan on a price *equilibrium*; in this case, the producers would no longer be price takers, and one would need some sort of theory of general equilibrium for monopolistic competition.

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Manuscript received June, 1970; revision received January, 1971.

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