Reprinted from

MATHEMATICAL TOPICS IN ECONOMIC THEORY AND COMPUTATION
Society for Industrial and Applied Mathematics
Philadelphia, Pennsylvania, 1972

Optimal Steady-state Behavior of an Economy with Stochastic Production and Resources*

Roy Radner†

Abstract. Existence of optimal "steady-state" programs is demonstrated for two models of an economy with stochastic production and resources. In both models, the source of uncertainty is a stochastic environment represented by a stationary stochastic process. In the first model, production is limited by the limited availability of unproduced primary resources, and the objective is to maximize the expected utility of consumption per period. In the second model, which is a generalization of von Neumann's model of an expanding economy, there are no limiting primary resources, and the problem is to characterize those stochastic processes of output that are "balanced" in some appropriate sense, and to determine which "balanced-growth" processes (if any) have the maximum rate of growth. Furthermore, in the first model it is shown that an optimal program can be sustained by a stationary stochastic process of prices such that, at each date, the optimal production program maximizes one-period expected profit, and the optimal consumption program is cost-minimizing in a certain sense.

1. Introduction. The characterization by prices of an economic plan that is optimal under conditions of uncertainty was introduced by Arrow (1953) for the case of exchange, and extended by Debreu (1953), (1959) to the case of production and exchange. The Arrow–Debreu theory postulates that the source of uncertainty in the economy is an uncertain environment, and shows that the prices that characterize an optimum must in general depend on the state of the environment. This implies that, if the uncertainty about the environment is described by a probability distribution, then the prices that characterize an optimum are random variables.

The Arrow–Debreu theory assumed that the set of states of the environment is finite, and that the planning horizon is also finite. With these assumptions, the formal theory of an economic optimum with a finite number of commodities can be applied directly. However, if we are interested in the optimal "steady-state" behavior of an economy with an uncertain environment, then we are led to consider a model with infinitely many states of the environment, even if there are only finitely many possible events at each date. This is so because the underlying

* Received by the editors October 22, 1971, and in revised form March 7, 1972. Presented by invitation at the Symposium on Mathematical Analysis of Economic Systems sponsored in part by a grant from the Office of Naval Research at the 1971 Fall Meeting of the Society for Industrial and Applied Mathematics, held at the University of Wisconsin, Madison, Wisconsin, October 11–13, 1971.

† Department of Economics, University of California, Berkeley, California 94720. This research was supported by the National Science Foundation under Grant GS-2078.

probability space takes the form of a stationary stochastic process, and each "state" of the environment corresponds to a particular realization (or trajectory) of the stochastic process.

In this paper, I shall present some recent results for two types of steady-state behavior. In both cases, the environment is a stationary stochastic process, with technology and resources depending upon the environment in a stationary way. In the first case, production is limited by the limited availability of (unproduced) primary resources, and the problem is to characterize those stationary stochastic processes of production and consumption that maximize the expected utility of consumption per period. In the second case, which corresponds to von Neumann's model of an expanding economy, there are no limiting primary resources, and the problem is to characterize those stochastic processes of output that are "balanced" in some appropriate sense, and to determine which "balanced growth" processes (if any) have the maximum rate of growth.

Most of the paper (§§ 2–5) will be devoted to the case of stationary production and consumption, and a detailed introduction to this case follows below. Section 6 discusses the case of balanced growth at the maximum rate, and § 7 provides a bibliographical note on related work.

One of the first problems considered in the theory of production and consumption over time was the characterization of "optimal," or "capital saturated," stationary states. In a deterministic world, a stationary state is a program of production and consumption such that the quantities of inputs, outputs and consumption are constant through time. A stationary state is optimal, or capital saturated, if no other stationary state provides a greater utility of consumption per period. Two basic questions about stationary states are: (i) under what conditions is capital saturation possible, and (ii) under what conditions can optimal stationary states be characterized, or sustained, by a price system, and in particular by a system of stationary prices, in which relative prices remain constant over time and the rate of interest is zero. The formulation and solution of these problems constituted a first step toward the successful treatment of more general problems of proportional and optimal growth.[1]

In a world with uncertain technology and resources, a natural extension of the concept of stationary state is a sequence of stochastic inputs, outputs and consumption that form a stationary stochastic process.[2] Sources of uncertainty in production for which the analysis of a (stochastically) stationary economy could be appropriate might include uncertainty about weather, about productivity, and—in the case of a nation or region—about prices on external markets. On the other hand, uncertainty about technological progress would not typically lead to

[1] For a history of this subject up to 1953, see Malinvaud's fundamental article of that date. The term "optimal" is used here in a more limited sense than that employed by Malinvaud; the term "capital saturation" was introduced by Koopmans (1957).

[2] A sequence $X(t)$ of random variables is *stationary* if for every finite sequence, $t_1, \cdots, t_n$ of "dates," and every nonnegative integer t, the joint distribution of the random variables $X(t_1 + t), \cdots, X(t_n + t)$ is the same as that of $X(t_1), \cdots, X(t_n)$. For a definition of, and facts about, stationary sequences, see, for example, Loève (1963, Chap. 30).

an analysis of stationary economies. However, in the certainty case, an interest in technological progress has often led to the consideration of "balanced" or "proportional" growth, and under certain assumptions the study of balanced growth can be formally reduced to the study of stationary economies, even in the case of uncertainty (see Radner (1971)).

In the present paper, uncertainty about technology and resources will be represented in terms of uncertainty about an (exogenous) environment, whose successive states form a stationary stochastic process, with probabilities that are unaffected by economic decisions. The successive states of the *economy*, of course, depend both on the environment and on the decisions taken with regard to production and consumption. I shall show that, under conditions that are natural extensions of "neoclassical" conditions in the case of certainty, (i) capital saturation is possible, i.e., an optimal stationary stochastic program exists, and (ii) an optimal program can be sustained by a price system that takes the form of a *stationary stochastic process* of price vectors. In other words, an optimal stationary program can be sustained by a stochastic "equilibrium," in which at each date the optimal production decisions maximize expected intertemporal profit, and the optimal aggregate consumption vector has minimum cost among all aggregate consumption vectors yielding no less (social) utility.

1.1. Review of the case of certainty. It may help the reader to review the problem of optimal stationary states in the case of certainty. The analysis in this case is simple, and will motivate the more complex treatment that seems required in the case of certainty.

Let aggregate inputs, outputs and consumption be represented by nonnegative vectors. At each date there is a set, say G, of technologically feasible input-output pairs, denoted by (a, b), where a denotes the vector of inputs at the beginning of the period, and b denotes the vector of corresponding outputs at the end of the period, which become available for consumption or production at the beginning of the next period, i.e., at the next date. The usual "neoclassical" assumptions lead us to describe G as a closed convex cone with vertex 0, such that if an input–output pair is technologically feasible, then any other pair with no smaller inputs or no larger outputs is also feasible (free disposal of inputs and outputs). Furthermore, we distinguish two types of commodities, *produced* and *primary*. Primary commodities cannot be produced, but are necessary for the production of the other commodities. Formally, if (a_1, a_2) represents the decomposition of an input vector into subvectors of produced and primary commodities, respectively, and (b_1, b_2) is the corresponding representation of an output vector, then we assume that for all technologically feasible input–output pairs, (i) $b_2 = 0$, and (ii) $a_2 = 0$ implies $b = 0$.

The phenomenon of primary commodities requires that there be an exogenous supply of primary commodities in order for nonzero production to take place. Let w_2 denote the vector of primary commodities that is exogenously supplied to the economy at each date (the same for all dates), and let $w = (0, w_2)$; w will be called the resource vector (note that the exogenous supply of *produced* commodities is assumed to be zero). If the input–output pair (a, b) is used at every date, then the resulting consumption is $c = b - a + w$.

A *stationary program*, (a, b), is *feasible*, given w, if (a, b) is in G (technological feasibility) and $b - a + w \geqq 0$ (consumption feasibility). Let u be a "utility" function defined on aggregate one-period consumption vectors. Given w, a feasible stationary program is *optimal* if it has maximum utility in the set of feasible stationary programs.

In the following discussion, let the resource vector, w, be fixed. First, it can be shown that if u is continuous, then there does exist an optimal program. (Indeed, the set of feasible programs is compact; see, for example, Radner (1967, Th. 2.1).) Let (a^*, b^*) denote an optimal input–output pair, and $c^* = b^* - a^* + w$ the corresponding optimal consumption vector, with $v^* = u(c^*)$.

Second, suppose in addition that u is concave, and that there exists a consumption vector that yields a utility strictly greater than v^* (nonsatiation). Let W denote the set of vectors of the form $(c + a - b)$ such that a, b and c are nonnegative, (a, b) is a *technologically* feasible input–output pair, and c is a consumption vector with utility strictly greater than v^*. W is a nonempty convex set, and the given resource vector, w, is not in W. Hence there is a hyperplane through w that supports W; i.e., there is a nonzero vector, p, such that

$$p(c + a - b) \geqq pw = p(c^* + a^* - b^*),$$

for all $(c + a - b)$ in W. It follows easily that

$$p(b - a) \leqq p(b^* - a^*) = 0 \quad \text{for all } (a, b) \text{ in } G,$$

$$pc \geqq pc^* \quad \text{for all} \quad c \geqq 0 \quad \text{such that} \quad u(c) \geqq u(c^*) = v^*.$$

(Recall that G is a cone.) From the assumption of free disposal, it further follows that p is nonnegative. The vectors a^*, b^*, c^* and p therefore have the "equilibrium" properties alluded to above, with p interpreted as a vector of prices. The fact that the same vector p is used to value both inputs and outputs implies that the rate of interest is zero.

1.2. Uncertainty in an activity analysis model. To illustrate the treatment of uncertainty and stationarity to be used in this paper, I first sketch the relevant concepts in the special case of a linear activity analysis model. Let A_t and B_t be sequences of input and output coefficient matrices, respectively, the elements of which are random variables. These random variables are all defined on a common basic probability space, S, which represents the environment. Each element, s, of S is a sequence, (s_t), where s_t represents the state of the environment at date t. The stationarity of the environment is expressed in terms of a shift operator, T, defined on S by

$$(1.1) \qquad (Ts)_t = s_{t+1};$$

it is assumed that both T and its inverse are measure preserving. Thus, for any event A, a subset of S, the probability of TA equals the probability of A.

Since we are interested in stationary economies, it is convenient to think of the history of the environment as extending infinitely far back into the past. In other words, we may imagine that we are observing the economy after it has been operating under a given policy for a very long time. In such an economy, we can generate stationary stochastic processes as follows. Let ξ be some function[3] on S that depends at most on the states $(\cdots, s_{-2}, s_{-1}, s_0,)$, i.e., on the history of the environment up through time 0; then the sequence of random variables (Z_t) defined by

$$Z_t = \xi(T^t s) \tag{1.2}$$

is a stationary process. By a slight abuse of notation, we can define the function T by

$$(T\xi)(s) = \xi(Ts). \tag{1.3}$$

Suppose, then, that the matrices A_t and B_t are generated in this way by

$$\begin{aligned} A_t &= \mathbf{A}(T^t s), \\ B_t &= \mathbf{B}(T^{t+1} s), \end{aligned} \tag{1.4}$$

where $\mathbf{A}$ and $\mathbf{B}$ are two given functions that depend at most on the states of the environment up through time 0. The two equations of (1.4) express the assumption that the input coefficients governing production in period t depend at most on the states of the environment up through date t, whereas the corresponding output coefficients may depend, in addition, on the state of the environment at date $(t + 1)$. Similarly, the exogenous sequence of resource vectors, (w_t), is generated by

$$w_t = \omega(T^t s). \tag{1.5}$$

The economic decisions at each date t result in a nonnegative vector l_t of activity levels; suppose that

$$l_t = \lambda(T^t s), \tag{1.6}$$

where λ also depends only on $(\cdots, s_{-2}, s_{-1}, s_0)$. This expresses the constraint that the decisions about activity levels at date t can at most depend upon the states of the environment up through date t. Of course, current decisions can depend upon the results of past production, but these results in turn depend upon past decisions, etc.

Given the sequences of activity levels, resources and input and output coefficients, the corresponding sequences of input, output and consumption vectors are determined by

$$a_t = A_t l_t, \quad b_{t+1} = B_t l_t, \quad c_t = b_t - a_t + w_t, \tag{1.7}$$

with the constraint that c_t must be nonnegative. Notice that b_{t+1} is the output

[3] In this heuristic introduction I omit any discussion of measurability and other technical matters, as far as possible.

vector that corresponds to the input and activity level vectors at date t, but is only available for consumption or further input into production at date $(t + 1)$.

One can express the input, output and consumption vectors directly as functions on S, as follows:

$$a_t = \alpha(T^t s), \quad b_{t+1} = \beta(T^t s), \quad c_t = \gamma(T^t s), \tag{1.8}$$

where

$$\alpha = \mathbf{A}\lambda, \quad \beta = (T\mathbf{B})\lambda, \quad \gamma = T^{-1}\beta - \alpha + \omega. \tag{1.9}$$

Since consumption must be nonnegative,

$$T^{-1}\beta - \alpha + \omega \geqq 0. \tag{1.10}$$

A pair (α, β) satisfying (1.9) for some nonnegative function λ is *technologically feasible*. In addition, if the nonnegativity constraint (1.10) is satisfied, the pair (α, β) forms a feasible stationary production program, and γ is the corresponding *stationary consumption program*. The set of technologically feasible pairs of functions, (α, β), corresponds to the set G of technologically feasible input–output pairs in the case of certainty.

To describe prices and intertemporal profit, let ψ be a function depending at most on the environment through date 0, and define

$$p_t = \psi(T^t s). \tag{1.11}$$

The intertemporal profit at date t is defined as the mathematical expectation

$$E[p_{t+1}b_{t+1} - p_t a_t]. \tag{1.12}$$

This expectation can be written in the form of two iterated expectations:

$$E(E[p_{t+1}b_{t+1} - p_t a_t | \cdots, s_{t-1}, s_t]), \tag{1.13}$$

but since p_t and a_t only depend on states of the environment up through date t, $E[p_t a_t | \cdots, s_{t-1}, s_t] = p_t a_t$, and the expression (1.13) for (expected) intertemporal profit can be rewritten as

$$E(E[p_{t+1}b_{t+1} | \cdots, s_{t-1}, s_t] - p_t a_t). \tag{1.14}$$

To express intertemporal profit in terms of the relevant functions on S, we can substitute (1.8) and (1.11) into (1.12) to get the expression

$$E[\psi(T^{t+1}s)\beta(T^t s) - \psi(T^t s)\alpha(T^t s)]. \tag{1.15}$$

By the assumption of stationarity of the environment process, expression (1.15) has the same value for all t, so that (setting $t = 0$) we can rewrite intertemporal profit as

$$E[(T\psi)\beta - \psi\alpha], \tag{1.16}$$

or, since $E[(T\psi)\beta] = E[\psi(T^{-1}\beta)]$, as

$$E[\psi(T^{-1}\beta - \alpha)]. \tag{1.17}$$

In this last form, the expression for intertemporal profit bears a suggestive resemblance to the expression $p(b - a)$ in the case of certainty.

2. Environment. The environment is represented by a probability space $(S, \mathscr{S}, \sigma)$ where:

(i) S is the set of all doubly infinite sequences $s = (s_t)$, $-\infty < t < \infty$, with each s_t belonging to the finite set $\{1, \cdots, n\}$. A particular s_t will be called the *environment at date t*, and s will be called an *environment sequence.*

(ii) $\mathscr{S}$ is the sigma-field generated by all cylinder sets[4] in S.

(iii) σ is a probability measure on $\mathscr{S}$.

Define a *shift transformation*, T, from S to itself by:

$$(Ts)_t = s_{t+1}, \qquad -\infty < t < \infty. \tag{2.1}$$

I shall assume:

E1. T and T^{-1} are measure preserving and ergodic.

E2. σ is nonatomic.[5]

Assumptions E1 and E2 would be satisfied if, for example, the environment were a regular stationary finite Markov chain.

For any function f defined on S, define Tf by

$$(Tf)(s) = f(Ts). \tag{2.2}$$

Let $\mathscr{S}_t$ denote the sigma-field generated by all "partial histories," $(\cdots, s_{t-1}, s_t)$, i.e., $\mathscr{S}_t$ is generated by all cylinder sets $\times A_m$ such that $A_m = \{1, \cdots, n\}$ for all $m > t$. Note that $(\mathscr{S}_t)$ is a monotone increasing sequence of sigma-fields and every $\mathscr{S}_t$ is contained in $\mathscr{S}$.

3. Stationary production and consumption. The *commodity space* (at each date) is represented by the nonnegative part, C_+, of a finite-dimensional Euclidean space C. The commodities are of two kinds, *produced* and *primary*. For a vector c in C, the corresponding decomposition of c will be denoted by (c_1, c_2), i.e., c_1 is the vector of produced commodities, and c_2 is the vector of primary commodities.

The *production possibilities* are represented by a correspondence G from S to C_+^{n+1}. Thus for $(a; b_1, \cdots, b_n)$ in $G(T^t s)$, the vector a is to be interpreted as the input at date t, and b_j is to be interpreted as the corresponding output, at date $(t + 1)$, if $s_{t+1} = j$.

A set F in C_+^{n+1} will be called *neoclassical* if it satisfies the following conditions:

(i) F is a closed convex cone, with vertex 0.

(ii) Free disposal: if $(a; b_1, \cdots, b_n)$ is in F, $a' \geqq a$, and $b'_j \leqq b_j$ for $j = 1, \cdots, n$, then $(a'; b'_1, \cdots, b'_n)$ is in F.

(iii) Primary inputs are necessary: if $(a; b_1, \cdots, b_n)$ is in F, and all the coordinates of a corresponding to primary commodities are zero, then $b_j = 0$ for $j = 1, \cdots, n$.

[4] A cylinder set in S is a Cartesian product $\times A_t$, where $A_t \subset \{1, \cdots, n\}$ for all t, and $A_t = \{1, \cdots, n\}$ for all but a finite set of t.

[5] For these concepts, see Friedman (1970).

(iv) Primary inputs are not producible: if $(a; b_1, \cdots, b_n)$ is in F, then for every $j = 1, \cdots, n$, all the coordinates of b_j corresponding to primary commodities are zero.

The following assumptions will be made concerning the production possibility correspondence G:

P1. There is a finite partition (A_i) of S such that for each i, A_i is in $\mathscr{S}_0$ and $G(s)$ is constant on A_i.

P2. For every s in S, $G(s)$ is neoclassical.

Let L denote the set of $\mathscr{S}_0$-measurable functions from S to C (where the measurable sets of C are the Borel sets). The set of technologically feasible stationary programs is represented by a set $\mathscr{G}'$, defined as follows:

(3.1) $\mathscr{G}'$ is the set of all pairs (α, β) such that, for some $\delta_1, \cdots, \delta_n$,

(i) α and each δ_j are in L,
(ii) $[\alpha(s); \delta_1(s), \cdots, \delta_n(s)]$ is in $G(s)$ a.s., and
(iii) $\beta(s) = \delta_{s_1}(s)$ a.s.

The $(n + 1)$-tuple $(\alpha; \delta_1, \cdots, \delta_n)$ will be called a *production decision function*, and the pair (α, β) will be called the corresponding *production program.*

The supply of primary resources will be assumed to be a bounded stationary stochastic process, generated by a function ω for which the subvector $\omega_1(s)$, corresponding to produced (i.e., nonprimary) resources, is always zero. This is summarized in the following assumption:

P3. ω is in L, is nonnegative and essentially bounded,[6] and $\omega_1(s) = 0$ a.s.

If consumption is constrained to be nonnegative, then to each primary resource program ω corresponds a set of *feasible* stationary production programs, which will be represented by $\mathscr{G}_\omega$:

(3.2) $$\mathscr{G}_\omega = \{(\alpha, \beta) : (\alpha, \beta) \text{ in } \mathscr{G}', \ \beta + T\omega \geqq T\alpha\}.$$

Note that $\mathscr{G}_\omega$ is convex, and that $\mathscr{G}'$ is a convex cone.

A pair (α, β) in $\mathscr{G}_\omega$ determines a stationary process of production and consumption as follows:

$(T^t\alpha)(s)$ is the vector of inputs at date t;
$(T^t\beta)(s)$ is the vector of outputs at date $t + 1$, resulting from inputs at date t;
$(T^{t-1}\beta - T^t\alpha + T^t\omega)(s)$ is the consumption at date t.

In the remainder of this paper I shall use the following notation:

L_1 = the set of $\mathscr{S}_0$-measurable, integrable functions from S to C.
L_1^2 = the set of pairs (f, g) such that f and $T^{-1}g$ are in the L_1.

The sets L_∞ and L_∞^2 are defined in an analogous way, with "integrable" replaced by "essentially bounded." (For definitions and properties of L_p spaces, see, for example, Dunford and Schwartz (1964, p. 121 ff., p. 241 ff.).) It is clear that L_∞ is contained in L_1 and that L_∞^2 is contained in L_1^2. In the discussion of prices (see § 5), it will be convenient to consider, instead of $\mathscr{G}'$, the set $\mathscr{G}$ of essentially bounded programs in $\mathscr{G}'$. The set $\mathscr{G}$ is, of course, a convex cone in L_∞^2.

[6] A function is essentially bounded if it is bounded on the complement of a set of measure zero.

4. Optimal stationary consumption. A stationary consumption program will be judged by the expected utility of consumption at each date; stationarity implies that this expected utility is the same for all dates. Let u, the *utility function*, be a real-valued function on C_+ satisfying:

U1. u is concave and continuous.

Let the resource program ω be given, and for any (α, β) in $\mathscr{G}_\omega$ define the corresponding *expected utility of consumption* by

$$U(\alpha, \beta) = \int u(T^{-1}\beta - \alpha + \omega)\, d\sigma. \tag{4.1}$$

It can be shown that $U(\alpha, \beta)$ is bounded on $\mathscr{G}_\omega$; hence,

$$v^* \equiv \sup \{U(\alpha, \beta) : (\alpha, \beta) \text{ in } \mathscr{G}_\omega\} \tag{4.2}$$

is finite. The first theorem states that the maximum expected utility is actually attained by a feasible stationary consumption program.

THEOREM 1. *$U(\alpha, \beta)$ attains the value v^* at some (α^*, β^*) in $\mathscr{G}_\omega$.*

The proof of this theorem makes use of the theory of weak convergence.[7] The following lemma is an important step in the proof, and has an independent interest. The proof of the lemma is suggested by that of Theorem 2.1 of Radner (1967), but essential use is made of the assumption that T is measure-preserving and ergodic, and that the underlying probability measure is nonatomic.

LEMMA. *For every ω satisfying Assumption* P3 *there is a compact set K in C_+ such that, for every (α, β) in $\mathscr{G}_\omega$, $\alpha(s)$ and $\beta(s)$ are in K a.s.*

5. Prices. In this section I show that, corresponding to an optimal stationary program (α^*, β^*), as defined in § 4, there exists a stationary process of nonnegative "price" vectors such that (i) the optimal production process maximizes expected one-period "profit" at each date, (ii) the optimal consumption process minimizes, at each date, the expected "cost" of consumption in the set of all stationary consumption processes with expected utility at least v^*, and (iii) for every date, the price vector is almost surely positive. To obtain this result, I make additional assumptions regarding monotonicity of preferences, positivity of primary resources, and the possibility of positive net output of produced commodities.

For a vector c in C, define $c > 0$ to mean that $c \geqq 0$, but $c \neq 0$, and $c \gg 0$ to mean that every coordinate of c is positive; further, for any number k define $c \geqq k$ to mean that every coordinate of c is greater than or equal to k.

Correspondingly, for ξ in L, define $\xi > 0$ to mean $\xi \geqq 0$ but $\xi \neq 0$, $\xi \gg 0$ to mean that $\xi(s) \gg 0$ a.s., and $\xi \ggg 0$ to mean that there exists a positive number k such that $\xi(s) \geqq k$ a.s.

I shall say that the utility function u is *strictly increasing* if, for c and c' in C_+, $c > c'$ implies $u(c) > u(c')$.

[7] For proofs of the theorems stated in §§ 4 and 5, see Radner (1972).

For γ in L_∞, define

$$V(\gamma) = \int u(\gamma)\, d\sigma. \tag{5.1}$$

Thus, if γ is a stationary consumption program, then $V(\gamma)$ is its expected one-period utility. If u is strictly increasing, and $\gamma > \gamma'$, then $V(\gamma) > V(\gamma')$.

In what follows, let ω be given, let (α^*, β^*) be an optimal stationary production program, as in § 4, and define

$$\begin{aligned} \gamma^* &= T^{-1}\beta - \alpha + \omega, \\ \eta^* &= T^{-1}\beta^* - \alpha^*, \\ \mathscr{Y} &= \{T^{-1}\beta - \alpha : (\alpha, \beta) \text{ is in } \mathscr{G}\}. \end{aligned} \tag{5.2}$$

Note that $\mathscr{Y}$ is the set of bounded stationary *net* output programs that are technologically feasible. $\mathscr{Y}$ is a convex cone with vertex at the origin.

For any ξ in L let ξ_1 and ξ_2 be the components corresponding to produced and primary commodities, respectively. Recall that $\omega_1 = 0$, and that $\beta_2 = 0$ for all (α, β) in $\mathscr{G}$.

THEOREM 5.1. *If u is strictly increasing, $\omega_2 \ggg 0$, and $\eta_1 \ggg 0$ for some η in $\mathscr{Y}$ with $\eta_2 = -\omega_2$, then there is a $\psi \gg 0$ in L_1 such that*

(i) $\int [(T\psi)\cdot\beta - \psi\cdot\alpha]\, d\sigma \leqq \int [(T\psi)\cdot\beta^* - \psi\cdot\alpha^*]\, d\sigma = 0$ *for all* (α, β) in $\mathscr{G}$;

(ii) $\int \psi\cdot(\gamma - \gamma^*)\, d\sigma \geqq 0$ *for all* $\gamma \geqq 0$ *in* L_∞ *such that* $V(\gamma) \geqq V(\gamma^*)$.

The proof is suggested by those of Theorem 6.4 of Debreu (1959, p. 95) and Theorem 4.3 of Bewley (1972, pp. 525–526). The proof is in two parts; first one finds a price system in L_∞^*, the dual of L_∞, and then one derives from this a price system in L_1. The method is based on the Yosida–Hewitt decomposition of a finitely additive measure into a countably additive part and a purely finitely additive part (see Yosida and Hewitt (1962)).

6. Balanced growth at the maximal rate. Another type of "steady-state' behavior that has been much studied in the case of certainty is that of "balanced" or "proportional" growth. In balanced growth, the outputs of all the commodities grow geometrically at the same rate, and the output vectors at all dates are proportional to a fixed vector. If there are primary resources, i.e., commodities that are essential for production but not producible (see § 1), then the rate of growth is limited by the rate of growth of the exogenous supply of primary resources (see Radner (1967)). However, if there are no primary resources, then the rate of growth is limited only by the "intrinsic productivity" of the technology. In this case, one may be interested in characterizing those balanced growth programs that have the maximum rate of growth, if indeed there are any such programs. It is intuitively clear that the maximum rate of growth would typically be achieved only if there is no consumption, i.e., only if all of each period's output is fed back into production at the next period. We might call such programs "pure accumulation programs." The study of balanced growth pure accumulation programs has been fundamental in the development of economic growth theory, beginning with the pioneering article by J. von Neumann (1945).[8] It should be emphasized that a pure accumulation program does not preclude consumption, if consumption is considered to be "technologically" determined, for example, by the "requirements" of feeding and clothing workers to ensure their productivity.

An attempt to extend the von Neumann model of balanced growth to the case of uncertainty immediately faces the problem that it is not typically desirable, and it may not even be possible, to guarantee that successive output vectors are proportional; furthermore, the rate of growth will typically not be the same from one period to the next. In this section, I outline a model of pure accumulation with balanced growth, in the presence of a stationary stochastic environment (as defined in § 2). (For a detailed presentation of these results in the case of a stationary *Markovian* environment, see Radner (1971).)

Consider now the production model of § 3, but with no primary resources, and suppose further that for every date t, the output from production at that date equals the input at date $(t + 1)$. Let (Z_t) denote a particular feasible growth process, i.e., for a particular sequence of decision functions, Z_t is both the output from production at date $(t - 1)$ and the input for production at date t (the exogenous supply of all commodities is zero at every date).

Let w be any strictly positive vector in the commodity space (except where noted, the vector w will remain fixed throughout the discussion). Define the *normalized growth process* (X_t), associated with (Z_t), by

$$X_t = \begin{cases} Z_t/w'Z_{t-1} & \text{if} \quad Z_{t-1} > 0, \\ 0 & \text{if} \quad Z_{t-1} = 0. \end{cases}$$

The process (Z_t) will be called *balanced* if the associated normalized growth process is stationary, or more precisely, if the stochastic process (s_t, X_t) is stationary. We may think of $w'Z_t$ as an index of output at date t. The *average rate of growth* R_T *from date* 0 *to date* T is defined as

$$R_T = \log\,[w'Z_T/w'Z_0]^{1/T} = \left(\frac{1}{T}\right) \log\,[w'Z_T/w'Z_0].$$

We can also express R_T in terms of the associated normalized growth process:

$$R_T = \frac{1}{T} \sum_{t=1}^{T} \log\,(w'X_t).$$

The *long-run average growth rate* of a balanced growth process (Z_t) is defined as

$$R = \lim_{T \to \infty} R_T.$$

[8] The original version of von Neumann's paper was published in German in 1937. His analysis was subsequently extended and simplified by Gale (1956) and others; for a recent treatment, with references to the literature, see Nikaido (1968, Chap. IV). It should be noted that, for balanced growth at the maximum rate, there may be some commodities that can be produced in greater quantities than are exactly needed for balanced growth at that rate; in particular, this phenomenon can be expected if there are fixed input and output coefficients. However, the existence of such "surplus outputs" is due to purely technological causes, not to any desire to consume these commodities.

One can show that this limit exists almost surely, in the sense that it is finite or minus infinity, since the process (X_t) is stationary. The long-run growth rate is, in general, a random variable. The *expected growth rate* is defined as $r = E \log(w'X_t)$; this expected value, which may be finite or minus infinity, is the same for all t, since (X_t) is stationary. One can also show that the expected growth rate r is equal to the expected value of the long-run growth rate R.

Suppose that production satisfies Assumptions P1 and P2 of § 3, with, however, parts (iii) and (iv) of the definition of neoclassical production replaced by:

(iii′) No output without input: if $(a; d)$ is in F, and $a = 0$, then $d = 0$.

(iv′) Positive output is possible: there exists $(a; d)$ in F such that $d \gg 0$.

One can show that the expected growth rate is bounded in the set of balanced growth processes. However, it is not known under what conditions the maximum rate of growth can actually be achieved within the class of balanced growth processes. Interestingly enough, this problem disappears if we allow *generalized growth processes*, in which the decision function at each date can be *randomized*. If we admit generalized growth processes, then one can show:

1. The expected growth rate r attains a maximum, say r^*, on the set of all balanced growth processes, and r^* is the same for all strictly positive w.

2. For any balanced growth process, $R \leqq r^*$ almost surely, and if $r = r^*$, then $R = r^*$ almost surely.

Thus a maximum rate of growth exists in the strong sense that it is independent of the particular linear index of output chosen, and for no balanced growth process can the long-run average growth rate exceed the maximum expected growth rate with positive probability.

It is not known under what conditions balanced growth at the maximum rate can be characterized by prices in the sense of § 1, although it is known that a price system can be obtained in the form of a continuous linear functional on L_∞. However, if this linear functional is strictly positive, then there does exist a price system in the form of a stochastic process (p_t), as in § 1; furthermore, *this price process is balanced* (in the same sense in which the growth process is balanced), and the rate of growth of the price process is equal to $(-r^*)$. In other words, for such a price process, *the rate of interest equals the rate of growth*. This result partially extends the corresponding theorems obtained by von Neumann and by Gale in the deterministic case.

7. Bibliographic note. The model of production used in this paper is suggested by the model of Radner (1971), in which, however, the environment is assumed to be Markovian. The latter paper is concerned with the existence of growth at the maximum rate in a stochastic version of von Neumann's model of balanced growth, and there is no consideration of prices. The characterization of stochastic balanced growth in a stationary environment by a stochastic process of prices will be considered in a forthcoming paper.

W. A. Brock and L. J. Mirman (1971), (1972) have studied optimal growth under uncertainty in a model with one commodity and a sequence of independent and identically distributed states of the environment. In particular, in the second paper they considered the problem of optimal stationary programs. P. Jeanjean (1972) has studied a multisector model with a Markovian environment. Jeanjean demonstrated the existence of "Lagrangian multipliers" associated with the constraint that programs be stationary, but these multipliers do not seem to lend themselves to an economic interpretation as prices in the usual sense. Nevertheless, Jeanjean's paper represents the most comprehensive treatment to date of the problem of optimal growth under uncertainty, in which the criterion is the total (discounted or undiscounted) expected utility of consumption.

From a general point of view, the present model can be considered as a special case of the Arrow–Debreu model of competitive equilibrium under uncertainty, with, however, an infinite-dimensional commodity space. The characterization of an economic optimum with infinitely many commodities by a price system represented as a continuous linear functional has been developed by Debreu (1954), and in the context of mathematical programming by Hurwicz (1958). However, unless the linear functional can be represented as an "inner product," the interpretation as a price system is not very natural. This problem has been explored in papers by Bewley, Majumdar, Peleg, Peleg and Yaari, Radner, and others; many of these results are reviewed in Radner and Majumdar (1972). In particular, the technique used in the present paper has been suggested by Bewley (1972).

REFERENCES

K. Arrow (1953), *Le rôle de valeurs boursières pour la répartition la meilleure des risques*, Econometrie (Centre National de la Recherche Scientifique), pp. 41–48; English transl.; *The role of securities in the optimal allocation of risk-bearing*, Rev. Economic Studies, 31 (1964), pp. 91–96.

T. Bewley (1972), *Existence of equilibria in economies with infinitely many commodities*, J. Economic Theory, 4, pp. 514–540.

G. Debreu (1953), *Une économie de l'incertain*, Electricité de France, Paris, mimeographed.

——— (1954), *Valuation equilibrium and Pareto optimum*, Proc. Nat. Acad. Sci. U.S.A., 40, pp. 588–592.

——— (1959), *Theory of Value*, John Wiley, New York.

W. A. Brock and L. J. Mirman (1971), *Optimal economic growth and uncertainty: the Ramsey–Weizsäcker case*, Mathematical Social Science Board Workshop on the Theory of Markets and Uncertainty, Working Paper 7, Department of Economics, University of California, Berkeley.

——— (1972), *Optimal economic growth and uncertainty: the discounted case*, J. Economic Theory, 4, pp. 479–513.

N. Dunford and J. T. Schwartz (1964), *Linear Operators, Part I*, Interscience, New York.

N. Friedman (1970), *Introduction to Ergodic Theory*, Van Nostrand-Reinhold, New York.

D. Gale (1956), *The closed linear model of production*, Linear Inequalities and Related Systems, H. W. Kuhn and A. W. Tucker, eds., Ann. of Math. Studies no. 38, Princeton, N.J., pp. 285–303.

L. Hurwicz (1958), *Programming in linear spaces*, Studies in Linear and Non-Linear Programming, K. J. Arrow, L. Hurwicz and H. Uzawa, eds., Stanford University Press, Stanford, Calif., pp. 38–102.

P. Jeanjean (1972), *Optimal growth in a multi-sector economy with stochastic technology*, Doctoral dissertation, Department of Economics, University of California, Berkeley (unpublished).

T. C. Koopmans (1957), *Three Essays on the State of Economic Science*, McGraw-Hill, New York.

M. LOÈVE (1963), *Probability Theory*, 3rd. ed., Van Nostrand, New York.

E. MALINVAUD (1953), *Capital accumulation and efficient allocation of resources*, Econometrica, 21, pp. 233–268.

H. NIKAIDO (1968), *Convex Structures and Economic Theory*, Academic Press, New York.

R. RADNER (1967), *Dynamic programming of economic growth*, Activity Analysis in the Theory of Growth and Planning, E. Malinvaud and M. O. L. Bacharach, eds., Macmillan, London, pp. 111–141.

R. RADNER (1971), *Balanced stochastic growth at the maximum rate*, Contributions to the von Neumann Growth Model, G. Bruckman and W. Weber, eds., Zeitschrift für Nationalökonomie, Suppl. 1, Springer-Verlag, Vienna, pp. 39–62.

R. RADNER AND M. MAJUMDAR (1972), *Shadow prices for infinite growth programs: the functional analysis approach*, Techniques of Optimization, A. V. Balakrishnan, ed., Academic Press, New York and London, 1972, pp. 371–390.

R. RADNER (1972), *Optimal stationary consumption with stochastic production and resources*, J. Economic Theory, forthcoming; also distributed as Tech. Rep. 14, Center for Research in Management Science, University of California, Berkeley.

J. VON NEUMANN (1945), *A model of general economic equilibrium*, Rev. Economic Studies, 13, pp. 1–9.

K. YOSIDA AND E. HEWITT (1962), *Finitely additive measures*, Trans. Amer. Math. Soc., 72, pp. 46–66.