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A STOCHASTIC DECENTRALIZED RESOURCE ALLOCATION PROCESS: PART II*

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This is the second part of a paper concerning an iterative decentralized process designed to allocate resources optimally in decomposable environments that are possibly characterized by indivisibilities and other nonconvexities. Important steps of the process involve randomization. In Part I we presented the basic models and results, together with examples showing that certain assumptions can be satisfied in both classical and nonconvex cases. Part II goes further with such examples in showing that our process yields optimal allocations in environments in which the competitive mechanism fails, and also shows how abstract conditions used in Part I can be verified in terms of properties of preferences and production functions that are familiar to economists.

INTRODUCTION

THIS IS Part II of a paper in which we construct an iterative decentralized process (to be called the B process because it involves bidding) designed to allocate resources optimally in environments that are decomposable, i.e., free of externalities, but possibly characterized by indivisibilities (in commodities) or nonconvexities (in preferences or production). We have largely confined ourselves to situations where either all goods are indivisible or all goods are divisible, although similar methods could also be applied to mixed cases. Important steps of the process involve randomization, hence the designation "stochastic" in the title.

Part II goes beyond Part I in showing that the B process yields optimal results in environments where the competitive mechanism often fails, specifically under circumstances in which some or all individuals have zero endowments of certain goods (Section 5.6). On the other hand, a counter-example (Section 5.5.6) shows that the B process may fail to converge under certain conditions involving incomplete endowments.

Further, Part II provides results designed to facilitate applications by showing how the abstract topological conditions, such as openness, used in the theorems of Part I can be verified in terms of properties of preferences and of production functions that are familiar to economists (especially Section 5.7).

Part I consists of Sections 1 through 5.4. General principles of notation and the definitions of equilibrium and optimality are found in Section 2, which also contains formulation of the rules governing the formation of bids and agreements in the B process. Section 3 contains a result on the stationarity of optimal distributions valid for arbitrary commodity spaces (hence, in particular, divisible, indivisible, and mixed commodity spaces). Section 4 contains the main results for the case of indivisible goods, Section 5 for divisible goods.

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In Section 5, devoted to divisible goods, the basic model and assumptions are formulated in Subsection 5.1. Important auxiliary propositions are established in 5.2 and the basic theorems on non-wastefulness and convergence are given in 5.3. Subsection 5.4 has examples showing how various assumptions (other than the "openness" assumption ED.6) underlying the theorems of 5.3 can be satisfied both in classical and non-convex models.

Part II of the paper begins with the case of incomplete endowments in Subsection 5.5. An analysis of an alternative form of the crucial "openness" assumption is found in Subsection 5.6, while 5.7 contains examples of applications, non-convex as well as convex, including one non-monotone, where commodities are divisible and utility functions continuous. Subsection 5.7.4 contains an example illustrating the existence of cases with nonconvex preferences and increasing returns in production satisfying all assumptions (including "openness") of the theorems in Subsection 5.3. Subsection 5.8 is a summary of the results in the complete Section 5.

Part II of the paper is not self-contained. The reader must consult Part I for much of the notation, definitions, etc., used in the second part.

5.5. *A Generalization Including Incomplete Endowments: A Weakening of Assumption ED.4*

5.5.0. *Introduction*

In an important class of cases, including that of pure exchange with some commodities absent from the initial aggregate endowment vector ω, Assumption ED.4 is not satisfied. Recall that this assumption required that there exist a y in Y_C such that every component y^i of y is in the interior of the corresponding set Y^i (see Subsection 5.1.1). The purpose of the present section is to show that Assumption ED.4 may be replaced by a weaker assumption, ED.4*, which could be applied in certain cases of pure exchange with incomplete initial endowment as well as many other cases (including those involving production); the case of Assumption ED.4 would still be covered.

Thus the results of the present section constitute generalizations of Theorems 5.1 and 5.2 (to be designated as Theorems 5.1* and 5.2*, respectively), retaining Assumptions ED.1 through ED.4 as well as Assumptions PD.1 and PD.2, but replacing Assumption ED.4 with ED.4*.

Since Assumption ED.4 was used only in proving Lemma 5.6, it is enough to establish its counterpart, to be designated as Lemma 5.6*.

5.5.1. *Assumption ED.4**

To state Assumption ED.4*, we first introduce the linear variety H in R^M defined by:[46]

$$H \equiv \mathscr{L}(Y_F^i) \quad \text{for all} \quad i = 1, 2, \ldots, N,$$

[46] Implicitly, we assume that $\mathscr{L}(Y_F^i)$ is the same for all $i = 1, 2, \ldots, N$. (Under pure exchange, if $C^i = \Omega$ for all i, even though the ω^i are different, $\mathscr{L}(Y_F^i)$ are all the same.)

where Y_F^i is the projection of Y_F into Y^i (see Subsection 5.2), and $\mathscr{L}(Y_F^i)$ is the smallest linear variety containing Y_F^i (see Subsection 5.1). We shall also write

$$H^* \equiv \underbrace{H \times \ldots \times H}_{N \text{ times}}.$$

Since $Y_F^i \subseteq Y^i$, it is necessarily the case that $H \subseteq \mathscr{L}(Y^i)$.

In Subsection 5.2.2 above we had[47]

$$H = \mathscr{L}(Y^i) = R^M,$$

but this is no longer assumed here. In fact, $H \subset \mathscr{L}(Y^i)$ is characteristic of the case of incomplete initial endowments under pure trade, assuming $\omega^i \in \Omega = C^i$,

$$H = \{x : x \in R^M, \quad x_j = 0, \quad \text{for } j \in \mathscr{M}^0\},$$

with $\mathscr{M}^0$ denoting the set of commodities for which $\omega_j \equiv \Sigma_{i=1}^N \omega_j^i = 0$; $\mathscr{L}(Y^i)$ may (though it need not) still equal R^M.

Assumption ED.4* will consist of three parts, to be denoted by ED.4*′, ED.4*″, and ED.4*‴. The first part, ED.4*′, is a generalization of ED.4, with the topological requirements relativized to the H topology:

ASSUMPTION ED.4*′: *There exists an N-tuple* $y^0 \equiv \langle y^{01}, \ldots, y^{0N} \rangle$, $y^{0i} \in R^M$, *such that* (i) $y^0 \in Y_C$, (ii) $y^{0i} \in Int_H(Y^i)$ (*i.e., there exists an* R^M*-open set* V^i *such that* $y^{0i} \in V^i \cap H \subseteq Y^i$), *where* $Int_H(Y^i)$ *denotes the interior of* Y^i *in the relative H topology.*

However, our earlier results would not be valid if Assumption ED.4 were simply replaced by ED.4*′. Additional conditions are needed. To state them we recall again that Lemmas 5.1 through 5.5 were proved without resort to Assumption ED.4 and are therefore valid in the present section as well.

Hence, by Lemma 5.5, there exists an L_F-open cube $S \in \mathscr{S}(y, \varepsilon)$ with center[48] $c(y, \varepsilon)$ and radius η.

As in Subsection 5.2.2 above, we define T as the open R^{MN} cube with the same center and radius and we write $T \equiv T^1 \times \ldots \times T^N$, where T^i is the R^M open cube with center c^i in Y^i and radius η.

In turn, we introduce subsets T^{*i} of the T^i by $T^{*i} \equiv T^i \cap Y^i$, and write $T^* \equiv T^{*1} \times \ldots \times T^{*N}$.

REMARK 1: In the case of Subsection 5.2.2, $T^i \subseteq Y^i$ (by (5.19), which follows from Assumption ED.4): therefore $T^{*i} = T^i$ and hence $T^* = T$.

[47] This follows from the fact that, by Assumption ED 4, the interior of Y^i (in the R^M Euclidean topology) is non-empty.

[48] We may note that $S \subset Y_F$, hence $c \in Y_C \cap H^* \cap (\times_{i=1}^N Y^i)$.

The remaining two parts of Assumption ED.4* can now be stated with reference to the sets T^{*i}. (The interpretation of these assumptions in the context of pure exchange with incomplete initial endowments is given in 5.5.5 below.)

First, it is essential for our argument that the set T^{*i} be of positive R^M-measure; in fact, we postulate that its M-dimensional measure $\mu(T^{*i})$ satisfies:

ASSUMPTION ED.4*″: *$\mu(T^{*i}) \geqslant \tilde{\mu} > 0$, where $\tilde{\mu}$ does not depend on $y = y_t$, the most recent "agreement" arrived at by the participants.*

REMARK 2: In the case of Subsection 5.2.2, Assumption ED.4*″ is satisfied because $\mu(T^{*i}) = (2\eta)^M$: on the other hand, when $C^i = \Omega$, and $\omega^i \in \Omega$ has M_i^0 zero components (hence $M_i^+ \equiv M - M_i^0$ positive components), we have

$$\mu(T^{*i}) = (2\eta)^{M_i^+}\eta^{M_i^0} = 2^{M_i^+}\eta^M \geqslant \eta^M,$$

since $M_i^+ \geqslant 0$, so that Assumption ED.4*″ is again satisfied. More generally, Assumption ED.4*″ is satisfied whenever $y^i \in Y^i$, $x \in \Omega$ implies $y^i + x \in Y^i$. (Cones other than Ω could also be used.)

Finally, when $H \neq R^M$, a restriction pertaining to the structure of preferences must also be imposed. We postulate

ASSUMPTION ED.4*‴: *For every $z^i \in T^{*i}$ there exists $x^i(z^i) \in T^{*i} \cap H$ such that $z^i \succsim_i x^i(z^i)$.*

REMARK 3: In the case of Subsection 5.2.2, $T^{*i} \cap H = T^{*i} = T^i$, so that we may take $x^i(z^i) = z^i$ and Assumption ED.4*‴ is automatically satisfied.

5.5.2. *Inevitability of Improvement from a Non-optimal Allocation*

We are ready to state the following counterpart of Lemma 5.6 of Subsection 5.2:

LEMMA 5.6*: *Given the assumptions of Subsection 5.1 above with Assumption ED.4 replaced by Assumption ED.4* (= ED.4*′ + ED.4*″ + ED.4*‴), for every $\varepsilon > 0$ there is a number $\gamma > 0$ (not depending on y) such that, for all y in Y_F,*

(5.15) $$\text{prob}\,\{y_{t+1} \in \hat{Y}_\varepsilon | y_t = y\} \geqslant \gamma.$$

PROOF: When $Y_F = \{0_x\}$ is a one-element set, $Y = \hat{Y}_\varepsilon = Y_F$ and $y_t \in \hat{Y}_\varepsilon$ for all t, so that the conclusion of Lemma 5.6* holds.[49] Now $Y_F = \{0_x\}$ is a one-element set when the dimension of H is zero or when $N = 1$. Hence without loss of generality, we shall henceforth assume that $N \geqslant 2$, and that Y_F has at least two elements, so that $\dim \mathscr{L}(Y_F) = 1$ and $M^+ \equiv \dim H \geqslant 1$.

Now, as noted earlier, in virtue of Lemmas 5.1 through 5.5, there exists an L_F-open cube $S \in \mathscr{S}(y, \varepsilon)$ with center $c(y, \varepsilon)$ and radius η. Hence

(5.17) $$S \subseteq G^+(y) \cap \hat{Y}_\varepsilon,$$

[49] Even without some of the other assumptions made in Lemma 5.6*.

and

(5.18) $z \in S$ implies $z^i \in G^{+i}(y^i) \cap Y^i_F$ for all $i \in \mathcal{N}$

where $\mathcal{N} \equiv \{1, 2, \ldots, N\}$.

It can be shown (see 5.5.3 below) that Assumption ED.4* implies

(5.18*) $L_F = Y_C \cap H^*$,

and hence, with T defined as above,

$$(5.20^*) \quad S = T \cap Y_C \cap H^* = \{x = \langle x^1, \ldots, x^N \rangle : \text{for each } i \in \mathcal{N}, x^i \in H;\ \sum_{i \in N} x^i = 0_x;\ \|x - c(y, \varepsilon)\| < \eta\}.$$

Now, for each $i \in \mathcal{N}$ a subset Y^{*i} of R^M is defined by

$$Y^{*i} \equiv \{x^i \in R^M : \|x^i - \tilde{x}^i\| \leqslant 2\eta \text{ for some } \tilde{x}^i \in Y^i_F;\ x^i \in Y^i\}.$$

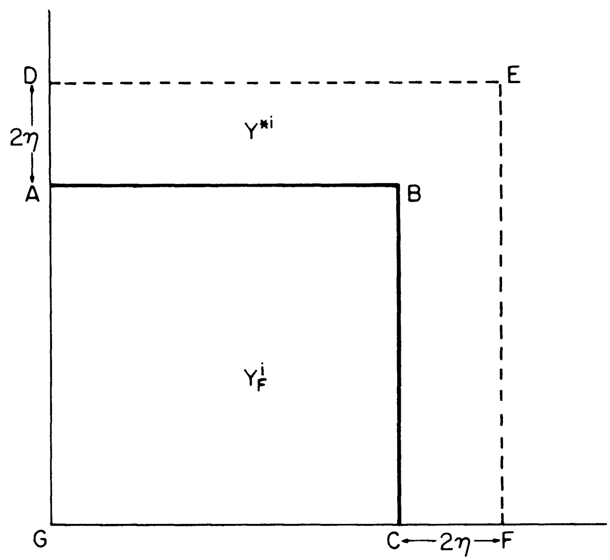


FIGURE 5.6

$M = 2$, $Y^i_F = GABC$, $Y^{*i} = GDEF$, $H = R^2$ (the whole plane).

REMARK 4: Even when $H = R^M$, $Y^{*i} \neq Y^i_F$: see Figure 5.6. This somewhat undesirable feature of the proof below can be avoided by using, instead of Y^{*i}, the set $Y^{**i} = (Y^i_F + H^{\perp}_{2\eta}) \cap Y^i$, in which case $Y^{**i} = Y^i_F$ when $H = R^M$ (so that $H^{\perp}_{2\eta} = 0_x$). (Here $H^{\perp}_{2\eta}$ denotes that subset of the complementary manifold $H^{\perp}$ of H whose points are within distance $\leqslant 2\eta$ of H.)

REMARK 5: We note that Y^{*i} is compact and is contained in Y^i: also, it contains Y^i_F.

It will be shown below that

$$(5.19^*) \quad T^{*i} \subseteq G^{+i}(y^i) \cap Y^{*i} \quad \text{for all} \quad i \in \mathcal{N}.$$

From this one can derive[50]

$$(5.27^*) \quad \text{prob}\{z_t \in T^* | y_t = y\} \geqslant \lambda \int_{T^*}^{N} dz^1 \dots dz^N = \lambda^N \tilde{\mu}^N \equiv \gamma_1^* > 0$$

by following the proof of (5.27) of Subsection 5.2, with T and Y_F^i, respectively, replaced by T^* and Y_F^{*i}. The remainder of the proof is unchanged except for replacing γ_1 by $\gamma_1^* \equiv \lambda^N \tilde{\mu}^N$ and T by T^*, and noting that $d \equiv \dim L_F \geqslant 1$ (since the case $d = 0$ was disposed of at the beginning of the proof).

PROOF OF (5.19*): For a fixed but arbitrary $i \in \mathcal{N}$, let $z \in T^{*i}$.

We use the R^M-vector $x^i(z^i)$ guaranteed by Assumption ED.4*‴ to construct, for $p \in \mathcal{N} \setminus \{i\}$,

$$x^p(z^i) = c^p - \frac{1}{N-1}[x^i(z^i) - c^i], \quad p \neq i,$$

and define

$$x(z^i) \equiv \langle x^1(z^i), \dots, x^i(z^i), \dots, x^N(z^i) \rangle.$$

We now show that $x(z^i) \in S$. Because of (5.20*) above, this can be done by proving that (i) $x(z^i) \in Y_C$, (ii) $x(z^i) \in M^*$, and (iii) $x(z^i) \in T$.

(i) By construction,

$$\sum_{p \neq i} x^p(z^i) = \sum_{p \neq i} c^p - [x^i(z^i) - c^i].$$

But $\Sigma_{q \in \mathcal{N}} c^q = 0_x$ since $c \in Y_C$. Hence $\Sigma_{q \in \mathcal{N}} x^q(z^i) = 0_x$, i.e., $x(z^i) \in Y_C$.

(ii) $x^i(z^i)$ is in H by Assumption ED.4*‴. For $p \neq i$, $x^p(z^i)$ are linear combinations of elements of H, hence also in H; hence $x(z^i) \in H^*$.

(iii) $x^i(z^i) \in T^{*i} \subseteq T^i$; for $p \neq i$, by construction,

$$\|x^p - c^p\| = \| -\frac{1}{N-1}[x^i(z^i) - c^i]\|$$

$$= \frac{1}{N-1}\|x^i(z^i) - c^i\| < \frac{1}{N-1}\eta \leqslant \eta,$$

so that $x(z^i) \in T$.

It follows that $x(z^i)$ is in S, and, therefore, by (5.18), that $x^i(z^i) \in G^{+i}(y^i) \cap Y_F^i$. By virtue of Assumption ED.4*‴, $z^i \succsim_i x^i(z^i)$: hence $z^i \in G^{+i}(y^i)$.

It remains to be established that $z^i \in Y^{*i}$.

We have just established that $x^i(z^i) \in Y_F^i \cap T^{*i}$. Since both z^i and $x^i(z^i)$ are in T^{*i}, $\|z^i - x^i(z^i)\| < 2\eta$. Now $x^i(z^i) \in Y_F^i$ together with the preceding inequality yield $z^i \in Y^{*i}$. Thus (5.19*) is established.

[50] Bearing in mind that $M^+ = \dim H \geqslant 1$ and that $N \geqslant 2$, and using Assumption ED.4*″ to guarantee the boundedness from below of probabilities.

5.5.3. *Completion of the Proof of Improvement*

In this section we show that Assumption ED.4*′ implies (5.18*) of Subsection 5.5.2.

Since $Y_F \subset H^*$ and $Y_F \subset Y_C$, it follows that Y_F is a subset of the linear variety $Y_C \cap H^*$, so that $L_F \subseteq Y_C \cap H^*$. It remains to be shown that $Y_C \cap H^* \subseteq L_F$. For this purpose, we define the Cartesian product $V = \bigtimes_{i=1}^{N} V^i$, where V^i is the R^M-open set (whose existence is guaranteed by Assumption ED.4*′) such that $y^{i0} \in V^i \cap H \subseteq Y^i$.

Since $V^i \cap H \subseteq Y^i$ and $\bigtimes_i (V^i \cap H) = V \cap H^* \subseteq \bigtimes_i Y^i$, we have $V \cap H^* \cap Y_C \subseteq (\bigtimes_i Y^i) \cap Y_C = Y_F$, so that $\mathscr{L}(V \cap H^* \cap Y_C) \subseteq L_F$.

The proof will be completed by showing that $Y_C \cap H^* \subseteq \mathscr{L}(V \cap H^* \cap Y_C)$. To do this, we take some $\tilde{y} \in Y_C \cap H^*$, and show that $\tilde{y} \in \mathscr{L}(V \cap H^* \cap Y_C)$.

Let $\tilde{y} \in H^* \cap Y_C$. Since $y^0 \in V \cap H^* \cap Y_C$, $\tilde{\tilde{y}} \equiv \alpha\tilde{y} + (1-\alpha)y^0 \in V \cap H^* \cap Y_C$ for sufficiently small $\alpha > 0$. Hence $\tilde{y} = (1/\alpha)\tilde{\tilde{y}} - ((1-\alpha)/\alpha)y^0 \in \mathscr{L}(V \cap H^* \cap Y_C)$.

Q.E.D.

5.5.4. *Generalizations of the Two Principal Theorems*

We have thus established the following generalizations of Theorems 5.1 and 5.2:

THEOREM 5.1*: *Under Assumptions PD.*1, *PD.*2, *ED.*1, *ED.*2, *ED.*5, *and ED.*6 *together with Assumption ED.*4* *of Subsection* 5.5.1, *if the bidding distribution is an equilibrium, then it is optimal.*

THEOREM 5.2*: *Under the assumptions of Theorem* 5.1*, *with probability one,* $\lim_{t\to\infty} u_t$ *is optimal.*

5.5.5. *The Case of Incomplete Initial Endowments Under Pure Exchange*

In this section we show how the preceding generalization can be applied in the case of pure exchange when some commodities are missing from the initial endowment. Specifically, we shall show that conditions (E) and (D) introduced below imply that Assumption ED.4* is satisfied: hence when the other assumptions of Theorem 5.1* hold, the conclusions of Theorems 5.1* and 5.2* follow.

To simplify matters we shall assume

$$\left.\begin{aligned} C^i &= \Omega \\ \omega^i &\in \Omega \end{aligned}\right\} \text{for all } i = 1, 2, \ldots, N,$$

so that (since we are only considering the pure exchange case)

$$\text{(E)} \qquad \left.\begin{aligned} Y^i &= \Omega - \{\omega^i\} \\ \omega^i &\in \Omega \end{aligned}\right\} \text{for all } i \text{ in } \mathscr{N}.$$

Hence Assumption ED.4*″ holds because $\omega^i \in \Omega = C^i$ (see Remark 2 in 5.5.1). Furthermore, as shown at the end of this section,

(I) Condition (E) implies Assumption ED.4*′.

In the proof of (I), and also in the formulation of condition (D) below, we shall find it convenient to introduce the following notation.

We write $\mathscr{M} \equiv \{1, \ldots, M\}$, for the set of all M commodities and partition $\mathscr{M}$ into two disjoint subsets $\mathscr{M}^+$, $\mathscr{M}^0$ in such a way that all commodities in $\mathscr{M}^+$ have positive initial *aggregate* endowments, i.e., $\omega_j \equiv \Sigma_{i=1}^N \omega_j^i > 0$, for $j \in \mathscr{M}^+$, while all commodities in $\mathscr{M}^0$ have zero initial endowments for *each* consumer, i.e., $\omega_j^i \equiv 0$ for $j \in \mathscr{M}^0$ and all $i \in \mathscr{N}$.

Now as was seen at the beginning of the proof of Lemma 5.6*, $\mathscr{M}^+$ may be taken to be non-empty; hence we may without loss of generality so number the commodities that $\mathscr{M}^+ \equiv \{1, \ldots, M^+\}$ for some M^+, $1 \leqslant M^+ \leqslant M$, and, for $M^+ < M$, $\mathscr{M}^0 \equiv \{M^+ + 1, \ldots, M\}$. Now in order to satisfy (5.2.b‴*) we make the further assumption:

(D) $a \equiv (a_1, \ldots, a_{s-1}, a_s, a_{s+1}, \ldots, a_M) \in \Omega - \{\omega^i\}$, for every integer $s \in \mathscr{M}^0$,

implies[51]

$$U^i(a) \geqslant U^i[(a_1, \ldots, a_{s-1}, 0, a_{s+1}, \ldots, a_M)],$$

which means that, other things being equal, a consumer likes (small) positive amounts of certain goods (viz., those absent from the initial endowment) at least as well as zero amounts.

Given (D), it can be seen that Assumption ED.4*‴ is satisfied for $x^i(z^i)$ defined by

$$x_r^i(z^i) \equiv \begin{cases} z_r^i & \text{for} \quad r \in \mathscr{M}^+, \\ 0 & \text{for} \quad r \in \mathscr{M}^0; \end{cases}$$

for, clearly, when $z^i \in T^{*i}$, $x^i(z^i)$ is in $T^{*i} \cap H$ (where $H = \{x \in R^M : x_j = 0$ for $j \in \mathscr{M}^0\}$) and, because of (D), $z^i \succsim_i x^i(z^i)$.

Thus conditions (E) and (D) imply ED.4*. Since they also imply Assumptions ED.1 and ED.3, the conclusions of Theorems 5.1* and 5.2* follow for a B process satisfying Assumptions PD.1 and PD.2, provided consumer preferences, in addition to (D), are also continuous (Assumption ED.5) and imply openness (Assumption ED.6). In particular, the conclusions of Theorems 5.1* and 5.2* hold under pure exchange, given (E), if preferences are continuous and monotone (see Lemma 5.9 and its corollary in Subsection 5.7 below).[52]

[51] In fact, it is sufficient that this implication hold for small values of a_s.

[52] It may be helpful to write out explicitly the formula for Y^{*i}. We have in this case

$$Y^{*i} = \{(a_1, \ldots, a_M) : -\omega_k^i \leqslant a_k \leqslant \omega_k^i \text{ for } k \in \mathscr{M}^+, 0 \leqslant a_j \leqslant 2\eta \text{ for } j \in \mathscr{M}^0\}$$

where $\omega_k \equiv \Sigma_i \omega_k^i$. It may be noted that, since $\omega_j^i = 0$ for $j \in \mathscr{M}^0$,

$$Y_F^i \equiv \{(a_1, \ldots, a_M) : -\omega_r^i \leqslant a_r \leqslant \omega_r - \omega_r^i \text{ for all } r \in \mathscr{M}\},$$

$\mathscr{M} \equiv \{1, \ldots, M\}$, is a *proper* subset of Y^{*i}.

It remains to provide a proof of proposition (I) stated at the beginning of 5.5.5. Defining

$$\bar{\omega} \equiv \frac{1}{N}\sum_{i=1}^{N} \omega^i \equiv (\bar{\omega}_1, \ldots, \bar{\omega}_M),$$

so that $\bar{\omega} \in H \cap \Omega$, we construct y^0 by $y^{0i} \equiv \bar{\omega} - \omega^i$ for each $i \in \mathcal{N}$. Clearly, $y^0 \in Y_C$ since

$$\sum_i y^{0i} = \sum_i (\bar{\omega} - \omega^i) = 0_x.$$

On the other hand, in view of (E),

$$\text{Int}_H\, Y^i = \{(x_1, \ldots, x_M) : x_j = 0 \text{ for } j \in \mathcal{M}^0, x_k > -\omega_k^i \text{ for } k \in \mathcal{M}^+\}.$$

It will follow that $y^{0i} \in \text{Int}_H\, Y^i$, if we can show that $\bar{\omega}_j - \omega_j^i = 0$, for $j \in \mathcal{M}^0$, and $\bar{\omega}_k - \omega_k^i > -\omega_k^i$, for $k \in \mathcal{M}^+$. But the preceding equalities hold because $\omega_j = 0$ and $\omega_j^i = 0$ for $j \in \mathcal{M}^0$, while the inequalities follow from $\omega_k > 0$ for $k \in \mathcal{M}^+$.

5.5.6. *A Counterexample*

In this section we give an example where the B process fails to converge to an optimum even though Assumptions ED.1–ED.3, ED.5, and ED.6 are satisfied as well as Assumptions ED.4*′ and ED.4*‴; thus Assumption ED.4*″ and also ED.4 are violated.

This shows that Assumption ED.4 cannot be dispensed with in Theorems 5.1 and 5.2 and in Lemma 5.6, and that Assumption ED.4*″ cannot be dispensed with in Theorems 5.1* and 5.2* and in Lemma 5.6*. It remains to be seen whether an alternative specification of the process would avoid such difficulties.

The example involves two traders, two commodities, and pure exchange. The initial endowments are respectively $\omega^1 = (1, 0)$ and $\omega^2 = (0, 0)$ so that the second commodity is absent. The admissible trading set for the second individual is the usual $Y^2 = \Omega$, while that for the first individual is $Y^1 = A \cup \Omega$, where

$$A = \{y_1^1, y_2^1) : -1 \leqslant y_1^1 \leqslant 0, y_2^1 = 0\},$$

as shown in Figure 5.6a. Y^1 and Y^2 are both closed; hence Assumption ED.1 is satisfied.

The feasible set Y_F constructed in accordance with Assumption ED.2, is given by

$$\begin{aligned} Y_F &= \{\langle (y_1^1, y_2^1), (y_1^2, y_2^2)\rangle : -1 \leqslant y_1^1 \leqslant 0, y_2^1 = 0, y_1^2 = -y_1^1, y_2^2 = 0\} \\ &= \{\langle y^1, y^2 \rangle : y^1 \in A, y^2 = -y^1\}, \end{aligned}$$

which corresponds to a closed segment on the boundary of the Edgeworth Box. Clearly, Assumption ED.3 is satisfied. Now

$$L_F = \{\langle (y_1^1, y_2^1), (y_1^2, y_2^2)\rangle : y_1^1 \text{ real}, y_1^2 = -y_1^1, y_2^1 = y_2^2 = 0\}$$

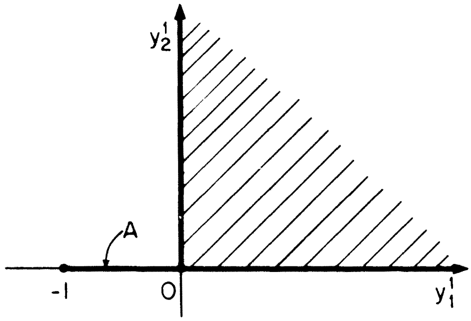


FIGURE 5.6a
Y^1 consists of shaded and heavy black areas.

is a line in 4-space, while

$$Y_C = \{\langle (y_1^1, y_2^1), (y_1^2, y_2^2)\rangle : y_1^1 \text{ real}, y_2^1 \text{ real}, y_1^2 = -y_1^1, y_2^2 = -y_2^1\}$$

is a two-dimensional set in 4-space. Thus $L_F \neq Y_C$ which contradicts (5.8); it is seen that Assumption ED.4 is violated.

The utility functions are given as follows:

$$U^1(y^1) \equiv \begin{cases} -|y_1^1 + \frac{1}{2}| + \frac{1}{2} & \text{for } y^1 \in A, \\ y_1^1 & \text{for } y^1 \in \Omega, \end{cases}$$

$$U^2(y^2) \equiv \begin{cases} -|y_1^2 - \frac{1}{2}| + \frac{1}{2} & \text{when} \quad 0 \leqslant y_1^2 \leqslant 1, \\ y_1^2 - 1 & \text{when} \quad y_1^2 \geqslant 1. \end{cases}$$

U^1 and U^2 are continuous; hence Assumption ED.5 is satisfied.

There is only one Pareto optimal distribution $\hat{y} \equiv \langle(-1/2, 0), (+1/2, 0)\rangle$, and we may write $\hat{Y} = \{\hat{y}\}$.

Therefore, for $0 < \varepsilon < 1/2$,

$$\hat{Y}_\varepsilon = \{\langle y^1, y^2\rangle : -\tfrac{1}{2} - \varepsilon < y_1^1 < -\tfrac{1}{2} + \varepsilon, y_1^2 - y_1^1, y_2^1 = y_2^2 = 0\},$$

and, as always, $\overline{Y}_\varepsilon = Y_F \setminus \hat{Y}_\varepsilon$.

We now verify Assumption ED.6 by constructing an L_F-open set S required by Assumption ED.6 as

$$S = \{\langle (y_1^1, y_2^1), (y_1^2, y_2^2)\rangle : c_1^1 - \eta \langle y_1^1 < c_1^1 + \eta\} \cap Y_F,$$

where $\eta > 0$, $c_1^1 - \eta > -1/2 - \varepsilon$ and $c_1^1 + \eta < -1/2 + \varepsilon$.

On the other hand, Assumption ED.4*″ is violated because the set T^{*1} appearing in Assumption ED.4*″ is of (two-dimensional) measure zero. Here $T^{*1} = T^1 \cap Y^1$, where

$$T^1 = \{(y_1^1, y_2^1) : c_1^1 - \eta < y_1^1 < c_1^1 + \eta, -\eta < y_2^1 < \eta\};$$

hence

$$T^{*1} = \{(y_1^1, y_2^1): c_1^1 - \eta < y_1^1 < c_1^1 + \eta, y_2^1 = 0\},$$

which is a segment in 2-space.

To show that there is no improvement from a non-optimal point $\bar{y} \in \bar{Y}_\varepsilon$, we note that (contrary to (5.15)) prob $\{y_{t+1} \in \hat{Y}_\varepsilon | y_t = \bar{y},\ \bar{y} \in \bar{Y}_\varepsilon\} = 0$, and $y_{t+1} \in \hat{Y}_\varepsilon$ can only occur when the center of the first individual's bidding cube is on the segment A.

But by Assumption PD.1, the probability distribution P^1 is absolutely contínous with regard to the two-dimensional Lebesgue measure and hence A, being one-dimensional, has probability zero.

Thus the conclusions of Theorems 5.1 and 5.2, 5.1* and 5.2*, and Lemmas 5.6 and 5.6* do not hold.

5.6. *Properties Related to the Assumption of Openness* (*Assumption ED.6*)

5.6.0. *Introduction*

In this section we state a property (Assumption ED.6a in Subsection 5.6.5 below) which, under certain conditions, is equivalent to the assumption of openness (Assumption ED.6 above). Roughly speaking, this property has the following meaning: given two allocations, one Pareto optimal and another Pareto inferior to it, there exists a third feasible allocation close to the given optimal one[53] and preferred to the given non-optimal one by *all* consumers.[54] Assumption ED.6a is easier to verify in terms of preference structures and production possibilities than is the assumption of openness. For instance, as shown in Subsection 5.7, Assumption ED.6a is satisfied for continuous utility functions that are strictly increasing in the interior of the consumption set, when the latter is the nonnegative orthant Ω of the commodity space.

The result concerning the equivalence of Assumptions ED.6 and ED.6a is Theorem 5.3 below. The equivalence holds if, in addition to Assumptions ED.1, ED.2, ED.3, and ED.5, it is assumed (Assumption ED.7 below) that the jointly feasible set is the closure of its interior (in the relative topology).

According to Lemma 5.8 below, Assumption ED.7 holds (when $\omega^i \in \Omega = C^i$ for all consumers) if near every jointly feasible point there are feasible points with positive components in all potentially available commodities for all consumers (Assumption ED.7a). Thus Theorems 5.1 and 5.2 hold with Assumptions ED.6a and ED.7 together replacing Assumption ED.6. In Subsection 5.6.3 an example is given to show that Assumption ED.7 cannot be completely dispensed with in Theorem 5.3, although it is conceivable that it might be weakened.

[53] Or, more precisely, close to some optimal allocation that is Pareto equivalent to the given Pareto optimal allocation.

[54] The given Pareto optimal allocation is, of course, preferred to the given non-optimal one by some consumers, while others may be indifferent between the two.

5.6.1. *Notation*

As in Subsection 2.7 above, it will be convenient to distinguish consumers from producers (if any). In Subsection 5.4 the two sets of agents were respectively labeled J and K. Here the set J of *consumers* will be assumed to consist of agents $1, 2, \ldots, H$, with $H \leqslant N$, so that the *producers* are numbered $H + 1, \ldots, N$. The case of $H = N$ is, of course, that of pure exchange (here the set K is empty).

It is also helpful to introduce the set of all commodity vectors that can be supplied by the production sector to the consumer sector. This *aggregate producers' supply set* X_S is defined by

$$X_S \equiv \begin{cases} \sum_{k=H+1}^{N} [X_P^k + \{\omega^k\}] = \sum_{k=H+1}^{N} (-Y^k), & \text{if } H < N, \\ \{0_x\}, & \text{if } H = N, \end{cases}$$

where X_P^k, ω^k, and Y^k are defined as in Subsections 2.7 and 5.4 above.

In subsequent formulations we shall use the notion of the *projection* of a set defined in R^{MN} into the HM-dimensional Euclidean space R^{HM} in which *consumers' allocations* are represented. Thus, let A be a set in R^{NM}, with a typical element $\langle a^1, \ldots, a^N \rangle$, the a^i being elements of R^M (the commodity space). Then the projection of A is denoted by A_H and is the set of $\langle b^1, \ldots, b^H \rangle$ such that $b^h \in R^M$, $h = 1, \ldots, H$, and $\langle b^1, \ldots, b^H, a^{H+1}, \ldots, a^N \rangle \in A$ for some $a^k \in R^M$, $k = H + 1, \ldots, N$.

It can be shown that:

(i) $$Y_H = \mathop{\times}_{i=1}^{H} Y^i \quad \text{for} \quad Y \equiv \mathop{\times}_{i=1}^{N} Y^i;$$

(ii) $$Y_{FH} = \left\{ y : y = \langle y^1, \ldots, y^H \rangle,\ y \in Y_H,\ \sum_{i=1}^{H} y^i \in X_S \right\}$$ [55]

(for the case of pure exchange, where $X_S = \{0_x\}$, we have

$$Y_{FH} = \left\{ y : y = \langle y^1, \ldots, y^H \rangle,\ y \in Y_H,\ \sum_{i=1}^{H} y^i = 0_x \right\} = Y_F);$$

(iii) $$L_{FH} = \mathscr{L}(Y_{FH});$$

(iv) $$G_H^+(y) = Y_{FH} \cap \mathop{\times}_{i=1}^{H} G^{+i}(y^i) \quad \text{for} \quad y = \langle y^1, \ldots, y^H \rangle, \quad y \in Y_{FH}.$$

5.6.2. *Assumption of Openness in Projection Form (Assumption ED.6H)*

It will now be shown that Assumption ED.6 of openness is equivalent to the following assumption:

[55] In this section symbols such as y, $\hat{y}$, and v may represent H-tuples of M-dimensional vectors rather than N-tuples of such vectors.

ASSUMPTION ED.4H: *For every $\varepsilon > 0$ and every $y \in \overline{Y}_{\varepsilon H}$, there exists an L_{FH}-open cube S_1 such that (i) $S_1 \subseteq G_H^+(y)$, and (ii) $S_1 \cap \hat{Y}_{\varepsilon H} \neq \varnothing$.*

LEMMA 5.7: *Assumptions ED.6 and ED.6H are equivalent.*

PROOF: (i) Assumption ED.6 implies Assumption ED.6H. For given $\varepsilon > 0$ and $\bar{y} = \langle \bar{y}^1, \ldots, \bar{y}^H, \bar{y}^{H+1}, \ldots, \bar{y}^N \rangle \in \overline{Y}_\varepsilon$, let S be the L_F-open cube whose existence was asserted in (5.12). Then there exists a set $\mathcal{O}$, open in R^{MN}, such that $S_H = L_{FH} \cap \mathcal{O}_H$, so that the projection S_H is an L_{FH}-open set.

Further, it is immediate that $S_H \subseteq G_H^+(\bar{y}_H), \bar{y} \equiv \langle \bar{y}^1, \ldots, \bar{y}^H \rangle$, and $S_H \cap \hat{Y}_{\varepsilon H} \neq \varnothing$. Take some $y \in S_H \cap \hat{Y}_{\varepsilon H}$. Then, since S_H is Y_{FH}-open, there is an L_{FH}-open cube S_1 with center y, such that $S_1 \subseteq S_H$.

(ii) *Assumption ED.6H implies Assumption ED.6.* Given $\varepsilon > 0$ and $y' = \langle y^1, \ldots, y^H \rangle$, $y' \in \overline{Y}_\varepsilon$, let S_1 be the L_{FH}-open cube whose existence is guaranteed by Assumption ED.6H. Let c_1 be the center of S_1. Then $c_1 \in L_{FH}$. Hence there exists $c = \langle c_1, c_2 \rangle \in L_F$. Let S be the L_F-open cube with center c and radius as that of S_1. Then, we have $S_1 = S_H$. It is immediate from the definitions of $G_H^+(y')$ and $G^+(\langle y', y'' \rangle)$, with $y'' = \langle y^{H+1}, \ldots, y^N \rangle$, that $S_H \subseteq G_H^+(y') \Rightarrow S \subseteq G^+(\langle y', y'' \rangle)$, and that $y' \in S_H \cap \hat{Y}_{\varepsilon H} \Rightarrow \langle y', y'' \rangle \in S \cap \hat{Y}_\varepsilon$, for $y'' \in R^{(N-H)M}$ such that $\langle y', y'' \rangle \in Y_F$.

5.6.3. *Example*

REMARK: If Y_{FH} is not the closure of its interior in L_{FH}, the openness property need not hold, as shown by the following example.

Let there be two consumers, $H = 2$, one commodity, $M = 1$, $\omega^i = 0$, $C^i = Y^i = \Omega$, $i = 1, 2$; $X_S = \{0, 1\}$, i.e., either none, or one unit of the commodity is suppliable by the production sector. Then Y_{FH} consists of the point (0, 0) together with the line segment

$$\{\langle y^1, y^2 \rangle : y^1 + y^2 = 1, y^1 \geqslant 0, y^2 \geqslant 0\},$$

as shown in Figure 5.7.

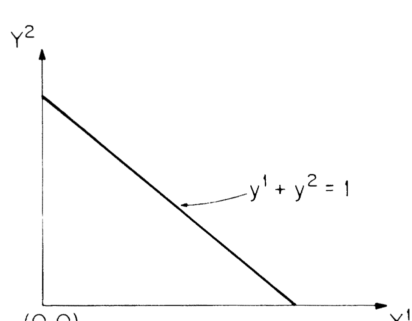


FIGURE 5.7

It follows that L_{FH} is the plane, and hence that L_{FH}-open cubes are squares.

Clearly no square can be a subset of Y_{FH} and so the openness assumption cannot be satisfied, regardless of the structure of the preferences. (Here the L_{FH} interior of Y_{FH} is empty, hence so is its closure; the latter is therefore unequal to the nonempty set Y_{FH}.)

5.6.4. *Definitions*

In order to state the main proposition of this section we need additional definitions.

First, for $y \in Y_{FH}$, write $\hat{Y}_H(y) \equiv \{\hat{y} \in \hat{Y}_H : \hat{y} \in G_H^+(y)\}$. That is $\hat{Y}_H(y)$ is the set of Pareto optimal points (allocations) that are at least as good as y in the Pareto ordering.

Given $\varepsilon > 0$, $\delta > 0$, $\bar{y} \in \bar{Y}_{\varepsilon H}$, $\bar{y} \equiv \langle \bar{y}^1, \ldots, \bar{y}^H \rangle$, and $\hat{y} \in \hat{Y}_H(\bar{y})$, $\hat{y} \equiv \langle \hat{y}^1, \ldots, \hat{y}^H \rangle$, define, for $i = 1, 2, \ldots, H$,

$$V_\delta^i(\bar{y}, \hat{y}) \equiv \{v^i \in R^M : \hat{y}^i + v^i \in G^{+i}(\bar{y}^i)\} \cap \mathcal{N}_\delta(0),$$

where $\mathcal{N}_\delta(0)$ is a neighborhood of the origin in R^M with radius δ.

Further, define

$$V_\delta(\bar{y}, y) \equiv \sum_{i=1}^{H} V_\delta^i(\bar{y}, \hat{y}).$$

Roughly speaking, the set $V_\delta^i(\bar{y}, \hat{y})$ consists of those small displacements of the component $\hat{y}^i$ of the distribution $\hat{y}$ (which is Pareto optimal and Pareto superior to $\bar{y}$) resulting in a point strictly preferred by consumer i to $\bar{y}^i$, while the set $V_\delta(\bar{y}, \hat{y})$ consists of the net (aggregate) resource requirements defined by such displacements of all consumers.

5.6.5. *The "Strict Dominance" Assumption* (*Assumption ED.6a*)

We now consider an alternative to the openness assumption (Assumption ED.6), which we shall call the strict dominance assumption (Assumption ED.6a). This assumption can be interpreted as follows.

Given a point (allocation) $\bar{y}$ which falls short in utility by at least ε of being Pareto optimal, and a point $\hat{y}$ which is Pareto optimal and superior to $\bar{y}$ in the Pareto ordering, there must exist an allocation $\hat{\hat{y}}$ Pareto equivalent to $\hat{y}$ and a small displacement $v \equiv \langle v^1, \ldots, v^H \rangle$ of $\hat{\hat{y}}$ resulting in a feasible point $\tilde{y} \equiv \hat{\hat{y}} + v$ which is preferred to $\bar{y}$ by *all* consumers.[56] Thus the allocation $\tilde{y}$ is near $\hat{\hat{y}}$; consumers indifferent between $\hat{y}$ and $\bar{y}$ strictly prefer $\tilde{y}$ to $\bar{y}$, while those who prefer $\hat{y}$ to $\bar{y}$ also prefer $\tilde{y}$ to $\bar{y}$. In the case of pure exchange, the displacement v is simply a redistribution of holdings among consumers.

We now present a formal statement of the strict dominance assumption.

ASSUMPTION ED.6a (Strict Dominance): *For every $\varepsilon > 0$, $\delta > 0$, $\bar{y} \in \bar{Y}_{\varepsilon H}$, $\bar{y} \equiv \langle \bar{y}^1, \ldots, \bar{y}^H \rangle$, and $\hat{y} \in \bar{Y}_H(\bar{y})$, $\hat{y} \equiv \langle \hat{y}^1, \ldots, \hat{y}^H \rangle$, there exists $\hat{\hat{y}} \sim \hat{y}$, ($\hat{\hat{y}} \equiv \langle \hat{\hat{y}}^1, \ldots, \hat{\hat{y}}^H \rangle$), with $\hat{\hat{x}} = \Sigma_{i=1}^H \hat{\hat{y}}^i$, such that $[X_S - \{\hat{\hat{x}}\}] \cap V_\delta(\bar{y}, \hat{\hat{y}}) \neq \varnothing$; i.e., there exists an alloca-*

[56] It may well be the case that $\hat{\hat{y}} = \hat{y}$.

tion $\hat{\hat{y}}$, H commodity space "displacement" vectors $v^1, v^2, \ldots, v^H$, and, if $H < N$, also $N - H$ commodity space vectors $x^{H+1}, \ldots, x^N$, with $x^k \in X_P^k$, $k = H + 1, \ldots, N$, such that

(i) $\quad \hat{y}^j \sim_j \hat{y}^j \qquad (j = 1, \ldots, H)$,

(ii) $\quad \hat{\hat{y}}^j + v^j \succ_i \bar{y}^j \qquad (j = 1, \ldots, H)$,

(iii) $\quad |v^j| < \delta \qquad (j = 1, \ldots, H)$,

and

(iv-a) $$\sum_{j=1}^{H} v^j = \sum_{k=H+1}^{N} (x^k + \omega^k) - \hat{\hat{x}} \quad \text{if} \quad H < N, \quad \text{or}$$

(iv-b) $$\sum_{j=1}^{H} v^j = 0_x - \hat{\hat{x}} = 0_x \quad \text{if} \quad H = N,$$

where the second equality in (iv-b) *follows from the fact that, for $H = N$, $\hat{\hat{x}} = 0_x$.*

5.6.6. *A Sufficient Condition for the Equivalence of Openness and Strict Dominance*

The set Y_{FH} will be called regularly closed if it is the closure of its interior in the L_{FH}-topology. The next theorem shows that, in the presence of Assumptions ED.1–ED.3 and ED.5, if Y_{FH} is regularly closed then openness is equivalent to strict dominance.

THEOREM 5.3: *If* (1*a*) *Y_F is compact* (*this follows from Assumptions ED.*1–*ED.*3), *and* (1*b*) *Y_{FH} is the closure of its interior in the L_{FH} topology* (*i.e., is regularly closed*), *and if* (2) *for each consumer $i = 1, 2, \ldots, H$,* (2*a*) *Y^i is closed* (*Assumption ED.*1), *and* (2*b*) *the preference ordering $\lesssim_i$ is representable by a continuous real-valued function U^i on Y^i* (*Assumption ED.*5) *then Assumption ED.*6*a* (*strict dominance*) *is equivalent to Assumption ED.*6 (*openness*).

REMARK: Y_{FH} is regularly closed under pure exchange with $C^i = \Omega$, since then it is the "Edgeworth Box" (see (ii) in 5.6.1 above).

PROOF OF THEOREM 5.3: 1. We first show that Assumption ED.6a implies Assumption ED.6H, which, by Lemma 5.7, implies Assumption ED.6. Given $\varepsilon > 0$ and $\bar{y} \in \bar{Y}_{\varepsilon H}$, let $\phi(y) \equiv \Sigma_{i=1}^{H} U^i(y^i)$. The set $G_H^+(\bar{y})$ is not empty, since $\bar{y} \in G_H^+(y)$, and, by (1a), it is compact. Since ϕ is continuous in y, it has a maximum on $G_H^+(\bar{y})$; let $\hat{y} \equiv \langle \hat{y}^1, \ldots, \hat{y}^H \rangle$ denote the maximizer of ϕ in $G_H^+(\bar{y})$.

Clearly, $\hat{y}^i \gtrsim_i \bar{y}^i$, $i = 1, \ldots, H$. There is no $y^* \in G_H^+(\bar{y})$ such that for some $i \in \{1, \ldots, H\}$, $y^{*i} \gtrsim_i \hat{y}^i$, for all $i = 1, \ldots, H$, with strict preference for some i; otherwise $\hat{y}$ would not be a maximizer of ϕ. Suppose there exists $y \notin G_H^+(\bar{y})$ such that $y^i \gtrsim_i \hat{y}^i$, for all $i = 1, \ldots, H$, with strict preference for some i; it would follow by transitivity of preference that $y \in G_H^+(\bar{y})$, which is a contradiction.

Therefore, there is no $y \in Y_{FH}$ which is Pareto superior to $\hat{y}$, and so $\hat{y}$ is Pareto optimal. Further, it cannot be the case that $\bar{y} \in G_H^+(\hat{y})$, since $\hat{y} \in \hat{Y}_H$ while $\bar{y} \in \bar{Y}_{\varepsilon H}$.

Since $\hat{y} \in G_H^+(\bar{y})$, it follows that $\hat{y}^i \succsim_i \bar{y}^i$ for some i as well as $\hat{y}^j \succsim_i \bar{y}^j$ for all $j = 1, \ldots, H$.

Thus $\hat{y} \in \hat{Y}_H(\bar{y})$, and by Assumption ED.6a for every $\varepsilon' > 0$, there exist $\hat{\hat{y}} \sim \hat{y}$ and $v \equiv \langle v^1, \ldots, v^H \rangle$ such that

$$\sum_{i=1}^{H} v^i \in [X_S - \{\hat{\hat{x}}\}] \cap V_{\varepsilon'}(\bar{y}, \hat{\hat{y}}).$$

Therefore, for ε' sufficiently small, continuity of utility functions implies that $\tilde{y} = \hat{\hat{y}} + v \in \hat{Y}_{\varepsilon/2,H}$, and $\tilde{y} \in G_H^+(\bar{y})$.

By hypothesis (1b) of the theorem, Y_{FH} is the closure of its interior in L_{FH}. Hence, there exists a sequence (z_n) of L_{FH}-interior points of Y_{FH} which converges to $\tilde{y}$. That is, for each $n = 1, 2, \ldots$ there exists an L_{FH}-open cube S_n with center z_n and positive radius such that $S_n \subset Y_{FH}$.

We shall show that, for sufficiently large n, S_n can serve as S_1 of Assumption ED.6H.

By continuity of utilities, $\tilde{y} \in G_H^+(\bar{y})$ implies the existence of $\delta > 0$ and $\mathcal{N}_\delta(\tilde{y}^i)$, such that $\mathcal{N}_\delta(\tilde{y}^i) \subseteq G^{+i}(\bar{y}^i)$, $i = 1, \ldots, H$.

Therefore, taking the radius of S_n not greater than δ, so that

$$S_n \subseteq \mathcal{N}_\delta(\tilde{y}^i) \times \ldots \times \mathcal{N}_\delta(\tilde{y}^H),$$

it follows, that for δ sufficiently small, we have $S_n \subset \bigtimes_{i=1}^{H} G^{+i}(\bar{y}^i)$; this, together with $S_n \subset Y_{FH}$, is equivalent to:

Assumption ED.6H(i): $S_n \subset G_H^+(\bar{y})$.

Also, again using the continuity of U^i, for n sufficiently large, $z_n \in \hat{Y}_{\varepsilon H}$, since $z_n \to \tilde{y}$ and $\tilde{y} \in \hat{Y}_{\varepsilon/2,H}$. But $z_n \in S_n$ and $z_n \in \hat{Y}_{\varepsilon H}$ imply:

Assumption ED.6H(ii): $S_n \cap \hat{Y}_{\varepsilon H} \neq \varnothing$.

Hence S_n can serve as S_1 in Assumption ED.6H.

By Lemma 5.7, Assumption ED.6H implies ED.6.

2. We now show that Assumption ED.6H (which, again by Lemma 5.7, is implied by Assumption ED.6) implies Assumption ED.6a.

Let there be given some $\varepsilon > 0$, $\delta > 0$, $\bar{y} \in \overline{Y}_{\varepsilon H}$, and $\hat{y} \in \hat{Y}_H(\bar{y})$.

Let Q denote the set of indices i for which $\hat{y}^i \succ_i \bar{y}^i$, and Q' the remaining indices, namely those for which $\hat{y}^i \sim_i \bar{y}^i$.

Since $\hat{y} \in \hat{Y}_H(\bar{y})$, when Q' is empty, the conclusions of Assumption ED.6a hold for $Q' = \varnothing$ if we set $\hat{\hat{y}} = \hat{y}$ and $v^j = 0_x$ for all $j = 1, \ldots, H$; for $H < N$, condition (iv-a) of ED.6a is satisfied in view of the feasibility of $\hat{y}$. So we need only consider the case in which $Q' \neq \varnothing$.

By Assumption ED.6H, there exists a cube S_H such that $S_H \subseteq G_H^+(\bar{y})$.

Taking $z \in S_H$, $z \equiv \langle z^1, \ldots, z^H \rangle$, for each $i = 1, \ldots, H$, define $\alpha_0^i \equiv U^i(z^i)$, and construct the sequence of real numbers $\alpha_0^i, \alpha_1^i, \ldots$ by

$$\alpha_k^i \equiv \alpha_{k-1}^i + \tfrac{1}{2}[U^i(\hat{y}^i) - \alpha_{k-1}^i].$$

Note that for $i \in Q'$, $\alpha_0^i = U^i(z^i) > U^i(\bar{y}^i) = U^i(\hat{y}^i)$, which, by construction of α_k^i, implies that $\alpha_k^i > U^i(\bar{y}^i)$ for all k. Further, for $i \in Q$, either $U^i(z^i) \leqslant \alpha_k^i \leqslant U^i(\hat{y}^i)$, or $U^i(\hat{y}^i) \leqslant \alpha_k^i \leqslant U^i(z^i)$; in view of the inequalities $U^i(z^i) > U^i(\bar{y}^i)$, and $U^i(\hat{y}^i) > U^i(\bar{y}^i)$, this implies $\alpha_k^i > U^i(\bar{y}^i)$, for all k. Hence, for all $i = 1, \ldots, H$, and all k,

$$(5.39) \qquad \alpha_k^i > U^i(\bar{y}^i).$$

We note that, for all i and k, α_k^i is in the interval of which the end points are $U^i(z^i)$ and $U^i(y^i)$. Hence, by continuity of $U = (U^1, \ldots, U^H)$ on $G_H^+(\bar{y})$ (which follows from continuity of U^i on Y^i for $i = 1, \ldots, H$), and using the intermediate value theorem of differential calculus, there exists an allocation y_k in $G_H^+(\bar{y})$ such that $U^i(y_k^i) = \alpha_k^i$.

By compactness of $G_H^+(\bar{y})$, there exists a subsequence of the y_k, say, $z_k \equiv (z_k^1, \ldots, z_k^H)$ which converges, say, to $z^* \in G_H^+(\bar{y})$.

We now show that $z^* \in I(\hat{y})$. This is so because, by construction, $U^i(z_k^i) = \alpha_k^i \to U^i(\hat{y}^i)$, for $i = 1, \ldots, H$, and hence, by continuity of U^i, $U^i(z^{*i}) = U^i(\hat{y}^i)$, $i = 1, \ldots, H$, which is equivalent to saying $z^* \in I(\hat{y})$. Taking $\mathcal{N}_\delta(z^*)$ to be an R^{HM}-neighborhood of z^* having the prescribed $\delta > 0$ as its radius, $z_k \in \mathcal{N}_\delta(z^*)$, for all k sufficiently large. In particular there is some value of k, say k', for which $z_{k'} \in \mathcal{N}_\delta(z^*)$. Taking $v \equiv z_{k'} - z^*$ and $\hat{\hat{y}} = z^*$, we will now show that there exist $\hat{\hat{x}}^{H+1}, \ldots, \hat{\hat{x}}^N$ in $X_P^{H+1}, \ldots, X_P^N$, respectively, which together with v and $\hat{\hat{y}}$ satisfy Assumption ED.6a.

We note first that $z_{k'} = v + z^* \in G_H^+(\bar{y})$ by (5.39) above, and secondly that $\|v\| = \|z_{k'} - z^*\| < \delta$ since $z_{k'} \in \mathcal{N}_\delta(z^*)$. Thus requirements (i), (ii), and (iii) of Assumption ED.6a are satisfied.

Finally, since $z^* \in G_H^+(\bar{y})$ and $z_{k'} \in G_H^+(\bar{y})$ and hence both are in Y_{FH}, in the case of production ($H < N$) there exist $\hat{\hat{x}}^{H+1}, \ldots, \hat{\hat{x}}^N$ and $x^{H+1}, \ldots, x^N$ in $X_P^{H+1}, \ldots, X_P^N$, respectively, such that

$$\sum_{i=1}^{H} z^{*i} = \sum_{i=H+1}^{H} (\hat{\hat{x}}^i + \omega^i) \equiv \hat{\hat{x}} \in X_S,$$

and

$$\sum_{i=1}^{H} z_{k'}^i = \sum_{i=H+1}^{N} (x^i + \omega^i) \equiv x \in X_S,$$

and, in the case of pure exchange,

$$\sum_{i=1}^{H} z_{k'}^i = \sum_{i=1}^{H} z^{*i} = 0_x \in X_S - \{0_x\} \equiv \{0_x\} - \{0_x\} = \{0_x\}.$$

Hence, in either case

$$\sum_{i=1}^{H} v^i = \sum_{i=1}^{H} (z_{k'}^i - z^{*i}) = x - \hat{\hat{x}} \in X_S - \{\hat{\hat{x}}\},$$

so that requirement (iv) of Assumption ED.6a is satisfied, and

$$\sum_{i=1}^{H} v^i \in [X_S - \{\hat{\hat{x}}\}] \cap V_\delta(\bar{y}, \hat{\hat{y}}).$$

So the theorem is proved.

5.6.7. *Circumstances under which Y_{FH} is Regularly Closed (Assumption ED.7)*

The condition that Y_{FH} be regularly closed is essential in Theorem 5.3, so it is natural to inquire as to the circumstances under which this condition would be satisfied. For ease of reference, we give this condition a label of its own.

ASSUMPTION ED.7: *Y_{FH} is regularly closed, i.e., it is the closure of its interior in the L_{FH}-topology.*

If Y_{FH} is closed and convex, then it is regularly closed, so that various "classical" cases are covered. (We omit the proof.) But we are particularly interested in non-convex cases. The following result (Lemma 5.8) defines one class of such situations. Its conditions are restrictive in that each C^i is assumed to be the nonnegative orthant Ω. On the other hand, the assumptions with regard to the supply set X_S do not imply its convexity. Generalizations which permit non-convex C^i's are also possible.

REMARK: To guarantee that Y_{FH} is the closure of its interior in the L_{FH}-topology it is not sufficient to postulate that X_S and Y^i are the respective closures of their interiors in appropriate relative topologies. This is shown by the example in Figure 5.8.

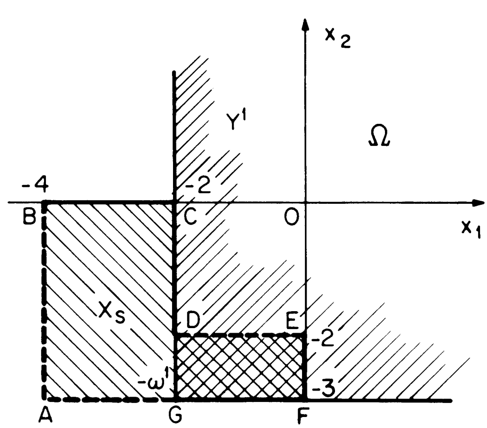


FIGURE 5.8

X_S = the set bounded by *ABCDEFGA* (shaded \\\\\\), $C^1 = \Omega$, $Y^1 = \Omega - \{\omega^1\}$ = all points east, north, and northeast of G (shaded ///), $Y_{FH} = DEFG$ and DC.

To state the result below, we first introduce some definitions. (These definitions can be specialized to the case of pure exchange by setting $X_S = \{0_x\}$.) From now on we shall be assuming that

$$\left.\begin{array}{l} C^i = \Omega \\ \omega^i \in \Omega \end{array}\right\} \text{ for all } i = 1, 2, \ldots, H.$$

Define first the ith feasible consumption set

$$C_F^i \equiv \{z^i \in R^M : \langle y^1, \ldots, y^i, \ldots, y^H \rangle \in Y_{FH}, \text{ for some } y^1, \ldots, y^{i-1}, y^i, \\ y^{i+1}, \ldots, y^H, z^i = \omega^i + y^i\}.$$

We may note that, with $\omega = \Sigma_{j=1}^{H} \omega^j$, $C_F^i = (\{\omega\} + X_S - \Omega) \cap \Omega$, for $H > 1$, and $c_F^1 = (\{\omega\} + X_S) \cap \Omega$, for $H = 1$.

Note also that C_F^i has the same value for all $i = 1, 2, \ldots, H$. We further define $E \equiv \mathscr{L}_0(C_F^i)$, the smallest linear *subspace*[57] containing C_F^i, and $K \equiv E \cap \Omega$.

Then E is a linear subspace of R^M, Ω is a closed convex cone, and hence so is K. We shall write, for $x \in R^M, x \geqslant_K 0_x$ to mean $x \in K$, and $x \gg_K 0_x$ to mean $x \in \text{Int}_E K$. Finally, define the feasible subset X_{SF} of X_S by

$$X_{SF} = \left\{x \in R^M : x \in X_S, x = \sum_{s=1}^{H} y^s, y \in \mathop{\times}_{i=1}^{H} Y^i\right\}$$
$$\equiv \left\{x \in R^M : x \in X_S, x = \sum_{s=1}^{H} y^s, y \in Y_{FH}\right\},$$

where $y \equiv \langle y^1, \ldots, y^H \rangle$, and $L_{SF} \equiv \mathscr{L}(X_{SF})$ = the smallest linear *variety* containing X_{SF}. We are now able to formulate Assumption ED.7a.

5.6.8. *A Positivity Assumption* (*Assumption ED.7a*).

ASSUMPTION ED.7a: *For every $x \in X_{SF}$ and every $\delta > 0$, there exists $x' \in X_{SF}$ such that*[58] (1) $\|x' - x\| < \delta$, (2′) $x' \in Int_{L_{SF}}(X_{SF})$, (2″) $x' + \omega \gg_K 0_x$.

Note that condition (2″) is equivalent to $x' + \omega \in \text{Int}_E K$, i.e., to $x' \in \text{Int}_E(K - \{\omega\})$. Hence conditions (2) can be stated more briefly as (2*) $x' \in \text{Int}_{L_{SF}}(X_{SF}) \cap \text{Int}_E(K - \{\omega\})$, and Assumption ED.7a is equivalent to:

ASSUMPTION ED.7a*: $X_{SF} = Cl_E[Int_{L_{SF}}(X_{SF}) \cap Int_E(K - \{\omega\})]$.

[57] For $H > 1$, condition (2″) in Assumption ED.7a below implies $0_x \in \{\omega\} + X_S - \Omega \equiv C_F^i$, so that the smallest linear *variety* containing C_F^i is necessarily a *subspace.* For $H = 1$, this need not be the case in general, but $0_x \in C_F^i$ would follow from the additional assumption of free disposal: $(X_S + (-\Omega)) \subseteq X_S$.

[58] In Example 2, this assumption is violated: $X_{SF} = Y_{FH}$ and if x were chosen in the middle of the segment CD, there does not exist a nearby x' such that (2′) is satisfied. (Here L_{SF} is the whole two-dimensional plane.)

REMARK 1: Given any feasible[59] input-output vector x, Assumption ED.7a guarantees the existence of a (not necessarily distinct) feasible input-output vector x' which is "surrounded" by other feasible vectors and which, together with the total initial endowment ω, contains positive amounts of all commodities that can be supplied by the economy in positive amounts.

REMARK 2: Condition (2″) of Assumption ED.7a is a generalization of (5.38′) in Example 2 of Subsection 5.4.2 above, since (2″) in Assumption ED.7a is equivalent to (5.38′) when $E = R^M$ (so that $K = \Omega$).

REMARK 3: Lemma 5.8 holds for the case of pure exchange, provided $C^i = \Omega$ and $\omega^i \in \Omega$. In this case $X_S = \{0_x\} = X_{SF} = L_{SF}$ and $E = \{x \in R^M : x_k = 0 \text{ for } k \notin J\}$, where $J = \{j \in \mathscr{M} : \omega_j > 0\}$ is the set of commodities available in positive amounts in the total initial endowment. (Recall that $\mathscr{M} \equiv \{1, 2, \ldots, M\}$ is the set of all commodities, and $\omega_s \geqslant 0$ for all $s \in \mathscr{M}$.) Assumption ED.7a is necessarily satisfied with $x' = x = 0_x$.

5.6.9. *A Sufficient Condition for Y_{FH} to be Regularly Closed*

LEMMA 5.8: *If $C^i = \Omega$, $\omega^i \in \Omega$ for all $i = 1, \ldots, H$, and Assumption ED.7a holds, then Y_{FH} is regularly closed (i.e., Assumption ED.7 holds).*

PROOF: We show that for every $\tilde{y} \in Y_{FH}$, $\varepsilon > 0$, there is a $\tilde{\tilde{y}} \in \text{Int}_{L_{FH}}(Y_{FH})$ such that $\|\tilde{\tilde{y}} - \tilde{y}\| < \varepsilon$.

Let $\tilde{y} \in Y_{FH}$ be given. Then there is $\tilde{x} \in X_{SF}$ such that $\tilde{x} = \Sigma_{i=1}^{H} \tilde{y}^i$. Hence, by Assumption ED.7a, there is $\tilde{\tilde{x}}$ arbitrarily close to $\tilde{x}$, with $\tilde{\tilde{x}} \in \text{Int}_{L_{SF}}(X_{SF})$, and $\tilde{\tilde{x}} + \omega \gg_K 0_x$.

We now construct $\tilde{\tilde{y}}$ by setting

$$\tilde{\tilde{y}} = -\omega^i + (1 - \lambda)(\tilde{y}^i + \omega^i) + \frac{\lambda}{H}(\omega + \tilde{\tilde{x}}) + \frac{1 - \lambda}{H}(\tilde{\tilde{x}} - \tilde{x}) \qquad (i = 1, 2, \ldots, H),$$

with $0 < \lambda < 1$ and λ sufficiently small.

Now it can be verified that

$$\text{(i)} \qquad \sum_{i=1}^{H} \tilde{\tilde{y}}^i = \tilde{\tilde{x}},$$

so that the $\tilde{\tilde{y}}^i$ are compatible with $\tilde{\tilde{x}}$.

Also,

$$(1 - \lambda)(\tilde{y}^i + \omega^i) + \frac{\lambda}{H}(\omega + \tilde{\tilde{x}}) \gg_K 0_x,$$

[59] In the sense that it can be supplied by producers and so distributed among consumers as to enable them to "survive" (stay in their respective consumption sets).

since $\tilde{y}^i + \omega^i \geqslant_K 0_x$, $\omega + \tilde{\tilde{x}} \gg_K 0_x$, and $\lambda > 0$. Since $\|\tilde{\tilde{x}} - \tilde{x}\|$ can be made arbitrarily small, it follows that

$$\text{(ii)} \qquad \tilde{\tilde{y}}^i + \omega^i \gg_K 0_x \qquad (i = 1, 2, \ldots, H).$$

Finally, by making λ small enough, we can make

$$\text{(iii)} \qquad \|\tilde{\tilde{y}}^i - \tilde{y}^i\| < \varepsilon \qquad (i = 1, 2, \ldots, H),$$

for any $\varepsilon > 0$.

Since $C^i = \Omega$, (ii) implies $\tilde{\tilde{y}}^i \in Y^i$. Hence, in view of (i), since $\tilde{\tilde{x}} \in X_S$, it follows that $\tilde{\tilde{y}} \in Y_{FH}$. More than that, by (ii), $\tilde{\tilde{y}}^i + \omega^i \in \mathrm{Int}_E(K)$.

Now consider any $y = \langle y^1, \ldots, y^H \rangle \in L_{FH}$. Clearly, $y^i + \omega^i \gg_K 0_x$ if $\|y^i - \tilde{\tilde{y}}^i\|$ is small enough. Also, $y \in L_{FH}$ implies $\Sigma_{i=1}^H y^i \in L_{SF}$, and if y is close enough to $\tilde{\tilde{y}}$, $\Sigma y^i \in X_{SF}$, since then $\|\Sigma y^i - \Sigma \tilde{\tilde{y}}^i\| = \|x - \tilde{\tilde{x}}\|$ is also small and $\tilde{\tilde{x}} \in \mathrm{Int}_{L_{SF}}(X_{SF})$.

Hence $\tilde{\tilde{y}} \in \mathrm{Int}_{L_{SF}}(Y_{FH})$, and $\tilde{y} \in \mathrm{Cl}_{L_{FH}} \mathrm{Int}_{L_{FH}}(Y_{FH})$.

REMARK 4: To take advantage of the fact that Lemma 5.8 covers the case of missing goods, we verify Assumption ED.4*. (Since Assumption ED.4*‴ pertains to preferences, it would have to be assumed separately. But we then have to check Assumptions ED.4*′ and ED.4*″.)

Under the assumptions of Lemma 5.8, we see that Assumption ED.4*″ holds, since $C^i = \Omega$ (see Remark 2, Subsection 5.5.1). Also, by Assumption ED.7a, there exists $y \in Y_{FH}$ such that

$$\tilde{\tilde{y}} \in \mathrm{Int}_{\mathscr{L}(Y^i_F)}(Y^i) \qquad (i = 1, 2, \ldots, H),$$

and

$$\tilde{\tilde{x}} = \sum_{i=1}^{H} \tilde{\tilde{y}}^i \in \mathrm{Int}_{\mathscr{L}(X_{SF})}(X_{SF}).$$

Under pure exchange ($N = H$) or when there is only one producer ($N = H + 1$), this implies that Assumption ED.4*′ is satisfied, since then $X_S = -Y^{H+1}$.

When $N > H + 1$ (two or more producers), a slight strengthening of Assumption ED.7a would again yield ED.4*′, viz, replacing (2′) of Assumption ED.7a by

$$(2'_1) \qquad x' = \sum_{k \in K} x'^k,$$

and

$$x'^k \in \mathrm{Int}_{\mathscr{L}(X^k_{SF})}(X^k_{SF}), \quad \text{for all} \quad k \in K.$$

(Here K is the set of all producers, i.e., $K = \mathscr{N} \setminus \{1, \ldots, H\}$ and $X^k_{SF} = X^k_P + \{\omega^k\}$, X^k_P denoting the production set of the kth producer and ω^k his initial endowment.)

(It would appear that the assumption in (5.37) of "free disposal" for every producer, i.e., $X^k_P + (-\Omega) \subseteq X^k_P$, together with Assumption ED.7a, might yield $(2'_1)$.)

REMARK 6: Possible generalizations of Lemma 5.8 might be obtained by replacing Ω with an arbitrary closed convex cone in R^M.

5.7. *Indirect Verification of the Assumption of Openness (Assumption ED.6) via Strict Dominance (Assumption ED.6a): Non-convex Examples*

5.7.0. *Introduction*

In this section we show that for certain classes of continuous utility functions Assumption ED.6a of Subsection 5.6 is satisfied when each of the individual consumption sets C^i, $i = 1, \ldots, H$, is the nonnegative orthant Ω of R^M. In particular, Lemma 5.9 below shows that this is so when the utility functions are strictly increasing.[60]

Now, since Ω is the closure of its interior, it follows that, under the hypotheses of Lemma 5.9, Theorem 5.3 is applicable provided its assumptions concerning the feasible set Y_{FH} are satisfied. (For example, they are satisfied for pure exchange since then Y_{FH} is compact and is the closure of its interior in $\mathscr{L}(Y_{FH})$.) Under such circumstances, therefore, the assumption of openness holds, and Theorems 5.1 and 5.2 are applicable provided that the environment satisfies Assumptions ED.1–ED.4 and that the B process satisfies Assumptions PD.1 and PD.2.

Notice that convexity of preferences is not required. Examples 3 and 4 below are cases of non-convex preferences satisfying the conditions of Theorem 5.3.

5.7.1. *A Sufficient Condition for Strict Dominance*

LEMMA 5.9: *If for each consumer $i = 1, 2, \ldots, H$, the utility function U^i is continuous and strictly increasing in each commodity y^i_k, $k = 1, \ldots, M$, and, if C^i is the nonnegative orthant Ω of R^M, then Assumption ED.6a is satisfied.*

REMARK: An example of a utility function satisfying the hypotheses of Lemma 5.9 is:

$$U(x_1, \ldots, x_M) = \prod_{j=1}^{M} (x_j + \beta_j)^{\alpha_j},$$

with α_j, β_j positive and x_j nonnegative for $j = 1, \ldots, M$. This is a translate of a Cobb-Douglas function. Examples with non-convex contour sets can also be constructed, e.g., $U(x_1, \ldots, x_M) = \Sigma_{j=1}^M x_j^2$.

PROOF: Let $\varepsilon > 0$, $\delta > 0$, $\bar{y} \in \bar{Y}_{\varepsilon H}$, and $\hat{y} \in \hat{Y}_H(\bar{y})$ be given. Let Q denote the subset of $\{1, 2, \ldots, H\} \setminus Q$, with $H_{Q'}$ elements, so that $i \in Q'$ if and only if $U^i(\hat{y}^i) = U^i(\bar{y}^i)$.

[60] Assumption ED.6a is also satisfied if utility is positive and strictly increasing in the interior of the nonnegative orthant but equals zero on the boundary (see below, Corollary to Lemma 5.9). This makes the results applicable to utility functions of the Cobb-Douglas type.

Examples in 5.7.3. below treat cases in which $C^i = \Omega$ and Assumption ED.6a is satisfied, even though the utility functions are not strictly increasing in the interior of C^i.

In models with production, the strict monotonicity of utility could probably be replaced by local non-saturation together with the postulate that any good can be produced in small amounts with small inputs.

Since $\hat{y} \in \hat{Y}_H(\bar{y})$ and $\bar{y} \in \bar{Y}_H$, Q is non-empty. If Q' is empty, then $\hat{\hat{y}} = \hat{y}$, $v^i = 0_x$ and $x^k = \hat{x}^k$, with $\Sigma_{k=H+1}^{N} \hat{x}^k = \hat{x}$, satisfy Assumption ED.6a. Hence it remains only to consider the case $Q' \neq \varnothing$.

Due to $C^i = \Omega$, for each $i \in Q$ there exists $j \in \{1, \ldots, M\}$ such that $\hat{y}^i_j > -\omega^i_j$, for otherwise $\hat{y}^i = -\omega^i$; since U^i is strictly increasing and $\bar{y}^i \geqslant -\omega^i$, this would contradict $U^i(\hat{y}^i) > U^i(\bar{y}^i)$, which follows from $i \in Q$.

Now fix $i \in Q$ and construct v as follows: for $h = 1, \ldots, H$ and $k = 1, \ldots, M$, let

$$v^h_k = \begin{cases} -\xi, \text{ with } 0 < \xi < y^i_j + \omega^i_j, & \text{for } k = j \text{ and } h = i, \\ \dfrac{\xi}{H'_Q} & \text{for } k = j \text{ and } h \in Q', \\ 0 & \text{otherwise,} \end{cases}$$

choosing ξ sufficiently small so that $U^i(\hat{y}^i + v^i) > U^i(\bar{y}^i)$ and $\|v^i\| < \delta$. (Such a choice is possible because of the continuity of U^i.) Then, since $v^h = 0_x$ for $h \in Q \setminus \{i\}$ and $\hat{y}^h + v^h > \hat{y}^h$ for $h \in Q'$, we have, by monotonicity of utility, $U^h(\hat{y}^h + v^h) > U^h(\bar{y}^h)$, for all $h = 1, \ldots, H$.

Hence $v^h \in V^h_\delta(\bar{y}, \hat{y})$, for all $h = 1, \ldots, H$, and therefore $\Sigma_{h=1}^H v^h \in V_\delta(\bar{y}, \hat{y})$. Further,

$$\sum_{h=1}^{H} v^h_j = -\xi + \sum_{h \in Q'} \frac{\xi}{H_{Q'}} = -\xi + \xi = 0,$$

and $v^h_k = 0$ for $k \neq j$, so that $\Sigma_{h=1}^H v^h = 0_x$. Since $\hat{y} \in Y_{FH}$, $\Sigma_{h=1}^H \hat{y}^h \equiv \hat{x} \in X_S$, and consequently

$$\sum_{h=1}^{H} v^h = 0_x \in [X_S - \{\hat{x}\}] \cap V_\delta(\bar{y}, \hat{y}).$$

(It is implicit that the x^k of Assumption ED.6a are the same $\hat{x}^k$ which have produced $\hat{y}$.)

5.7.2. *A Second Sufficient Condition for Strict Dominance*

Lemma 5.9 does not cover the Cobb-Douglas[61] utility functions, since these are not strictly increasing on the boundary of their domain. However, the following corollary to Lemma 5.9 covers all utility functions that are strictly increasing in the interior of the nonnegative orthant, constant on the boundary, with any point of the interior preferred to every point of the boundary; the corollary is therefore applicable to Cobb-Douglas functions.

COROLLARY TO LEMMA 5.9: *If, for each consumer* $i = 1, \ldots, H$, $C^i = \Omega$ *(the nonnegative orthant of* R^M*), and the continuous utility function* U^i *satisfies, for* $x', x'', x''' \in \Omega$, *the two conditions,*

(i) $$\left.\begin{matrix} x' \gg 0_x \\ x' > x'' \end{matrix}\right\} \Rightarrow U^i(x') > U^i(x''),$$

(ii) $$x' \not\gg 0_x,\ x'' \not\gg 0_x,\ x''' \gg 0_x \Rightarrow U^i(x') = U^i(x'') < U^i(x'''),$$

[61] A Cobb-Douglas utility function on Ω is of the form: $U(x_1, \ldots, x_M) \equiv \Pi_j x_j^{\alpha_j}$ with $\alpha_j > 0$ and $x_j \geqslant 0$ for $j = 1, 2, \ldots, M$.

then Assumption ED.6a is satisfied. (For vectors, $x \gg 0$ means that every coordinate of x is strictly positive.)

PROOF: (We shall find it convenient to set $U^i(x) = 0$ for $x \not\gg 0$, so that $U^i(x) > 0$ for all $x \gg 0$.) A slightly altered version of the proof of Lemma 5.9 can be used to establish the corollary. First note that for $i \in Q$ we just have $y^i_j > -\omega^i_j$ for all $j = 1, \ldots, M$. Otherwise $U^i(\hat{y}^i) = 0$. Since $U^i(\bar{y}^i) \geqq 0$, this contradicts $U^i(\hat{y}^i) > U^i(\bar{y}^i)$ for $i \in Q$.[62] Now fix $i \in Q$ and construct v, with $0 < \xi < \hat{y}^i_j + \omega^i_j$, as follows:

$$v^h_j = \begin{cases} \left.\begin{array}{ll} -\xi_j, & \text{for } h = i, \\ \dfrac{\xi_j}{H_{Q'}}, & \text{for } h \in Q', \end{array}\right\} & j = 1, \ldots, M, \\ 0 \quad \text{otherwise.} \end{cases}$$

We may take ξ_j so small that $U^i(\hat{y}^i + v^i) > U^i(\bar{y}^i)$, and $\|v^i\| < \delta$. Then $\hat{y}^h_j + v^h_j > \hat{y}^h_j$, for $h \in Q'$ and $j = 1, \ldots, M$, and hence $U^h(\hat{y}^h + v^h) > U^h(\hat{y}^h) = U^h(\bar{y}^h)$, for $i \in Q'$, so that $v^h \in V^h_\delta(\bar{y}, \hat{y})$, for all $h = 1, \ldots, H$, and $\Sigma^H_{h=1} v^h \in V_\delta(\bar{y}, \hat{y})$. As before,

$$v_j \equiv \sum_{i=1}^{H} v^i_j = -\xi_j + \sum_{i \in Q'} \frac{\xi_j}{H_{Q'}} = -\xi_j + \xi_j = 0 \qquad (j = 1, \ldots, M),$$

so that $\Sigma^H_{h=1} v^h = 0_x$, and

$$0_x = \sum_{h=1}^{H} v^h \in [X_S - \{\hat{x}\}] \cap V_\delta(\bar{y}, \hat{y}).$$

5.7.3. *Non-convex Examples: Pure Exchange*

The following examples give continuous utility functions which are neither of Cobb-Douglas type, nor strictly increasing in each commodity,[63] but which satisfy Assumption ED.6a. Hence by Theorem 5.3 in the (pure exchange) case $X_S = \{0_x\}$ with $C^i = \Omega$ for all i, the openness property (Assumption ED.6) is also satisfied in these examples. It can readily be seen that the other conditions of Theorems 5.1 and 5.2 are also satisfied.

EXAMPLE 1: Pure exchange, two persons, two commodities: $H = N = M = 2$, $\omega = (a, b)$, $a > b > 0$; $X_S = \{(0, 0)\}$, $Y_F = Y_{FH} = \{\langle y^1, y^2 \rangle : y^1_1 + y^2_1 = a,\ y^1_2 + y^2_2 = b,\ y^1 \geqq 0,\ y^2 \geqq 0\}$, $U^1(y^1_1, y^1_2) = \min(y^1_1, y^1_2)$, $U^2(y^2_1, y^2_2) = y^2_1 + y^2_2$.

Figure 5.9 shows this example in the form of the Edgeworth Box: $\bar{y}$ is a typical non-optimal point[64] in $\bar{Y}_\varepsilon$ (the set $\bar{Y}_\varepsilon$ is omitted from the diagram); $\hat{y}$ is a non-inferior Pareto optimal point, $v = (v^1, v^2)$, $v^1 + v^2 = 0_x$ is an element of $V_\delta(\bar{y}, \hat{y})$, and $\tilde{y} = \hat{y} + v$. Thus Assumption ED.6a is satisfied. We can also see directly that openness (Assumption ED.6) is satisfied since, e.g., the triangle EVD has a non-empty interior. Here u^1 is not strictly increasing in either commodity.

[62] The corollary does apply to cases covered by Lemma 5.6* above.

[63] Not even in the interior of Ω.

[64] More precisely, $\bar{y} \equiv \langle \bar{y}^1, \bar{y}^2 \rangle$ is an element of the 4-dimensional space, while the point labeled $\bar{y}$ on our diagram has the coordinates of $\bar{y}^1$ when referred to 0^1 and those of $\bar{y}^2$ when referred to 0^2. A similar interpretation applies to other vector symbols in Figures 5.9 through 5.11.

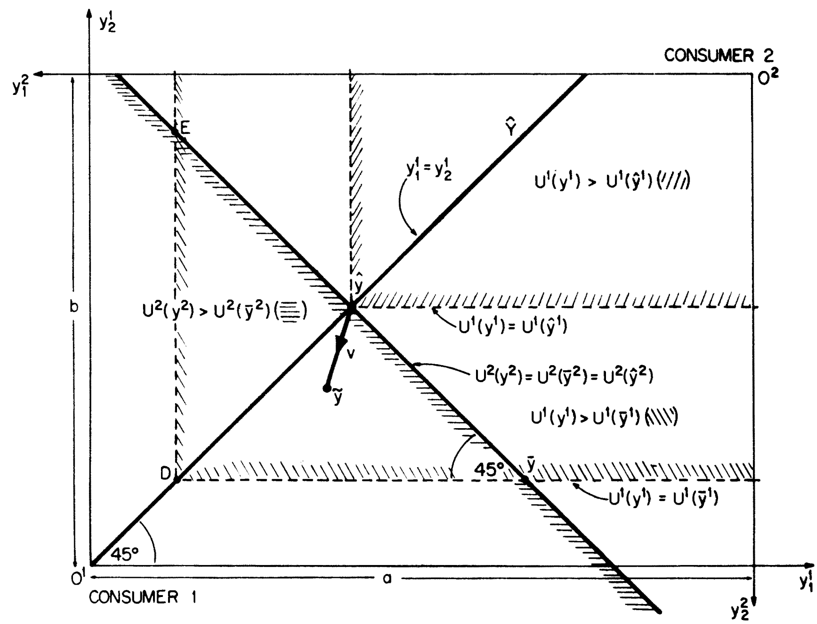


FIGURE 5.9

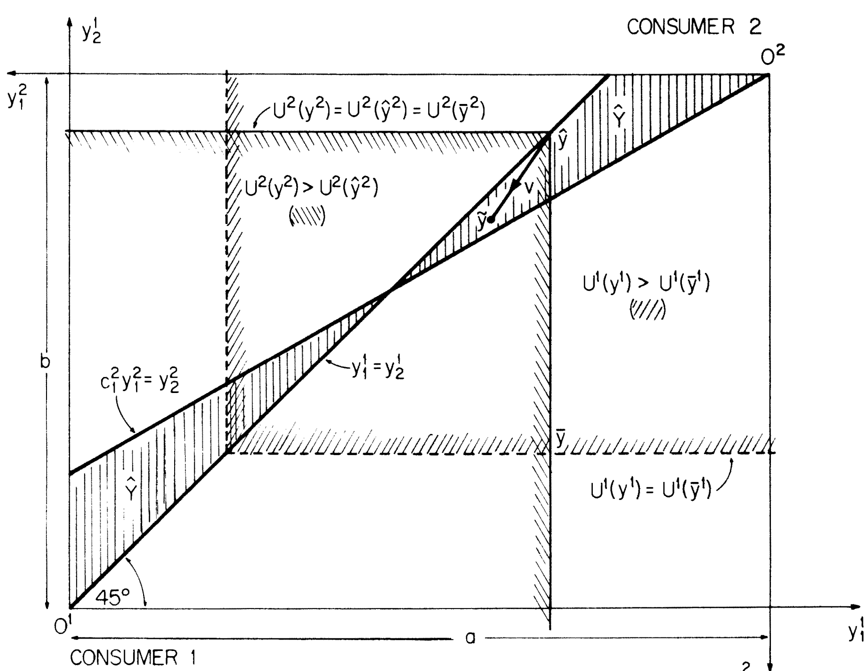
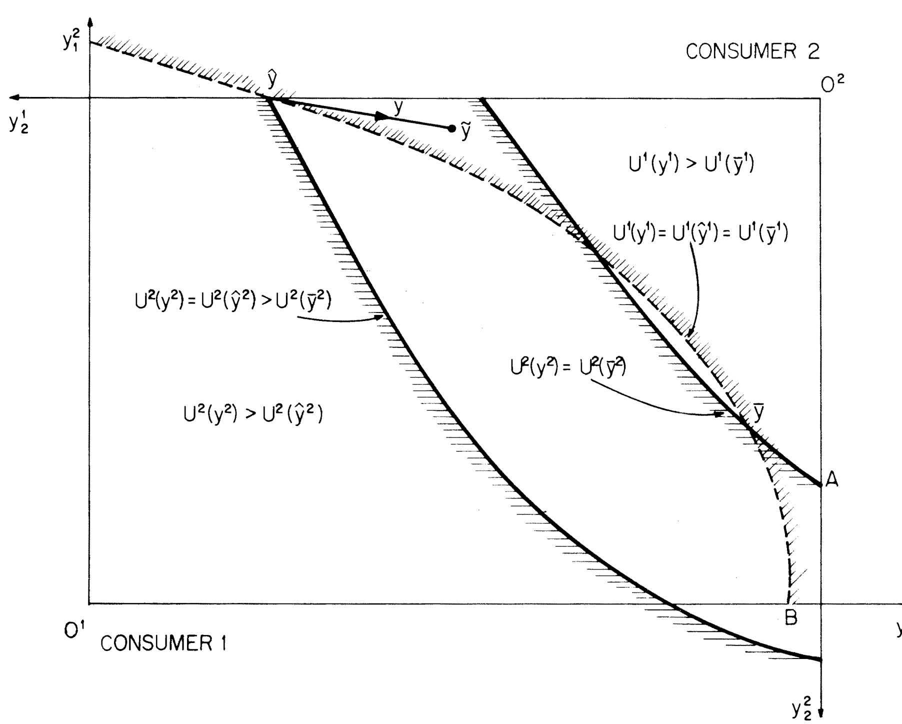


FIGURE 5.10

EXAMPLE 2: This example is the same as the preceding one, except that U^2 of Example 1 is replaced by $U^2(y_1^2, y_2^2) \equiv \min(cy_1^2, y_2^2)$, where $c < 1$ is a constant, so that both goods are strict complements for both consumers. Figure 5.10 shows this example in Edgeworth Box form. In this case the Pareto optimal set $\hat{Y}$ consists of two triangles formed by the lines along which the vertices of indifference curves for consumers 1 and 2 respectively are located. The set $\bar{Y}_\varepsilon$ is omitted from the figure but the point $\bar{y}$ is supposed to be an element of $\bar{Y}_\varepsilon$; $\hat{y}$ is a Pareto optimal point preferred to $\bar{y}$ by consumer 1 and indifferent to it for consumer 2. In this case $\tilde{y}$ is also Pareto optimal, but preferred to $\bar{y}$ by both consumers. Clearly many choices of (v^1, v^2) satisfying Assumption ED.6a are possible.

EXAMPLE 3: Pure exchange, two persons, two goods, non-convex preferences. Figure 5.11 (again in Edgeworth Box form) shows a non-optimal point $\bar{y}$ and a Pareto optimal point $\hat{y}$ preferred to $\bar{y}$ by consumer 2 but indifferent to it for consumer 1. The set of points preferred to $\bar{y}$ by consumer 1 is the (non-convex) set lying to the right of the curve through $\bar{y}$ labeled B. The set of points preferred by consumer 2 to $\bar{y}$ is the (non-convex) set to the left of the curve through $\bar{y}$ labeled A. $\hat{y}$ is a Pareto optimal point preferred by consumer 2 to $\bar{y}$, and indifferent to $\bar{y}$ for consumer 1. The displacement (v^1, v^2) satisfies Assumption ED.6a.



158.

FIGURE 5.11

EXAMPLE 4: Pure exchange, two persons, one good, non-convex preferences. In this slightly unconventional example there is one good measured in units so chosen that the available amount $\omega = 2\pi$. The utility functions are $U^i(y^i) \equiv |\sin 2y^i|$, $i = 1, 2$. We may think of this as an idealization of the case of a good which, while in principle available in any amount, is indivisible in consumption, so that multiples of a certain amount are preferred to intermediate quantities.

Figure 5.12 shows this example in product-space form (rather than in an Edgeworth Box). In this picture the feasible set Y_F is the line of slope 1 shown. The graph of the utility function of consumer 1 is shown (superimposed), but that of consumer 2 (which is identical) is omitted. The Pareto optimal set consists of the four isolated points labeled A, B, C, and D, respectively. Assumption ED.6a is satisfied, since any optimal point is preferred by both consumers to every non-optimal point.

5.7.4. *An Example with Increasing Returns in Production*

As a simple illustration with regard to non-convexity, one can show that the following environment would satisfy all assumptions of Theorems 5.1 and 5.2 (hence, a fortiori, those of Theorems 5.1* and 5.2*): (i) there are two goods ($M = 2$);

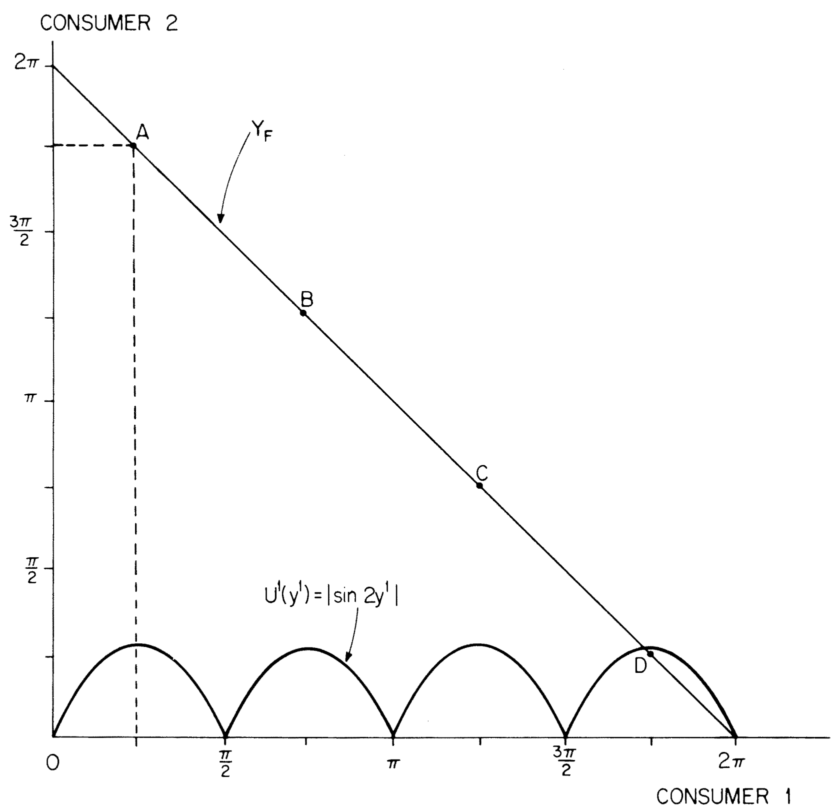


FIGURE 5.12

(ii) there are $H \geqslant 1$ consumers with consumption sets $C^i = \Omega$ and utility functions $U^i(x^i) \equiv \Sigma_{j=1}^{H} \alpha_j(x_j^i)^2$, all $\alpha_j > 0$, for all consumers, $i \in \{1, \ldots, H\}$; the total endowment $\omega_j = \Sigma_{i=1}^{H} \omega_j^i$ is positive for each commodity (i.e., $\omega_1 > 0$, $\omega_2 > 0$); (iii) there is one producer (so that $N = H + 1$) whose production possibility set is $X_P^{H+1} = \{(x_1, x_2): x_2 \leqslant (x_1)^2, x_1 \leqslant 0\}$.

Here preferences are non-convex and production is subject to increasing returns (see Figures 5.13 and 5.14).

The openness assumption is satisfied in virtue of Theorem 5.3 and Lemmas 5.8 and 5.9. That the other assumptions of Theorems 5.1, 5.2 are satisfied follows from the discussion in Subsection 5.4.3.

(The assumption $\omega \gg 0$ could be weakened so as to make Theorems 5.1* and 5.2* applicable via Lemma 5.6*, even though Assumption ED.6 of Theorems 5.1 and 5.2 was violated.)

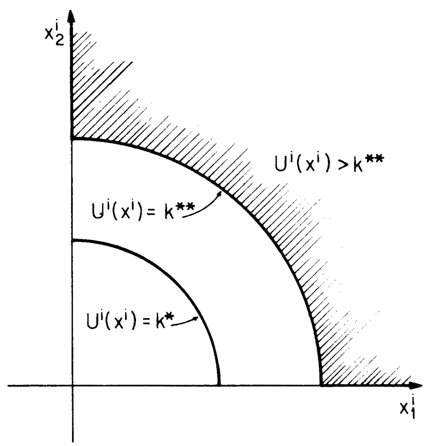


FIGURE 5.13
Preferences of the ith consumer; $k^{**} > k^*$.

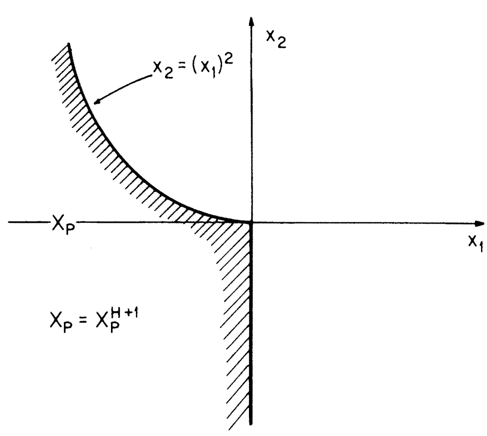


FIGURE 5.14
The production possibility set (the closed set shaded ⑊⑊).

5.8. *A Summary of the Results for Divisible Commodities*

Because of the complexity of the case of divisible commodities, we provide a summary of the results of, and cases covered by, Section 5.

The two basic results are contained in Theorems 5.1 and 5.2 of Subsection 5.3. These theorems respectively assert the optimality and stability (in a probabilistic sense) of the equilibrium of the B process, given that the process has properties given in Assumption PD.1 and PD.2 and that the environment satisfies conditions ED.1–ED.6.

Theorems 5.1* and 5.2* in Subsection 5.5 are generalizations of Theorems 5.1 and 5.2 of Subsection 5.3, respectively, obtained by replacing Assumption ED.4 with a weaker Assumption ED.4*. This relaxation makes the B process work in situations where some of the commodities are missing from the initial endowment (Subsection 5.5.5); it may also turn out to be useful in dealing with resource allocation in economies with public goods. The counterexample in Subsection 5.5.6 shows that Assumption ED.4*″ (hence *a fortiori* Assumption ED.4) cannot be dispensed with.

Going back to Theorems 5.1 and 5.2, the environment assumptions ED.1–ED.5 are not too difficult to verify. Assumption ED.5 simply states that preferences are representable by continuous utility functions (i.e., that upper and lower contour sets are closed). Assumption ED.1 requires that the admissible trading sets Y^i be closed. For consumers, this is equivalent to the closedness of admissible consumption sets C^i and holds in the familiar special case where $C^i = \Omega$ (the nonnegative orthant). For producers it is equivalent to the closedness of the production sets X_P^i.

Assumption ED.3 requires that the feasible trading set Y_F be bounded; since we are dealing with finite dimensional Euclidean commodity spaces, Assumptions ED.1–ED.3 imply the compactness of Y_F.

Under pure exchange, Y_F is bounded when the consumption sets C^i are bounded from below, in particular again when $C^i = \Omega$. When production is present, and especially when there are two or more producers, it is somewhat more difficult to assure the boundedness of Y_F. A set of conditions sufficient to insure that Assumptions ED.1–ED.4 are satisfied is given in Example 2 of Subsection 5.4.2; these conditions do not imply the convexity of preferences, although in other respects they are of the conventional type used in the theory of competitive equilibrium. As seen in Example 3 of Subsection 5.4.3, one can also dispense with the requirement of convexity of the aggregate production set.

Assumption ED.4 requires that there exist a vector of compatible trades each of which is interior to its respective admissible trading set Y^i. For pure exchange with $C^i = \Omega$ (all i) this will be the case if (and only if) the total endowment in each commodity is positive (Example 1, Subsection 5.4). But the latter assumption is rather stringent. In Subsection 5.5, therefore, we establish the more general Theorems 5.1* and 5.2* (using the weaker Assumption ED.4* in place of Assumption ED.4); their application to the pure exchange case with $C^i = \Omega$ where some goods may be absent from the initial endowment is given in Subsection 5.5.5 on the assumption that preferences are monotone with regard to the absent goods.

When production is present, Assumption ED.4 can be satisfied with the help of conventional postulates such as free disposal in production (Examples 2 and 3, Subsections 5.4.2 and 5.4.3) but without resort to convexity. Again a weakening of those assumptions along the lines of Theorems 5.1* and 5.2* is possible.

Finally, one must verify the least transparent condition, that of "openness" (Assumption ED.6). To facilitate this verification we show that Assumption ED.6 is equivalent to the more easily verified ED.6a, provided that a projection Y_{FH} of the feasible set Y_F has the topological property of being the closure of its interior (Assumption ED.7). (The topology used is relative to a certain linear manifold and this makes the requirement less restrictive than it would otherwise be.)

Now the topological requirement in Assumption ED.7 is fulfilled in the usual pure exchange case with $C^i = \Omega$. A more general result concerning the circumstances (Assumption ED.7a) where Y_{FH} satisfies Assumption ED.7 with production present, is given in Lemma 5.8 (Subsection 5.6.9).

Supposing that Y_{FH} is the closure of its interior (Assumption ED.7), "openness" is guaranteed by Assumption ED.6a. In turn, Lemma 5.9 and its corollary (Subsection 5.7) shows that monotonicity and continuity of the utility function, together with $C^i = \Omega$, are enough to imply Assumption ED.6a, hence the "openness" Assumption ED.6.

Examples 1 through 4 in Subsection 5.7.3 show how the preceding results may be applied to verify the assumption of openness. These examples involve pure exchange with preferences that are not strictly monotone (Example 1), non-convex (Examples 3 and 4), and (Example 4) even oscillatory! A non-convex example involving production (increasing returns) is given in Subsection 5.7.4.

Looking now at the totality of these conditions, it is easy to satisfy them all with non-convex preferences and non-convex production sets. In particular, Theorems 5.1* and 5.2* are applicable in the case of pure exchange with continuous monotone preferences and $C^i = \Omega$, regardless of whether there is convexity and regardless of whether the initial total endowment is strictly positive.

When production is present, assuming the situation for consumers to be as in the preceding paragraph, one must additionally make sure that the jointly feasible set Y_F is compact and that the production Y_{FH} is the closure of its own interior; Example 3 of Subsection 5.4 and Lemma 5.8 of Subsection 5.6.7. may be of help in this task. It can be seen that non-convexity and absence of goods from the initial endowment do not pose a problem for the applicability of the B process.

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REFERENCES

[1] ARROW, K. J.: "An Extension of the Basic Theorems of Classical Welfare Economics," in *Proceedings of the Second Berkeley Symposium on Mathematics, Statistics, and Probability*, ed. by J. Neyman, Berkeley, Calif.: University of California Press, 1959.

[2] ARROW, K. J., AND L. HURWICZ: "Gradient Method for Concave Programming, III: Further Global Results and Applications to Resource Allocation," in *Studies in Linear and Non-Linear Programming* by K. J. Arrow, L. Hurwicz, and H. Uzawa. Stanford, Calif.: Stanford University Press, 1958.

[3] ———: "Decentralization and Computation in Resource Allocation," in *Essays in Economics and Econometrics*. Chapel Hill, N.C.: University of North Carolina Press, 1960.

[4] DAVIS, O. A., AND A. WHINSTON: "Externalities, Welfare, and the Theory of Games," *Journal of Political Economy*, 70 (1962), 241–262.

[5] DEBREU, G.: *Theory of Value*. New York: Wiley, 1959.

[6] ———: "New Concepts and Techniques for Equilibrium Analysis," *International Economic Review*, 3 (1962), 257–273.

[7] FENCHEL, W.: *Convex Cones, Sets, and Functions* (from notes by D. W. Blackett), Department of Mathematics, Princeton University, 1953.

[8] HURWICZ, L.: "Optimality and Informational Efficiency in Resource Allocation Processes," in *Mathematical Methods in the Social Sciences, 1959*, ed. by K. J. Arrow, S. Karlin, and P. Suppes. Stanford, Calif.: Stanford University Press, 1960.

[9] HURWICZ, L., AND S. REITER: "On the Boundedness of the Feasible Set Without Convexity Assumptions," *International Economic Review*, 14 (1973), 580–586.

[10] KANEMITSU, H.: "On the Stability of an Adjustment Process in Non-Convex Environments—A Case of the Commodity Space (Strong) Inertia/Greed Process," unpublished paper presented at the Second World Congress of the Econometric Society, Cambridge, England, 1970.

[11] KOOPMANS, T. C.: *Three Essays on the State of Economic Science*. New York: McGraw-Hill, 1957.

[12] LUCE, D., AND H. RAIFFA: *Games and Decisions*. New York: Wiley, 1957.

[13] MARSCHAK, T. A.: "Centralization and Decentralization in Economic Organizations," *Econometrica*, 27 (1959), 399–430.

[14] QUIRK, J., AND R. SAPOSNIK: *Introduction to General Equilibrium Theory and Welfare Economics*. New York: McGraw-Hill, 1968.

[15] RADNER, R.: "The Evaluation of Information in Organizations," in *Proceedings of the Fourth Berkeley Symposium on Mathematics, Statistics, and Probability*, Vol. I, ed. by J. Neyman. Berkeley, Calif.: University of California Press, 1961.

[16] REITER, S.: "A Market Adjustment Mechanism," Discussion Paper No. 1, Institute for Quantitative Research in Economics and Management, Purdue University, 1959.

[17] STARRETT, D. A.: "Fundamental Non-Convexities in the Theory of Externalities," *Journal of Economic Theory*, 4 (1972), 180–199.

[18] VIND, K.: "A Theorem on the Core of an Economy," *Review of Economic Studies*, 32 (1965), 47–48.

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References

[4] **Externalities, Welfare, and the Theory of Games**
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