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A STOCHASTIC DECENTRALIZED RESOURCE  
ALLOCATION PROCESS: PART I

BY LEONID HURWICZ, ROY RADNER, AND STANLEY REITER

This is Part I of a paper concerning an iterative decentralized process designed to allocate resources optimally in decomposable environments that are possibly characterized by indivisibilities and other nonconvexities. Important steps of the process involve randomization. In Part I we present the basic models and results, together with examples showing that certain assumptions can be satisfied in both classical and nonconvex cases. Part II will go further with such examples in showing that our process yields optimal allocations in environments in which the competitive mechanism fails, as well as show how abstract conditions used in Part I can be verified in terms of properties of preferences and production functions that are familiar to economists.

1. INTRODUCTION AND SUMMARY<sup>1</sup>1.0. *General Introduction*

IN THIS PAPER we construct an iterative decentralized process (to be called the *B* process because it involves *bidding*) designed to allocate resources optimally in environments that are decomposable, i.e., free of externalities, but possibly characterized by indivisibilities (in commodities) or nonconvexities (in preferences or production). We have largely confined ourselves to situations where either all goods are indivisible or all goods are divisible, although similar methods could also be applied to mixed cases.<sup>2</sup> Important steps of the process involve randomization, hence the designation "stochastic" in the title.<sup>3</sup>

To understand the motivation for constructing such a new process, one must look at the limitations of the known allocation processes.

The decade of the fifties saw a rigorous formulation of the relationship between Pareto optima and competitive equilibria. In the realm of statics, the results due to Arrow [1], Debreu [5], and Koopmans [11] provide conditions under which a competitive equilibrium is Pareto optimal<sup>4</sup> ("nonwastefulness") and a Pareto optimum is capable of sustaining a competitive equilibrium<sup>5</sup> ("unbiasedness"). These two results are, e.g., Koopmans' Propositions 4 and 5, respectively [11].

Both results presuppose the absence of externalities (i.e., external economies or diseconomies) and local nonsaturation of preferences. The second result

<sup>1</sup> This research was supported by the National Science Foundation, the Office of Naval Research, and the General Electric Company.

<sup>2</sup> Indivisible goods are available only in integer-valued amounts. (Actually, all that is required is that there be only a finite number of feasible allocations.) In fact, one of our results, that of Section 3, does apply to mixed cases as well.

<sup>3</sup> The use of randomization in adjustment processes was studied in Reiter [16] and provided a stimulus for the development of the *B* process. Radner first formulated and studied the *B* process for indivisible goods in an unpublished note in 1960.

<sup>4</sup> In the terminology of [8], the competitive process is *non-wasteful*.

<sup>5</sup> In the terminology of [8], the competitive process is *unbiased*.

(unbiasedness) uses additional assumptions, in particular the convexity of preferences and production. Further assumptions are needed to assure that the competitive equilibrium is stable.

If one believes that these restrictive *classical* assumptions are significantly violated in situations of some importance, and if one believes that economics should be concerned with the possibility of having satisfactory economic institutions for non-classical environments, one is naturally interested in resource allocation mechanisms for environments in which these restrictive classical assumptions are not all satisfied. Pursuit of this interest in a normative spirit requires study of mechanisms other than perfect competition, for there are examples showing that these classical assumptions cannot be dispensed with if competitive equilibria are to have their desirable performance properties. For instance, it has been shown that competitive equilibrium need not be Pareto optimal when all goods are indivisible. Similarly, there are examples in which non-convexity of preferences or production sets results in optimal allocations that cannot be attained through the competitive process.<sup>6</sup>

Economists are, of course, familiar with processes other than perfectly competitive, which operate under conditions of non-convexity or indivisibility. Many of these, classified as monopolistic, lack the attributes of non-wastefulness and unbiasedness because their equilibria are typically non-optimal. Others fail to qualify as informationally decentralized.<sup>7</sup>

One is then led to try to design new allocation mechanisms that would meet our standards of performance (non-wastefulness and unbiasedness) in non-classical environments and still qualify as informationally decentralized.

The design and analysis of such mechanisms form part of the theory of economic institutions. For, just as the competitive model is an abstract specification of a system of institutions (markets together with other economic relationships), so the abstract specification of another allocation mechanism characterizes an alternative, possibly new, set of economic institutions.

Clearly, to avoid dreaming up mere utopias in designing new mechanisms with good theoretical performance characteristics, one must take account of those informational and incentival attributes without which an allocation mechanism would be infeasible, extremely costly, or otherwise undesirable. The present paper is focused on designing a mechanism possessing the desired optimality characteristics for a broad class of *decomposable* environments, i.e., those free of externalities, but also qualifying as informationally decentralized.<sup>8</sup> Now a mechanism of this type, suggested in [8] under the name of "greed process," suffers from several defects; in particular, on the side of performance, it lacks stability, and on the informational side its structure is quite burdensome for those participating

<sup>6</sup> Such examples appear, for instance, in Chapter 4 of Quirk and Saposnik [14, pp. 134 and 139].

<sup>7</sup> See [8] for a definition of informational decentralization. Alternative definitions have been proposed in [15] and elsewhere.

<sup>8</sup> The modified Lagrangian gradient process proposed by Arrow and Hurwicz in [2 and 3] constitutes such a mechanism for environments with non-convexities (increasing returns) but without indivisibilities. (See, in particular, "Optimal Allocation through Imperfect Competition," [3, Section IV.D, pp. 95-100].)

in the process. Thus in designing the *B* process, our objective is to overcome some of the defects of the “greed process” while retaining most of its advantages.<sup>9</sup>

While deferring a more detailed comparison to Section 1.1.3 below, we may note that the *B* process constitutes an improvement on the “greed process” in that it does converge to equilibrium (i.e., has stability), although in a probabilistic sense only. Also, its messages are significantly simpler than those of the “greed process” so that it is informationally less burdensome. On the other hand, the class of environments covered is slightly narrower, but it still includes indivisibilities and nonconvexities.

The probabilistic nature of the *B* process accounts for a difference in the nature of the relevant equilibrium concept. While in nonstochastic processes (such as that of the Walrasian competitive model) equilibrium is characterized by constancy (over time) of proposals or messages, the equilibrium of a stochastic process only involves the constancy of a probability distribution based on past agreements. Thus at equilibrium in the *B* process there may be different proposals still coming in, but these proposals cannot overcome the standing agreements which do remain constant, at least in utility terms. Viewed in computational terms, processes such as the competitive mechanism “print out” the final (equilibrium) result—if and when reached—and stop; others, like the *B* process, continue searching for a “better” solution indefinitely, but when in equilibrium keep “printing out” the answer already found.

Another noteworthy feature of the *B* process is the generality of the class of objects whose allocation it can help determine. Specifically, the indivisible case of the *B* process can be applied not only to conventional “commodity bundles” but to any collection of entities that can be described as a discrete set; in particular, there need not be any concept of (“quantitative”) measurability associated with these objects, nor need these objects be orderable except in terms of the participants’ preferences. Although individual participants may consider as admissible an infinity of such objects of choice, the discreteness of the set, together with our other assumptions, requires that only a finite number of objects be jointly feasible. As an example of such objects of choice one might think of alternative clauses in a proposed labor-management contract referring to qualitatively distinct characteristics.

As already indicated, our results for the *B* process, as those for the competitive and the “greed processes,” are valid for environments that are decomposable, i.e., free of externalities. (Indeed, the *B* process, like the competitive process, is not even well-defined when “non-separable” externalities are present.)<sup>10</sup> Whether one can hope for the existence of any informationally decentralized process for non-decomposable environments is as yet not completely resolved and depends very critically on the concept of informational decentralization.

<sup>9</sup> See [8] for the definition of the “greed process” and a proof of its non-wastefulness and unbiasedness for all environments free of externalities (decomposable); its definition implies that it is informationally decentralized. Since this was written, Kanemitsu [10] has constructed a convergent deterministic process for divisible environments which is closely related to the “greed process.”

<sup>10</sup> See [4] for a definition of “non-separable” externalities.

The unbiasedness of the  $B$  process is established for all decomposable environments. Its non-wastefulness and convergence, on the other hand, have so far only been obtained separately for the indivisible (discrete) case and for the divisible (continuous) case. It is remarkable, in view of the difficulties encountered in the competitive process, that the  $B$  process has the desirable performance properties (unbiasedness, non-wastefulness, and stability) for all indivisible decomposable environments. Among the divisible decomposable environments, where the commodity space is a continuum and preferences continuous, the class of environments covered is very broad: it includes all cases of monotone preferences whether convex or not (provided that the feasible set has certain topological properties), but non-monotone preferences are not excluded. Nor is it necessary to assume the convexity of production possibility sets either for individual units or in the aggregate when dealing with models that involve production as well as exchange.<sup>11</sup>

The applicability of our results to situations involving indivisibilities and non-convexities on the production side makes the  $B$  process, or a similar mechanism, an attractive possibility for handling allocation problems involving capital goods where both indivisibility and nonconvexity are typically of the essence. A further extension of our results to "mixed" (divisible-indivisible) cases would be helpful.

Situations characterized by the absence of some goods from the initial aggregate endowment,<sup>12</sup> troublesome in perfectly competitive processes, are covered by suitable extensions of the main results for the divisible case; the issue does not arise in the indivisible case. This is significant from the viewpoint of informational decentralization, since an individual agent is only assumed to know that he lacks a given good, but not that everyone else is in the same boat. The  $B$  process is capable of handling allocation under such circumstances. The mathematical situation characteristic of initially absent goods is similar to that involving public goods, viz., the allocations are confined to a linear variety (a flat) in the commodity space. Hence, our techniques, and possibly some of the more general results in the section on absent commodities, should be useful in connection with allocation problems involving public goods.

Among problems we have not as yet looked into is the rate of convergence of the  $B$  process. If speeding up convergence were our objective, one might try varying the probability distributions as the process goes on, thus introducing what one might think of as "learning." Mathematically, such variations are not ruled out, since our proofs do not require that the probability distributions be constant over time.

Before proceeding to a somewhat more detailed outline of the results of the paper, the readers' attention may be drawn to the possible methodological, as distinct from substantive, interest of the techniques used here. In particular, the stochastic aspects are of interest, as well as the exploration of the implications of

<sup>11</sup> This is of particular interest in situations described by Starrett [17] in which the formal elimination ("internalization") of externalities accomplished by the introduction of fictitious commodities generates non-convexities in the expanded commodity space.

<sup>12</sup> The "Arrow case" (see [1, Fig. 3, p. 528]).

assumptions that are topological in nature, as contrasted with the customary reliance on the algebraic (convexity) properties of economic models.

### 1.1. *Pure Exchange*

While the results obtained cover both production and exchange, it seems best to start by providing an informal description of the workings of the  $B$  process for the case of pure exchange, a case usually treated in terms of the familiar “Edgeworth Box.” We shall first present the process as it operates when all commodities are indivisible (see footnotes 1 and 3).

#### 1.1.1. *Pure Exchange of Indivisible Commodities*

The process consists of a sequence of bids (hence “ $B$  process”) and exchanges. Before bidding begins, each participant selects a probability distribution over all those exchanges that would leave him at least as well off as he is with his initial endowment<sup>13</sup> (this is a subset of his admissible trading set, determined by his admissible consumption set and initial endowment); using a randomized device governed by these probabilities, he makes a bid proposing to trade goods with other participants. At this point a “referee” enters the picture to check whether the bids made by the participants are compatible, i.e., whether the aggregate net demand equals zero. If not, the participants must make new bids choosing from the same class of possible trades according to the same probabilities; if the new bids are compatible, the participants carry out the proposed trades, thus reaching new endowment positions. The bidding process is then repeated with reference to the new endowment. Since the set of exchanges that are not inferior to the new endowment is in general smaller, the domain from which the bid is picked by the randomized device is smaller and the probabilities will be scaled up by a proportionality factor.

There may, of course, occur “no deal” phases due to incompatible bid combinations,<sup>14</sup> but when the commodity space is discrete (indivisible commodities), a compatible set of bids will occur, with probability one, in a finite number of tries. In fact, again with probability one, the process will reach a Pareto optimal allocation in a finite time and will remain stationary (at equilibrium) thereafter (Theorem 4.2). Moreover, every Pareto optimum is an equilibrium of the process (Theorem 3.1) and vice versa (Theorem 4.1).<sup>15</sup> These results presuppose selfish preferences (absence of externalities) but no other assumptions need be made concerning the preference patterns. The situation is strikingly different from that prevailing under

<sup>13</sup> This together with other rules of the process, assures “individual rationality” (Luce and Raiffa [12]).

<sup>14</sup> The frequency of occurrence of “no deal” phases is in part dependent on the specific probabilities used. We have ignored the speed of convergence of the process, and hence also this issue, confining attention only to convergence itself.

<sup>15</sup> In the terminology of [8], the  $B$  process is Pareto satisfactory, since it also is essentially single valued.

perfect competition where indivisibilities result in difficulties, including the occurrence of non-optimal equilibria.

These attributes of the  $B$  process are largely due to the fact that traders ordinarily go up (and never down) on their preference scales when exchanges take place. Hence, if the initial allocation happens to be Pareto optimal, the utility levels are bound to remain stationary. If the initial allocation is non-optimal, the new allocation will be at or above the respective previous satisfaction levels.

### 1.1.2. *Pure Exchange of Divisible Commodities*

When commodities are *indivisible*, the force that “drives” the system toward optimality is the *positive* probability associated with every trade leading to a Pareto superior allocation (as against *zero* probability associated with Pareto inferior allocations). But for *divisible* commodities an application of the same bidding rules would result in a fiasco, since the probability of “hitting” any particular point (trade) within the commodity continuum is *zero*, and so is the probability of occurrence of a compatible set of bids. Therefore, to avoid stagnation at a non-optimal allocation, it is necessary to modify the rules for the divisible case. As in the indivisible case, the participant still picks, according to an appropriate probability distribution, a trade, the *central bid*, from among those not inferior to his current endowment. But there is a change in what he is required to communicate to the referee. He conveys not merely this central bid, but also—as alternatives—all trades within a specified “distance” from the central bid that are not inferior to the present endowment. Because the bids now contain a continuum of alternatives, the referee may be faced with a multiplicity of compatible bid combinations. He will then pick one of them at random to serve as the basis for trades to take place.

More explicitly, the participant first constructs a “bidding cube” (in a two-dimensional commodity world, a square) centered on the probabilistically selected central bid; the “radius” (half-width) of the cube is chosen arbitrarily, but remains fixed throughout the bidding process.

The participant then communicates to the referee as his bid those trades within the bidding cube that are at least as good as his current endowment, using the current endowment point as the origin.

As an illustration, in Figure 1.1, let  $I$  be the current endowment and  $\alpha'\alpha''$  the indifference curve through  $I$ , with points above it being preferred. Then the central bid will be selected probabilistically from among the points on or above  $\alpha'\alpha''$ . Let the fixed size of the bidding square (“cube”) be that shown at the right in Figure 1.1.

If the central bid drawn happens to be  $A$ , the whole square  $BCDE$  (interior and perimeter) becomes the bid. On the other hand, if the central bid drawn is  $K$ , the participant’s bid conveyed to the referee is the set bounded by  $RMNPSR$  (interior and perimeter); the points of  $LRS$  that are below  $\alpha'\alpha''$  are excluded because they are inferior to the current endowment  $I$ . (Since the bids are always conveyed to the referee with  $I$  as the origin, the referee and the other participants learn neither

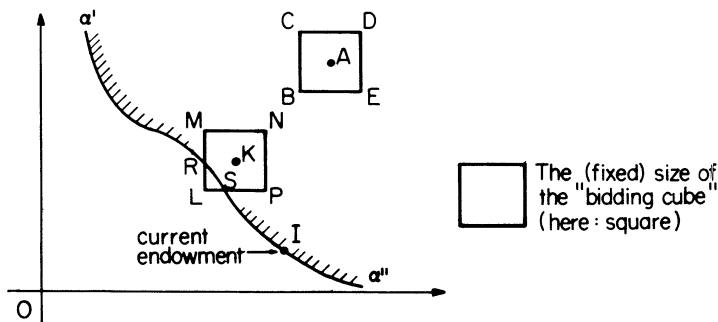


FIGURE 1.1

the bidder's current endowment nor the location of his indifference curve relative to the commodity space origin  $O$ ; this is a feature of informational decentralization.)<sup>16</sup>

The referee's role is illustrated in Figures 1.2a and 1.2b. In Figure 1.2a the two participants' bidding cubes are completely above their current utility levels. If the respective central bids are  $A_1$  and  $A_2$ , the compatible bid combinations constitute the rectangle  $KLMN$  (the shaded area) and the referee will select at random some point of this rectangle.

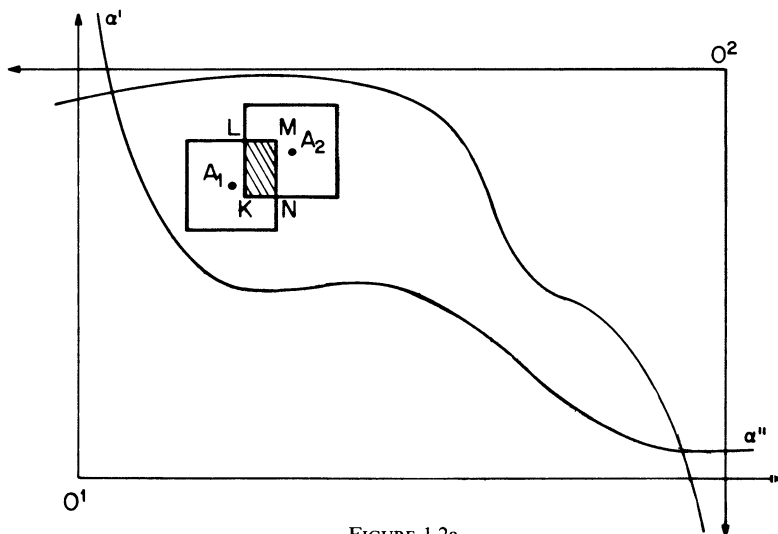


FIGURE 1.2a

<sup>16</sup> This admittedly cumbersome form of bidding has the merit of producing positive probabilities of compatibility essential for driving the process toward optimality in the divisible case.

Among the various alternatives we considered was a variant of the process in which the whole bidding cube would always become the bid, even if a part of it were inferior to the current endowment; thus, in Figure 1.1, if the central bid were  $K$ , the complete bid would be the square  $LMNP$  (rather than the set bounded by  $RMNPSR$ ). This modification would result in an informational simplification, but the process would not necessarily result in raising the participant's utility at every stage. Thus one would not expect convergence to Pareto optimality, although some form of convergence to "approximate optimality" might be established.



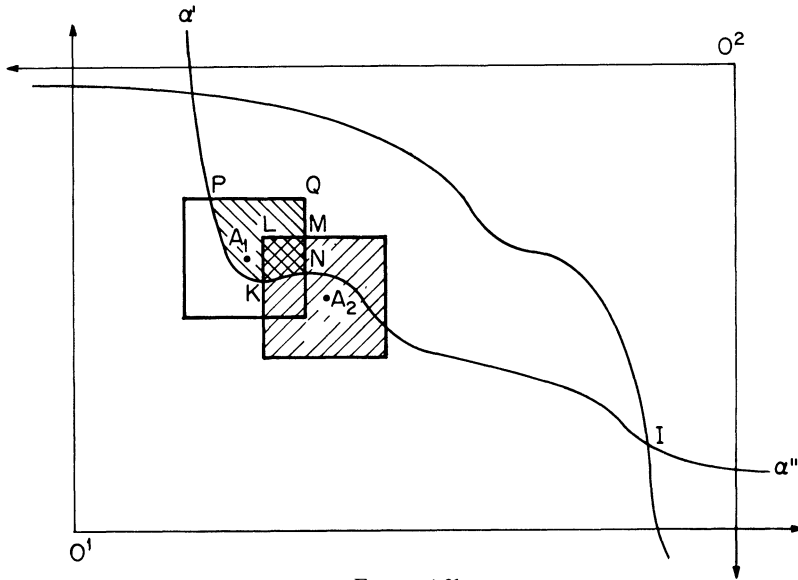


FIGURE 1.2b

In Figure 1.2b, the bid of the first participant is truncated to the set bounded by  $PQNK$  and the referee will select at random a point from the set bounded by  $KLMNK$  which constitutes the compatible bid combinations.

As in the indivisible case, no assumptions other than decomposability (absence of externalities) are needed to guarantee the stationarity of an optimum in the divisible case; Theorem 3.1 applies here as well. But to assure the optimality of an equilibrium (Theorem 5.1) and the convergence<sup>17</sup> toward an optimum (Theorem 5.2) when goods are divisible, additional restrictions must be imposed on the environment. These are topological in nature.

First, preferences are assumed representable by continuous utility functions. Second, the individually feasible trade sets are assumed closed, and such that the jointly feasible set of the economy as a whole is bounded.<sup>18</sup> Third, there is an assumption ("openness," labeled Assumption ED.6) postulating the existence of certain open sets in the space of allocations Pareto superior to a given one. The assumption of openness will be satisfied, under certain conditions, for utility functions that are strictly increasing in the interior of the consumption set, as, for instance, for the Cobb-Douglas utility functions, but also for many functions lacking the usual convexity properties. Finally, there is an assumption guaranteeing the possibility of small displacements from the initial position that would leave all participants in the respective interiors (rather than at the boundaries) of their individually feasible sets. (This requirement is satisfied in the pure exchange case

<sup>17</sup> Again with probability one, but (for non-optimal initial resource allocations) not in finite time.

<sup>18</sup> This will be the case for pure exchange when the nonnegative orthant constitutes the consumption set. Sufficient conditions, free of convexity, when production is present are given in Example 3, Section 5.4.3.

if the consumption set consists of all nonnegative commodity points and there is a positive aggregate initial endowment of each good. The latter requirement can be weakened, but (as shown by the counterexample of Section 5.5.6) not dispensed with, so as to permit the absence of some goods, if, e.g., consumer preferences are assumed monotone (see Lemma 5.6\*).

Examples given in Sections 5.4, 5.7, and 5.8 illustrate the range of applicability of the results, including cases with nonconvex and disconnected contour sets.

### 1.1.3. Comparison with Other Processes

Thus the  $B$  process is Pareto satisfactory and stochastically convergent to an optimum in most cases in which the perfectly competitive process is Pareto satisfactory and stable (see Section 5.4), but it also retains these properties in cases (continuous, non-convex, or indivisible, or the “Arrow case”) where the competitive equilibrium might fail to exist at an optimum or where the competitive equilibrium (in a tâtonnement type process) may lack stability.<sup>19</sup>

The convergence property of the  $B$  process, due to its utility-monotone character, points up its close relationship to nontâtonnement processes. In the case of pure exchange, the  $B$  process may be considered to be of the nontâtonnement type, since trades take place while the process is going on. However, unlike in the more usual nontâtonnement processes, in the  $B$  process the exchanges only occur when demand and supply are in balance. On the other hand, the exchanges may also be interpreted as “virtual” rather than real, thus placing the  $B$  process in the tâtonnement category.<sup>20</sup>

The pure exchange version of the  $B$  process, aside from its stochastic nature, does conform to the requirements of informational decentralization as defined in [8]. As compared with the “greed process” in [8], the  $B$  process, while applicable to a narrower class of environment, shows two points of superiority: first, the  $B$  process is not merely *statically* Pareto satisfactory, but (unlike the “greed process”) it *converges* to an optimum; second, it is informationally better, since its messages (bids) can be formed by looking in detail at only a small portion of the participant’s preferences map (it is informationally “localized”) while the “greed process” requires a “global” look at the preferences at every stage.<sup>21</sup>

Still, the construction of the bid in the divisible commodity case can be quite complex if the “central bid” falls near points inferior to the current endowment, thus requiring a detailed inspection of the local preference pattern.<sup>22</sup> It is an open

<sup>19</sup> For example, in cases involving gross complementarity with more than two goods.

<sup>20</sup> It is this latter interpretation that will be appropriate for the  $B$  process when production is introduced, since production activities are treated as virtual rather than real. A nontâtonnement variant of the  $B$  process including production would merit analysis.

<sup>21</sup> In fact, the messages of the “greed process” may be regarded as corresponding to a limiting case of bids of the  $B$  process when the radius of the “bidding neighborhood” becomes infinitely large. But it is the finiteness of the radius that makes the  $B$  process convergent, while the “greed process,” even in the two-person case, is characterized by oscillations of constant amplitude.

<sup>22</sup> As in Figure 1.1, when the central bid drawn is  $K$ , the participant must determine the shape of the curve segment  $RS$  and then convey it (using the current endowment point  $I$  as origin) to the referee. Since convexity of preferences is not assumed, this shape might be very complex even when preferences are continuous and monotone.

question to what extent one could hope to simplify the structure of bids in the divisible case and still retain the optimality and convergence properties for non-convex environments.<sup>23</sup>

We have remarked earlier that an abstract resource allocation process may be viewed as a specification of a set of economic institutions which realize the process. Such institutions would not, in general, be uniquely specified by the abstract process. It is interesting to consider possible institutional arrangements that realize the  $B$  process, especially with a view to the informational demands of the process.

We may imagine that each participant is equipped with a random point generator which is capable of selecting points of his admissible trade set according to a prescribed probability density. The participant makes the binary comparison between the random point and the current agreement, deciding thereby whether or not he prefers the new point at least as well as the current agreement. If so, it becomes the center of his bid and this may also require him to compute his contour set locally. If not, he rejects that point and considers another presented by his random mechanism.

According to this realization of the  $B$  process, each participant makes only binary preference comparisons, possibly also computing a small section of his contour set in the bidding neighborhood, and transmits only that information.

The referee is required to test whether bids add up to zero and to select one bid from such a set of bids. This is similar to tests of market clearing.

### 1.2. *Models with Production*

So far we have been discussing the case of pure exchange. When production is present, the process must be so designed that the outcome is technologically feasible, both individually and in the aggregate. In the spirit of informational decentralization, each producer is only required to check the technological feasibility from his own viewpoint of the production plan underlying the exchange proposal it conveys to the referee, just as each consumer checks on the feasibility of his proposal. (The feasibility is checked by the producers or consumers from their individual points of view only, while compatibility of their proposals is checked by the referee.)

From the formal point of view, a producer can be considered as a special kind of consumer characterized by indifference among all trades. A producer's set of feasible trades would be a "translate" of his production set (i.e., the set of sums of feasible input-output vectors plus the initial endowment), while that of a conventional consumer is the admissible consumption set (e.g., all nonnegative commodity vectors) translated by the initial endowment.

### 1.3. *Guide to the Remainder of the Paper*

General principles of notation and the definitions of equilibrium and optimality are found in Section 2, which also contains the formulation of rules governing the formation of bids and agreements in the  $B$  process.

<sup>23</sup> See footnote 16 above.

Section 3 contains a result on the stationarity of optimal distributions valid for arbitrary commodity spaces (hence, in particular, divisible, indivisible, and mixed commodity spaces).

Section 4 contains the main results for the case of indivisible goods, Section 5 for divisible goods.

In Section 5, devoted to divisible goods, the basic model and assumptions are formulated in Subsection 5.1. Important auxiliary propositions are established in 5.2 and the basic theorems on non-wastefulness and convergence are given in 5.3; an extension covering the case where some goods may be missing from the total initial endowment is provided in 5.5. Subsection 5.4 has examples showing how various assumptions (other than the “openness” assumption, ED.6) underlying the theorems of 5.3 can be satisfied both in classical and non-convex models. An analysis of an alternative form of the crucial “openness” assumption is found in Subsection 5.6, while 5.7 contains examples of applications, non-convex as well as convex, including one non-monotone, where commodities are divisible and utility functions continuous. Subsection 5.7.4 contains an example illustrating the existence of cases with non-convex preferences and increasing returns in production satisfying all assumptions (including “openness”) of the theorems in Subsection 5.3. Subsection 5.8 is a summary of the results in Section 5.

## 2. THE GENERAL NOTATION AND TERMINOLOGY

### 2.1. The $B$ Process

The  $B$  process involves  $N$  participants (persons, agents) who, at successive dates indexed as  $t = 0, 1, \dots$ , make bids and arrive at agreements. The *bid* (proposal) made by agent  $i$  at date  $t$  is a subset  $B_t^i$  of the set  $X^i$  of  $i$ 's conceivable actions.<sup>24</sup> (Elements of  $X^i$ , i.e., conceivable actions of agent  $i$  (whether or not they constitute agreements) are generically denoted by  $y^i$  or  $x^i$  or, sometimes,  $x$ .)

Given the array (ordered  $N$ -tuple),

$$B_t \equiv \langle B_t^1, \dots, B_t^N \rangle,$$

of bids made at date  $t$ , an agreement  $y_t$  for date  $t$  results.<sup>25</sup> This agreement is also an ordered  $N$ -tuple,

$$y_t \equiv \langle y_t^1, \dots, y_t^N \rangle,$$

<sup>24</sup> As will be seen below in the present section, the set  $X^i$  may include actions of agent  $i$  that are not feasible, either individually or jointly. However, the rules of the process formulated in Section 2.3 will require that the bid  $B_t^i$ , be a subset of the set  $Y^i$  (to be introduced later in this section) of actions that are individually feasible,  $Y^i \subseteq X^i$ . It may be helpful to keep in mind an interpretation of this model, in which actions are trades among participants; here  $Y^i$  might be the subset of the commodity space that consists of trades leaving person  $i$ 's consumption within his subsistence set (e.g., a translate of the nonnegative orthant), while  $X^i$  might conveniently be taken to be the whole commodity space, so that  $B_t^i$  is a subset of the commodity space. In this interpretation,  $y_t^i$  (defined in this section) is a point of the commodity space.

<sup>25</sup> The agreement  $y_t$  for date  $t$  is either identical with that for the previous date  $t - 1$ , or selected on date  $t$ , and according to the rules of Section 2.4, from a subset of  $\bigcap_{i=1}^N B_t^i$ . Thus  $y_t$  is the agreement currently in force, but it may have been formulated at an earlier date and continued since then.

where  $y_t^i$  is the *action of agent  $i$  agreed for date  $t$* . This action is an element of set  $X^i$  of  $i$ 's *conceivable actions*. Thus the agreement  $y_t$  belongs to the Cartesian product

$$X \equiv \prod_{i=1}^N X^i,$$

the set of *conceivable joint actions*. Elements of  $X$ , i.e., conceivable joint actions (whether or not they constitute agreements) are generically denoted by  $y = \langle y^1, \dots, y^N \rangle$ , or sometimes by  $x = \langle x^1, \dots, x^N \rangle$ .

A sample realization of the process is a sequence  $(y_0, B_0), (y_1, B_1), \dots$ , written briefly as  $\{(y_t, B_t)\}_{t=0}^\infty$ . The ordered pair  $(y_t, B_t)$  constitutes the *state* of the process at date  $t$ .

Not all joint actions in  $X$  are feasible. There are both "individual" and "global" feasibility constraints. To each agent we shall associate a set of actions that are *individually feasible* for him. The set of actions individually feasible for agent  $i$  will be denoted by  $Y^i$ , a subset of  $X^i$ . We shall write

$$Y = \prod_{i=1}^N Y^i.$$

We shall further suppose that there is some set  $Y_C \subseteq X$  (not necessarily a subset of  $Y$ ) of joint actions representing those  $N$ -tuples  $\langle y^1, \dots, y^N \rangle$  from  $X$  that are *compatible*, i.e., satisfy certain conditions of global feasibility. The set of *feasible joint actions*, to be denoted by  $Y_F$ , is the set of  $N$ -tuples  $\langle y^1, \dots, y^N \rangle$ , such that the  $y^i$  are both individually feasible and compatible, i.e.,

$$Y_F \equiv \left( \prod_{i=1}^N Y^i \right) \cap Y_C = Y \cap Y_C.$$

In Subsection 2.7, we shall provide a specific interpretation of the sets  $X^i$ ,  $Y^i$ , and  $Y_C$  in terms of trade, consumption, and production.

### 2.2. Preferences

Agent  $i$  has a preference preordering  $\succsim_i$  on  $Y^i$ . For each  $y^i \in Y^i$ , the *contour set*  $G^i(y^i)$  is the set  $\{y^i : y^i \succsim_i \bar{y}^i, \bar{y}^i \in Y^i\}$  of actions in  $Y^i$  that are at least as good as  $y^i$  for agent  $i$ . For  $y$  in  $Y$ ,  $G(y) \equiv Y_F \cap \prod_{i=1}^N G^i(y^i)$ .<sup>26</sup>

### 2.3. Bidding

Roughly speaking, bids are generated as follows.

For every  $i$  there is given (and left fixed throughout the process) a probability distribution<sup>27</sup>  $P^i$  on  $Y^i$ . For every  $y^i$  in  $Y^i$  define  $P^i(\cdot | y^i)$  to be the conditional probability distribution induced by  $P^i$  on  $G^i(y^i)$ ,<sup>28</sup> and let

$$P(\cdot | y) \equiv [P^1(\cdot | y^1), \dots, P^N(\cdot | y^N)].$$

<sup>26</sup> Usually, preferences are postulated on outcomes of actions rather than the actions themselves. However, we shall be primarily interested in cases in which preferences with regard to outcomes can be used to define preferences with regard to actions.

<sup>27</sup>  $P^i$  is called the "initial" distribution.

<sup>28</sup> This means that  $P^i(\cdot | y^i)$  gives zero probability to subsets of  $Y^i$  that are outside  $G^i(y^i)$ , probability one to the set  $G^i(y^i)$ , and that for the subsets of  $G^i(y^i)$  the ratio of probabilities according to  $P^i(\cdot | y^i)$  is the same as according to  $P^i$ .

Initially, i.e., for  $t = 0$ , we set  $y_0 = \{0_x\}$ . Given that  $y_t \equiv \langle y_t^1, \dots, y_t^N \rangle$ ,  $t \geq 0$ , is the current (most recent) agreement,  $z_t^1, \dots, z_t^N$  are independent "random points" in  $Y^1, \dots, Y^N$ , respectively, with  $z_t^i$  selected according to the distribution  $P^i(\cdot | y_t^i)$ .

Now for every  $z^i$  in  $Y^i$ , there is given a set  $\zeta^i(z^i) \subseteq Y^i$ , the *bidding neighborhood* of  $z^i$ . Then  $i$ 's bid is specified as

$$B_t^i = \zeta^i(z_t^i) \cap G^i(y_t^i).$$

In other words, at date  $t \geq 0$ , agent  $i$  chooses, according to a distribution  $P^i(\cdot | y_t^i)$ , an action  $z^i$  from among those actions that are at least as good as  $y_t^i$ , and makes as his bid all those actions in the neighborhood  $\zeta^i(z_t^i)$  of  $z_t^i$  that are for him at least as good as  $y_t^i$ .

In the discrete case,  $\zeta^i(z^i)$  will consist merely of the single point  $z^i$ . In the divisible case, illustrated by Figure 2.1,  $\zeta^i(z^i)$  will be the cube of fixed "radius" centered at  $z^i$ , intersected with  $Y^i$ , the admissible set.<sup>29</sup>

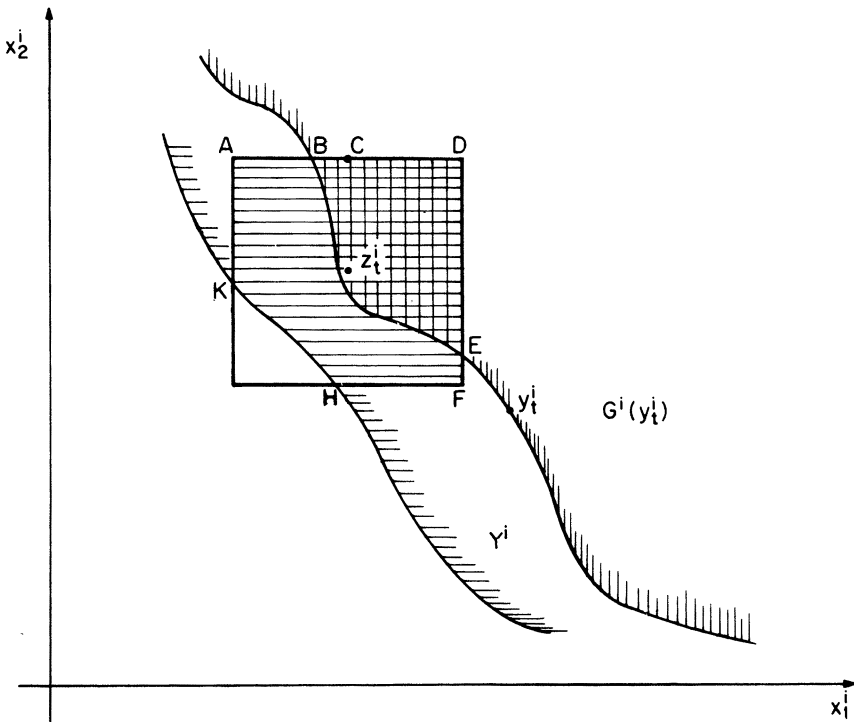


FIGURE 2.1.  $\zeta^i(z_t^i) = ADFHK$  (shaded  $\equiv$ ); the "radius" is the length of  $AC$ ;  $B_t^i = BDE$  (shaded  $\#$ ).

<sup>29</sup> The reasons for using such bidding neighborhoods in the divisible case are discussed in Section 1.1.2 above.

2.4. Agreement

An array  $B \equiv \langle B^1, \dots, B^N \rangle$  of bids is *feasible* if  $B^* \equiv (\bigcap_{i=1}^N B^i) \cap Y_C$  is not empty. If  $B_i \equiv \langle B_i^1, \dots, B_i^N \rangle$  is *not* feasible, then  $y_{i+1} = y_i$ . If  $B_i$  is feasible, then  $y_{i+1}$  is a joint action chosen “at random” from  $B_i^* \equiv (\bigcap_{i=1}^N B_i^i) \cap Y_C$ . (One may imagine that an “umpire” checks the feasibility of  $B_i$ , and if  $B_i^*$  is not empty, chooses  $y_{i+1}$  from  $B_i^*$  according to the uniform probability distribution.)

2.5. Optimum

As usual, a feasible  $N$ -tuple  $\hat{y}$  (in  $Y_F$ ) is *Pareto optimal* (or simply *optimal*) if for all  $y$  in  $Y_F$ ,

$$\hat{y}^i \succsim_i y^i \quad (i = 1, \dots, N),$$

implies

$$\hat{y}^i \succsim_i y^i \quad (i = 1, \dots, N)$$

(where  $\succsim_i$  denotes indifference with respect to the preordering  $\succsim_i$ ).<sup>30</sup>

A bidding distribution  $P = P(\cdot | y)$  is *optimal* if  $y$  is optimal.

Suppose all individuals’ preferences are representable by real-valued utility functions, and denote the  $i$ th utility function by  $U^i$ . An  $N$ -tuple  $u \equiv \langle u^1, \dots, u^N \rangle$  of the process through time  $t$ ,  $P_t = \bar{P}$  implies that with probability one,  $y^N \rangle$ ,  $u^i \equiv U^i(y^i)$ ,  $i = 1, \dots, N$ , i.e., if  $u$  is the image of some optimal joint action  $y$ .

2.6. Equilibrium and Stability

Define the sequence of bidding (probability) distributions  $\{P_t\}_{t=0}^\infty$  associated with the process  $\{(y_t, B_t)\}_{t=0}^\infty$ , by

$$P_t \equiv [P^1(\cdot | y_t^1), \dots, P^N(\cdot | y_t^N)] = P(\cdot | y_t).$$

A bidding distribution  $\bar{P}$  is an *equilibrium bidding distribution* if, given the history of the process through time  $t$ ,  $P_t = \bar{P}$  implies that with probability one,

$$P_s = \bar{P} \quad \text{for all } s > t.$$

To define *stability* we note that corresponding to the stochastic process  $\{y_t\}_{t=0}^\infty$  of agreements, there is also a stochastic process  $\{P_t\}_{t=0}^\infty$  of bidding distributions, and when preferences are representable, a stochastic process  $\{u_t\}_{t=0}^\infty$  of utility  $N$ -tuples.

A process could be defined as stable if the sequence  $\{y_t\}_{t=0}^\infty$  converged almost surely to some point of  $Y_F$ . But such a definition would be too strong for our purposes and we define stability in terms of almost sure convergence of  $\{P_t\}_{t=0}^\infty$

<sup>30</sup> If  $\hat{y}^i \succsim_i \tilde{y}^i$  for all  $i = 1, \dots, N$ , we say that  $\hat{y} \equiv \langle \hat{y}^1, \dots, \hat{y}^N \rangle$  is noninferior to (at least as good as)  $\tilde{y} \equiv \langle \tilde{y}^1, \dots, \tilde{y}^N \rangle$  in Pareto’s sense, and write  $\hat{y} \succsim \tilde{y}$ . We write  $\hat{y} \sim \tilde{y}$  (Pareto equivalence) if  $\hat{y} \succsim \tilde{y}$  and  $\tilde{y} \succsim \hat{y}$ ; we write  $\hat{y} \prec \tilde{y}$  ( $\hat{y}$  is Pareto superior to  $\tilde{y}$ ) if  $\hat{y} \succ \tilde{y}$  but not  $\tilde{y} \succ \hat{y}$ .

as topologized in terms of the corresponding  $u_t$ . Thus, when preferences are representable, and hence the sequence  $\{u_t\}_{t=0}^\infty$  is defined, the  $B$  process is said to be stable if  $\bar{u} \equiv \lim_{t \rightarrow \infty} u_t$  exists almost surely (with probability one). This limit is a random (vector-valued) variable.

In the *discrete* case ( $Y^i$  denumerable and  $Y_F$  finite), the limit is almost surely attained in a *finite time*; i.e., with probability one, there is some  $t' < \infty$  such that  $u_t = u_{t'}$  for all  $t > t'$ . Of course, here  $\bar{u} = u_{t'}$ . (In the discrete case one may avoid any reference to the utilities and define *finite-time stability* of the  $B$  process as the almost sure existence of  $t' < \infty$  such that  $P_t = P_{t'}$  for all  $t > t'$ . This finite-time stability is a stronger property than ordinary stability as defined above.)

A stable process is said to be *optimally stable* if the  $\bar{u}$  is almost surely optimal. (In the discrete case, the process is optimally (as well as finite-time) stable and the distribution  $P_{t'}$  is optimal.)

### 2.7. Trade, Consumption, and Production

We now present a particular specification of our model that relates it to the usual models of resource allocation. Suppose that there are  $M$  commodities, each commodity being either "indivisible" or "divisible." An "indivisible" commodity can be traded, consumed, and produced only in integer-valued amounts; the corresponding quantities for "divisible" commodities may be any real numbers. Thus a point in the *commodity space*,  $\mathcal{X}$ , is an ordered  $M$ -tuple  $(x_1, \dots, x_M)$ , where  $x_j$  represents the quantity of commodity  $j$ , and is either an integer-valued variable or a real-valued variable. In either case, points in the commodity space can be added or subtracted in the obvious way:

$$(x_1, \dots, x_M) \pm (z_1, \dots, z_M) = (x_1 \pm z_1, \dots, x_M \pm z_M).$$

Using the notation of Subsection 2.1, let  $X^i \equiv \mathcal{X}$  for  $i = 1, \dots, N$ . A point  $x^i \equiv (x_1^i, \dots, x_M^i)$  in  $\mathcal{X}$  represents a *trade* for agent  $i$ ; a positive coordinate represents a net receipt by  $i$ , and a negative coordinate a net delivery.

In general, an agent may be a consumer and a producer, as well as a resource holder. Let  $\omega^i$ , a point in the commodity space, denote the *initial endowment* of agent  $i$ . A subset  $C^i$  of the commodity space is  $i$ 's *consumption set*.<sup>31</sup> A subset  $X_p^i$  of the commodity space is  $i$ 's *production possibility set*. Since the actions of particular interest are trades, we define the set of actions individually feasible for agent  $i$  as the set of *trades*

$$Y^i \equiv C^i - X_p^i - \{\omega^i\}.$$

Since usually preferences are postulated on  $C^i$ , it is desirable to show how a preference relation, say  $\succsim_i^*$ , defined on  $C^i$ , induces a preference relation  $\succsim_i$  on  $Y^i$ . This is accomplished by a formulation due to Trout Rader (see Vind [18, p. 47]): given two elements  $x', x''$  in  $Y^i$ , it is postulated that

$$x' \succsim_i x'',$$

<sup>31</sup> As is often done in resource allocation models, we shall, in certain special cases (Sections 5.5 and 5.7), sometimes take  $C^i$  to be the nonnegative orthant of the commodity space.



if and only if there exists  $\tilde{x} \in X_p^i$ , such that

$$x' + \tilde{x} + \omega^i \succsim_i^* x'' + x + \omega^i, \text{ for all } x \in X_p^i.$$

$\succsim_i$  is a transitive reflexive complete relation on  $Y^i$ , if  $\succsim_i^*$  has the corresponding properties on  $C^i$ .

As in conventional competitive equilibrium theory, we often find it convenient to rule out the “mixed” consumer-producer agent and to divide agents into “pure” consumers and “pure” producers. A *consumer* has

$$X_p^i = \{0_x\}, \text{ so that}$$

$$Y^i = C^i - \{\omega^i\},$$

and trade  $\bar{y}^i$  is preferred (or indifferent) to trade  $\bar{y}^i$  (in terms of  $\succsim_i$ ) if and only if  $\bar{y}^i = \bar{c}^i - \omega^i$ ,  $\bar{y}^i = \bar{c}^i - \omega^i$ , and consumption  $\bar{c}^i$  is preferred (or indifferent) to consumption  $\bar{c}^i$  (in the sense of  $\succsim_i^*$ ). A *producer* has  $C^i = \{0_x\}$ , so that  $Y^i = -X_p^i - \{\omega^i\}$ , and the preference relation  $\succsim_i$  is such that he is indifferent among all trades, i.e., such that

$$x' \sim x'' \text{ for all } x', x'' \in Y^i.$$

More particularly, a resource custodian may be represented as a producer for whom  $X_p^i$  is the set of all points in  $\mathcal{X}$  with nonpositive coordinates (see, e.g., Koopmans [11]).

Although a producer is, by definition, indifferent among all trades, we do not rule out the possibility that a consumer also exhibits such indifference. Such a consumer could still be distinguished from a producer in terms of the properties that are subsequently postulated for the set  $C^i$  of a consumer as against the set  $-X_p^i$  of a producer.

In order for an  $N$ -tuple  $\langle y^1, \dots, y^N \rangle$  of conceivable trades to be *compatible*, one must have

$$\sum_{i=1}^N y^i = 0_x,$$

where  $0_x$  is the origin  $(0, \dots, 0)$  of  $\mathcal{X}$ ; the set of such compatible trades will be  $Y_C$ . The set  $Y_F$  of *feasible* trades is the set of compatible  $N$ -tuples  $\langle y^1, \dots, y^N \rangle$ , such that  $y^i$  is in  $Y^i$  for each  $i$ .

The development of the theory in Sections 3 and 4 is without reference to this particular model of resource allocation, whereas Section 5 deals specifically with the resource allocation model of this section in the case in which all commodities are divisible. Section 4 deals with the general model, but with the added assumptions that the  $Y^i$  are denumerable and that  $Y_F$  is finite (the *discrete* case). This would cover, in particular, the resource allocation model in the case in which all commodities are indivisible and the set of attainable actions of the economy is bounded.

### 3. THE EQUILIBRIUM PROPERTY OF OPTIMA

#### 3.1. An Optimal Bidding Distribution Is an Equilibrium

Without further specification of the bidding process it is already clear that successive agreements  $y_t$  do not get worse from the point of view of any of the individuals, i.e.,  $y_t \preceq y_{t+1}$  for all  $t$ .<sup>32</sup> Hence if  $y_t$  is optimal, then  $y_s \sim y_t$  for all  $s \geq t$ . It follows that if  $y_t$  is optimal, then  $P(\cdot | y_s) = P(\cdot | y_t)$  for all  $s \geq t$ , since if  $x \sim y$  (i.e.,  $x^i \preceq y^i$  for all  $i$ ) then  $P(\cdot | x) = P(\cdot | y)$ . Thus we have proved (without any specializing assumptions concerning the environment of the bidding process):

**THEOREM 3.1:** *If a bidding (probability) distribution is optimal, then it is an equilibrium.*

#### 3.2. Other Propositions

As indicated in the Introduction, we shall prove two additional propositions under alternative sets of additional assumptions. These propositions are: (i) every equilibrium bidding distribution is optimal; (ii) the sequence of bidding distributions  $P_t$  converges, in a sense to be made precise, to an equilibrium. The basic tool in proving both of these propositions will be the fact that every non-optimal bidding distribution is sure to be improved upon eventually.

In the next sections we consider successively the discrete case (denumerable  $Y^i$  and finite  $Y_F$ , Section 4) and the case of divisible commodities (Euclidean commodity space, Section 5). In each case we first prove the "inevitability of improvement" starting from a non-optimal agreement, and then prove the other two propositions.

## 4. THE DISCRETE CASE: THE OPTIMALITY AND STABILITY OF EQUILIBRIA

### 4.1. Assumptions

Of the following four assumptions,<sup>33</sup> Assumptions EN.1 and EN.2 define the class of environments covered by the results of Section 4, while Assumptions PN.1 and PN.2 specify the nature of the bidding process of Section 4.

We make the following assumptions:

ASSUMPTION EN.1: *Each set  $Y^i$  is denumerable ( $i = 1, \dots, N$ ).*

ASSUMPTION EN.2:  *$Y_F$  is finite.*

Assumption EN.1 is, of course, satisfied in the case where the commodity space is finite-dimensional and all components of the commodity vector are integer-valued, i.e., where there are finitely many indivisible goods. In this case the probability distribution  $P^i$  can be described in terms of a frequency function  $p^i$ .

<sup>32</sup> This is the case because  $P^i(\cdot | y')$  gives zero probability to any subset of  $Y^i$  that is outside  $G^i(y')$ .

<sup>33</sup> In the symbols designating assumptions, E refers to environment, P to process, N to the discrete (non-divisible) case, and D to the divisible case.

ASSUMPTION PN.1: Each of the distributions  $P^i$  is strictly positive, i.e.,

$$(4.1) \quad p^i(y^i) > 0 \quad \text{for all } y^i \in Y^i \quad (i = 1, \dots, N).$$

ASSUMPTION PN.2: The bidding neighborhood  $\zeta^i(z^i)$  consists of the point  $z^i$  alone.

#### 4.2. Improvement

Consider any feasible joint action  $\bar{y}$ . Let  $I(\bar{y})$  be the set of feasible joint actions that are indifferent to  $\bar{y}$  for all individuals (i.e., Pareto equivalent), and  $H(\bar{y})$  be the set of all feasible joint actions that are as good as  $\bar{y}$  for all individuals and better for at least one (i.e., Pareto superior to  $\bar{y}$ ); i.e., for any  $\bar{y} \in Y_F$ , write

$$(4.2) \quad I(\bar{y}) \equiv \{y : y \in Y_F \cdot y \sim \bar{y}\},$$

and

$$(4.3) \quad H(\bar{y}) \equiv \{y : y \in Y_F \cdot y \succ \bar{y}\}.$$

Clearly, if  $y_t = \bar{y}$ , then any future action  $y_s (s > t)$  must be either in  $I(\bar{y})$  or in  $H(\bar{y})$ , since  $G(\bar{y}) \equiv I(\bar{y}) \cup H(\bar{y})$ .

By Assumption PN.1 and the definition of the bidding distributions (Sections 2.2 and 2.3), the bidding distributions  $P(\cdot | y)$  are strictly positive on  $G(\bar{y})$ , and are identical for all  $y \in I(\bar{y})$ . That is,

$$(4.4) \quad q(y|\bar{y}) \equiv \text{prob} \{y_{t+1} = y | y_t = \bar{y}\} > 0, \quad \text{for all } \bar{y} \in Y_F, y \in G(\bar{y}),$$

and

$$(4.5) \quad q(y|\bar{y}) \equiv q(y|\bar{y}') \quad \text{if } \bar{y}, \bar{y}' \in Y_F, \bar{y} \sim \bar{y}', \text{ and } y \in G(\bar{y}).$$

Now if  $\bar{y}$  is optimal, then  $H(\bar{y})$  is empty, and given that  $y_t = \bar{y}$ , it follows that  $y_s \in I(\bar{y})$  for all  $s \geq t$ , which is a restatement of Theorem 3.1. On the other hand, if  $\bar{y}$  is not optimal, then  $H(\bar{y})$  is not empty, and therefore

$$(4.6) \quad \sum_{y \in H(\bar{y})} q(y|\bar{y}) > 0.$$

Inequality (4.6) states that if the most recent joint action is not optimal, then there is a positive probability that the next action of the process will constitute improvement for at least one individual and not harm anyone.

#### 4.3. An Equilibrium Is an Optimum, and the Process Is Stable

From (4.6) it is immediate that if  $\bar{y}$  is not optimal, then  $P(\cdot | \bar{y})$  is not an equilibrium.

**THEOREM 4.1:** *Under Assumptions EN.1, EN.2, PN.1, and PN.2, if a bidding distribution is an equilibrium, then it is optimal.*

The sequence  $\{y_t\}_{t=0}^\infty$  of agreements is a Markov process, as is the sequence  $\{P_t\}_{t=0}^\infty$  of bidding distributions. Since the  $y_t$  sequence is monotone with respect to the preferences, i.e.,  $\text{prob}\{y_t \preceq y_s \text{ for all } t \leq s\} = 1$ , it follows by (4.6) that if  $y \in Y_F$  is not optimal, then  $y$  is a transient state of the  $y_t$  process, and correspondingly  $P(\cdot | y)$  is a transient state of the  $P_t$  process. On the other hand, if  $y$  is optimal, then  $I(y)$  is an absorbing set of states. Since  $Y_F$  is finite, the  $y_t$  process must eventually enter some indifference set  $I(y)$  for which  $y$  is optimal, and correspondingly  $P_t$  must eventually reach an optimal distribution, and stay there. We summarize this by:

**THEOREM 4.2:** *Under Assumptions EN.1, EN.2, PN.1, and PN.2, with probability one, there is some  $t < \infty$  such that (i)  $P_t$  is optimal; and (ii)  $P_s = P_t$  for all  $s \geq t$ .*

*Remark:* We have shown that in the *discrete* case the  $B$  process is *finite-time* stable, and in fact, *optimally* so. The finite-time feature is due to discreteness and will not appear when goods are assumed to be divisible.

## 5. THE CASE OF DIVISIBLE COMMODITIES: THE OPTIMALITY AND STABILITY OF EQUILIBRIA

### 5.1. Assumptions and Notation

#### 5.1.0. Introduction

In this section we present an analysis of the case of divisible commodities that is analogous to the discussion of the discrete case. However, as indicated in the Introduction, the argument is considerably more complicated, and the equilibrium is not attained in finite time. Furthermore, several additional assumptions are made to obtain results corresponding to those of Section 4.

The basic results for the divisible case are stated in Theorems 5.1 and 5.2 of Section 5.3. Section 5.2 contains the fundamental lemmas underlying these theorems. The new assumptions are given in 5.1.1; those pertaining to the process are Assumptions PD.1 and PD.2, while assumptions labelled ED.1 through ED.6 pertain to the environment. (Recall that P refers to the "process," E to the "environment," and D to the "divisible" case.)

The results are generalized in Section 5.5; illustrative examples are provided in Sections 5.4 and 5.7, and a counterexample in Section 5.5.6. Sections 5.6 and 5.7 provide more readily verifiable counterparts of one of the assumptions (Assumption ED.6). A more detailed summary of Section 5 is found in Section 5.8.

The development in this section takes explicit account of trade, consumption, and production, using the model of Section 2.7, with the *commodity space*  $\mathcal{X}$  taken to be the  $M$ -dimensional Euclidean space  $R^M$ . The reader is referred to Sections 2.1 and 2.7 for the general notation and terminology.

To define *distance* in the commodity space we find it convenient to use the norm

$$\|w\| = \max \{|w_j| : j = 1, \dots, M\}, \quad w \equiv (w_1, \dots, w_M) \in R^M,$$

sometimes also written as  $|w|$ .

### 5.1.1. Assumptions about the Environment and the Bidding Process

We now state the assumptions to be made in our analysis of the case of divisible commodities. (The reader is again referred to Section 2 for general notation and terminology.)

ASSUMPTION ED.1:  $Y^i$  is a closed subset of  $X \equiv R^M$ , for  $i = 1, 2, \dots, N$ .

ASSUMPTION ED.2: As in Section 2.1, the feasible set is given by

$$Y_F \equiv \left( \bigtimes_{i=1}^N Y^i \right) \cap Y_C = Y \cap Y_C,$$

with the set  $Y_C$  of (jointly) compatible bids given, as in Section 2.7, by

$$(5.1) \quad Y_C \equiv \{y : y = \langle y^1, \dots, y^N \rangle \cdot y \in Y, \sum_{i=1}^N y^i = 0_x\}.$$

ASSUMPTION ED.3:  $Y_F$  is bounded.

ASSUMPTION ED.4: There is a  $y \equiv \langle y^1, \dots, y^N \rangle$  in  $Y_C$  such that,<sup>34</sup> for each  $i = 1, \dots, N$ ,  $y^i \in \text{Int}(Y^i)$ .

(Note that Assumption ED.4 implies  $Y_F \neq \emptyset$  where  $\emptyset$  denotes the empty set.)

Examples in which the preceding assumptions are satisfied are given in Section 5.4 below.

We make the further assumptions:

ASSUMPTION ED.5: For each  $i = 1, \dots, N$ , the preference ordering  $\succsim_i$  is representable by a continuous real-valued function  $U^i$  on  $Y^i$ .<sup>35</sup>

ASSUMPTION PD.1: For each  $i = 1, \dots, N$ , the distribution  $P^i$  is (i) absolutely continuous (with respect to Lebesgue measure), and has a corresponding density function  $p^i$  that is (ii) continuous and (iii) strictly positive on  $Y^i$ ;

ASSUMPTION PD.2: For each  $i = 1, \dots, N$  and each  $z^i \in Y^i$ , the "bidding neighborhood"  $\zeta^i(z^i)$  is given by

$$\zeta^i(z^i) \equiv \{x^i : x^i \in Y^i, \|x^i - z^i\| \leq \delta\},$$

<sup>34</sup> For a set  $A$ , the interior of  $A$  is denoted by  $\text{Int}(A)$ .

<sup>35</sup> Decomposability (absence of externalities) is implied by the requirement in Assumption ED.2 that  $Y_F = (\bigtimes_{i=1}^N Y^i) \cap Y_C$ , and the fact that the  $U^i$  are defined on  $Y^i$  by Assumption ED.5.

where the “radius”  $\delta$  is a positive number, fixed throughout the process, and independent of  $i$  and of  $z^i$ .

Our final assumption (Assumption ED.6 below, called “openness”) again pertains to the environment. Before stating it we introduce some additional notation. Define

$$(5.2) \quad \hat{Y} \equiv \{y : y \in Y_F, y \text{ is Pareto optimal}\},$$

and, for  $\varepsilon > 0$ ,

$$(5.3) \quad \hat{Y}_\varepsilon \equiv \{y : y \in Y_F. \text{ There exists } z \in \hat{Y} \text{ such that, for all } i, U^i(z^i) - U^i(y^i) < \varepsilon\},$$

$$(5.4) \quad \bar{Y}_\varepsilon \equiv \{y : y \in Y_F, y \notin \hat{Y}_\varepsilon\} = Y_F \setminus \hat{Y}_\varepsilon,$$

where  $\setminus$  denotes the set-theoretic difference. Thus  $\hat{Y}$  is the set of optimal allocations; for any positive number  $\varepsilon$ ,  $\hat{Y}_\varepsilon$  is the set of all allocations that fall short (in terms of everyone’s utility) of some optimal allocation by less than  $\varepsilon$ . If  $y$  is in  $\bar{Y}_\varepsilon$ , then for every optimal allocation  $z$  there is some individual  $i$  for whom  $U^i(y^i) \leq U^i(z^i) - \varepsilon$ . Note that  $Y_F$  is a closed bounded set in  $Y$ , and therefore, by the continuity of preferences,

$$(5.5) \quad \begin{cases} \text{(i) } \hat{Y} \text{ is closed in } Y_F, \text{ and hence compact in } Y; \\ \text{(ii) } \hat{Y}_\varepsilon \text{ is open in } Y_F \text{ for each } \varepsilon > 0; \\ \text{(iii) } \bar{Y}_\varepsilon \text{ is closed in } Y_F, \text{ and hence compact in } Y, \text{ for each } \varepsilon > 0. \end{cases}$$

The utility space images of the sets  $Y_F$ ,  $\hat{Y}$ ,  $\hat{Y}_\varepsilon$ , and  $\bar{Y}_\varepsilon$  are illustrated in Figure 5.1. In the figure,  $\bar{U}(Y_F)$  is the utility space image of  $Y_F$ , etc.<sup>36</sup>

Let  $K$  be the set of those agents  $i$  for whom  $U^i$  is constant on  $Y^i$ , i.e., for whom  $x \sim_i y$  for all  $x$  and  $y$  in  $Y^i$ . ( $K$  includes or equals the set of “producers.”) For every  $i = 1, \dots, N$ , every  $y^i$  in  $Y^i$ , and every  $\bar{y}$  in  $Y_F$ , define

$$(5.6) \quad \begin{cases} G^{+i}(y^i) \equiv \begin{cases} \{x : x \in Y^i, U^i(x) > U^i(y^i)\}, & \text{for } i \notin K; \\ Y^i, & \text{for } i \in K; \end{cases} \\ \text{and} \\ G^+(\bar{y}) \equiv Y_F \cap \left( \bigcap_{i=1}^N G^{+i}(\bar{y}^i) \right). \end{cases}$$

Note that  $G^+(y)$  is open in  $Y_F$  for all  $y \in Y_F$ .

For any subset  $A$  of a Euclidean space, by an  $A$ -open cube in  $A$  we mean a set  $S$  of the form

$$(5.7) \quad S = \{w : w \in A, \|w - c\| < \rho\},$$

<sup>36</sup> For typographical reasons, the symbol  $\varepsilon$  is reproduced in this and subsequent figures as  $\epsilon$ .

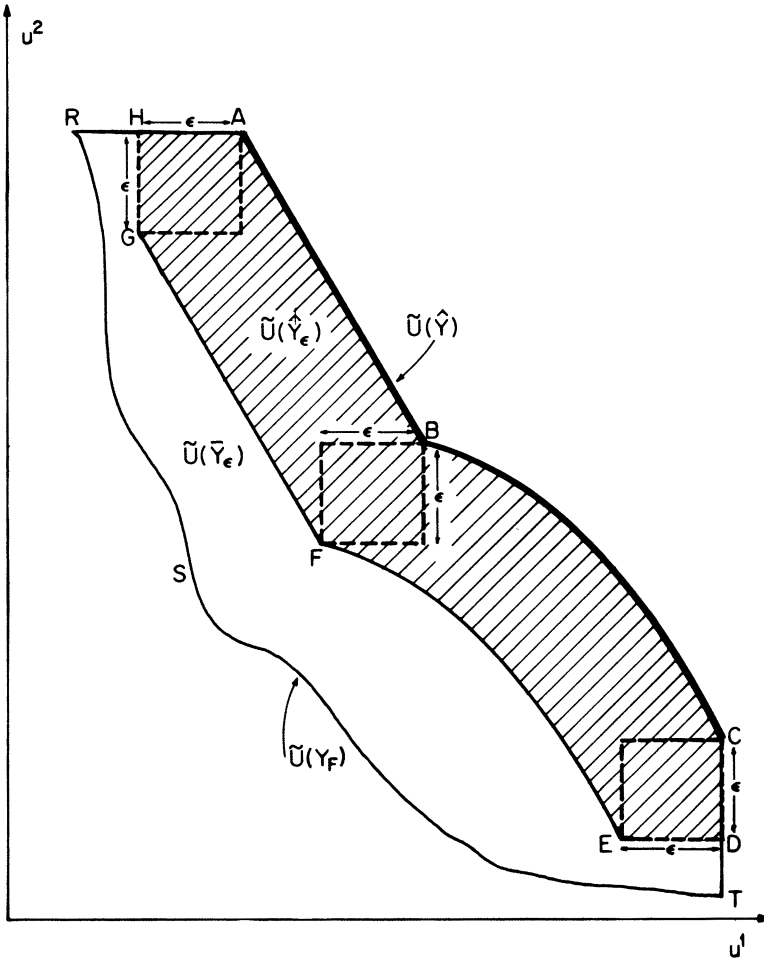


FIGURE 5.1

For a set  $W$ ,  $\tilde{U}(W)$  is the utility space image of  $W$ .  $\tilde{U}(Y_F)$ : the (closed) area bounded by the curve  $ABCDTSRHA$ .  $\tilde{U}(\bar{Y}_\epsilon)$ : the (closed) area bounded by  $RHGFEDTSR$ .  $\tilde{U}(\hat{Y})$ : the curve  $ABC$  (the thick curve).  $\tilde{U}(\hat{Y}_\epsilon)$ : the shaded area bounded by  $ABCDEFHGA$  including the perimeter, except for the points  $H$  and  $E$  and the curve  $HGFE$  joining them.

for some  $c \in A$  and some (finite) positive number  $\rho$ . The point  $c$  is called the *center* of the cube, and the number  $\rho$  its *radius*. In particular,  $A$  may be the whole Euclidean space. Corresponding to (5.7) we have the definition of an *A-closed cube* in  $A$ , for which the strict inequality in (5.7) is replaced by the weak inequality,  $\|w - c\| \leq \rho$ . For example, if  $A$  is the Euclidean plane, the interior of the dotted square in Figure 5.2 constitutes an *A-open cube* with center at the origin  $0_A$ , and radius  $\rho$ .

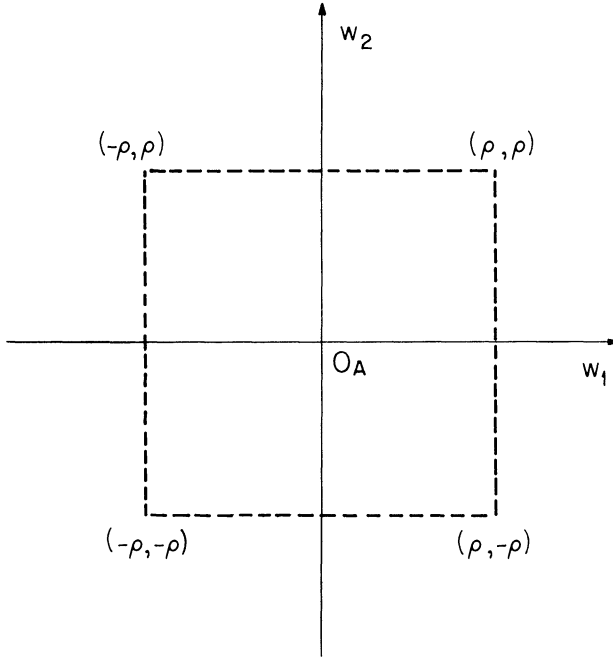


FIGURE 5.2

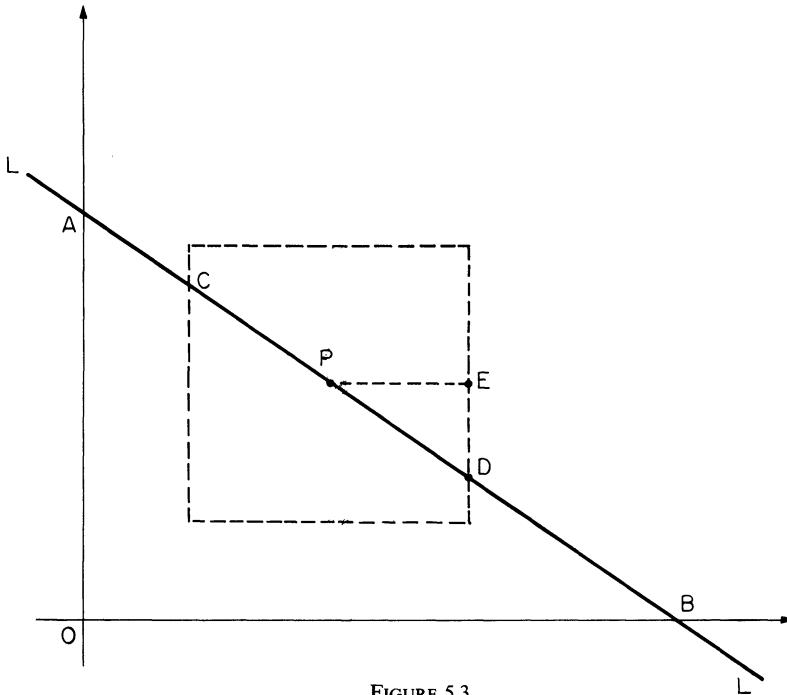


FIGURE 5.3



For any subset  $W$  of Euclidean space let  $\mathcal{L}(W)$  denote the smallest linear variety<sup>37</sup> containing  $W$ . We shall be dealing with  $\mathcal{L}(W)$ -open cubes. For example, in Figure 5.3, the Euclidean space is the plane,  $W$  is the line segment  $AB$ ,  $\mathcal{L}(W)$  is the (infinite) line  $LL$ , and the *open* segment  $CD$  is an  $\mathcal{L}(W)$ -open cube with center  $P$  and radius equal to the length of the segment  $PE$ .

We now state the final assumption of this section, which we shall call the assumption of *openness*.

ASSUMPTION ED.6: For every  $\varepsilon > 0$  and  $y \in \bar{Y}_\varepsilon$ , there exists an  $\mathcal{L}(Y_F)$ -open<sup>38</sup> cube  $S$  such that (i)  $S \subseteq G^+(y)$  and (ii)  $S \cap \hat{Y}_\varepsilon \neq \emptyset$ .

Assumption ED.6 (illustrated in Figure 5.4) will play a crucial role in guaranteeing the inevitability of improvement from a non-optimal allocation.

Properties related to the assumption of openness as well as examples of special cases are presented in Sections 5.6 and 5.7.

### 5.1.2. Bounds on the Content of a Cube in a Linear Variety

We conclude this section with two facts that will be useful in what follows. First, it is easy to show (using Assumption ED.4) that

$$(5.8) \quad \mathcal{L}(Y_F) = Y_C.$$

Second, we shall need bounds for the content of a cube in a linear variety. Let  $E$  be a Euclidean space of dimension  $D$ ; then the  $E$ -content of an  $E$  cube of radius  $\rho$  is  $(2\rho)^D$  (where  $E$ -content means length for  $D = 1$ , area for  $D = 2$ , volume for  $D = 3$ , Lebesgue measure in general). Let  $L$  be a linear variety of dimension  $d$  in  $E$ , and let  $S$  be an  $L$  cube (open or closed) with radius  $\rho$ ; then the  $L$ -content  $\mu(S)$  of  $S$  (i.e., the content of  $S$  considered as a subset of  $L$ ) satisfies the inequalities

$$(5.9) \quad k_d(2\rho)^d \leq \mu(S) \leq (2\rho\sqrt{D})^d,$$

where  $k_d$  is a positive number depending only on  $d$ .

PROOF OF (5.9): Without loss of generality we may center the cube  $S$  at the origin. No two points in  $S$  have a greater *Euclidean distance* than the points ( $D$ -tuples)  $(-\rho, -\rho, \dots, -\rho)$  and  $(+\rho, +\rho, \dots, +\rho)$ . The Euclidean distance of these two points is

$$\sqrt{D \cdot [\rho - (-\rho)]^2} = \sqrt{D} \cdot (2\rho).$$

<sup>37</sup> A linear variety is a translate of a linear subspace. Let  $\bar{w}$  be any point in  $W$ ; then  $\mathcal{L}(W)$  may be characterized as the set of all points of the form

$$\bar{w} + \sum_h \alpha_h(w^h - \bar{w}),$$

where  $H \geq 1$ ;  $w^1, \dots, w^H$  are any points in  $W$ ; and  $\alpha_1, \dots, \alpha_H$  are any real numbers.

<sup>38</sup> We shall sometimes write  $L_F$  for  $\mathcal{L}(Y_F)$ .

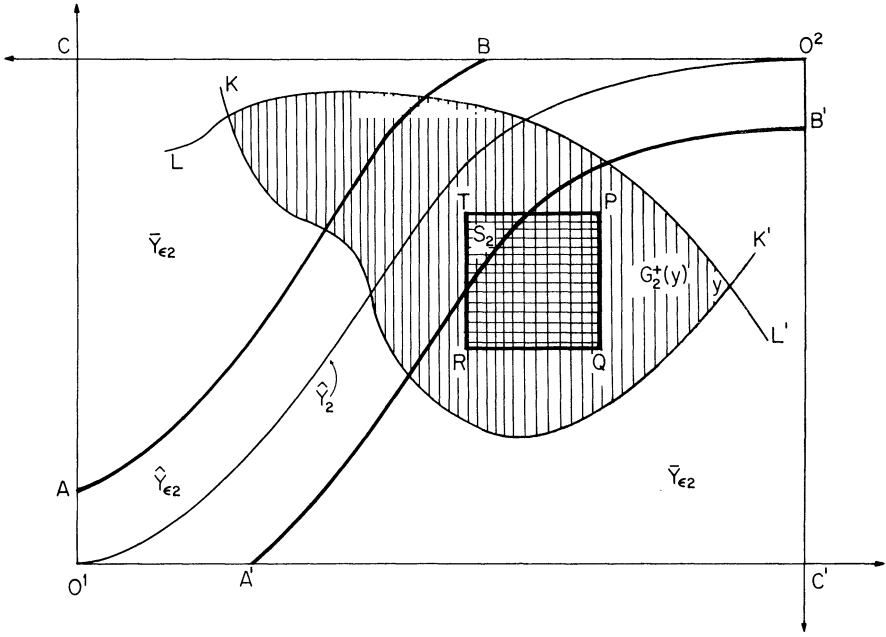


FIGURE 5.4

$M = 2, N = 2, Y^i = \Omega$ : conventional preference maps;  $\hat{Y}_2$ : curve  $0^1 0^2$ ;  $\hat{Y}_{\epsilon 2}$ : "strip" below  $AB$  but above  $A'B'$ ;  $\bar{Y}_{\epsilon 2}$ : closed "triangles"  $ABC$  and  $A'B'C'$ ;  $G_2^+(y)$ : the lens-shaped area (shaded |||) above 1's indifference curve  $KK'$  but below 2's indifference curve  $LL'$ ;  $S_2$ : the interior of the square  $TPQR$  (shaded #).

Note:  $\hat{Y}_2, \hat{Y}_{\epsilon 2}, \bar{Y}_{\epsilon 2}, G_2^+(y)$ , and  $S_2$  are 2-dimensional (in the "Edgeworth Box") representations of the corresponding 4-dimensional sets  $\hat{Y}, \hat{Y}_{\epsilon}, \bar{Y}_{\epsilon}$ , and  $G^+(y)$ .

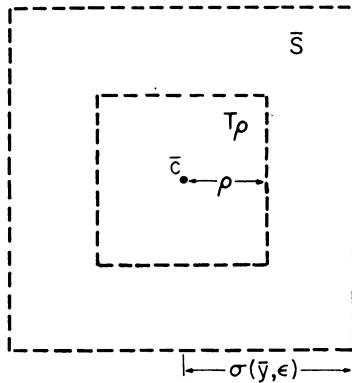


FIGURE 5.5

Hence

$$\mu(S) \leq [\sqrt{D}(2\rho)]^d.$$

On the other hand, any point  $w = (w_1, \dots, w_k) \in L$  such that

$$\sqrt{\sum_{k=1}^K w_k^2} < \rho,$$

is an element of  $S$ , i.e.,  $S$  is a superset of the open  $d$ -dimensional Euclidean sphere whose  $L$ -content is  $k_d(2\rho)^d$ , where  $k_d$  is a constant depending on the dimension  $d$  only (e.g.,  $k_1 = 1, k_2 = \pi/4$ , etc.). Hence,

$$\mu(S) \geq k_d(2\rho)^d. \tag{Q.E.D.}$$

### 5.2. Improvement

#### 5.2.0. Introduction

As in Section 4, the key step in the argument is the proposition that improvement from a non-optimal allocation is inevitable. In the divisible case this proposition takes the form of Lemma 5.6 in Section 5.2.2 below, and we lead up to it with a series of lemmas in Section 5.2.1.

#### 5.2.1. Preliminary Lemmas

LEMMA 5.1: *If  $y \in Y_F$  is not optimal, then there is a positive  $\varepsilon$  such that  $y \in \bar{Y}_\varepsilon$ .*

The proof of this lemma is left to the reader.  
From now on, to simplify notation, we write

$$(5.10) \quad L_F \equiv \mathcal{L}(Y_F).$$

For every  $\varepsilon > 0$  and  $y \in \bar{Y}_\varepsilon$  define  $\mathcal{S}(y, \varepsilon)$  to be the set of all  $L_F$ -open cubes  $S$  such that

$$S \subseteq G^+(y) \cap \hat{Y}_\varepsilon.$$

LEMMA 5.2: *For every  $\varepsilon > 0$  and  $y \in \bar{Y}_\varepsilon$ ,  $\mathcal{S}(y, \varepsilon)$  is not empty.*

PROOF: By Assumption ED.6 there exists an  $L_F$ -open cube  $S_1$  with center  $c \in L_F$  and radius  $\rho > 0$  such that

$$(5.11) \quad \begin{cases} \text{(i)} & S_1 \subseteq G^+(y) \quad \text{and} \\ \text{(ii)} & S_1 \cap \hat{Y}_\varepsilon \neq \emptyset. \end{cases}$$

Hence there exist  $z_1$  and  $z_2$  such that

$$(5.12) \quad \begin{cases} \text{(i)} & \|z_1 - c\| < \rho, \quad z_1 \in L_F; \\ \text{(ii)} & z_2 \in \hat{Y}; \quad \text{and} \\ \text{(iii)} & U^i(z_1^i) > U^i(z_2^i) - \varepsilon, \quad i = 1, \dots, N. \end{cases}$$

It is easy to show that, for some sufficiently small positive number  $\gamma$ , the  $L_F$ -open cube with center  $z_1$  and radius  $\gamma$  belongs to  $\mathcal{S}(y, \varepsilon)$ .

LEMMA 5.3: For any  $\varepsilon > 0$  and  $y \in \bar{Y}_\varepsilon$ , there is a cube  $S$  in  $\mathcal{S}(y, \varepsilon)$  of maximum radius.

PROOF: For any  $\varepsilon > 0$  and  $y$  in  $\bar{Y}_\varepsilon$ , consider the set of pairs  $(c, \rho)$  such that the  $L_F$ -open cube with center  $c$  and radius  $\rho$  is in  $\mathcal{S}(y, \varepsilon)$ . Using the compactness of  $Y_F$  (from Assumptions ED.1 and ED.3), it is straightforward to show that the set of such  $(c, \rho)$  pairs is compact, from which the statement of the lemma follows immediately.

We write:

$$(5.13) \quad \sigma(y, \varepsilon) \equiv \max \{ \rho : S \in \mathcal{S}(y, \varepsilon), \quad S \text{ has radius } \rho \}.$$

LEMMA 5.4: For every fixed  $\varepsilon > 0$ ,  $\sigma(\cdot, \varepsilon)$  is lower semicontinuous on  $\bar{Y}_\varepsilon$ .

PROOF: Suppose that  $\varepsilon > 0$  is fixed, and that

$$\bar{y} = \lim_{n \rightarrow \infty} y_n,$$

where  $y_n \in \bar{Y}_\varepsilon$ ,  $n = 1, 2, \dots$ . Since  $\bar{Y}_\varepsilon$  is closed,  $\bar{y} \in \bar{Y}_\varepsilon$  also. We wish to show that

$$(5.14) \quad \sigma(\bar{y}, \varepsilon) \leq \liminf_{n \rightarrow \infty} \sigma(y_n, \varepsilon)$$

(where  $\sigma(y, \varepsilon)$  is defined by (5.13)).

By Lemma 5.3, there exists an  $L_F$ -open cube  $\bar{S} \in \mathcal{S}(\bar{y}, \varepsilon)$  with the (maximal for  $\bar{y}$  and  $\varepsilon$ ) radius  $\bar{\rho} = \sigma(\bar{y}, \varepsilon)$ . Let  $\bar{c}$  be the center of  $\bar{S}$ . ( $\bar{c}$  depends on  $\bar{y}$  and  $\varepsilon$ , but the notation will not indicate this.)

For any number  $\rho$ ,  $0 < \rho < \sigma(\bar{y}, \varepsilon)$ , denote by  $T_\rho$  the  $L_F$ -open cube of radius  $\rho$  with center also at  $\bar{c}$ . (See Figure 5.5.)

Using the continuity of preferences (Assumption ED.5), it is easy to show that there exists a positive integer  $H_\rho$  such that

$$T_\rho \in \mathcal{S}(y_n, \varepsilon), \quad \text{for } n \geq H_\rho;$$

and so, because  $\sigma(y_n, \varepsilon)$  is defined as the maximum radius of cubes in  $\mathcal{S}(y_n, \varepsilon)$ ,

$$\sigma(y_n, \varepsilon) \geq \rho, \quad \text{for } n \geq H_\rho.$$

In summary, for every  $\rho$  such that  $0 < \rho < \bar{\rho}$ , there exists an integer  $H_\rho$  such that  $n \geq H_\rho$  implies  $\sigma(y_n, \varepsilon) \geq \rho$ . Inequality (5.14) follows immediately.

LEMMA 5.5: For any  $\varepsilon > 0$  there exists a  $\rho > 0$  such that for any  $y \in \bar{Y}_\varepsilon$  there is a cube in  $\mathcal{S}(y, \varepsilon)$  of radius  $\rho$ .

PROOF: Let  $\varepsilon > 0$  be fixed.  $\bar{Y}_\varepsilon$  is compact and, by Lemma 5.4,  $\sigma(\cdot, \varepsilon)$  is lower semicontinuous; hence,  $\sigma(\cdot, \varepsilon)$  attains a minimum in  $\bar{Y}_\varepsilon$ , say at  $\bar{y}$ . By Lemma 5.3,  $\rho \equiv \sigma(\bar{y}, \varepsilon) > 0$ . Again by Lemma 5.3, for any  $y \in \bar{Y}_\varepsilon$ , there is an  $S \in \mathcal{S}(y, \varepsilon)$  with

radius  $\sigma(y, \varepsilon)$  and center, say,  $c$ . But  $\sigma(y, \varepsilon) \geq \rho$ , so that the  $L_F$ -open cube  $S'$  with center  $c$  and radius  $\rho$  is also in  $\mathcal{S}(y, \varepsilon)$ , which completes the proof of the lemma.

### 5.2.2. Inevitability of Improvement from a Nonoptimal Allocation

LEMMA 5.6: For every  $\varepsilon > 0$  there is a  $\gamma > 0$  such that for all  $y, y \in Y_F$ ,

$$(5.15) \quad \text{prob} \{y_{t+1} \in \hat{Y}_\varepsilon | y_t = y\} \geq \gamma.$$

(Note that the number  $\gamma$  does not depend on  $y$ .)

PROOF: When  $N = 1$ ,  $Y_F = \{0_x\}$  is a one element set. Hence in this case  $Y = \hat{Y}_\varepsilon = Y_F$  and  $y_t \in \hat{Y}_\varepsilon$  for all  $t$ , so that the conclusion of Lemma 5.6 holds with  $\gamma = 1$ . Hence, without loss of generality, we shall henceforth assume that  $N > 1$ .

Let  $\varepsilon > 0$  be fixed. If  $y_t = y \in \hat{Y}_\varepsilon$ , then the probability in (5.15) equals unity, because the utility of successive agreements is non-decreasing. It remains to consider the case  $y \in \bar{Y}_\varepsilon$ .<sup>39</sup>

Lemma 5.5 guarantees the existence of a cube  $S^1$  in  $\mathcal{S}(y, \varepsilon)$  whose radius  $\rho > 0$  is independent of  $y$  (but dependent on  $\varepsilon$ ) and whose center shall be denoted by  $c(y, \varepsilon)$ . We now construct the  $L_F$ -open cube  $S$  with the same center  $c(y, \varepsilon)$  and the radius

$$(5.16) \quad \eta \equiv \min \left( \rho, \frac{\delta}{2} \right),$$

where  $\delta$  is the radius of the bidding neighborhood (see Assumption PD.2). It follows that  $S$  is also in  $\mathcal{S}(y, \varepsilon)$ , so that

$$(5.17) \quad S \subseteq G^+(y) \cap \hat{Y}_\varepsilon.$$

Let  $Y_F^i$  denote the projection of  $Y_F$  into  $Y^i$ . By (5.17),

$$(5.18) \quad z \in S \text{ implies } z^i \in G^+(y^i) \cap Y_F^i \quad (i = 1, \dots, N).$$

Now let  $T$  be the  $R^{MN}$ -open cube with center  $c(y, \varepsilon)$  and radius  $\eta$ . ( $S$  and  $T$  have the same center and radius, but  $S = T \cap L_F$ .) We shall show that

$$(5.19) \quad z \in T \text{ implies } z^i \in G^+(y^i) \cap Y_F^i \quad (i = 1, \dots, N).$$

We first note that, using (5.8) and ED.4,

$$(5.20) \quad \begin{aligned} S &= T \cap L_F = T \cap Y_C \\ &= \{x : x = \langle x^1, \dots, x^N \rangle, \sum_{i=1}^N x^i = 0_x, \|x - c(y, \varepsilon)\| < \eta\}. \end{aligned}$$

<sup>39</sup> Note that  $y \in \bar{Y}_\varepsilon$  implies  $N > 1$ .

Write  $c(y, \varepsilon) \equiv \langle c^1, \dots, c^N \rangle$ .

Let  $z$  be any vector in  $T$ , and  $i$  any integer between 1 and  $N$ . Define  $\bar{z}^k$  by<sup>40</sup>

$$(5.21) \quad \bar{z}^k \equiv \begin{cases} c^k + \frac{1}{N-1}(c^i - z^i), & \text{for } k \neq i, \\ z^i, & \text{for } k = i. \end{cases}$$

One can easily verify that

$$\sum_{k=1}^N \bar{z}^k = \sum_{k=1}^N c^k = 0_x,$$

$$\|\bar{z} - c(y, \varepsilon)\| < \frac{\eta}{N-1} \leq \eta.$$

Hence, by (5.20),  $\bar{z}$  is in  $S$ . But  $\bar{z}^i = z^i$ , so by (5.18),

$$(5.22) \quad z^i \in G^{+i}(y^i) \cap Y_F^i.$$

Thus we have shown that (5.22) holds for any  $z \in T$  and  $i = 1, \dots, N$ ; i.e., we have demonstrated (5.19).

We shall now show that there exists a number  $\lambda > 0$  such that

$$(5.23) \quad \prod_{i=1}^N p^i(z^i|y) \geq \lambda^N, \quad \text{for all } z \in T.$$

First, each  $Y_F^i$  is a coordinate projection of the compact set  $Y_F$ , and is therefore compact. Hence there exists  $\lambda > 0$  such that

$$(5.24) \quad p^i(z^i) \geq \lambda \quad \text{for all } z^i \in Y_F^i,$$

where  $p^i$  is the "initial" density function. To see this, for each  $i$  let  $\lambda_i \equiv \min [p^i(z^i): z^i \in Y_F^i]$ ; this minimum exists and is positive because of the compactness of  $Y_F^i$  (Assumptions ED.1 and ED.3) and the continuity and positivity of  $p^i$  (Assumption PD.1).

Take  $\lambda \equiv \min \{\lambda_i: i = 1, \dots, N\}$ . Since  $G^i(Y^i) \subseteq Y^i$ , the construction of conditional probabilities (see Section 2.3) ensures that

$$(5.25) \quad p^i(z^i|y) \geq p^i(z^i), \quad \text{for all } z^i \in G^i(y^i).$$

Therefore, (5.24) and (5.25) together yield

$$(5.26) \quad p^i(z^i|y) \geq \lambda, \quad \text{for all } z^i \in G^i(y^i) \cap Y_F^i.$$

Hence, by (5.19), for all  $i$ ,

$$p^i(z^i|y) \geq \lambda, \quad \text{for all } z = \langle z^1, \dots, z^N \rangle \in T,$$

so that (5.23) follows.

<sup>40</sup> Recall that  $N > 1$ .

Now note that

$$\text{prob} \{Z_t \in T | y_t = y\} = \int_T \prod_{i=1}^N p^i(z^i | y) dz^1 \dots dz^N.$$

Hence, by (5.23),

$$\begin{aligned} (5.27) \quad \text{prob} \{Z_t \in T | y_t = y\} &\geq \lambda^N \int_T dz^1 \dots dz^N \\ &= \lambda^N \cdot (2\eta)^{MN} \\ &\equiv \gamma_1 > 0, \end{aligned}$$

where the term  $(2\eta)^{MN}$  is the content of the  $R^{MN}$ -cube  $T$  of radius  $\eta$ .

Adopting temporarily a more explicit notation in which  $B_t^* = B^*(Z_t)$  and  $B_t = B(Z_t)$ , we shall show that

$$(5.28) \quad Z_t \in T \text{ implies } B^*(Z_t) \supseteq S.$$

Let  $Z_t \in T$  and  $x \in S$ . Then, by the definition of  $S$ ,  $T$ , and  $\eta$ ,

$$\begin{aligned} \|x - z_t\| &\leq \|x - c\| + \|c - z_t\| \\ &\leq \frac{\delta}{2} + \frac{\delta}{2} = \delta. \end{aligned}$$

Since  $Z_t$  is the ‘‘center’’ of the set  $\zeta(Z_t)$ , and  $\|x - Z_t\| < \delta$ , it follows that  $x \in \zeta(Z_t)$ . But  $x \in S \subseteq Y_C \cap G(y)$ ; hence  $x \in Y_C \cap B(Z_t) \equiv B^*(Z_t)$ , which completes the proof of (5.28).

From (5.28) it follows that

$$(5.29) \quad \text{prob} \{B_t^* \supseteq S | y_t = y\} \geq \text{prob} \{Z_t \in T | y_t = y\}.$$

In conjunction with (5.27), this implies

$$(5.30) \quad \text{prob} \{B_t^* \supseteq S | y_t = y\} \geq \gamma_1.$$

Denoting by  $W_t$  the umpire’s choice, we have

$$\text{prob} \{W_t \in S | B_t^* \supseteq S\} = \frac{\text{content of } S}{\text{content of } B_t^*} \geq \frac{s}{b} \equiv \gamma_2 > 0,$$

where by (5.9),

$$\begin{aligned} s &\equiv k_d(2\eta)^d \leq L_F\text{-content of } S, \quad k_d > 0, \\ b &\equiv [\sqrt{MN}(2\delta)]^d \geq L_F\text{-content of } B_t^*, \\ d &\equiv \text{dimension of } L_F. \end{aligned}$$

We observe that by (5.8) and Assumption ED.1,  $d = (N - 1)M \geq 1$ . Therefore,

$$\begin{aligned} \text{prob} \{W_t \in S | y_t = y\} &\geq \text{prob} \{W_t \in S | B_t^* \supseteq S\}, \\ \text{prob} \{B_t^* \supseteq S | y_t = y\} &\geq \gamma_1 \gamma_2 \equiv \gamma > 0. \end{aligned}$$

A fortiori, since  $S \subseteq \hat{Y}_\varepsilon$ ,

$$\begin{aligned} \text{prob } \{y_{t+1} \in \hat{Y}_\varepsilon | y_t = y\} &\geq \text{prob } \{W_t \in S | y_t = y\} \\ &\geq \gamma > 0. \end{aligned}$$

(Note that  $\gamma$  does not depend on  $y$ , although it does depend on  $\varepsilon$ .)  
 This completes the proof of Lemma 5.6.

5.3. *An Equilibrium is an Optimum, and the Process is Stable*

5.3.1. *An Equilibrium Bidding Distribution is Optimal*

**THEOREM 5.1:** *Under Assumptions PD.1, PD.2, and ED.1 through ED.6 of Section 5.1, if a bidding distribution is an equilibrium, then it is optimal.*

**PROOF:** Proof is immediate from Lemmas 5.1 and 5.6.

5.3.2. *The Utility Process*

As in the finite case, the sequence  $\{y_t\}_{t=0}^\infty$  of agreements is a Markov process, and so is the sequence  $\{P_t\}_{t=0}^\infty$  of bidding distributions. We define

$$(5.31) \quad \begin{cases} u_t^i \equiv U^i(y_t^i), \\ u_t \equiv \langle u_t^1, \dots, u_t^N \rangle. \end{cases}$$

The sequence  $\{u_t\}_{t=0}^\infty$  is also a Markov process, and is (coordinate-wise) non-decreasing. Furthermore, since  $Y_F$  is compact, and the utility functions are continuous, the sequence  $\{u_t\}_{t=0}^\infty$  is bounded. Hence

$$(5.32) \quad \lim_{t \rightarrow \infty} u_t \equiv \bar{u}$$

exists almost surely. This limit  $\bar{u}$  is a random (vector-valued) variable.

5.3.3. *The Utility Process Converges to an Optimum*

Our stability theorem, corresponding to Theorem 4.2, is that  $\bar{u}$  is almost surely optimal (see definition at end of Section 2.5 above).

**THEOREM 5.2:** *Under Assumptions PD.1, PD.2, and ED.1 through ED.6 of Section 5.1,  $\bar{u} \equiv \lim_{t \rightarrow \infty} u_t$  is optimal with probability one.*

**PROOF:** Define, for any point  $y \in Y$ ,  $y \equiv \langle y^1, \dots, y^N \rangle$ , its utility space image

$$U(y) \equiv [U^1(y^1), \dots, U^N(y^N)];$$

and, for any set  $W \subseteq Y$ , its utility space image

$$\tilde{U}(W) \equiv \{u : u = U(y) \text{ for some } y \in W\}.$$



Corresponding to  $Y_F$ ,  $\hat{Y}$ , and  $\bar{Y}_\varepsilon$  we define

$$V_F \equiv \tilde{U}(Y_F),$$

$$\hat{V} \equiv \tilde{U}(\hat{Y}),$$

$$\bar{V}_\varepsilon \equiv \tilde{U}(\bar{Y}_\varepsilon).$$

By Lemma 5.1,

$$V_F \setminus \hat{V} = \bigcup_{n=1}^{\infty} \bar{V}_{1/n}.$$

From Lemma 5.6 one easily sees that, for any  $n$ ,

$$\text{prob} \{ \bar{u} \in \bar{V}_{1/n} \} = 0,$$

since

$$\text{prob} \{ u_t \in \bar{V}_{1/n} \} \leq [1 - f(1/n)]^t,$$

where  $f(\varepsilon) = \gamma$  of Lemma 5.6, and  $u_t \notin \bar{V}_{1/n}$  implies  $u_s \notin \bar{V}_{1/n}$  for all  $s \geq t$  (almost surely). Hence

$$\text{prob} \{ \bar{u} \in (V_F \setminus \hat{V}) \} = 0.$$

#### 5.4. Verification of Assumptions ED.1 through ED.4: Some Special Cases

##### 5.4.0. Introduction

It is important to verify that the  $B$  process, while designed to handle cases where some of the conventional assumptions are violated, does work in the conventional cases as well. We therefore provide in this section examples in which Assumptions ED.1 through ED.4 are satisfied when some of the traditional postulates are adopted.<sup>41</sup> We use here the notation and terminology of Section 2.7, referring specifically to consumers, producers, and trade. Let  $J$  be the set of consumers, and  $K$  be the set of producers, so that  $J \cup K = \{1, \dots, N\}$ ; recall that  $C^j$  is the consumption set of consumer  $j \in J$ ;  $X_p^k$  is the production set of producer  $k \in K$ ;  $(\omega_1^i, \dots, \omega_M^i) \equiv \omega^i$  is the initial resource endowment of agent  $i$  ( $i$  in  $J$  or  $K$ ), and  $\omega_m \equiv \sum_{i=1}^N \omega_m^i$  ( $m = 1, \dots, M$ ).

We define further:

$$\omega \equiv \sum_{i=1}^N \omega^i \quad (\text{aggregate initial resource endowment}),$$

$$X_P \equiv \sum_{k \in K} X_P^k \quad (\text{aggregate production set}),$$

$$\Omega \equiv \text{nonnegative orthant of } R^M,$$

$$(\omega_1, \dots, \omega_M) \equiv \omega \gg 0_x \text{ means that every component } \omega_m \text{ is positive, } m = 1, 2, \dots, M.$$

<sup>41</sup> The conditions under which the assumption of openness (Assumption ED.6) is satisfied are discussed in Sections 5.6 and 5.7.

5.4.1. *Example 1: Pure Exchange*

Suppose that there are no producers ( $K = \emptyset$ ),  $N \geq 1$ , and that  $C^j = \Omega$ , all  $j$ ,  $\omega \gg 0_x$ .<sup>42</sup> It is clear that  $Y^j = C^j - \{\omega^j\}$  is closed for each  $j$ : hence Assumption ED.1 is satisfied. Let<sup>43</sup>

$$S \equiv \{x: 0_x \leq x \leq \omega\}.$$

Note that, constructing  $Y_F$  according to Assumption ED.2,

$$Y_F \subseteq \bigtimes_{i=1}^N (S - \{\omega^i\});$$

hence  $Y_F$  is bounded (Assumption ED.3). To show that Assumption ED.4 is satisfied, let

$$\bar{\omega} \equiv \frac{1}{N} \sum_i \omega^i;$$

then, for each  $i$ ,

$$\bar{\omega} - \omega^i \in \text{Int}(C^i - \{\omega^i\}).$$

Thus Assumptions ED.1 through ED.4 are satisfied even though no convexity or non-saturation assumptions are made with regard to consumer preferences.

5.4.2. *Example 2: Production and Exchange*

This example covers the previous one as a special case with  $K = \emptyset$ . The assumptions include some of those commonly made in the theory of competitive equilibrium. Let  $N \geq 1$  and assume

(5.33) (i)  $C^j$  is closed and (ii)  $C^j$  has a lower bound<sup>43</sup> for  $\leq$ , for all  $j$  in  $J$ ;

(5.34) (i)  $X_p$  is (i') closed and (i'') convex; (ii)  $X_p^k$  is closed for every  $k$  in  $K$ ;

(5.35)  $X_p \cap \Omega = \{0_x\}$  ("impossibility of outputs without inputs");

(5.36)  $X_p \cap (-X_p) = \{0_x\}$  ("irreversibility of production") or  $K$  is a one-element set (only one producer);

(5.37)  $X_p^k + (-\Omega) \subset X_p^k$ , all  $k$  in  $K$  ("free disposal in production");

(5.38) there exists a  $c$  in  $\bigtimes_{j \in J} \text{Int } C^j$  such that  $\sum_{j \in J} c^j \in X_p + \{\omega\}$ .

Because of (5.33a) and (5.34b), Assumption ED.1 is clearly satisfied. In view of (5.33b), (5.34a), (5.35), and (5.36), and for  $Y_F$  constructed according to Assumption ED.2, the boundedness of  $Y_F$  (Assumption ED.3) follows from Proposition (2), Chapter 5, Section 4 of Debreu [5].

<sup>42</sup> In Section 5.5 below, Assumption ED.4 is replaced by Assumption ED.4\* which permits the replacement of the requirement  $\omega \gg 0_x$  by  $\omega \geq 0_x$  in the presence of other assumptions.

<sup>43</sup> Here  $\leq$  represents the vectorial ordering of the commodity space.

That Assumption ED.4 follows from (5.37) and (5.38) can be shown by using the following fact: if  $A$  is a set in  $R^M$  such that  $A + (-\Omega) \subseteq A$ ,  $x$  is a point in  $A$ , and  $z$  is a point in  $R^M$ ,  $z \gg 0_x$ , then  $x - z$  is in the interior of  $A$ .

Thus, again, Assumptions ED.1 through ED.4 hold, even though no assumptions are made as to convexity or non-saturation of consumer preferences. The assumption that the aggregate production set is convex can be relaxed: see Example 3 of Section 5.4.3.

A special case of this example is obtained by replacing (5.33) and (5.38) respectively by the following:

$$(5.33') \quad C^j = \Omega, \quad \text{all } j \text{ in } J.$$

$$(5.38') \quad \text{There exists an } x \text{ in } X_p \text{ such that } x + \omega \gg 0_x.$$

An analogue of Example 1 is obtained when it is further assumed that  $X_p = -\Omega = X_p^k$  for all  $k \in K$ .

### 5.4.3. Example 3: A Model Free of Convexity Requirements

The model of Example 2 can be so modified as to avoid any convexity requirements. To accomplish this, it suffices to replace (5.34i''), which is the only condition of Example 2 involving convexity.<sup>44</sup> In fact, (5.34'') is used, in conjunction with the assumptions in (5.33b), (5.35), and (5.36), to assure (along the lines of (5.4.2) in Debreu [5, pp. 77–78]) the boundedness of the set  $Y_F$ .

It can be shown, however, that the above set of assumptions could be replaced by the following set of conditions:<sup>45</sup>

$$(5.35') \quad \text{the set } X_p \cap [C - \omega] \text{ is bounded};$$

$$(5.33b') \quad AC \cap (-AC) = \{0\} \text{ or there is only one consumer};$$

$$(5.36') \quad AX_p \cap (-AX_p) = \{0\} \text{ or there is only one producer.}$$

Condition (5.35') is implied by, but not equivalent to,

$$(5.35'') \quad AX_p \cap AC = \{0\}.$$

The following example shows that (5.35'') is in fact stronger than (5.35'):

*Example 3<sub>1</sub>*:  $M = 2$  (two goods); one consumer,  $C = C^1 = \Omega$ ; one producer,

$$X_p = X_p^1 = \{(x_1, x_2) : x_1 \leq 0, x_2 \leq x_1^2\},$$

with  $x_2$  as the output and  $x_1$  as the (negative) input. Here we have increasing returns and  $X_p$  is non-convex. (5.35'') is violated because the positive half-axis  $0x_2$  is in  $AX_p$ , but (5.35') is satisfied. (The theorems used in this section concerning the

<sup>44</sup> Example 3 also replaces (5.38b) by its generalization of (5.38b'), although (5.33b) does not imply convexity. Cf. Debreu [6, p. 259, condition (2.1)].

<sup>45</sup> We write  $C \equiv \sum_{j \in J} C^j$ .  $AS$  denotes the asymptotic cone (see Debreu [5, 1.9], and Fenchel [7]) of the set  $S$ .

boundedness of the feasible set without the assumption of convexity have been published.)

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