A behavioral model of cost reduction

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Three aspects of bounded rationality seem important for decision theory: (1) the existence of goals, (2) the search for improvement, and (3) long-run success. Two important criteria of long-run success are (a) the probability of survival, and (b) the long-run average rate of growth of performance (relative to one or more goals). One obstacle to the development of mathematical theories of resource allocation based on bounded rationality has been the absence of a clear and precise formulation of what is meant by "satisficing." In this paper I develop a few related mathematical models of satisficing in an uncertain environment, and apply them to the analysis of cost-reduction and technical change. An additional theme is that the allocation of resources in an organization is significantly influenced by the allocation of decision-making effort.

1. Introduction

■ Although the concepts of "bounded rationality" and "satisficing" have been familiar to organization theorists, and a few economists, for some time, most mathematical theories of resource allocation are based on some postulate of optimizing behavior on the part of individual economic decision makers.[1] One obstacle to the development of alternative mathematical theories of resource allocation has been the absence of a clear and precise formulation of what is meant by "satisficing." My goal here is twofold: (1) to develop a few related mathematical models of satisficing in an uncertain environment, and (2) to apply these models to the analysis of a simple process of cost-reduction and technical change. An additional theme, which relates the foregoing two points, is that the allocation of resources in an organization is significantly influenced by the allocation of decision-making effort.

According to Simon, "Theories that incorporate constraints on the information-processing capacities of the actor may be called theories of bounded rationality."[2] I shall not attempt here

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[1] See, however, Winter [9] and [10].

[2] In [7]. See, also, Simon [6] for a review of theories of decision-making in economics and behavioral science.

to give a precise definition of bounded rationality. However, three aspects of bounded rationality do seem important for decision theory: (1) the existence of goals, (2) the search for improvement, and (3) long-run success. Two important criteria of long-run success are (a) the probability of survival, and (b) the long-run average rate growth of performance (relative to one or more goals). These aspects will be central to the theories I shall develop here.[3]

In Section 3, I present a general formulation of a "thermostat" type of satisficing process, based on the theory of semimartingales, together with a specialization to the case of random walks. The main idea is to describe a decision-maker's intermittent search for improvement with respect to a single objective. The analysis centers on estimating the probability of survival and the long-run average rate of change of performance, as measured by the single objective.

A manager usually supervises more than one activity. For any given level of total search effort per unit of time, the opportunity cost of searching for improvement in one activity is the neglect of others. Section 4 describes a model of the allocation of effort among several activities. A particular behavior, called "putting out fires," is intimately related to satisficing for a single activity, and has a number of interesting operating characteristics. In particular, putting out fires (1) equalizes the long-run rates of growth of performance, (2) keeps the levels of performance of the different activities "close together," in some statistical sense, and (3) achieves a positive probability of survival (appropriately defined) if there exists any behavior with this property.

In Section 5, I explore the consequences of a behavior in the multiactivity case in which activities become candidates for attention according to the thermostat rule of Section 3, and in which the manager's attention is allocated among the current candidates (if any) at every date according to putting out fires. As compared with pure putting out fires, the introduction of thermostat behavior in determining the candidates for attention will, in general, radically alter the behavior of the system. Nevertheless, the theory of putting out fires provides a simple criterion for determining, in this more complicated case, whether the capacity of the manager is adequate for the given set of activities.

These various modes of "boundedly rational" behavior are first introduced in the context of a process of cost reduction under uncertainty (analogous to processes studied by Kennedy, von Weizsäcker, and Samuelson in the case of certainty). A complete analysis of this process is given in Section 2, using the theoretical results of the subsequent sections. Further remarks on possible generalizations are offered in Section 6.

□ **Acknowledgements.** The mathematical analysis in this paper makes heavy use of an inequality in the theory of semimartin-

[3] In another paper [3] I hope to offer a more elaborate discussion of these ideas.

gales, due to Freedman,[4] and I am indebted to him for helpful discussions and insights on how to apply his inequality to the problems that arise in the theory of satisficing. The theory of putting out fires (Section 4) is taken from a paper by Radner and Rothschild,[5] and the important influence on the entire paper of many discussions with Michael Rothschild is obvious to me, and I hope to him.

2. A process of cost reduction

■ Various modes of search for improvement can be illustrated in the context of a process of cost reduction like those studied by Kennedy, von Weizsäcker, and Samuelson.[6] Consider a single production process with a single output and I inputs. At any date t, the required input of factor i per unit output is $R_i(t)$, and the (unit) price of factor i is p_i. For the time being, I assume that the factor prices p_i do not depend on t. Thus the unit cost of output at date t is $\Sigma_i p_i R_i(t)$.

The use of each input is itself a complicated activity, whose effectiveness is subject to stochastic variation. Accordingly, I shall assume that the vector of input coefficients, $R_i(t)$, is a stochastic process whose evolution is influenced by, but not completely determined by, the allocation of managerial effort among the input activities. I shall make the model precise, below, but an initial heuristic description may be helpful. Roughly speaking, if at date t the manager devotes all of his effort to the reduction of the input of factor i, then at that date he can expect that the input coefficient $R_i(t)$ will be *reduced* by some fraction, say $\bar{\rho}_i$. On the other hand, if he does not devote any effort to the input of factor i, then he can expect $R_i(t)$ to *increase* by a fraction $\bar{\delta}_i$. Assume that $\bar{\rho}_i > 0$ and that $\bar{\delta}_i \geqslant 0$.

It is central to the spirit of the present inquiry that the consequences of devoting managerial effort to cost reduction are uncertain. Realized changes in the input coefficients will differ stochastically from the expected changes, and the realized consequences of cost-reduction effort may on occasion be unfavorable, even though they are favorable on the average. Similarly, inattention to a particular input need not result in deterioration of the input coefficient, but it is not unusual for stochastic changes in input quality, behavior of workers, and other "environmental" factors, to result in a gradual, if uneven, increase in the input coefficients in the absence of managerial attention. Thus I assume that $\bar{\rho}_i$ is strictly positive, but only that $\bar{\delta}_i$ is nonnegative.

Suppose that the manager's behavior is to allocate, at each date, all of his effort to cost reduction for an input that promises the largest expected cost reduction. This expected cost reduction might be perceived as equal to $p_i\bar{\rho}_iR_i(t)$, or as $p_i(\bar{\rho}_i + \bar{\delta}_i)R_i(t)$. In either case we may write the expected cost reduction as $p_i\gamma_iR_i(t)$ (where γ_i is either $\bar{\rho}_i$ or $\bar{\rho}_i + \bar{\delta}_i$). I shall

[4] See [1].

[5] See [4].

[6] See von Weizsäcker [8] and the references given there. The present treatment was stimulated by a seminar given by Winter at Berkeley; see [10].

show that, with certain assumptions about the stochastic process of input coefficients, this process of cost reduction has the following properties: (1) the input coefficients will all asymptotically decrease at the same geometric rate, and (2) the vector $[C_1(t), \ldots, C_i(t)]$ of *relative* shares of input costs is ergodic, with the same long-run distribution for all (strictly positive) vectors of input prices.

Property (1) is shared by other effort allocation behaviors, as I shall show, but conclusion (2) is much more powerful, and represents a nontrivial implication of such cost-reduction behavior in the case of uncertainty. This behavior is an example of what has elsewhere been called *putting out fires.*[7] A general model of putting out fires is described in Section 4.

Another device for rationing a manager's attention is analogous to a thermostat. Suppose that the manager has been allocating his attention to input i at date $(t - 1)$, but switches attention to some other input at date t. Suppose further that, whatever the current costs of the other inputs, input i does not become a candidate for the manager's attention again until its cost rises by some preassigned fraction, say β_i, above $p_i R_i(t)$. At such a date there may be other inputs that are also candidates for attention. Among all the candidate inputs, attention is allocated according to the putting-out-fires rule. At the first subsequent date at which the cost of input i falls to the previous level, $p_i R_i(t)$, input i ceases to be a candidate for attention, and the "thermostat" cycle is restarted. If we assume that the logarithms of the input coefficients can change by at most one step in any one period, and can never decrease without attention, and never increase with attention, it follows that each input coefficient will fluctuate between the corresponding values $R_i(0)$ and $R_i(0)(1 + \beta_i)$.

In summary, each input i becomes a candidate for attention whenever its cost rises to $p_i R_i(0)(1 + \beta_i)$, and remains a candidate until its cost falls again to $p_i R_i(0)$. At every date, attention is allocated among the candidates at that date according to putting out fires. If there are no candidates, the manager is "idle" at that date, or turns his attention to some different set of activities.

I shall show that, with these assumptions, the consequence of following this combination of thermostat and putting-out-fires behavior is that the stochastic process of actual (nonnormalized) costs, $[p_1 R_1(t), \ldots, p_I R_I(t)]$, is ergodic, with a limiting distribution that is independent of the input prices, *provided a certain parameter of the process, $\bar{\zeta}$, is strictly positive.* [See equation (9), below, for the definition of $\bar{\zeta}$.] In particular, the long-run average rate of change of each input coefficient will be zero, and there will be no "technical change." In other words, if $\bar{\zeta} > 0$, then the manager's capacity is adequate to the task of keeping costs stable. On the other hand, if $\bar{\zeta} \leq 0$, then in the long run costs will rise indefinitely.

If the logarithms of the input coefficients can fall by more than one step in any one period (as a result of attention), then

[7] See Radner and Rothschild [4].

unit cost can actually decline over time, provided the thermostat for each input is "reset" as a function of past experience. Thus, let T_1 be the first date at which input i becomes a candidate for attention, and let T_2 be the first subsequent date at which the cost $p_iR_i(t)$ is not more than $p_iR_i(0)$. Suppose that at date T_2 the thermostat for input i is reset so that $p_iR_i(T_2)$ becomes its new "initial value"; $p_iR_i(T_2)$ will be either equal to $p_iR_i(0)$ or strictly less than it. In the latter case, which will happen from time to time, there will be a permanent reduction in the aspiration level for the cost of input i. As these reductions take place intermittently for all of the inputs, the total unit cost will decline, on the average.

Finally, I shall consider a more general situation in which there exist alternative techniques (vectors of input coefficients) for the production of the given (single) output. If the activity with the smallest cost per unit output is used at each date, and if cost-reduction effort is devoted only to the technique currently being used, and allocated according to putting out fires, then from some date onwards the same technique will be used all the time; however, that date and the corresponding technique will be uncertain. Furthermore, each technique will have a positive probability of being eventually selected, and therefore there is a positive probability that this process of technique selection and allocation of cost-reduction effort will *not* result in the eventual selection of the technique with the highest potential rate of cost reduction.

I shall now give a precise description of the stochastic properties of the cost-reduction process. I consider first the case in which the logarithm of each input coefficient can change by at most one step at each date. The allocation of attention at each date t will be indicated by the vector $a(t)$, with coordinates $a_1(t)$, . . . , $a_I(t)$, where $a_i(t)$ equals 1 or 0 according as attention is or is not allocated to input i at date t. One and only one coordinate of $a(t)$ will equal 1 at each date. (The notation is meant to suggest generalizations to the case in which fractional allocations of attention are feasible. Such generalizations will be discussed briefly later.) If at date t the manager pays attention to input i [$a_i(t) = 1$], then the corresponding input coefficient will be reduced by a fraction ρ_i with probability ϕ_i, and will remain unchanged with probability $(1 - \phi_i)$. Correspondingly, if at date t the manager does not pay attention to input i [$a_i(t) = 0$], then the input coefficient will be increased by a fraction δ_i with probability ψ_i, and will remain unchanged with probability $(1 - \psi_i)$. More formally, the *conditional* distribution of the input coefficients at date $(t + 1)$, given the history of the process up to and including date t, is given by:

$$\left.\begin{array}{l}\text{Prob}\{R_i(t+1) = (1-\rho_i)R_i(t)\} = \phi_i \\ \text{Prob}\{R_i(t+1) = R_i(t)\} = 1 - \phi_i\end{array}\right\} \text{ if } a_i(t) = 1,$$

$$\left.\begin{array}{l}\text{Prob}\{R_i(t+1) = (1+\delta_i)R_i(t)\} = \psi_i \\ \text{Prob}\{R_i(t) = R_i(t)\} = 1 - \psi_i\end{array}\right\} \text{ if } a_i(t) = 0; \quad \text{(1a)}$$

the transitions of the individual input coefficients are mutually independent. (1b)

The difference in conditional expected costs of input i (per unit output), as between not paying attention to i and doing so at date t, is $\gamma_i p_i R_i(t)$, where

$$\gamma_i \equiv \phi_i \rho_i + \psi_i \delta_i. \tag{2}$$

Define

$$U_i(t) \equiv -\log \gamma_i - \log p_i - \log R_i(t). \tag{3}$$

We may take $U_i(t)$ as a measure of "performance" of the activity of using input i, since it is a strictly monotone decreasing function of the cost of input i. Define

$$\begin{aligned} \eta_i &\equiv -\phi_i \log(1-\rho_i), \\ \xi_i &\equiv \psi_i \log(1+\delta_i). \end{aligned} \tag{4}$$

While attention is allocated to input i, $U_i(t)$ is a random walk with mean increments η_i per period, and while attention is not allocated to input i, $U_i(t)$ is a random walk with mean increments $-\xi_i$ per period. Note that the increments in $U_i(t)$ are

$$U_i(t+1) - U_i(t) = \log \frac{R_i(t)}{R_i(t+1)}. \tag{5}$$

Define

$$M(t) \equiv \min_i U_i(t). \tag{6}$$

Allocating attention to the (an) input with the largest expected cost reduction is equivalent to allocating attention to the (an) input for which $U_i(t)$ is smallest, i.e., equal to $M(t)$. I shall call this behavior putting out fires.[8]

One can easily verify that, under putting out fires, the stochastic process $[a(t-1), U(t)]$ is a Markov chain with countably many states, where $U(t)$ is the vector with coordinates $U_i(t)$. Without significant loss of generality, I shall assume that for each i:

$$\begin{aligned} &\text{(a) } U_i(t) \text{ is integer-valued,}^{9} \\ &\text{(b) } 0 < \phi_i < 1,\ 0 < \rho_i < 1. \end{aligned} \tag{7}$$

Define:

$$W_i(t) \equiv U_i(t) - M(t). \tag{8}$$

$$\bar{\zeta} \equiv \left(1 - \sum_i \frac{\xi_i}{\eta_i + \xi_i}\right) \Big/ \left(\sum_i \frac{1}{\eta_i + \xi_i}\right). \tag{9}$$

$$\bar{a}_i \equiv \frac{\bar{\zeta} + \xi_i}{\eta_i + \xi_i}, \quad i = 1, \ldots, I. \tag{10}$$

It is also straightforward to verify that $[a(t-1), W(t)]$ is a Markov chain, where $W(t) = [W_1(t), \ldots, W_I(t)]$.

It follows from Theorem 5 of Section 4 that, if $\bar{\zeta} > 0$, then the Markov chain $[a(t-1), W(t)]$ is ergodic. The long-run average of $a_i(t)$ is $\bar{a}_i$; thus $\bar{a}_i$ is the long-run relative frequency of dates at which attention is allocated to input i. Furthermore,

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[8] To make the rule precise, one must decide how to break ties; this can be done in any fixed manner. See equation (29).

[9] This assumption will be satisfied after a suitable choice of units if $\log \gamma_i$, $\log p_i$, $\log(1-\rho_i)$, $\log(1+\delta_i)$, and $\log U_i(0)$ are all rational numbers.

Theorem 5 implies that, for *every* i,

$$\lim_{t\to\infty}\left(\frac{1}{t}\right)U_i(t) = \bar{\zeta}. \tag{11}$$

This last is equivalent to

$$\lim_{t\to\infty}\,[R_i(t)]^{1/t} = e^{-\bar{\zeta}}. \tag{12}$$

In other words, in the long run every input coefficient approaches zero on the average like $e^{-\bar{\zeta}t}$.

Let $C_i(t)$ denote the relative share of total unit cost that is due to the use of input i, that is,

$$C_i(t) = \frac{p_i R_i(t)}{\sum_j p_j R_j(t)}. \tag{13}$$

Another implication of the ergodicity of $[a(t-1), W(t)]$ is that the vector $C(t) = [C_1(t), \ldots, C_I(t)]$ is ergodic. Since the limit distribution of $W(t)$ depends only on the distribution of the increments of $U(t)$, the same is true of the limit distribution, say Γ, of $C(t)$. In particular, this limit distribution does not depend on the input prices.[10]

Hence the ratios of the input coefficients, say

$$\frac{R_2(t)}{R_1(t)}, \ldots, \frac{R_I(t)}{R_1(t)}, \tag{14}$$

will have a limit distribution that *does* depend on relative input prices. A change in prices will result in a change in the limit distribution of coefficients so as to leave Γ unchanged. In particular, an increase in the price of input i relative to the prices of the other inputs will result in a long-run decline of the ratios of $R_i(t)$ to the other coefficients $R_j(t)$.

It is interesting to note that the independence of the limit distribution Γ from the relative input prices is also consistent with a model in which the output is produced according to a Cobb-Douglas production function (with random disturbances and neutral technical change), and in which the producer chooses the input proportions at each date so as to minimize cost per unit output!

A second direct implication of Theorem 5 is: for any given number c, if $\bar{\zeta} > 0$ and $M(0) > c$, then the probability that $M(t) > c$ *forever* is positive; *a fortiori*, the probability that total unit cost never falls below c is also positive. To say it more colorfully, if keeping total unit cost below some critical level (forever) is the definition of "survival," then $\bar{\zeta} > 0$ is a necessary and sufficient condition for the probability of survival to be positive.

The common long-run average rate of decline of input coefficients, equal to $\bar{\zeta}$, can be achieved by other allocation behaviors. For example, suppose that fractional allocations of effort are feasible, and that the expected reduction in the

[10] Strictly speaking, this statement follows from the preceding one only for changes in prices such that the coordinates of $U(t)$ remain integer-valued.

logarithm of an input coefficient is linear in the amount of effort devoted to it. To be precise, suppose that $a(t)$ can be any vector satisfying

$$a(t) \geqslant 0, \sum_i a_i(t) = 1, \tag{15}$$

and suppose that the conditional probability distribution of $R_i(t+1)$, given $a(t)$ and the history of the process up to and including date t, is given by the following table:

$R_i(t+1)$	Conditional Probability
$(1-\rho_i)R_i(t)$	$a_i(t)\phi_i$
$R_i(t)$	$1 - a_i(t)\phi_i - [1 - a_i(t)]\psi_i$
$(1+\rho_i)R_i(t)$	$[1 - a_i(t)]\psi_i$

(16)

As before, assume that the transitions of the different coefficients are mutually independent. It is easily verified from (15) and (4) that the conditional expectation of the logarithm of $R_i(t)/R_i(t+1)$, given $R_i(t)$ and $a(t)$, is

$$E\{\log R_i(t)/R_i(t+1) | R_i(t), a(t)\} = a_i(t)\eta_i - [1 - a_i(t)]\xi_i. \tag{17}$$

Now consider an allocation behavior in which each $a_i(t)$ is constant and equal to $\bar{a}_i$, as defined in (10). Then each $U_i(t)$ is a random walk, with mean increments equal to $\bar{\zeta}$ (the same for all i). Therefore, by the law of large numbers, for each i,

$$\lim_{t\to\infty} \frac{U_i(t)}{t} = \bar{\zeta},$$

almost surely, so that (12) is also satisfied. However, in contrast with putting out fires, for any i and j the difference $[U_i(t) - U_j(t)]$ is a random walk with increments that have zero mean and positive variance. Thus the variance of this difference increases linearly with t, and the successive values fluctuate around zero with increasing deviations, and with no limiting probability distribution. Correspondingly, $\log [C_i(t)/C_j(t)]$ is also a random walk with increments that have zero mean and positive variance, so that the relative input cost shares will exhibit extreme statistical instability in the long run, under the "constant-proportions" allocation behavior.

A more general model of putting-out-fires behavior is presented in Section 4. A more complete discussion of the relationship between putting-out-fires and constant-proportions behaviors, in the context of the model of Section 4, is given elsewhere.[11]

The consequences of introducing "thermostat" behavior to determine when an input activity becomes a candidate for attention have already been described above. A more systematic discussion of thermostat behavior is deferred to Sections 3 and 5.

Let us now return to the case of pure putting-out-fires behavior. I now extend the model by supposing that at all dates there are finitely many alternative techniques available for pro-

[11] See Radner and Rothschild [4].

ducing the (single) output. Let $R_i^k(t)$ denote the input of factor i per unit of output if technique k is used at date t, and let p_i^k be the corresponding input prices. The unit cost of using technique k at date t is

$$c^k(t) = \sum_i p_i R_i^k(t). \tag{18}$$

At each date the manager must make two decisions: (1) the choice of which technique to use in production, and (2) the choice of which input will be the object of cost-reduction effort. I shall suppose that at each date (1) the manager chooses a technique that has the minimum unit cost at that date, i.e., that minimizes (18), and (2) for the technique used in production at that date, he allocates his cost-reduction effort according to putting out fires. I shall show that, provided an additional condition on the effectiveness of all techniques is satisfied, there will exist some finite date T and some technique J such that technique J is used from date T on. The date T and the activity J are random variables, and every activity k has a positive probability of being the activity J eventually used forever. Thus there is a positive probability that activity J does not turn out to be an activity with the minimum long-run average rate of cost reduction.

In order to avoid uninteresting complications, I shall rule out cases in which a technique has no possibility of "catching up" once it falls behind. For this purpose, I shall say that a *technique k can compete* if, for any date t, any allocation of cost-reduction effort at that date, and any other technique j, there is a positive probability that the decrease in unit cost for technique k is greater than the decrease in cost for technique j.

In what follows I shall assume that every technique can compete. I shall also assume that, for any given allocation of effort, the changes of input coefficients of different techniques are mutually independent (stochastically). Finally, I assume that, for every technique k, the corresponding parameter $\bar{\zeta}^k$, given by (9), is strictly positive.

I give only a sketch of the proof.[12] Suppose that at some date s technique k has a lower unit cost than any other technique. As long as technique k is being used, for every other technique j and every input i, $\log R_i^j(t)$ is a random walk with positive drift. Hence there is a positive probability that $R_i^j(t)$ exceeds $R_i^j(s)$ for all $t > s$, and therefore a positive probability that

$$C^j(t) > C^j(s) \quad \text{for all } t > s. \tag{19}$$

Since the cost processes of the different techniques are statistically independent, this implies that there is a positive probability that (19) is satisfied simultaneously for all j different from k.

As long as technique k is being used, the individual terms $\log R_i^k(t)$ are not random walks. However, we know from (11) that, for each i, $\log R_i^k(t)$ diverges to minus infinity, and one can show[13] that there is a positive probability that

[12] The argument is similar to that used in Radner and Rothschild [4] with regard to the behavior called there "staying with a winner."

[13] See the proof of Theorem 3 in Radner and Rothschild [4].

$$C^k(t) < C^k(s) \quad \text{for all} \quad t > s. \tag{20}$$

Let $K(t)$ denote the technique that is being used at date t. Combining (19) and (20), one can then show that there is a *positive* number q such that, for all dates s and all techniques k, if $K(s) = k$, then the conditional probability that $K(t) = k$ for all $t > s$, given the process up through date s, is at least q.

Let $R(t)$ denote the matrix with entries $R_i^j(t)$. The process $\{a(t - 1), K(t), R(t)\}$ is Markovian. Consider the sequence of dates, T_n, at which $K(t)$ changes; i.e., $T_0 = 0$, $T_1 =$ the first $t > 0$ such that $K(t) \neq K(t - 1)$, etc. Let $T = \sup_n T_n$; we want to show that T is finite with probability one. Given the state of the Markov chain at any date T_n, the probability is at least q that T_n is the last one of the sequence; hence the probability that there are at least n such dates in the sequence is no greater than $(1 - q)^n$, which converges to 0 as n increases without limit. Hence T is finite with probability one. The technique J that is eventually used forever is, of course, equal to $K(T)$. The fact that the probability distribution of J assigns positive probability to every technique follows from the assumption that every technique can compete. Notice that this implies that the technique with the largest $\overline{\zeta}^k$ need not be the one that is eventually chosen.

3. Thermostat behavior: satisficing with a single activity and objective

■ **General formulation of a satisficing process.** I start with a general formulation of a process of intermittent search for improvement with respect to a single objective. Consider a basic probability space, (X, F, P), where F is a sigma-field of subsets of X, and P is a probability measure on F. Let (F_t), $t = 0, 1, 2, \ldots$, be an increasing sequence of subfields of F; F_t is to be interpreted as the set of observable events through date t. Let $\{U(t)\}$ be a corresponding sequence of integer-valued random variables on X, such that $U(t)$ is F_t-measurable; $U(t)$ will be called the *performance at t*, relative to a given single objective. Finally, let (T_n), $n = 0, 1, 2, \ldots$, be a nondecreasing sequence of random times, possibly taking on the value plus infinity, such that $T_n < T_{n+1}$ if T_n is finite; for n odd, T_n is to be interpreted as a date at which a period of search for improvement begins, and T_{n+1} as the date at which that period ends. (A random time T is an integer-valued random variable, possibly equal to plus infinity, such that the event $(T = t)$ is F_t-measurable.) Take $T_0 = 0$.

An interval $(T_n \leqslant t < T_{n+1})$ will be called a *search period* if n is odd and a *rest period* if n is even. To capture the idea of intermittent search for improvement I assume: for $T_n \leqslant t < T_{n+1}$,

$$\begin{aligned} E[U(t + 1)|F_t] &\geqslant U(t), \quad \text{if } n \text{ is odd;} \\ E[U(t + 1)|F_t] &\leqslant U(t), \quad \text{if } n \text{ is even.} \end{aligned} \tag{21}$$

In other words, $U(t)$ is a submartingale during the search periods, and a supermartingale during the rest periods.

To capture the idea of "satisficing" let $\{S(t)\}$ be a sequence of random variables such that $S(t)$ is F_t-measurable; $S(t)$ is to be interpreted as the "satisfactory level of performance" at date t.

The random times, T_n, are determined by: for n even,

$$\begin{aligned} &T_{n+1} \text{ is the first } t > T_n \text{ such that } U(t) < S(t), \\ &T_{n+2} \text{ is the first } t > T_{n+1} \text{ such that } U(t) \geqslant S(t); \end{aligned} \tag{22}$$

this is qualified by the convention that, for any n, if T_n is infinite, then so is T_m for every $m > n$.

In the following subsections more specific assumptions will be made about the processes $U(t)$ and $S(t)$.

□ **A favorable satisficing process.** Let $Z(t)$ be the successive increments of the process, $U(t)$; thus $Z(t + 1) = U(t + 1) - U(t)$. Let ξ, η, and β be given positive numbers. For $T_n \leqslant t < T_{n+1}$, assume

$$\begin{aligned} &\text{(i) for } n \text{ even (rest),} \\ &E[Z(t + 1)|F_t] \leqslant -\xi, \\ &S(t) = U(T_n) - \beta + 1; \end{aligned} \tag{23a}$$

$$\begin{aligned} &\text{(ii) for } n \text{ odd (search),} \\ &E[Z(t + 1)|F_t] \geqslant \eta, \\ &S(t) = U(T_{n-1}). \end{aligned} \tag{23b}$$

Thus, if a search period ends with $U(T_n) = u$, then the next search period begins as soon as $U(t)$ reaches or falls below $(u - \beta)$, and ends thereafter as soon as $U(t)$ reaches or exceeds u again. During such a search period, u may be called the "aspiration level." For technical reasons assume further that there is a number b such that

$$\text{(iii) } |Z(t)| \leqslant b, \quad \text{for all } t. \tag{23c}$$

Theorem 1. The random times, T_n, have uniformly bounded expectations, i.e., there is a finite number, μ, such that, for all n,

$$E[T_{n+1} - T_n|F_{T_n}] \leqslant \mu.$$

The proof of Theorem 1 is obtained immediately by applying the following lemma separately to the search and rest periods.

Lemma 1. Hypothesis:

(1) η, b, and λ are strictly positive numbers, with $\eta \leqslant b$.
(2) $\{Y_t\}$ is a sequence of random variables, with $Y_t F_t$-measurable.
(3) $E(Y_t|F_{t-1}) \geqslant \eta$, and $|Y_t| \leqslant b$.

Define $S_n \equiv Y_1 + \ldots + Y_n$ and $T \equiv \inf\{n: S_n \geqslant \lambda\}$. Conclusion:

There exist H and K strictly positive such that, for every $t \geqslant 2\lambda/\eta$, $\text{Prob}\{T > t\} \leqslant He^{-Kt}$.

(The proofs of Lemma 1 and other propositions in this section, unless otherwise noted, will be found in the Appendix.)

The lemma can be applied directly to the search periods; recall that, for n odd, $U(T_n) \geqslant U(T_{n-1}) - \beta - b$, so that in the application of the lemma, $\lambda \leqslant \beta + b$. For the rest periods one must change the appropriate signs and parameters.

For any nonnegative integer k, let $V_k = U(T_{2k})$. The V_k are the performance levels at which successive search periods end, and each V_k is the aspiration level for the next succeeding search period. It is clear that the V_k form a nondecreasing sequence. If, during search, performance can (with positive probability) increase by more than one unit at a time, then V_k will actually increase from time to time. I shall say that the process is *strictly favorable* if there is a (strictly) positive number v such that, for every k,

$$E[V_{k+1}|F_{2k}] \geqslant V_k + v. \tag{24}$$

Theorem 2. If the process is strictly favorable, then

$$\liminf_{k\to\infty} \frac{V_k}{k} \geqslant v, \text{ almost surely.}$$

To prove Theorem 2, define

$$M_k \equiv E(V_{k+1}|F_{2k}) - V_k,$$
$$N_k \equiv M_1 + \dots + M_k.$$

By (24), $k \leqslant N_k/v$, so that $\lim_{k\to\infty} N_k = +\infty$. Hence, by Freedman,[14]

$$\lim_{k\to\infty} \frac{V_k}{N_k} = 1, \text{ almost surely,}$$

and, therefore,

$$\liminf_{k} \frac{V_k}{k} \geqslant \liminf_{k} \frac{V_k v}{N_k} = v.$$

□ **Random-walk search and rest.** In the model of the previous subsection, assume further that during rest the increments $Z(t+1)$ are independent and identically distributed, with mean $-\xi$, and during search they are also independent and identically distributed, with mean η. In other words, during rest the performance process is a random walk with negative drift, and during search it is a random walk with positive drift. To minimize technical complications assume further that these random walks are integer-valued and aperiodic.[15]

Let $a(t) = 1$ during search, and 0 during rest. The process $\{a(t-1), U(t), S(t)\}$ is a Markov chain with countably many states and a single class. To see this, first consider four cases:

(1) If $a(t-1) = 0$ and $U(t) \geqslant S(t)$, then $a(t) = 0$ and $S(t+1) = S(t)$.
(2) If $a(t-1) = 0$ and $U(t) < S(t)$, then $a(t) = 1$ and $S(t+1) = S(t) + \beta - 1$.
(3) If $a(t-1) = 1$ and $U(t) \geqslant S(t)$, then $a(t) = 0$ and $S(t+1) = U(t) - \beta + 1$.
(4) If $a(t-1) = 1$ and $U(t) < S(t)$, then $a(t) = 1$ and $S(t+1) = S(t)$.

Hence, $[a(t-1), U(t), S(t)]$ determines $a(t)$ and $S(t+1)$. Furthermore, the conditional distribution of $U(t+1)$, given F_t, is

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[14] In [1], p. 912.
[15] A random variable is aperiodic if 1 is the greatest common divisor of its support.

determined by $a(t)$ and $U(t)$. Hence, $\{a(t - 1), U(t), S(t)\}$ is Markovian. The fact that there is a single class follows from the assumptions that ξ and η are strictly positive and that the two random walks are aperiodic.

Let $D(t) = U(t) - S(t)$. The process $\{a(t - 1), D(t)\}$ is also Markovian, with a single class. To see this, consider the four cases for $[a(t - 1), D(t)]$ that correspond to the four cases above. By the definition of $D(t)$, $a(t)$ is in each case determined by $a(t - 1)$ and $D(t)$. Since

$$D(t + 1) = D(t) + Z(t + 1) - [S(t + 1) - S(t)],$$

in each case $D(t + 1)$ is, respectively, equal to

(1) $D(t) + Z(t + 1)$,
(2) $D(t) + Z(t + 1) - (\beta - 1)$,
(3) $Z(t + 1) + \beta - 1$,
(4) $D(t) + Z(t + 1)$.

Since the conditional distribution of $Z(t + 1)$, given F_t, depends only on $a(t)$, this shows that $[a(t - 1), D(t)]$ is Markovian.

Theorem 3. The process $\{a(t - 1), D(t)\}$ is positive recurrent. Let $\bar{a}$ denote the long-run frequency with which $a(t) = 1$, and let $\bar{\zeta} = \bar{a}\eta - (1 - \bar{a})\xi$; then, almost surely,

$$\lim_{t\to\infty} \frac{U(t)}{t} = \bar{\zeta}. \tag{25}$$

If $\bar{\zeta} > 0$, then the process is strictly favorable, in the sense of Theorem 2. This will be the case if and only if during search performance can increase by more than one unit at a time. In the present case the sequence (V_k) is a random walk. However, *the sequence $U(t)$ is not a random walk, nor even a submartingale.* Nevertheless, one can prove[16] for $\{U(t)\}$ the following result.

Theorem 4. If the process is strictly favorable ($\bar{\zeta} > 0$), then there exist positive numbers H and K such that, if $U(0) \equiv u > \beta + b$, then

$$\text{Prob}\{U(t) \leqslant 0 \text{ for some } t | F_0\} \leqslant He^{-Ku}.$$

If $\bar{\zeta} = 0$, then the above probability is 1.

Let us say that the process *survives* if the performance $U(t)$ remains positive for all t. Taken together, Theorems 3 and 4 assert that, for a strictly favorable process, with random-walk rest and search, in the long run performance increases at a positive average rate per unit time, and the probability of survival approaches unity exponentially or faster as a function of the initial performance level, $U(0)$. This implies further that, if the process has "survived" for a long time, then the performance level is probably very high, and therefore the conditional probability of subsequent survival is close to unity. If the process is not strictly favorable, then the probability of survival is zero.

[16] For a proof of Theorem 4, see Radner [3].

4. Putting out fires

■ A manager usually supervises more than one activity. For any given level of search effort per unit time, the opportunity cost of searching for improvement in one activity is the neglect of others. Consider a stochastic process $\{U(t), F_t\}$, as in the last subsection of Section 3, but let $U(t)$ be a vector with coordinates $U_i(t), i = 1, \ldots, I$, where $U_i(t)$ is a measure of performance of activity i at date t. An *allocation behavior* is a sequence, $\{a(t)\}$, where $a(t)$ is an F_t-measurable random vector with coordinates $a_i(t), i = 1, \ldots, I$, such that, for any date t, exactly one coordinate of $a(t)$ is 1, and the other coordinates are 0. If $a_i(t) = 1$, this is interpreted as a search for improvement in activity i at date t.

Concerning the process $U(t)$, I shall make assumptions analogous to those of the previous subsection. As before, let

$$Z(t + 1) = U(t + 1) - U(t). \tag{26}$$

For the conditional distribution of $Z(t + 1)$, given F_t, assume:

The distribution of $Z(t + 1)$ depends only on $a(t)$. (27a)

$$EZ_i(t + 1) = a_i(t)\eta_i - [1 - a_i(t)]\xi_i, \tag{27b}$$

where ξ_i and η_i are given positive parameters.

$$\operatorname{Var} Z_i(t + 1) = s_i(a_i[t]), \tag{27c}$$

where $s_i(0)$ and $s_i(1)$ are given positive parameters.

The coordinates of $Z(t + 1)$ are mutually independent. (27d)

To minimize technical complications, I also assume:

The coordinates of $Z(t + 1)$ are integer-valued, uniformly bounded by b, and aperiodic.[17] (27e)

A common managerial behavior is to pay attention only to those activities that are giving the most trouble; this is colloquially called "putting out fires." Formally, let

$$M(t) = \min_i U_i(t), \tag{28}$$

and define *putting out fires* by

(a) if $U_i(t) > M(t)$, then $a_i(t) = 0$;
(b) if $U_i(t) = M(t)$ and $a_i(t - 1) = 1$, then $a_i(t) = 1$; (29)
(c) if neither (a) nor (b) holds, then $a_i(t) = 1$ for $i =$ the smallest j such that $U_j(t) = M(t)$.

To compare putting out fires with the satisficing model of Section 2, roughly speaking, the satisfactory level of performance of any activity is here defined to be equal to $M(t) + 1$.

To describe the properties of the performance process under putting out fires, I first define

$$\bar{\zeta} = \left(1 - \sum_i \frac{\xi_i}{\eta_i + \xi_i}\right) \Big/ \left(\sum_i \frac{1}{\eta_i + \xi_i}\right). \tag{30}$$

$$\bar{a}_i = \frac{\bar{\zeta} + \xi_i}{\eta_i + \xi_i}, \quad i = 1, \ldots, I. \tag{31}$$

[17] See footnote 15.

If the limit, as t increases, of $U_i(t)/t$ exists, I shall call this limit the *rate of growth of activity i*. If $M(t) > 0$ for all t, I shall say that the performance process *survives*. Define $W(t) = U(t) - M(t)$.

Theorem 5. Under putting-out-fires behavior, if $\bar{\zeta} > 0$, and if

$$\mathcal{P}\{Z_i(t+1) = 0 | a_i(t)\} > 0,$$
$$\mathcal{P}\{Z_i(t+1) = 1 | a_i(t) = 1\} > 0,$$
$$\mathcal{P}\{Z_i(t+1) = -1 | a_i(t) = 0\} > 0,$$

then the Markov chain $\{a(t-1), W(t)\}$ is ergodic, and for each activity i,

(a) the long-run frequency with which $a_i(t) = 1$ is almost surely equal to $\bar{a}_i$;
(b) the rate of growth of $U_i(t)$ is almost surely $\bar{\zeta}$ (the same for all activities);

furthermore, if $M(0) > 0$, then

(c) the probability of survival is positive.

In the context of the model defined by (27a)-(27e) one could explore other allocation behaviors, but the limitation of space does not permit that here. I mention, however, that a necessary and sufficient condition that there exist *any* allocation behavior with positive probability of survival is $\bar{\zeta} > 0$. In other words, *survival is possible with positive probability if and only if it is possible with putting out fires*.

In the special case of two activities ($I = 2$) the conclusions (a) and (b) of Theorem 5 are true also if $\bar{\zeta} \leq 0$.

For proofs of the facts mentioned in this section and for an analysis of other allocation behaviors see Radner and Rothschild.[18]

5. Queuing and organizational slack

■ Under putting out fires (Section 4), an activity becomes a candidate for the manager's attention whenever its level of performance is less than or equal to the performance levels of all other activities. By contrast, under the thermostat behavior described in Section 3, the activity does not claim the manager's attention until its performance falls sufficiently far below the level of performance established at the end of the previous period of search for improvement, but then it continues to get his attention until the performance reaches or exceeds that level.

What happens in the multiactivity case if the manager tries to follow the thermostat behavior of Section 3 with respect to each activity? He must then face the problem that two or more activities may claim his attention at the same time. In that circumstance, all but one of the activities that claim his attention must wait until some future date for consideration, unless the manager can split his attention among several activities. In any case, the dynamic behavior of the system will depend upon the

[18] [4].

queue (waiting line) discipline, or other rule for allocating attention among the candidate activities.

In this section I explore the consequences of a behavior in the multiactivity case in which activities become candidates for attention according to the thermostat rule of Section 3, and in which the queue discipline is putting out fires. This means that an activity does not even become a *candidate for the manager's attention* unless it falls by at least a given amount, which may depend upon the activity. This device for screening out "small" or "unimportant" problems would appear to be common in practice; it is also related to quality control of industrial processes. With such behavior, at some times there may be several activities queuing for the manager's attention, and at other times none. The latter times might be interpreted as periods of "organizational slack," at least as regards the economy of managerial attention.

To be more precise, consider a multiactivity model of performance as described in Section 4, but with a different allocation behavior. For each activity, define the sequence of satisfactory levels of performance as in (23a-23b). I shall say that an activity is a *candidate for the manager's attention* (search effort) whenever and as long as its performance $U_i(t)$ is less than its current satisfactory level, $S_i(t)$. At every date t, the manager allocates his attention to one of the candidate activities (if there is one) according to the putting-out-fires rule.

Let $S(t)$ denote the vector of satisfactory levels of performance at date t. For each i let $B_i(t)$ be 1 or 0 according as activity i is or is not a candidate for attention at date t, respectively; let $B(t)$ be the vector with coordinates $B_i(t)$. It is easy to see that the process $[B(t-1), a(t-1), U(t), S(t)]$ is a Markov chain.

Let $D(t) = U(t) - S(t)$, and $W(t) = U(t) - M(t)$, where $M(t)$ is defined as in Section 4 as the minimum performance at date t. It is easy to see that the process $[B(t-1), a(t-1), D(t), W(t)]$ is also Markovian.

An interesting special case is the one in which no performance $U_i(t)$ can change by more than one unit at a time. Following random-walk terminology, I shall call this the *continuous case*. (The detailed model of cost reduction is an example of a continuous case; see conditions (1a).) In this case, assuming that every activity starts at rest, no performance can rise above its initial level. Using the methods of proof of Theorem 5, one can prove:

Theorem 6. In the continuous case, if assumptions (27a)-(27e) are satisfied, and $\zeta > 0$, then the Markov chain $[B(t-1), a(t-1), U(t), S(t)]$ is ergodic.

In the continuous case, if $\bar{\zeta} > 0$, then the manager's "capacity" is in a precise sense adequate to his task. When all of the activities are candidates for the manager's attention, the system will behave like putting out fires for all of the I activities. If a subset, say J, of activities are candidates for attention, then that subset will behave like putting out fires, with a parameter, $\bar{\zeta}_J$, corresponding to that subset. One can show that if $\bar{\zeta} > 0$, then for any proper subset J of activities, $\bar{\zeta}_J > \bar{\zeta}$; hence, whenever the candidates for attention form a proper subset of the whole set of activities, the behavior of that subset is even more favorable than in the case of putting out fires for all activities taken together.

In the case in which performance may "jump," i.e., change by more than one unit at a time (but still with $\bar{\zeta} > 0$), the process need not be ergodic, and a variety of asymptotic behaviors are possible. For some configurations of the parameters, the Markov chain $[B(t), a(t), D(t), W(t)]$ may be ergodic, with the performance of all activities growing asymptotically at the same long-run average rate, which will be nonnegative but less than $\bar{\zeta}$. For other configurations, different activities will grow at different long-run average rates, again nonnegative but less than $\bar{\zeta}$; in this case, the activity with the lowest long-run average rate of growth (if there is a unique one) will eventually always get the manager's attention whenever it is a candidate.

To summarize, the introduction of thermostat behavior in conjunction with putting out fires will, in general, radically change the behavior of the system, although the condition that $\bar{\zeta}$ be strictly positive may still be interpreted as a condition that the manager's capacity be adequate to prevent the system from getting out of control.

6. Some directions for generalization

■ If fractional allocations of managerial effort were feasible at each date, then the model of Sections 2 and 4 would be analogous to a linear-activity-analysis model in the case of certainty, where effort is the input, and the expected change in performance of the corresponding activity is the output. To make the model precise for the case of fractional allocations of effort (with uncertainty), consider assumptions (27a)-(27e) of Section 4, with (27c) replaced by:

$$\text{Var}\, Z_i(t) = s_i(a_i[t]), \qquad (27c')$$

where s_i is a strictly positive continuous function of a_i.

Note that (27c′) is verified in the case of the special model (16) of cost reduction in Section 2.

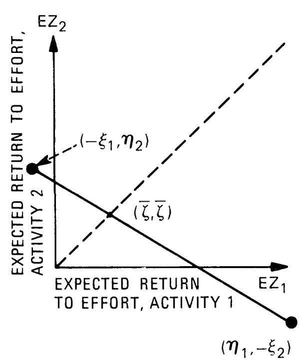
An allocation of effort at date t is a vector $a = (a_1, \ldots, a_I)$ that satisfies:

$$a \text{ is nonnegative, and} \sum_i a_i = 1. \qquad (32)$$

Corresponding to the set of all allocations of effort at any given date, there is a convex set of vectors of expected values, $EZ = (EZ_1, \ldots, EZ_I)$, of increments of performance. For example, in the case of two activities, this convex set is the solid line segment between the two points $(\eta_1, -\xi_2)$ and $(-\xi_1, \eta_2)$, as in Figure 1. One might describe this as the case of constant marginal productivity of effort per unit time for each activity.

FIGURE 1

CONSTANT MARGINAL PRODUCTIVITY OF EFFORT



Incidentally, Figure 1 illustrates the significance of the condition that $\bar{\zeta}$ be strictly positive. The point $(\bar{\zeta}, \bar{\zeta})$ is the one on the line segment where the two coordinates are equal. Thus $\bar{\zeta}$ is strictly positive if and only if the line segment intersects the strictly positive quadrant, i.e., if and only if there is any fractional allocation of effort that makes the expected increments in the performances of both activities strictly positive.

The line segment in Figure 1 might be interpreted as the "technological possibility frontier" for expected changes of performance. There are at least two other interesting hypotheses concerning the expected returns to effort per unit time. First, for any one activity, the marginal productivity of effort per unit time might be either decreasing or increasing. These two cases are illustrated, for the case of two activities, in Figures 2 and 3, respectively. It is clear that, in the case of decreasing marginal productivity of effort, there could be an advantage to fractional allocations of effort, whereas in the case of increasing marginal productivity, there would be an advantage to all-or-nothing allocations (as in the present paper), at least as regards the long-run average rate of change in performance.

A second hypothesis is that there are increasing returns to *uninterrupted* effort allocated to any single activity, or equivalently, there are costs of interrupting the allocation of effort to any one activity. The implications of this hypothesis have been studied by Rothschild.[19]

Another pair of alternative hypotheses concerns the statistical dependence of increments of performance of different activities at the same date, or of the same activity at different dates. First, the returns to effort for different activities at the same date may be statistically dependent. Second, the increments of performance at different dates (for the same activity) may not be statistically independent, nor even identically distributed, as was assumed in Sections 2 and 4. A more general model is that defined in Section 3, which only postulates properties of the *expected* returns to effort. There, condition (23) postulates that, if all the effort at a given date is allocated to activity i, then the expected increment in performance is at least η_i, and if no effort is allocated to that activity, then the expected *decrement* in performance is at least ξ_i, where η_i and ξ_i are positive parameters. However, in the present paper such a model was studied only with regard to the implications of thermostat behavior with a single activity (Section 3). Rothschild[20] has provided a generalization of Theorem 5, concerning the implications of putting out fires. Other models of thermostat behavior have been explored by Radner.[21]

FIGURE 2

DECREASING MARGINAL PRODUCTIVITY OF EFFORT

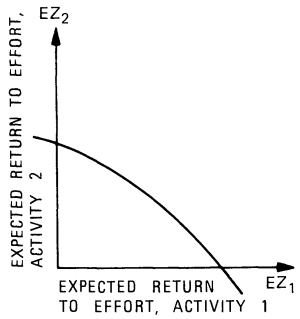
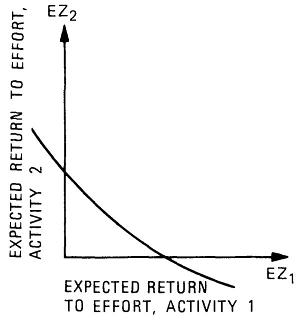


FIGURE 3

INCREASING MARGINAL PRODUCTIVITY OF EFFORT



Appendix

■ **Proof of lemma 1.** For the purposes of this proof, let

$$Y'_n \equiv \frac{Y_n + b}{2b},$$

[19] [5].
[20] [5].
[21] [3].

$$S'_n \equiv Y'_1 + \ldots + Y'_n = \frac{S_n + nb}{2b},$$

$$M'_n \equiv E(Y'_n | F_{n-1}).$$

By Freedman,[22] if $0 \geq A \geq B$, then for any random time τ,

$$\text{Prob}\{S'_\tau \leq A \text{ and } \sum_1^\tau M'_n \geq B\} \leq \exp\left[-\frac{(A-B)^2}{2B}\right]. \quad (33)$$

To apply Freedman's inequality (33), take

$$A \equiv \frac{\lambda + tb}{2b}, B \equiv t\left(\frac{\eta + b}{2b}\right), \tau \equiv t.$$

Note that, by hypothesis 3, $\Sigma_1{}^t M'_n \geq B$, and if $T > t$, then $S'_t > A$. Hence, $\text{Prob}\{T > t\}$ does not exceed the probability on the left side of Freedman's inequality. Further, if $t \geq 2\lambda/\eta$, then $A \leq B$, and by the inequality (33),

$$\text{Prob}\{T > t\} \leq \exp\left[-\frac{(A-B)^2}{2B}\right].$$

If $t \geq 2\lambda/\eta$, then $\eta - (\lambda/t) \geq (\eta/2)$, and

$$\frac{(A-B)^2}{2B} = \left(\eta - \frac{\lambda}{t}\right)\frac{(t\eta - \lambda)}{4b(\eta + b)} \geq \frac{\eta(t\eta - \lambda)}{8b(\eta + b)}.$$

To complete the proof of the lemma, take

$$H \equiv \exp\left[\frac{\lambda\eta}{8b(\eta + b)}\right],$$

$$K \equiv \frac{\eta^2}{8b(\eta + b)}.$$

□ **Proof of theorem 3.** Let D^* be the set of states for which $|D(t)| \leq 2b$. Since D^* is finite, and the Markov chain $\{a(t-1), D(t)\}$ has a single class, in order to prove that the chain is positive recurrent it is sufficient to show that from any state the expected time to return to D^* is finite.[23] This follows from Theorem 1 and the bound (23c (iii)) on the increments of $U(t)$. Therefore, the long-run frequency with which $a(t) = 1$ (search) exists and is unique, almost surely; call this frequency $\bar{a}$. The expected lengths of all rest periods are the same; the expected lengths of all search periods are at least 1. Hence $\bar{a} > 0$. Define

$$\begin{aligned}\zeta(t+1) &\equiv E\{Z(t+1) \mid F_t\} \\ &= a(t)\eta - [1 - a(t)]\xi.\end{aligned}$$

Then

$$\frac{1}{t}\sum_1^t \zeta(n) \to \bar{a}\eta - (1 - \bar{a})\xi > -b \quad \text{as} \quad t \to \infty. \quad (34)$$

Therefore, by a generalized law of large numbers[24]

$$\frac{tb + Z(1) + \ldots + Z(t)}{tb + \zeta(1) + \ldots + \zeta(t)} \to 1 \text{ a.s.} \quad \text{as} \quad t \to \infty.$$

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[22] [1], (4a) on p. 911.

[23] See Kemeny, Snell, and Knapp [2], Proposition 6.24, p. 143.

[24] [1], (40) on p. 921.

This and (34) imply that

$$\frac{1}{t}\sum_{1}^{t} Z(n) \rightarrow \bar{a}\eta - (1 - \bar{a})\xi \quad \text{as} \quad t \rightarrow \infty,$$

which, in turn, implies (25), and completes the proof of Theorem 3.

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