Stochastic Stability of Market Adjustment in Disequilibrium

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1. INTRODUCTION

In a system of markets in which prices, demands, and supplies are repeatedly subject to random disturbances, one cannot typically expect excess demands to be exactly zero at all times. Strictly speaking, such a system of markets might be said to be always in disequilibrium. On the other hand, one would want to distinguish situations in which prices and quantities fluctuated in some "steady" manner around long-run averages from situations in which prices or quantities, or both, fluctuated with greater and greater variance or increased without bound. To describe the first situation, it is natural to use the concept of a *stationary stochastic process*, which is the generalization to the case of uncertainty of the concept of a deterministic equilibrium. However, it is important to emphasize that the stationarity of a stochastic process does not rule out fluctuations of varying period and amplitude.

In a large number of markets, inventories and back-orders serve as buffers against random variations in supply and demand. We use the term *stock* to denote the level of an inventory (positive stock) or back-orders (negative stock). The levels and movements of stocks provide important signals about past and current excess demands, and these signals in turn influence the movements of prices. On the other hand, prices are influenced by signals other than stocks, and themselves serve as signals that influence supplies and demands, and thus also the stocks. In this paper we study conditions under which the stochastic process of interdependent prices and stocks converges to a stationary process; i.e., we study conditions sufficient for *stochastic stability*.

Our assumptions about price and stock adjustment are similar in spirit to the diagonal dominance assumption used by Lionel McKenzie to study the stability of a tâtonnement adjustment process in the deterministic case. (See [1, 5].) Our assumptions are not directly comparable with diagonal dominance, however, because of the difference in the model.[1] Our assumptions about adjustment can be paraphrased as follows: For every commodity when its stock is sufficiently high (low), its own money price will fall (rise) *on the average*, and when its price is sufficiently high (low), its stock will rise (fall) on the average.

We also make use of two other assumptions: (B) The (one-period) increments of prices and stocks are uniformly bounded, and (M) the stochastic process of prices and stocks is Markovian, with stationary transition probabilities and a discrete state space. With these assumptions we can show that the Markov process converges to a stationary process, which, as we have noted, is the stochastic analog of an "equilibrium". Our assumptions do not preclude a multiplicity of equilibria, but they do ensure that there are only finitely many of them. In general, from any starting state there will be a probability distribution on the set of ultimate equilibria, i.e., on the set of alternate stationary processes to which the process may converge.

Even without the Markovian assumption (M) we can prove a stabilitylike result: There is a bounded set D such that from every initial state, every trajectory (almost surely) enters D, and the expected time to the first entry is finite.

Our results can be compared with those of Green and Majumdar [4]. Green and Majumdar consider a tâtonnement-type Markovian adjustment process in which the state variables are the prices and excess demands (stocks are not explicitly represented in the system). They give sufficient conditions for the process to be ergodic, i.e., to converge to a unique stationary process. However, the stationary process (equilibrium) may have the property that the mean excess demand is different from zero. This contrasts with the situation in our model in which the stationarity of stocks implies that the mean excess demand (flow) is zero (unless, of course, there is asymmetric attrition of inventories and back-orders).

[1] In addition to being deterministic, the model used by McKenzie (and others in that literature) differs from ours in that the state variables are prices and *excess demands*, rather than prices and stocks. For a general discussion of such tâtonnement models see, e.g., [2, Chaps. 11, and 12].

2. THE STOCHASTIC STABILITY THEOREM

We consider a system of H interrelated markets, operating at dates $t = 0, 1, \ldots,$ *ad infinitum*. At each date t, the state of market h is characterized by the price p_t^h of commodity h and the stock (inventory) v_t^h of that commodity. We shall assume that $p_t^h \in R_+$ and $v_t^h \in R$. Negative stocks are interpreted as accepted but unfilled orders. Let

$$p_t = (p_t^1, \ldots, p_t^H) \quad \text{and} \quad v_t = (v_t^1, \ldots, v_t^H)$$

denote the H – tuples of prices and stocks, respectively, at date t. The *state* s_t of the economy at date t is characterized by the pair $s_t = (p_t, v_t)$.

We make four assumptions about the *process* (s_t), which are listed below. The first is that (s_t) is a *stochastic process*. In the second part of the theorem we strengthen this assumption by assuming that the stochastic process (s_t) is *Markovian* with stationary transition probabilities and discrete state space. The second assumption (B) is technical, but essential; it requires the increments in the variables p_t^h, v_t^h to be uniformly bounded. The third and fourth assumptions (PA and SA) concern price and stock adjustment, respectively. These last two assumptions can be paraphrased as follows: For every commodity when *its* stock is sufficiently high (low), its own price will fall (rise) *on the average*, and when its price is sufficiently high (low), its stock will rise (fall) *on the average*.

We now give the precise statements of these four assumptions.

(S) *The process* (p_t, v_t) *is a stochastic process defined on some probability space* $(\Omega, \mathscr{F}, P)$.

Let $\mathscr{F}_t$ denote the sub-σ-field of $\mathscr{F}$ generated by partial histories of the process up through date t; i.e., $\mathscr{F}_t$ is the smallest sub-σ-field of $\mathscr{F}$ such that the random variables $s_1, \ldots, s_t$ are measurable. All statements about random variables defined on $(\Omega, \mathscr{F}, P)$ are to be understood as holding P-almost surely.

(B) *The coordinates of* p_t *and* v_t *have uniformly bounded increments. This bound will be denoted by* γ; i.e., *for every* h *and* t, $|p_{t+1}^h - p_t^h| \le \gamma$ *and* $|v_{t+1}^h - v_t^h| \le \gamma$.

(PA) *For every commodity* h *there exist numbers* $v^{*h} > 0, \alpha^{*h} > 0, v_*^h \le 0,$ and $\alpha_*^h > 0$ *such that*

1. $E(p_{t+1}^h | \mathscr{F}_t) \le \max\{0, p_t^h - \alpha^{*h}\}$ *on* $\{v_t^h \ge v^{*h}\}$,
2. $E(p_{t+1}^h | \mathscr{F}_t) \ge p_t^h + \alpha_*^h$ *on* $\{v_t^h \le v_*^h\}$.

(SA) *For every commodity* h *there are positive numbers* $p^{*h}, \beta^{*h}, p_*^h, \beta_*^h$ *with* $p_*^h < p^{*h}$ *such that*

1. $E(v_{t+1}^h | \mathscr{F}_t) \ge v_t^h + \beta^{*h}$ *on* $\{p_t^h \ge p^{*h}\}$,
2. $E(v_{t+1}^h | \mathscr{F}_t) \le \max(0, v_t^h - \beta_*^h)$ *on* $\{p_t^h \le p_*^h\}$.

THEOREM *Assumptions (S), (B), (PA) and (SA) imply that there exists a bounded set $D \subset R_+^H \times R^H$ of states of the stochastic process (p_t, v_t) such that P-almost every trajectory intersects D, and the expected time for the first intersection is finite*; i.e.,

$$E(T) < \infty, \quad \text{where} \quad T(\omega) = \inf\{t \mid (p_t(\omega), v_t(\omega)) \in D\}.$$

Furthermore, if the stochastic process (p_t, v_t) is Markovian with stationary transition probabilities and discrete state space, then there is a partition of the state space into finitely many positive recurrent classes and a set of transient states that contains no closed set.

We would like to mention that the first part of the theorem can be strengthened. Using a result of Föllmer [3], who generalized and strengthened considerably an argument that we used in an earlier version of this paper,[2] one can assert that the set D is positive recurrent in the sense that every trajectory spends a positive fraction of time in D; i.e., there exists a number $c > 0$ such that

$$\liminf_t \, (1/t) \sum_{k=0}^{t-1} 1_D \circ s_k \geq c, \qquad \text{P-a.s.}$$

(For any subset A of a set B, the symbol 1_A denotes the function that is 1 on A and 0 otherwise, i.e., the "indicator function" for the set A.) For a proof of this stronger result we refer to Föllmer [3].

Proof of the Theorem

The complex part of the proof consists in showing that there exists a positive function L (Liapunov) defined on the state space S of the process (s_t) such that for $d \geq 0$ the set $\{s \in S \mid L(s) \leq d\}$ is bounded, and that the stochastic process $(L \circ s_t)$ has the following "conditional strict supermartingale" property: (SSM) *There exist $\varepsilon > 0$ and $d > 0$ such that*

$$E(L \circ s_{t+1}) \mid \mathscr{F}_t) \leq L \circ s_t - \varepsilon \qquad \text{on} \quad \{L \circ s_t \geq d\}.$$

In the first part of the proof we show that the existence of a function L with the above properties implies the theorem.[3] In the second part we construct the function L.

I. Lemma 1 proves the first assertion of the theorem.

[2] Presented at the Third World Congress of the Econometric Society, Toronto, 1975.

[3] In this part of the proof we use several suggestions of Föllmer, which are gratefully acknowledged. Our proof in an earlier version of this paper was less elementary.

LEMMA 1 *Given a probability space $(\Omega, \mathscr{F}, P)$, an increasing sequence $(\mathscr{F}_t)$ of sub-σ-fields of $\mathscr{F}$, and a sequence (x_t) of positive random variables such that x_t is $\mathscr{F}_t$-measurable and $E(x_0) < \infty$, assume that there exist $\varepsilon > 0$ and $d > 0$ such that*

$$E(x_t - x_{t+1} \mid \mathscr{F}_n) \geq \varepsilon \qquad \text{on} \quad \{\omega \mid x_t(\omega) \geq d\}.$$

Let $T(\omega) = \inf\{t \mid x_t(\omega) \leq d\}$. Then $E(T) < \infty$. Hence, in particular, $T(\omega) < \infty$, P-a.s.

Proof For every integer N let $T_N(\omega) = \min\{T(\omega), N\}$, and let x_{T_N} be the random variable defined by $x_{T_N}(\omega) = x_t(\omega)$, when $t = T_N(\omega)$. Then

$$\begin{aligned} E(x_0 - x_{T_N}) &= E\left(\sum_{t=0}^{N} (x_t - x_{t+1}) \cdot 1_{\{T_N > t\}}\right) \\ &= \sum_{t=0}^{N} E[E(x_t - x_{t+1}) \cdot 1_{\{T_N > T\}} \mid \mathscr{F}_t] \\ &= \sum_{t=0}^{N} E[E(x_t - x_{t+1} \mid \mathscr{F}_t) \cdot 1_{\{T_N > t\}}], \end{aligned}$$

since $1_{\{T_N > t\}}$ is $\mathscr{F}_t$-measurable. By the above assumption (SSM) we obtain

$$E[E(x_t - x_{t+1} \mid \mathscr{F}_t) \cdot 1_{\{T_N > t\}}] \geq \varepsilon \cdot P\{T_N > t\}.$$

Hence

$$E(x_0) \geq E(x_0 - x_{T_N}) \geq \varepsilon \cdot E(T_N) \qquad \text{for all} \quad N.$$

By the monotone convergence theorem this implies $E(x_0) \geq \varepsilon E(T)$. Hence $E(T) < \infty$, since $E(x_0) < \infty$. Q.E.D.

We shall now assume that (s_t) is a Markov process with stationary transition probabilities and *discrete* state space S (i.e., S is a countable subset of $R_+^H \times R^H$ such that every bounded set is finite). Then every state in S is either recurrent or transient according to the classification of states for Markov processes. We shall show that

(1) *every recurrent state of (s_t) is positive recurrent.*

Let the state s be recurrent. Then its communicating class is closed; hence, by the first part of the theorem, the class intersects the set D. Since recurrence is a class property, we can assume without loss of generality that $s \in D$.

Let $R(\omega) = \inf\{t \geq 1 \mid s_t(\omega) = s\}$. We have to show that the expected time to return to s is finite; i.e., $E_s(R) < \infty$.

We now consider the Markov process "on" the finite set D. This process is formally defined as follows: Let d_t denote the successive times the process

(s_t) spends in D; thus, at time $d_t(\omega)$ the trajectory $s_t(\omega)$ is for the tth time in D. The process (s_t^D) "on" D is defined by $s_t^D = s_{d_t}$. The state s, being recurrent for the process (s_t), is also recurrent for the process on D. Since this process is a finite Markov chain it follows that s is positive recurrent for the process on D; i.e.,

$$E_s(R^D) < \infty, \qquad \text{where} \quad R^D = \inf\{t \geq 1 \,|\, s_t^D = s\}.$$

Let r_t denote the waiting time from d_{t-1} until the next return to D; i.e., $r_t = d_t - d_{t-1}$, or equivalently,

$$r_t(\omega) = \inf\{t' \geq 1 \,|\, s_n(\omega) \in D, n = t' + d_t(\omega)\}.$$

With this notation we obtain

$$R = r_1 + \cdots + r_{R^D}.$$

$$E_s(R) = E_s\left[\sum_{t=1}^{\infty} r_t \cdot 1_{\{R^D \geq t\}}\right] = \sum_{t=1}^{\infty} E_s[E_s(r_t \cdot 1_{\{R^D \geq t\}} \,|\, \mathscr{F}_{d_t})]$$

$$= \sum_{t=1}^{\infty} E_s[E_s(r_t \,|\, \mathscr{F}_{d_t}) \cdot 1_{\{R^D \geq t\}}].$$

Now, $E_s(r_t \,|\, \mathscr{F}_{d_{t-1}}) = E_s d_{t-1}$ (return time to D). Hence we obtain from the first part of the theorem that $E_s(r_t \,|\, \mathscr{F}_{d_{t-1}})$ is bounded. Since D is finite, we obtain that there exists a bound b such that

$$E_s(r_t \,|\, \mathscr{F}_{d_{t-1}}) \leq b \qquad \text{for all} \quad t.$$

Hence it follows that $E_s(R) \leq b \cdot E_s(R^D) < \infty$.

Let C be a closed set of states. By the first part of the theorem and the strong Markov property, every sample path in C intersects D infinitely often. Hence $C \cap D$ must contain a recurrent state. This proves

(2) *there are recurrent, and hence by* (1) *positive recurrent, classes; since D is finite, there are at most finitely many recurrent classes;*

(3) *there is no closed set of transient states; hence every transient state is inessential.*

II. *Existence of a Liapunov function L.* First we make some notational simplifications. Define

$$p^* = \max_h p^{*h}, \qquad p_* = \min_h p_*^h,$$

$$\beta = \min_h \min\{\beta^{*h}, \beta_*^h\},$$

$$v^* = \max_h v^{*h}, \qquad v_* = \min_h v_*^h$$

$$\alpha = \min_h \min\{\alpha^{*h}, \alpha_*^h\}.$$

Assumptions (PA) and (SA) are now satisfied with p^{*h}, p_*^h, v^{*h}, v_*^h replaced by p^*, p_*, v^*, v_*, respectively, with β^{*h} and β_*^h replaced by β, and with α^{*h} and α_*^h replaced by α. Note that α, β, and p_* are positive. We may assume that $p^* > p_*$, $v^* > v_*$, $\alpha < p_*$, and $\beta < v^*$.

From now on we fix a commodity, say h, and consider the stochastic process (p_t^h, v_t^h) on the state space $S_h = R_+ \times R$. We prove the following lemma:

LEMMA 2 *There is a function L^h of S_h into R that has the following properties: there are numbers $d^h > 0$ and $\varepsilon^h > 0$ such that*

(i) *the set $\{(p, v) \in S_h \,|\, L^h \circ (p, v) \leq d^h\}$ is bounded;*
(ii) $E(L^h \circ (p_{t+1}^h, v_{t+1}^h) \,|\, \mathscr{F}_t) \leq L^h \circ (p_t^h, v_t^h) - \varepsilon^h$, *on* $\{L^h \circ (p_t^h, v_t^h) \geq d^h\}$.

Proof To simplify the notation, we suppress the index h.

The function $L(p, v)$ shall be of the form

$$\max_{0 \leq m \leq M} K_m(p, v),$$

where the functions K_m (to be specified later) are of the form:

if $0 \leq m \leq N$,

$$K_m(p, v) = a_m p - v + c_m, \qquad a_0 < a_1 < \cdots < a_{N-1} < 0, \qquad \text{and} \qquad a_N > 0;$$

if $N \leq m \leq M$,

$$K_m(p, v) = a_N p + b_m v + c_m,$$

$$-1 = b_N < b_{N+1} < \cdots < b_{M-1} < 0, \qquad \text{and} \qquad b_M > 0.$$

We shall show that the function L will have the required property provided the numbers N and M and the coefficients a_m, b_m, and c_m are suitably chosen. Figure 1 illustrates two level curves of the function L. With the notation used in Fig. 1 we have

$$K_m(p^* + \gamma(1 + 2m), v) = K_{m+1}(p^* + \gamma(1 + 2m), v), \qquad 0 \leq m < N;$$

(4) *hence* $c_{m+1} - c_m = -(a_{m+1} - a_m)(p^* + \gamma(1 + 2m))$ *and*

$$K_m(p, v^* + \gamma(1 + 2(m - N))) = K_{m+1}(p, v^* + \gamma(1 + 2(m - N))), \qquad N \leq m < M;$$

hence $c_{m+1} - c_m = -(b_{m+1} - b_m)(v^* + \gamma(1 + 2(m - N)))$.

We now determine the integer N and the slope of the level curve $K_m =$ constant, i.e., a_m for $m \leq N$.

(a_0): Let a_0 be such that $\varepsilon_0 =_{Df} -a_0\alpha - \gamma > 0$; hence $a_0 < 0$.
(a_N): Let $\bar{a} > 0$, but such that $\bar{\varepsilon} =_{Df} \beta - \bar{a}\gamma > 0$.

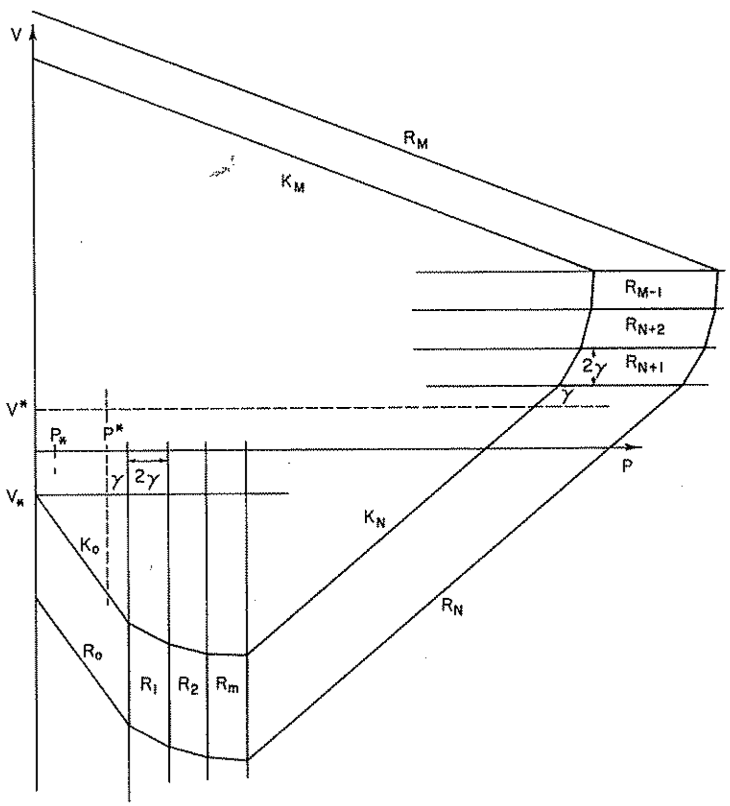

FIGURE 1

(N) The integer N is then determined by

$$\frac{\bar{a}-a_0}{N} < \frac{1}{\gamma}\min\{\varepsilon_0, \bar{\varepsilon}\}.$$

Now we define

(a_m): $a_1 = a_0 + \Delta$, $a_m = a_0 + m\Delta$, $a_N = \bar{a}$, where $\Delta = (\bar{a} - a_0)/N$.

Let $\varepsilon =_{Df} -\Delta\gamma + \min\{\varepsilon_0, \bar{\varepsilon}\}$; hence $0 < \varepsilon < \min\{\varepsilon_0, \bar{\varepsilon}\}$. We now prove the strict supermartingale property for $L(s_t)$ on $\bigcup_{m=0}^{N} R_m$; i.e.,

(SSM) $E(L(p_{t+1}, v_{t+1}) \mid \mathscr{F}_t) \le L(p_t, v_t) - \varepsilon$ *on* $(p_t, v_t) \in \bigcup_{m=0}^{N} R_m$, *where the regions* R_m *are defined as in Fig.* 1.

1. Let $R_0^- = \{(p, v) \in R_0 \mid p \le p^*\}$. On $\{s_t \in R_0^-\}$ then follows

$$E(L(p_{t+1}, v_{t+1}) \mid \mathscr{F}_t) = E(K_0(p_{t+1}, v_{t+1}) \mid \mathscr{F}_t)$$
$$= a_0 E(p_{t+1} \mid \mathscr{F}_t) - E(v_{t+1} \mid \mathscr{F}_t) + c_0.$$

By assumption (PA2), $E(p_{t+1} \mid \mathscr{F}_t) \ge p_t + \alpha$ and by assumption (B), $v_{t+1} \ge v_t - \gamma$. Hence, on $\{s_t \in R_0^-\}$,

$$E(L(s_{t+1} \mid \mathscr{F}_t) \le a_0 p_t + a_0\alpha - v_t + \gamma + c_0 = K_0 \circ (p_t, v_t) + a_0\alpha + \gamma$$
$$\le L \circ (p_t, v_t) - \varepsilon_0 \le L \circ s_t - \varepsilon.$$

2. Let $R_m^+ = \{(p, v) \in R_m \mid (p - \gamma, v) \in R_m,\ 0 \le m < N$. On $\{s_t \in R_m^+\}$ it follows

$$E(L \circ s_{t+1} \mid \mathscr{F}_t) = E(\max\{K_m \circ s_{t+1}, K_{m+1} \circ s_{t+1}\} \mid \mathscr{F}_t)$$
$$= E(K_m \circ s_{t+1} \mid \mathscr{F}_t) + \int_{R_{m+1}} [K_{m+1}(s) - K_m(s)]\hat{P}(ds \mid \mathscr{F}_t),$$

where $\hat{P}(s \mid \mathscr{F}_t)$ is a regular conditional distribution for s_{t+1} given $\mathscr{F}_t$. By definition of K_m we obtain from (4)

$$K_{m+1}(s) - K_m(s) = (a_{m+1} - a_m)p + (c_{m+1} - c_m)$$
$$= (a_{m+1} - a_m)(p - (p^* + \gamma(1 + 2m))).$$

Hence

$$\int_{R_{m+1}} [K_{m+1}(s) - K_m(s)]\hat{P}(ds \mid \mathscr{F}_t) \le \Delta \cdot \gamma.$$

By assumptions (PA2), (SA1), and (B) for $0 \le m < N$, we obtain on $\{s_t \in R_m^+\}$,

$$E(K_m \circ s_{t+1} \mid \mathscr{F}_t) \le a_m E(p_{t+1} \mid \mathscr{F}_t) - E(v_{t+1} \mid \mathscr{F}_t) + c_m \le K_m \circ s_t + \alpha a_m - \beta.$$

Hence, since $a_m < a_N$ and $\alpha < \gamma$,

$$E(L \circ s_{t+1} \mid \mathscr{F}_t) \le L \circ s_t + \alpha a_m - \beta + \Delta\gamma \le L \circ s_t + \gamma a_N - \beta$$
$$\le L \circ s_t - \varepsilon \quad \text{on} \quad \{s_t \in R_m^+\}.$$

3. Let $R_m^- = \{(p, v) \in R_m \mid (p + \gamma, v) \in R_m\}$, $1 \le m \le N$. The proof is similar to the previous case. On $\{s_t \in R_m^-\}$ it follows that

$$E(L \circ s_{t+1} \mid \mathscr{F}) = E(K_m \circ s_{t+1} \mid \mathscr{F}_t) + \int_{R_{m-1}} [K_{m-1}(s) - K_m(s)]\hat{P}(ds \mid \mathscr{F}_t)$$
$$\le (K_m \circ s_t + \alpha a_m - \beta) + \Delta\gamma$$
$$\le L \circ s_t - \varepsilon.$$

4. Let $R_N^+ = \{(p, v) \in R_N \mid (p - \gamma, v) \in R_N\}$. On $\{s_t \in R_N^+\}$ we obtain by assumptions (SA1) and (B) and property (a_N),

$$E(L \circ s_{t+1} \mid \mathscr{F}_t) = E(K_N \circ s_{t+1} \mid \mathscr{F}_t) \le K_N \circ s_t + a_N\gamma - \beta$$
$$\le L \circ s_t - \varepsilon.$$

This completes the proof of property (SSM) on $\{s_t \in \bigcup_{m=0}^{N} R_m\}$.

We shall not give the details for the functions K_m, $m > N$. The arguments are similar. One has to increase the coefficient b_m slowly enough (recall $b_N = -1$) until it becomes positive but not greater than $(\alpha/\gamma) \cdot a_N$. We just define the last function K_M. The slow turning around from K_N to K_M is done analogously as from K_0 to K_N.

Let $\bar{b} > 0$ such that $\bar{b} < (\alpha/\gamma)a_N$. Consider the function $K_M(p, v) = a_N p + \bar{b}v + \bar{c}$. On $\{s_t \in R_M$ and $p_t \geq p_*\}$ we obtain from assumption (PA1) (note that $\alpha < p_*$) and (B) that

$$E(K_M \circ s_{t+1} | s_t) = a_N E(p_{t+1} | s_t) + \bar{b}E(v_{t+1} | s_t) + \bar{c}$$
$$\leq K_M \circ s_t - a_N\alpha + \bar{b}\gamma = K_M \circ s_t - \varepsilon_2$$

with $\varepsilon_2 = a_N\alpha - \bar{b}\gamma > 0$.

On $\{s_t \in R_M$ and $p_t < p_*\}$ we obtain from assumption (SA2) that

$$E(K_M \circ s_{t+1} | s_t) \leq a_N p_t + \bar{b}(v_t - \beta) + \bar{c} = K_M \circ s_t - \bar{b}\beta.$$

In summary, we showed that there is $\varepsilon > 0$ such that

$$E(L \circ s_t | \mathscr{F}_t) \leq L \circ s_t - \varepsilon \qquad \text{on} \quad s_t \in \textstyle\bigcup_{m=0}^{M} R_m, \quad \text{P-a.s.}$$

The shape of the level curves of L clearly shows that the set $\{s \in S | L(s) \leq d\}$ is bounded. This proves Lemma 2.

The function L is now defined by

$$L(s^1, \ldots, s^H) = \sum_{h=1}^{H} (L^h(s^h))^2.$$

It remains to verify that the stochastic process $L \circ s_t$ has the "conditional strict supermartingale" property. For every stochastic process $(L^h \circ s_t^h)$, $h = 1, \ldots, H$, there are, by Lemma 2, numbers $d^h > 0$ and $\varepsilon^h > 0$ such that properties (i) and (ii) of Lemma 2 hold. Let $\delta = \max_h d^h$ and $\bar{\varepsilon} = \min_h \varepsilon^h$.

LEMMA 3 *For every* $0 < \varepsilon \leq \bar{\varepsilon}$ *there is* $d > 0$ *such that*

$$E(L \circ s_{t+1} | \mathscr{F}) \leq L \circ s_t - \varepsilon \qquad \textit{on} \quad L \circ s_t \geq d.$$

Proof To simplify the notation, $l_t^h = L^h(s_t)$. Since

$$E((l_{t+1}^h)^2 | \mathscr{F}_t) = E(l_{t+1}^h | \mathscr{F}_t)^2 + \operatorname{var}(l_{t+1}^h | \mathscr{F}_t),$$

we obtain from Lemma 2 that

$$E((l_{t+1}^h)^2 | \mathscr{F}_t) \leq (l_t^h - \varepsilon)^2 + \lambda^2 \qquad \text{on} \quad \{l_t^h > \delta\},$$

where λ denotes a uniform bound on the increments of the processes $L^h \circ s_t^h$, which exists according to assumption (B). If $l_t^h \leq \delta$, then, clearly, $l_{t+1}^h \leq \delta + \lambda$. Hence

$$E(L \circ s_{t+1} | \mathscr{F}_t) = \sum_{h=1}^{H} E((l_{t+1}^h)^2 | \mathscr{F}_t)$$
$$\leq \sum_{\{h | l_t^h > \delta\}} [(l_t^h - \varepsilon)^2 + \lambda^2] + \sum_{\{h | l_t^h \leq \delta\}} (\delta + \lambda)^2$$
$$\leq L \circ s_t - \sum_{\{h | l_t^h > \delta\}} 2\varepsilon l_t^h + H(\varepsilon^2 + \lambda^2 + \delta^2 + 2\delta\lambda).$$

The third term is a given number and the second term can be made as negative as we want. Indeed, $L(s_t) \geq d$ implies for at least one h that $l_t^h > \sqrt{d/H}$. Hence, given the numbers $0 < \varepsilon \leq \bar{\varepsilon}$, λ, and δ, there is a number d such that

$$E(L \circ s_{t+1} | \mathscr{F}_t) \leq L \circ s_t - \varepsilon \qquad \text{on} \quad L \circ s_t \geq d. \quad \text{Q.E.D.}$$

Remark As the proof for the existence of the function L^h shows, the assumpions (PA) and (SA) can be slightly weakened.

(PA) 1. $E(p_{t+1}^h | \mathscr{F}_t) \leq p_t^h - \alpha^*$ *on* $\{v_t^h \geq v^*$ *and* $p_t^h \geq p_*^h\}$
$\leq p_t^h$ *on* $\{v_t^h \geq v^*\}$;
2. $E(p_{t+1}^h | \mathscr{F}_t) \geq p_t^h + \alpha_*$ *on* $\{v_t^h \leq v_*$ *and* $p_t^h \leq p^*\}$
$\geq p_t^h$ *on* $\{v_t^h \leq v_*\}$.

(SA) 1. $E(v_{t+1}^h | \mathscr{F}_t) \geq v_t^h + \beta^*$ *on* $\{p_t^h \geq p^*$ *and* $v_t^h \leq v^*\}$
$\geq v_t^h$ *on* $\{p_t^h \geq p^*\}$;
2. $E(v_{t+1}^h | \mathscr{F}_t) \geq v_t^h - \beta_*$ *on* $\{p_t^h \leq p_*$ *and* $v_t^h \geq v^*\}$
$\geq v_t^h$ *on* $\{p_t^h \leq p_*\}$.

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