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Collusive Behavior in Noncooperative Epsilon-Equilibria of Oligopolies with Long but Finite Lives

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In a game of a finite number of repetitions of a Cournot-type model of an industry, if firms are satisfied to get close to (but not necessarily achieve) their optimal responses to other firms' sequential strategies, then in the resulting noncooperative "equilibria" of the sequential market game, (1) if the lifetime of the industry is large compared to the number of firms, there are equilibria corresponding to any given duration of the cartel, whereas (2) if the number of firms is large compared to the industry's lifetime, all equilibria will be close (in some sense) to the competitive equilibrium.

1. INTRODUCTION

In 1838 Augustin Cournot introduced his model of market equilibrium, which has become known in modern game theory as a noncooperative (or Nash) equilibrium [1]. Cournot's model was intended to describe an industry with a fixed number of firms with convex cost functions, producing a homogeneous product, in which each firm's action was to choose an output (or rate of output), and in which the market price was determined by the total industry output and the market demand function. A Cournot–Nash equilibrium is a combination of outputs, one for each firm, such that no firm can increase its profit by changing its output alone.

Cournot thought of his model as describing "competition" among firms; this corresponds to what we call today the "noncooperative" character of the equilibrium. He showed that, if the number of firms is regarded as a parameter of the market, a larger number of firms would lead to a larger industry output and a lower price (in equilibrium), and that as the number of firms increased without limit, the corresponding equilibria would converge to the situation he called "unlimited competition," in which marginal cost equaled price.

If there are at least two firms, then they can make more profit than in the Cournot–Nash equilibrium (CNE) by forming a cartel in which the total industry output is chosen to maximize total industry profit, and this profit

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is shared equally among the firms. This corresponds to what would be called a cooperative solution of the game.

What determines whether there will be a cooperative rather than a non-cooperative outcome in the market situation? If the market situation were repeated a number of periods, then, even in the absence of some institution (such as regulation) to enforce cooperation, it would seem that the firms would have opportunities to signal their willingness to cooperate. Furthermore, the larger the number of periods, the greater would be the relative loss due to defection from the cartel and a reversion to the CNE outputs and profits. On the other hand, the larger the number of firms, the greater would be the difficulty of holding the cartel together, at least according to conventional wisdom.

If the market situation is repeated, say T times, then this gives rise to a game in which the strategies available to the players (firms) are sequential. A sequential strategy is a sequence of functions, one for each period, each of which determines the output in that period as a function of the outputs of all firms in the previous periods. One can show that every perfect Cournot–Nash equilibrium of the T-period game results in each firm producing its one-period CNE output in each period.[1] (For this result, one assumes that each firm's objective is to maximize its average, or total discounted, profit over the T periods.) I should emphasize that this property of T-period perfect CNE's is satisfied for any finite T, no matter how large. On the other hand, one can show that if T is infinite, and each firm's objective is to maximize its long-run-average profit, then there is a perfect CNE of the (infinite-period) game that results in indefinite survival of the cartel.[2] The result for infinite T goes part of the way towards confirming our intuition about the determinants of cooperation, but has the unsatisfactory feature that the infinite case is not well approximated by the case of large but finite T. Another unsatisfactory feature of the result is that it holds for any number of firms, no matter how large.

In this paper I explore the consequences of weakening the strict rationality of the Cournot–Nash equilibrium concept, so that each player is satisfied to get close to (but not necessarily achieve) his best response to the other players' strategies. Formally, an *epsilon-equilibrium* is a combination of strategies, one for each player, such that each player's strategy is within epsilon in utility (e.g., average profit) of the maximum possible against the other players' strategies. I shall show that, for any fixed positive epsilon,

[1] A perfect equilibrium of a sequential game is defined in Section 3. As will be seen there, the restriction that the equilibrium be perfect excludes equilibria based on unconvincing threats. The statement in the text is no longer correct if one eliminates the condition that equilibria be perfect.

[2] This is a special case of a more general theorem on perfect equilibria of infinite supergames, due to Aumann and Shapley (unpublished) and Rubinstein [5, 6].

any given number of firms, and any integer k, there is an integer T_0 such that, for all T exceeding T_0, there is a perfect epsilon-equilibrium of the T-period game in which the cartel lasts exactly k periods. In choosing T_0, it is sufficient to make $(T_0 - k)$ larger than some number that depends on epsilon and the number of firms. In particular, one can (approximately) achieve any desired fraction, k/T, by taking T large enough.

The effect on perfect epsilon-equilibria of varying the number of firms depends on the relationship between the industry demand function and the number of firms. (The latter is a parameter of the game.) Suppose first that, as we compare markets with different numbers of firms, the demand price is the same function of the *average* industry output *per firm*; call this the *replication case*. This would be the situation that would obtain if, when we doubled the number of firms, we also duplicated the population of consumers. In the replication case, one can show that, for every fixed positive epsilon and T, as the number of firms increases without limit all perfect epsilon-equilibria approach competitive equilibrium (Cournot's "unlimited competition") in the following sense. For every positive ϵ, T, and N (>1), there is a number $B(\epsilon, T, N)$ such that, in every perfect ϵ-equilibrium of the T-period game, each firm's output and total industry output at each date are all within $B(\epsilon, T, N)$ of their corresponding one-period CNE values. The bounds $B(\epsilon, T, N)$ can be chosen so that, (1) for every ϵ and T, the ratios $B(\epsilon, T, N)/N^{1/2}$ are uniformly bounded in N, and (2) for every T and N, $B(\epsilon, T, N)$ approaches zero as ϵ approaches zero. One can further show that, for fixed ϵ and T, as N increases without limit, at every date (1) average output per firm approaches the competitive equilibrium output per firm, (2) market price approaches marginal cost, and (3) every firm's relative share of total industry output approaches zero. Finally, under the same conditions, the difference between each firm's profit and its one-period CNE profit is uniformly bounded in N, and approaches zero as ϵ approaches zero.

Leaving the replication case, suppose now that the industry demand function is fixed as we compare markets with different numbers of firms; call this the *fixed-demand* case. In this case, one can show that, for any fixed number of periods, T, there is a number B (depending on epsilon and T) such that, in all perfect epsilon-equilibria of the T-period game, the outputs and profits of all firms, total industry output and profit, and market price are all within B of their corresponding one-period CNE values, *uniformly in the number of firms*. Furthermore, the bound B tends to zero as epsilon tends to zero, with T fixed. This implies that, in the fixed-demand case, for any fixed number of periods and *small* epsilon, all perfect epsilon-equilibria are "close" to competitive equilibrium in terms of outputs, profits, and prices, for a sufficiently large number of firms.

Note that the effect of increasing the number of firms on reducing the possibilities for cooperation is observed in the replication case, but not in the fixed-demand case. In both cases, reducing epsilon reduces the possibilities for cooperation (i.e., keeps all perfect epsilon-equilibria closer to the CNE), and increasing the number of firms brings the CNE closer, of course, to the competitive equilibrium. Thus, for the replication case, one can paraphrase the results of this paper as follows: if firms are satisfied to get close to (but not necessarily achieve) their optimal responses to other firms' strategies, then in the resulting noncooperative "equilibria" of the sequential market game, (1) if the lifetime of the industry is large compared to the number of firms, there are equilibria corresponding to any given duration of the cartel, whereas (2) if the number of firms is large compared to the lifetime of the industry, all equilibria will be close (in some sense) to the competitive equilibrium.

Although the replication case is the one of central interest here, I shall use the fixed demand case (Section 5) as a stepping-stone in the analysis of the replication case (Section 6).

As will be seen below, several alternative definitions of perfect epsilon-equilibria may be reasonably considered. I shall begin the analysis with the simplest one (Section 4). A more satisfactory definition, is introduced in Section 7. An important behavioral implication of this second definition is that cooperation will tend to break down as the industry approaches the horizon T. In Section 8, I discuss some alternative interpretations of epsilon.

The entire analysis in the present paper is carried out only for a special model, in which the market demand function is linear, and all firms have the same linear homogeneous cost function (average and marginal costs are equal and constant). However, the analysis can be extended easily to the case in which each firm has a fixed (setup) cost of production, and in which there is free entry. In this case, the number of firms in the industry is endogenous.[3] This extension will be discussed in a forthcoming paper.

2. The One-Period Cournot Game

Consider an industry producing a single homogeneous product, with N firms. The cost to a single firm of producing a quantity Q is γQ. If firm j produces quantity Q_j, the market-clearing price P is determined by the industry demand function

$$P = \alpha - \beta \sum_{j=1}^{N} Q_j, \tag{2.1}$$

[3] For this situation, Cournot–Nash equilibria in the one-period game have been studied by Novshek [2] and by Novshek and Sonnenschein [3].

if this is positive; otherwise it is zero. The profit to firm i is therefore

$$PQ_i - \gamma Q_i = \left(\alpha - \beta \sum_{j=1}^{N} Q_j\right) Q_i - \gamma Q_i$$

$$= \left(\alpha - \gamma - \beta \sum_{j \neq i} Q_j\right) Q_i - \beta Q_i^2.$$

Assume that $\alpha > \gamma$, and define

$$Q_i' \equiv \sum_{j \neq i} Q_j , \qquad \delta \equiv \alpha - \gamma; \tag{2.2}$$

then firm i's profit can be expressed as

$$p(Q_i , Q_i') \equiv (\delta - \beta Q_i') Q_i - \beta Q_i^2. \tag{2.3}$$

Equations (2) and (3) define a game with N players, in which the pure strategy of player i is a nonnegative number Q_i, and his utility is $p(Q_i , Q_i')$.

It is easily verified that, given Q_i', the Q_i that maximizes i's profit is

$$r(Q_i') \begin{cases} \equiv \dfrac{\delta - \beta Q_i'}{2\beta}, & \text{if } \textit{this is nonnegative}, \\ \equiv 0, & \textit{otherwise}. \end{cases} \tag{2.4}$$

I shall call $r(Q_i')$ firm i's *best response* to Q_i'. If the best response is positive, firm i's corresponding maximum profit is

$$g(Q_i') = \frac{(\delta - \beta Q_i')^2}{4\beta}; \tag{2.5}$$

otherwise it is zero. A *Cournot–Nash equilibrium* (CNE) is an N-tuple (Q_i) of outputs such that, for each i, Q_i is the best response to $\sum_{j \neq i} Q_j$. In other words, a CNE is a solution (Q_i) of

$$r\left(\sum_{j \neq i} Q_j\right) = Q_i , \qquad i = i,..., N.$$

It is easily verified that the unique CNE is given by

$$Q_i = Q_N^* = \frac{\delta}{(N+1)\beta}, \qquad i = 1,..., N, \tag{2.6}$$

and the corresponding CNE profit per firm is

$$\frac{\delta^2}{\beta(N+1)^2}. \tag{2.7}$$

Note that the total industry CNE output is

$$\frac{N\delta}{(N+1)\,\beta}, \tag{2.8}$$

and the total industry CNE profit is

$$\frac{N\delta^2}{(N+1)^2\,\beta}. \tag{2.9}$$

Therefore, as N increases without bound, total CNE industry output approaches δ/β, total industry CNE profit approaches zero, and CNE price approaches $\alpha - \delta = \gamma$ (i.e., marginal cost), all of which conditions characterize a competitive equilibrium.

If the industry acts as a cartel to maximize total industry profit then the cartel output is $(\delta/2\beta)$, and the corresponding cartel profit is $(\delta^2/4\beta)$. If the cartel output and profit were divided equally among the firms, then the corresponding cartel output per firm would be

$$\hat{Q}_N = \frac{\delta}{2\beta N}, \tag{2.10}$$

and the cartel profit per firm would be

$$\frac{\delta^2}{4\beta N}. \tag{2.11}$$

Note that if $N > 1$, then the cartel profit per firm is strictly greater than the CNE profit per firm, so that (from the point of view of the firms) the CNE is not Pareto optimal.

If the capacities of the firms are sufficiently large, then no coalition of fewer than N firms can guarantee itself more than a zero profit. That is to say, for any output of the coalition, there is an output of the other firms such that the coalition's profit is not greater than zero. Hence in this case, the core is the set of all nonnegative allocations of the cartel profit among the firms.[4] Given the symmetry among the firms, the equal division of the cartel profits is a "reasonable" target for cooperation, and in any case is the one to which attention will be given in this paper.

In what follows, I shall simplify the formulas by taking $\beta = 1$ and $\delta = 1$, unless notice is given to the contrary. This normalization will not entail any essential loss of generality.

[4] For a characterization of the core with arbitrary capacities, see Radner [4].

3. The Several-Period Cournot Game

Consider now a sequential, *T-period*, game in which the one-period game is repeated T times (T finite). The resulting utility to a firm is assumed to be the average of the T one-period profits. Let Q_{it} denote the output of firm i at date $t(1 \leqslant t \leqslant T)$, i.e., during the ith one-period game. A pure strategy for firm i is a sequence of functions, σ_{it}, one for each date t; the function for date t determines i's output at t as a function of the outputs of all firms at all previous dates. A Cournot–Nash equilibrium of the T-period game is a combination of strategies, one for each firm, such that each firm's strategy is a best response to the combination of the other firms' strategies.

The concept of *perfect* equilibrium of the T-period game has been introduced by Selten (1975) to rule out equilibria in which players use threats that are not credible. For every data t, let H_t denote the *history* of all the firms' outputs through t, i.e., the array of outputs Q_{ik}, $i = 1,..., N$, $k = 1,..., t$. For any sequential strategy σ_i for firm i, any date t and any history H_{t-1} let $\sigma_i[t, H_{t-1}]$ denote the *continuation* of σ_i from date t on, given the history $H_{\geqslant -1}$. A strategy combination (σ_i) is a *perfect* CNE if, for every date t and history H_{t-1}, the strategy combination $(\sigma_i[t, H_{t-1}])$ is a CNE of the sequential game corresponding to the remaining dates $t,..., T$. Note that in the definition of a perfect CNE one must test, for each t, whether the combination of continuations is a CNE for *all* possible histories H_{t-1}, not just the history that would be produced by the strategy combination (σ_i).

It is easy to verify that in every perfect CNE of the T-period game, each firm produces output Q^* at each date, where Q^* is given by (2.6). This can be seen by "working backwards," since at the end of period t the firms face a $(T - t)$- period game. The resulting utility to each firm is, from (2.7), $1/(N + 1)^2$. (Recall that $\beta = \delta = 1$.)

On the other hand, if each firm were to produce its cartel output $\hat{Q}$ at each date (see (2.10)), then the resulting utility to each player would be $1/4N$. Since there are several periods, the firms have the opportunity to react differently to cooperative and noncooperative moves by the other firms. For example, consider the following strategy: firm i produces output $\hat{Q}$ in each period as long as every other firm has been doing the same; thereafter firm i produces Q^* in each period. Call this strategy C_T. Formally, define D_i as follows:

$$D_i \begin{array}{l} = \infty, \quad \text{if} \quad Q_{jt} = \hat{Q} \quad \text{for all } t \text{ and all } j \neq i, \\ = \min\{t\colon Q_{jt} \neq \hat{Q} \text{ for some } j \neq i\}, \quad \text{otherwise.} \end{array} \tag{3.1}$$

The pure strategy C_T is defined by:

$$Q_{it} \begin{array}{ll} = \hat{Q} & \text{if} \quad t \leqslant D_i, \\ = Q^* & \text{if} \quad t > D_i. \end{array} \tag{3.2}$$

More generally, for any integer k between 0 and T define the pure strategy C_k by:

$$Q_{it} \begin{array}{ll} = \hat{Q} & \text{if} \quad t \leqslant \min(D_i, k), \\ = Q^* & \text{if} \quad t > \min(D_i, k). \end{array} \tag{3.3}$$

Note that if i uses the strategy C_0, then he always produces the CNE output Q^*.

The strategy C_k is a special case of a slightly more general class, which I shall call *trigger strategies of order k*. Let D_i be defined again as in (3.1) and let Q^D be some output. If $D_i \geqslant k$, then

$$\begin{array}{lll} & = \hat{Q} & \text{if} \quad t \leqslant k, \\ Q_{it} & = Q^D & \text{if} \quad t = k+1, \\ & = Q^* & \text{if} \quad t \geqslant k+2. \end{array} \tag{3.4}$$

If $D_i \leqslant k$, then

$$Q_{it} \begin{array}{ll} = \hat{Q} & \text{if} \quad t \leqslant D_i, \\ = Q^* & \text{if} \quad t > D_i. \end{array} \tag{3.5}$$

One might call Q^D the *defection output*, which i uses once only if all other firms have stayed with the cartel for (at least) k periods.

Suppose now that all firms other than i use the same trigger strategy of order $k > 0$, with some defection output $Q^D > \hat{Q}$. I shall show that i's best response is a trigger strategy of order $(k - 1)$, with a defection output equal to

$$\tilde{Q} \equiv r[(N-1)\,\hat{Q}] = \frac{N+1}{4N}. \tag{3.6}$$

Note that $\tilde{Q}$ is the best *one-period* response to a total output of $(N-1)\hat{Q}$ by all the other firms, and yields a one-period profit of

$$\frac{(N+1)^2}{16N^2}. \tag{3.7}$$

To prove this, first observe that if at some date t any firm i produces an output different from $\hat{Q}$, then at all subsequent dates all firms other than i will produce Q^*, and it will be optimal for firm i to do the same. Hence firm i's best response has the property that

$$\begin{array}{ll} Q_{jt} \neq \hat{Q} & \textit{for some } j \textit{ implies} \\ Q_{it'} = Q^* & \textit{for all } t' > t. \end{array} \tag{3.8}$$

It follows that firm i's best response to the given trigger strategies of order k is a trigger strategy of some order n. It is straightforward to verify that if $n < k$ then the optimal defection output is $\tilde{Q}$, with resulting *total* profit

$$\frac{n}{4N} + \frac{(N+1)^2}{16N^2} + \frac{T-n-1}{(N+1)^2}. \tag{3.9}$$

If $n = k$, then the optimal defection output is $r[(N-1)Q\tilde{Q}^D]$, and the corresponding total profit is

$$\frac{k}{4N} + g[(N-1)\, Q^D] + \frac{T-k-1}{(N+1)^2}. \tag{3.10}$$

Finally, if $n > k$, then i's defection output is irrelevant and the corresponding profit is

$$\frac{k}{4N} + p[\hat{Q}, (N-1)\, Q^D] + \frac{T-k-1}{(N+1)^2}. \tag{3.11}$$

In (3.10) and (3.11), if $k = T$, then it is to be understood that the final term in the equation is zero. Since $Q^D > \hat{Q}$,

$$\left(\frac{N+1}{4N}\right)^2 = g[(N-1)\, \hat{Q}] > g[(N-1)\, Q^D] \geqslant p[\hat{Q}, (N-1)\, Q^D].$$

Hence since (3.9) is increasing in n, it follows that i's optimal response has $n = (k-1)$. This completes the proof that i's *optimal response to a trigger strategy of order $k > 0$ with defection output $Q^D > \hat{Q}$ is a trigger stragegy of order $k-1$ with defection output $\tilde{Q}$. The resulting average* profit for i is

$$\left(\frac{1}{T}\right)\left[\frac{k-1}{4N} + \left(\frac{N+1}{4N}\right)^2 + \frac{T-k}{(N+1)^2}\right]. \tag{3.12}$$

Note that neither i's optimal response nor his resulting average profit depend on the other firms' defection output.

In particular, if all firms $j(\neq i)$ use the trigger strategy C_T, then i's best response gives him an average profit of

$$\frac{1}{T}\left[\frac{T-1}{4N} + \left(\frac{N+1}{4N}\right)^2\right], \tag{3.13}$$

whereas if *all* firms use C_T, then every firm's average profit is $(1/4N)$, which is the cartel profit (per firm). The difference between (3.13) and the cartel profit per firm is

$$\left(\frac{1}{T}\right)\left(\frac{N-1}{4N}\right)^2. \tag{3.14}$$

Hence, as T increases without limit, the advantage to any one firm of defecting from the cartel one period before the end of the game approaches zero.

4. Epsilon-Equilibria

In the previous section it was noted that, in the T-period game, all perfect Cournot–Nash equilibria have the property that each firm produces the one-period CNE output in each period. On the other hand, the advantage to any one firm of defecting from the cartel approaches zero as T gets large, provided the other firms use trigger strategies of order T.

These considerations suggest a weakened form of the Cournot–Nash equilibrium concept. For any positive number ϵ, an *ϵ-equilibrium* is an N-tuple of strategies, one for each firm, such that each fiirm's average profit is within ϵ of the maximum average profit it could obtain against the other firms' strategies. In this and the following sections I shall explore some of the properties of epsilon-equilibria in the T-period Cournot game. I do not, however, have a complete characterization of epsilon-equilibria in this game.

The first candidate for an epsilon-equilibrium is the situation in which each firm uses the trigger strategy C_k (with defection output Q^*). From (3.10) and (3.12) we see that the difference in average profit between the best response and C_k, for an individual firm, is

$$\left(\frac{1}{T}\right)\left[\left(\frac{N+1}{4N}\right)^2 - \frac{1}{4N}\right] = \left(\frac{1}{T}\right)\left(\frac{N-1}{4N}\right)^2. \tag{4.1}$$

Hence the N-tuple (C_k) is an ϵ-equilibrium with $k > 0$ if and only if

$$\left(\frac{1}{T}\right)\left(\frac{N-1}{4N}\right)^2 \leqslant \epsilon \quad \text{or} \quad T \geqslant \left(\frac{1}{\epsilon}\right)\left(\frac{N-1}{4N}\right)^2. \tag{4.2}$$

Note that (4.2) is independent of k, so that either (C_k) is an ϵ-equilibrium for all $k = 1,..., T$, or for none. Note, too, that for fixed ϵ, (4.2) is satisfied uniformly in N for sufficiently large T. Of course, (C_0) is a Cournot–Nash equilibrium, so it is an ϵ-equilibrium for all ϵ.

The concept of perfect CNE can be extended to epsilon-equilibria as follows. A strategy combination (σ_j) is a *perfect* ϵ-equilibrium if for every date t, every history H_{t-1}, and every firm i, the continuation of i's strategy from date t on, given the history H_{t-1}, is within ϵ of being the best response to the corresponding continuations of the other firms' strategies. In this definition, the utility of a continuation of a strategy is the average of the profits in *all* T periods. (For an alternative definition, see Section 7.)

It is easy to verify that (4.2) is a necessary and sufficient condition for the combination (C_k) of trigger strategies to be a perfect ϵ-equilibrium. Hence, *for any $\epsilon > 0$ there is a T_ϵ such that, for all $T \geqslant T_\epsilon$ and all $k = 0,..., T$, there is a perfect ϵ-equilibrium in which each firm produces its cartel output for* *exactly k periods*. Furthermore, one can take T_ϵ to be independent of the number of firms.

An examination of (3.9)–(3.11) shows that one can get similar results for N-tuples of trigger strategies that use defection outputs other than Q^*, and even for N-tuples of trigger strategies that differ among firms in both the orders of the trigger strategies and the defection outputs. No attempt will be made here to characterize all such perfect epsilon-equilibria, but it is clear that for fixed ϵ, the larger T is the larger, in some sense, is the set of perfect ϵ-equilibria.

In the rest of this paper, all epsilon-equilibria are to be understood as perfect, unless specific notice is given to the contrary.

5. Large Number of Firms: The Fixed-Demand Case

In the last section I considered how the set of (perfect) ϵ-equilibria varied with T, the number of periods, with the number of firms fixed. In this section I consider the effect of increasing N, the number of firms, with the horizon, T, fixed. I first analyze the case in which the total industry demand function remains fixed as the number of firms varies. The results for this case will be used to analyze the more interesting, but slightly more complicated, replication case (Section 6).

To begin the analysis of the fixed-demand case, first observe that, by (4.2), for fixed T, there is no (C_k) ϵ-equilibrium ($k \geqslant 1$) for ϵ sufficiently small.

I shall prove, in addition, a result that is in some ways stronger. Roughly speaking, I shall show that if ϵ is small then all ϵ-equilibria are close to the CNE, *uniformly in N*. To be precise I shall show: for every $\epsilon > 0$ and $T \geqslant 1$ there is a number $B(\epsilon, T)$ such that for every $N > 1$ and every ϵ-equilibrium the following are all bounded by $B(\epsilon, T)$:

$$\begin{gathered} | Q_{it} - Q_N^* |, \\ \left| \sum_{i=1}^{N} Q_{it} - NQ_N^* \right|, \\ \left| p\left(Q_{it}, \sum_{j \neq i} Q_{jt}\right) - \frac{1}{(N+1)^2} \right|, \end{gathered} \tag{5.1}$$

for $i = 1,\dots, N$, $t = 1,\dots, T$. In addition, for every T

$$\lim_{\epsilon \to 0} B(\epsilon, T) = 0. \tag{5.2}$$

The first line of (5.1) is the difference between firm i's output in period t and CNE output; the second line is the difference between total industry output in period t and industry CNE output; and the third line is the difference between firm i's profit in period t and CNE profit per firm. It follows that, for any positive number d, and any horizon T, there is a positive ϵ such that, for all N and all ϵ-equilibria of the T-period game with N firms, industry outputs and market prices in all periods $t = 1,..., T$ will be within d of their corresponding one-period CNE values (note that ϵ does not depend on N). In particular, as N increases without limit, the corresponding one-period CNE values will converge to their respective competitive equilibrium values, and hence any corresponding sequence of ϵ-equilibria will approach the "neighborhood" of competitive equilibrium defined by the distance d, with respect to total industry output and price.

To prove (5.1) and (5.2), I shall first do the one-period case, and then make an induction on T. For the one-period case, an ϵ-equilibrium is an N-tuple $(Q_1 ,..., Q_N)$ satisfying

$$p\left(Q_i, \sum_{j\neq i} Q_j\right) \geqslant g\left(\sum_{j\neq i} Q_j\right) - \epsilon, \qquad i = 1,..., N. \tag{5.3}$$

Define new variables x_i and x_i' by

$$\begin{aligned} x_i &\equiv Q_i - Q_N^* = Q_i - \frac{1}{N+1}, \\ x_i' &\equiv \sum_{j\neq i} x_j . \end{aligned} \tag{5.4}$$

Using (2.3) and (2.5) one easily verifies that (5.3) is equivalent to

$$(x_i + (x_i'/2))^2 \leqslant \epsilon, \qquad i = 1,..., N,$$

or to

$$| x_i + x. | \leqslant 2h, \qquad i = 1,..., N,$$

where

$$x. \equiv \sum_{j=1}^{N} x_j , \qquad h \equiv (\epsilon)^{1/2}. \tag{5.5}$$

Summing the inequalities (5.5) one gets

$$| x. | \leqslant \frac{2Nh}{N+1}; \tag{5.6}$$

this and (5.5) imply

$$| x_i | \leqslant \frac{2(2N+1)\, h}{N+1}, \tag{5.7}$$

$$| x_i' | \leqslant \frac{2(3N+1)\, h}{N+1}. \tag{5.8}$$

LEMMA. *If* $|x| \leqslant b$ and $|x'| \leqslant b$, then

$$|p[Q_N^* + x, (N-1)\, Q_N^* + x'] - p[Q_N^*, (N-1)\, Q_N^*]| \leqslant \frac{b}{N+1} + 2b^2. \tag{5.9}$$

Proof. The inequality can be verified using (2.3), (2.6), and (2.7).

Note that the right-hand sides of inequalities (5.6)–(5.8) are all dominated by $6h$. Hence, by the lemma, one can take

$$B(\epsilon, 1) = \max\left[6h, \frac{6h}{N+1} + 2(6h)^2\right]. \tag{5.10}$$

For ϵ sufficiently small ($(\epsilon)^{1/2} \leqslant 1/18$), the right side of (5.10) is $6h = 6(\epsilon)^{1/2}$.

Now make the induction hypothesis that the main result is true for any number of periods up to and including T. Fix ϵ, and consider any $(T+1)$-period ϵ-equilibrium. Given the initial outputs Q_{i1}, $i = 1,..., N$, the remaining T-period strategies constitute, *a fortiori*, a T-period, ϵ'-equilibrium, where

$$\epsilon' \equiv \frac{(T+1)\,\epsilon}{T}. \tag{5.11}$$

Let i be any particular firm, which will remain fixed for the time being, let $Q = Q_{i1}$ be i's output in period 1, and let S denote the strategy that i follows during periods 2 through $(T+1)$. Given the strategies of the other firms, one may denote i's average profit as

$$\frac{p(Q, Q')}{T+1} + V(Q, S) \equiv \pi(Q, S), \tag{5.12}$$

where Q' is the total output in period one of all the firms other than i.

The induction hypothesis is that all of the quantities in (5.1), for $i = 1,...,$ N and $t = 2,..., T+1$, are bounded by $b \equiv B(\epsilon', T)$. Actually, I shall strengthen the induction hypothesis to add the following inequalities:

$$\left| \sum_{j\neq i} Q_{jt} - (N-1)\, Q_N^* \right| \leqslant b, \qquad i = 1,..., N, \quad t = 2,..., T+1. \tag{5.13}$$

The definition of epsilon-equilibrium implies that

$$\pi(Q, S) \geqslant M - \epsilon,$$

where (5.14)

$$M \equiv \max_{q,s} \pi(q, s).$$

(In the definition of M, (q, s) ranges over all $(T+1)$-period strategies for firm i, holding constant the ϵ-equilibrium strategies of the other firms.) This last is equivalent to

$$\frac{p(Q, Q')}{T+1} + V(Q, S) \geqslant M - \epsilon,$$

or

$$p(Q, Q') \geqslant (T+1)[M - \epsilon - V(Q, S)]. \tag{5.15}$$

By the induction hypothesis,

$$\frac{(T+1)\, V(Q, S)}{T} - \frac{1}{(N+1)^2} \leqslant b. \tag{5.16}$$

To get a bound on M, note that

$$(T+1)M \geqslant g(Q') + Tm,$$

where

$$m \equiv \max_X \min_Y \{p(X, Y): |\, Y - (N-1)\, Q_N^* \,| \leqslant b\}. \tag{5.17}$$

One easily verifies that

$$m = g[(N-1)\, Q_N^* + b] = \frac{1}{(N+1)^2} - \frac{b}{(N+1)} + \frac{b^2}{4}. \tag{5.18}$$

Putting together (5.15)–(5.18), one has

$$p(Q, Q') \geqslant g(Q') - \epsilon'',$$

where

$$\epsilon'' \equiv \frac{T(N+2)\, b}{(N+1)} + (T+1)\, \epsilon. \tag{5.19}$$

The inequality (5.19) holds for all firms i. In other words, in a $(T+1)$-period ϵ-equilibrium, the first-period outputs constitute a one-period ϵ''-equilibrium, where ϵ'' is given by (5.19) and

$$b = B\left[\frac{(T+1)}{T}\, \epsilon, T\right]. \tag{5.20}$$

Hence, by the one-period case, the quantities in (5.1) and (5.13), with $t = 1$, are all bounded by

$$b' \equiv 6(\epsilon'')^{1/2} = 6\left[\frac{T(N+2)\, b}{(N+1)} + (T+1)\, \epsilon\right]^{1/2}, \tag{5.21}$$

provided that $\epsilon''^{1/2} \leqslant 1/18$. If we take

$$B(\epsilon, T+1) = \max\{b', B(\epsilon, T)\}, \tag{5.22}$$

then the induction step is completed, as far as the inequalities (5.1) and (5.13) are concerned. Furthermore, (5.20)–(5.22) determine a recursion formula for the bounds $B(\epsilon, T)$, which with the formulas (5.5) and (5.10) for $B(\epsilon, 1)$ prove (5.2), i.e., that $B(\epsilon, T)$ tends to zero as ϵ tends to zero, for fiyed T.

6. Large Numbers of Firms: The Replication Case

In the replication case, the demand function depends on the number of firms, so that corresponding to (2.1) we have

$$P = \alpha - \frac{\beta_1 Q}{N}, \tag{6.1}$$

where Q is the total industry output, P is the demand price, and β_1 is a parameter that is independent of N, the number of firms. One may motivate this formulation in terms of replicating an industry. Consider an industry with 1 firm and a given population of M potential buyers, whose demand function is given by

$$Q = \frac{\alpha - P}{\beta_1}. \tag{6.2}$$

The N-fold replication of this industry is made up of N firms, together with a population of NM potential buyers with the same per capita demand at any price as the original population. The resulting demand function is then

$$\frac{Q}{N} = \frac{\alpha - P}{\beta_1},$$

which is equivalent to (6.1). Without essential loss of generality, I shall take $\beta_1 = 1$ and $\delta = \alpha - \gamma = 1$.

Corresponding to the formulas of Section 2, one has the following formulas, obtained by everywhere replacing β by $1/N$. Firm i's profit function is

$$p(Q_i, Q_i') = \left(1 - \frac{Q_i'}{N}\right) Q_i - \frac{Q_i^2}{N}. \tag{6.3}$$

Its optimal response to Q_i' is

$$r(Q_i') \begin{cases} = \dfrac{N - Q_i'}{2N}, & \text{if } \textit{this is positive}, \\ = 0, & \textit{otherwise}, \end{cases} \tag{6.4}$$

and its corresponding profit is

$$g(Q_i') = \frac{(N - Q_i')^2}{4N}, \tag{6.5}$$

if this is nonnegative, and zero otherwise.

The one-period CNE output and profit per firm are given by

$$Q_N^* = \frac{N}{N+1}, \tag{6.6}$$

$$g[(N-1)\,Q_N^*] = \frac{N}{(N+1)^2}. \tag{6.7}$$

The one-period cartel output and profit per firm are

$$\hat{Q}_N = \tfrac{1}{2}, \tag{6.8}$$

$$p[\hat{Q}_N, (N-1)\hat{Q}_N] = \tfrac{1}{4}. \tag{6.9}$$

Firm i's best response if each other firm produces the cartel output is

$$r[(N-1)\,\hat{Q}_N] = \frac{N+1}{4N}, \tag{6.10}$$

with corresponding profit

$$g[(N-1)\,\hat{Q}_N] = \frac{(N+1)^2}{16N}. \tag{6.11}$$

Turning to the T-period game, one easily verifies that the strategy combination in which each firm has the trigger strategy C_k (cf. Section 4) is an ϵ-equilibrium if and only if

$$T \geqslant \left(\frac{N}{2}\right)\left(\frac{N-1}{4N}\right)^2; \tag{6.12}$$

compare this with (4.2). Hence, as in the fixed-demand case, for every ϵ and N, every (C_k) strategy combination is an ϵ-equilibrium for all sufficiently large T. However, in this case one gets the result that, for fixed ϵ and T, no (C_k) strategy combination ($k > 0$) is an ϵ-equilibrium for sufficiently large N. In other words, *for any fixed ϵ and number of periods, the cartel cannot survive at all if the number of firms is large enough.*

Corresponding to the rest of the analysis in Section 5, one has the following results. For every ϵ, T, and $N(>1)$, there is a number $B(\epsilon, T, N)$ such that, in every ϵ-equilibrium, the following quantities are bounded by $B(\epsilon, T, N)$:

$$\begin{gathered} |Q_{it} - Q_N^*|, \\ \left|\sum_j Q_{it} - NQ_N^*\right|, \end{gathered} \tag{6.13}$$

for $i = 1,\ldots, N$ and $t = 1,\ldots, T$; the bounds $B(\epsilon, T, N)$ may be chosen so that, for every ϵ and T,

$$\frac{B(\epsilon, T, N)}{N^{1/2}} \text{ is uniformly bounded in } N, \tag{6.14}$$

and for every T and N,

$$\lim_{\epsilon\to 0} B(\epsilon, T, N) = 0. \tag{6.15}$$

It follows from (6.6) that average output per firm approaches 1, and market price approaches $1 - \alpha = \gamma$ (marginal cost) as N increases without limit. In addition, in every period every firm i's *relative* share of total industry output, which is

$$\frac{Q_{it}}{\sum_j Q_{jt}} = \frac{(1/N)\, Q_{it}}{(1/N) \sum_j Q_{jt}},$$

converges to zero as N gets large. Finally, one can show that, in every period, every firm i's profit is within

$$\frac{B(\epsilon, T, N)}{N + 1} + \frac{2[B(\epsilon, T, N)]^2}{N} \tag{6.16}$$

of the one-period CNE profit, which is $N/(N + 1)^2$, and this bound is uniformly bounded in N, and goes to zero with ϵ.

Thus, in these various ways, for large N, ϵ-equilibria are close to competitive equilibrium.

I shall omit the proof of these results, which parallels the argument in Section 5. The key facts are that, for the one-period case, Eqs. (5.5)–(5.8) are still valid, but with

$$h = (N\epsilon)^{1/2}, \tag{6.17}$$

and in the Lemma of Section 5, the right side of (5.9) must be replaced by

$$\frac{b}{N + 1} + \frac{2b^2}{N}, \tag{6.18}$$

which is less than b for

$$b < \frac{N^2}{2(N + 1)}. \tag{6.19}$$

In particular, for the one-period case one can take

$$B(\epsilon, 1, N) = (N\epsilon)^{1/2}, \tag{6.20}$$

provided that

$$(\epsilon)^{1/2} \leqslant \frac{N}{12(N + 1)}, \tag{6.21}$$

7. AN ALTERNATIVE DEFINITION OF PERFECT EPSILON-EQUILIBRIUM

As an alternative to the definition of perfect epsilon-equilibrium given in Section 4, one can take the utility of the continuation of a strategy to be the average of the profits in the remaining periods, rather than the average of the profits in all T periods.[5] This change leads to a definition of perfect epsilon-equilibrium that is more restrictive, in the sense that, for every positive epsilon, the set of perfect epsilon-equilibria is smaller. Results analogous to those of Sections 5 and 6 can be derived for this definition; I omit the details. However, the results in Section 4, on trigger strategies, are changed in an interesting way. For a fixed positive epsilon, and a fixed number of firms, the combination (C_k) of trigger strategies is a perfect epsilon-equilibrium for all sufficiently large horizons T and all k not too close to T, namely, if and only if

$$T - k \geqslant \left(\frac{1}{\epsilon}\right)\left(\frac{N-1}{4N}\right)^2 - 1.$$

Thus, for this alternative definition of epsilon-equilibrium, *a cartel held together by trigger strategies will break down as the industry approaches the horizon T.*

8. INTERPRETATIONS OF EPSILON

Why should a firm be satisfied with a less-than-optimal response to the strategies of other firms? One type of answer refers to the various costs of discovering and using alternative strategies, and alludes to the possibility that a truly optimal response might be more costly to discover and use than some alternative, "nearly optimal" strategy. In this interpretation, the "epsilon" for a particular firm represents a judgement of the firm that the additional benefits from improving its strategy would be outweighed by the additional costs. (In the present analysis, all firms were assumed to have the same epsilon, but this simplification is not strictly needed for the results.) It would be consistent with the spirit of the model for this judgement to be in part subjective, rather than necessarily based on some precise calculation of benefits and costs.

A second interpretation of epsilon might be based on the supposition that the firms realize that strict optimization of each firm's response to the other firms' strategies would lead to a breakdown of the cartel. This approach is intuitively appealing, but I am not aware of any satisfactory formal model of rational behavior on which it could be based.

Recall that, in the model of the present paper, it is assumed that each firm uses the criterion of average profit per period to compare strategies; thus

[5] This alternative was suggested to me by Sanford Grossman and Robert Rosenthal.

epsilon is measured in "dollars per period." This scale of measurement would be consistent with the cost-of-decision interpretation of epsilon-equilibrium. In a more general treatment, epsilon would be measured in units of utility. However, epsilon-equilibria would not be invariant under transformations of the utility functions of the firms that change the unit of measurement of utility. For example, if the preferences of a firm were scaled in terms of a von Neumann–Morgenstern utility function, then epsilon-equilibria would not be invariant under all transformations of the utility function that leave the firm's preferences invariant. A solution to this last problem would be to adopt a "canonical" utility representation for each firm, and then to interpret epsilon with reference to those canonical utility functions. This would be equivalent, for each firm, to interpreting epsilon as a given *percentage* of the difference in utility between two reference profits. Within this framework, the interpretation of the results of the present paper is straightforward, with the proviso that, for those situations in which the number of firms increases without limit, some condition of "similarity" of the epsilons of the different firms would have to be satisfied (as would naturally occur in the replication case).

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