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PRIVATE INFORMATION AND PURE-STRATEGY EQUILIBRIA*

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In this paper it is proved that in games with a finite number of players and a finite number of moves, if each player observes a private information random variable which has an atomless distribution and is independent of the observations and payoffs of all other players, then the game possesses a pure-strategy equilibrium. Examples are presented that illustrate the importance of the assumptions.

1. Introduction. One of the reasons why game-theoretic ideas have not found more widespread application is that randomization, which plays a major role in game theory, seems to have limited appeal in many practical situations. If it were the case, however, that for the kinds of games that exist in the "real world" randomization could be shown to play no useful role, then this difficulty, at least, would be insignificant. It is sometimes claimed that in realistic situations games tend to have incomplete information (i.e., the rules and/or the payoffs are not completely known by all players). It is also sometimes claimed that when information in games is sufficiently disparate among the players and when its distribution is sufficiently diffuse, the players might as well restrict their attention to pure strategies. If these claims were verified then in real-world situations well-modeled by noncooperative games with such information features, the case for randomization would not be as compelling as students of game theory might otherwise suspect.

The purpose of this paper is to examine conditions under which diffuse and disparate information leads to the existence of pure-strategy equilibria. To illustrate, consider a game in which each of a finite set of players privately observes the outcome of a personal random variable. Following this the players play a finite game Γ^*. In a normal-form-game model Γ of this situation, the pure strategies for each player are functions from the set of his possible observations into the set of his pure strategies in Γ^*. (To avoid confusion, from now on the pure strategies of Γ^* will be referred to as moves.) If the personal random variables are mutually independent with atomless distributions, and if the payoffs of Γ^* are independent of these random variables, then it is well known (e.g., Aumann (1974)) that Γ has a pure-strategy Nash equilibrium. To see this, it is sufficient to observe that the independent randomizations needed for a Nash equilibrium of Γ^* can be reproduced by pure strategies in Γ. Such a pure-strategy combination is an equilibrium of Γ.

After establishing notation and supplying definitions in §2, we show in §3 that pure-strategy equilibria still exist when the situation just discussed is generalized so that each player's own payoff in Γ^* may be correlated with his personal observation. (This more general situation does not exclude the possibility that the players may have additional, correlated, information.) Under the hypotheses of our main result (Theorem 2), we show that to every mixed-strategy equilibrium there corresponds a

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pure-strategy equilibrium that gives to every player (respectively) the same expected payoff. One can also demonstrate that, with the same hypotheses, there exists at least one equilibrium (Theorem 3). (These results also generalize some implications of Theorems 2 and 4 in Harsanyi (1973). Harsanyi's methods do not extend to our setting, however.) In §4, we give three examples of games in which the information variables have atomless distributions, and in which there are mixed-strategy equilibria but no pure-strategy equilibria.

If there are any surprises in this paper, we feel that they are in the strength of the hypotheses needed to establish existence of pure-strategy equilibria and the ease with which examples may be constructed which do not possess such equilibria. Nevertheless, our opinion is that there is some truth in the imprecise claims mentioned in our first paragraph. In particular we note that the conditions needed to establish the existence of pure-strategy approximations to mixed-strategy equilibria are significantly weaker than what we will be assuming here. This subject is treated in Aumann *et al.* (1981).

2. Notation and definitions. The game Γ is played by a finite number, I, of players. Each player i first observes the realization of a (secret) random variable, Z_i, and then selects a *move*, a_i, from a finite set, $A_i = \{1, \ldots, K_i\}$. The choice of this move may depend on Z_i. The resulting payoff to player i is $u_i(a, X_i)$, where $a = (a_i)$ is the combination of moves, and X_i is a random variable (the "payoff-relevant" state of nature for i). Each player is interested in maximizing his expected utility.

In order to describe a consistent framework for our probability calculations, we need some additional notation. For each i, let $(\mathbf{Z}_i, \mathfrak{Z}_i)$ and $(\mathbf{X}_i, \mathfrak{X}_i)$ be measurable spaces where $\mathfrak{Z}_i$ and $\mathfrak{X}_i$ are the sigma-fields of measurable subsets of $\mathbf{Z}_i$ and $\mathbf{X}_i$, respectively, Let Ω denote the Cartesian product of the sets $\mathbf{Z}_1, \mathbf{X}_1, \ldots, \mathbf{Z}_I, \mathbf{X}_I$, with the corresponding product sigma-field $\mathfrak{F}$ of measurable sets, and let μ be a probability measure on $\mathfrak{F}$. For a point $\omega = (z_1, x_1, \ldots, z_I, x_I)$ in Ω, define the coordinate projections

$$Z_i(\omega) = z_i,$$
$$X_i(\omega) = x_i. \tag{2.1}$$

(In ordinary probability parlance, the "random variable" Z_i is the function defined in (2.1).)

In the game Γ, a pure strategy for player i is a measurable function, say g_i, from $\mathbf{Z}_i$ to A_i. If the players use the strategy combination $g_1, \ldots, g_I$, the resulting expected utility to i is

$$\int_\Omega u_i\{ g_1[Z_1(\omega)], \ldots, g_I[Z_I(\omega)], X_i(\omega)\} \mu(d\omega). \tag{2.2}$$

In order for (2.2) to be well-defined for every pure strategy combination, we assume that, for every $a = (a_i)$ in $A \equiv \times_i A_i$, $u_i[a, X_i(\cdot)]$ is a (real-valued) μ-integrable function on Ω. (This would be the case, for example, if, for every a in A, $u_i(a, \cdot)$ were a bounded measurable function on $\mathbf{X}_i$.) We shall usually replace expression (2.2) by the less cumbersome notation

$$\mathcal{E} u_i[g_1(Z_1), \ldots, g_I(Z_I), X_i], \tag{2.3}$$

and similarly use the "expectation" operator $\mathcal{E}$ in other expressions.[1]

[1]For explanatory material on products of measurable spaces see, for example, Chapter VII of Halmos (1950), or any other standard reference on measure theory. Chapter IX of the same book provides a measure-theoretic approach to probability.

We now introduce randomization over pure strategies. One approach would be to endow each player's set of pure strategies with a sigma-field of measurable sets; a mixed strategy would then be a probability measure on the set of pure strategies. An essentially equivalent approach, but one that involves fewer technical difficulties, is to use the concept of a behavior strategy.[2] For each i, let S_i denote the unit simplex of K_i-dimensional Euclidean space (recall that K_i is the number of moves in A_i available to player i). A *behavior strategy* for i is a measurable function, say b_i, from $\mathbf{Z}_i$ to S_i. We interpret the kth coordinate $b_{ik}(z_i)$ as the probability that i uses move k, given that he has observed z_i. We say that b_i is *pure* if $b_i(z_i)$ is a unit vector for almost every z_i. Let B_i denote the set of behavior strategies for player i.

In order to calculate the expected utility to a player resulting from a combination of behavior strategies, first note that if, for each player i, s_i is a point in the simplex S_i, and a_i is a move in A_i, then the corresponding probability of the combination $a = (a_i)$ of moves is

$$\pi(a,s) \equiv \prod_i s_{ia_i}, \tag{2.4}$$

where $s = (s_i)$. Letting S denote the Cartesian product $\times_i S_i$, we can (with a slight abuse of notation) extend the utility function u_i to $S \times \mathbf{X}_i$ by

$$u_i(s,x_i) = \sum_{a \in A} u_i(a,x_i)\pi(a,s). \tag{2.5}$$

If, for every j, player j uses behavior strategy b_j, and if we denote $b = (b_j)$ and $b(Z) = (b_1[Z_1], \ldots, b_I[Z_I])$, then we can express the resulting expected utility to player i as

$$U_i(b) = \mathcal{E} u_i[b(Z), X_i]. \tag{2.6}$$

An equilibrium (Nash) in behavior strategies is defined in the usual way: it is a combination such that no player can increase his expected payoff by changing his own behavior strategy alone (i.e., each player uses a *best response* to the behavior-strategy combination of all other players). If b^* and b are equilibria, we shall say that b is a *purification* of b^* if, for every player i,

$$b_i \text{ is pure}, \tag{2.7a}$$

$$U_i(b) = U_i(b^*). \tag{2.7b}$$

Thus a purification of an equilibrium b^* is an equilibrium that gives every player the same expected utility that b^* does. Our main positive result (§3) is that, under certain hypotheses about the random variables $Z_1, X_1, \ldots, Z_I, X_I$, every equilibrium has a purification; furthermore, the purification b of b^* can be chosen so that, for every i,

$$\mathcal{E} b_i(Z_i) = \mathcal{E} b_i^*(Z_i). \tag{2.7c}$$

In fact, the same method of proof generates the stronger conclusion that every strategy combination $b' = (b_i')$ such that, for every i, b_i' is either b_i or b_i^*, is also an equilibrium.

3. Equilibria in pure strategies. Our main result is Theorem 2, but since the hypotheses of that theorem are somewhat complicated, we first introduce the main ideas in a simpler setting. This preliminary result (Theorem 1) can, in fact, be used to prove almost directly the ostensibly more general result in Theorem 2.

[2]For a discussion of mixed and behavior strategies, see Aumann (1964), (1974). However, Aumann's terminology differs somewhat from ours. Harsanyi (1973) calls our behavior strategies "normalized strategies."

Recall that a probability measure is *atomless* if every set of positive measure has a subset of strictly smaller, but still positive, measure. We shall say that the distribution of Z_i is atomless if the probability measure induced by μ on $\mathbf{Z}_i$ is atomless.

THEOREM 1. *If, for every player i,*
(a) *the distribution of Z_i is atomless,*
(b) *the random variables $\{Z_j : j \neq i\}$ together with the random variable $Y_i \equiv (Z_i, X_i)$ form a mutually independent set, then every equilibrium has a purification.*

Notice that hypothesis (b) implies that the random variables $Z_1, \ldots, Z_I$ are mutually independent, but does *not* require that Z_i and X_i be mutually independent, nor that $X_1, \ldots, X_I$ be mutually independent. The following example satisfies hypothesis (b). Suppose that $X_0', X_1', \ldots, X_I', E_1, \ldots, E_I$ are mutually independent real random variables, and that, for $i = 1, \ldots, I$,

$$Z_i = X_i' + E_i, \qquad X_i = (X_0', X_i').$$

In this example, one may interpret X_0' as a common payoff-relevant variable, X_i' as a payoff-relevant variable "private" to player i ($i > 0$), and Z_i as a measurement of X_i' with error E_i. In Theorem 2 we shall show that hypothesis (b) can be weakened in such a way that, in the context of this example, each player could also have information about X_0'.

PROOF[3] OF THEOREM 1. Using (2.4)–(2.6) we can rewrite the expected utility to player i, resulting from the strategy combination b, as

$$U_i(b) = \mathcal{E} \sum_{a \in A} u_i(a, X_i) \prod_{j=1}^{I} b_{ja_j}(Z_j). \tag{3.1}$$

By hypothesis (b) of the theorem, (3.1) is equal to

$$U_i(b) = \sum_{a \in A} \mathcal{E}\{u_i(a, X_i) b_{ia_i}(Z_i)\} \prod_{j \neq i} \mathcal{E}\{b_{ja_j}(Z_j)\}. \tag{3.2}$$

Write b^{-i} for the $(I-1)$-tuple of behavior strategies $b_j (j \neq i)$, and write $\mathcal{E} b_j \equiv \mathcal{E} b_j(Z_j)$. From (3.2) one sees that the set of best responses by player i to b^{-i} depends only on $\mathcal{E} b^{-i}$.

Suppose that b is an equilibrium, and consider a particular player i. In order that b_i be a best response to b^{-i} it is necessary and sufficient that, for almost every z_i, $b_i(z_i)$ maximize player i's conditional expected payoff given $Z_i = z_i$, or equivalently, from (3.2), that $b_i(z_i)$ be a point s_i in the simplex S_i that maximizes

$$\sum_{a \in A} s_{ia_i} \mathcal{E}\{u_i(a, X_i) | Z_i = z_i\} \prod_{j \neq i} \mathcal{E}\{b_{ja_j}(Z_j)\}. \tag{3.3}$$

This expression is linear in s_i, so $b_i(z_i)$ is a convex combination of the set of those *unit vectors* in S_i that maximize (3.3); call this set $\phi_i(z_i)$. (Keep in mind that $\phi_i(z_i)$ also depends on b^{-i}.) Since the number of unit vectors in S_i is finite, there is a (measurable) partition $\{\mathbf{Z}_{in} : n = 0, 1, \ldots, N_i\}$ of $\mathbf{Z}_i$ such that:

$$\operatorname{Prob}(\mathbf{Z}_{in}) > 0, \qquad n = 1, \ldots, N_i; \tag{3.4a}$$

$$\operatorname{Prob}(\mathbf{Z}_{i0}) = 0; \tag{3.4b}$$

$$\text{for each } n > 0, \qquad \phi_i \text{ is constant on } \mathbf{Z}_{in}. \tag{3.4c}$$

[3]The method of proof of Theorem 1 was suggested by Schmeidler (1973). It is also reminiscent of Dvoretzky *et al.* (1950).

Because of (3.4c), one can, with some abuse of notation, write $\phi_i(\mathbf{Z}_{in}) = \phi_i(z_i)$ for z_i in $\mathbf{Z}_{in}$. Since the distribution of Z_i is atomless, for every $n > 0$ one can, by (3.4a), partition $\mathbf{Z}_{in}$ into measurable sets $\mathbf{Z}_{ink}$, one for each unit vector v_k in $\phi_i(\mathbf{Z}_{in})$, such that for each v_k in $\phi_i(\mathbf{Z}_{in})$,

$$\operatorname{Prob}(\mathbf{Z}_{ink}) = \int_{\mathbf{Z}_{in}} b_{ik}(Z_i(\omega))\mu(d\omega). \tag{3.5}$$

(Use, e.g., Example 7, p. 100, of Loève (1960).) Define a pure behavior strategy $\hat{b}_i$ by:

$$\hat{b}_i(z_i) = k\text{th unit vector in } S_i \quad \text{if } z_i \text{ is in } \mathbf{Z}_{ink}. \tag{3.6}$$

To complete the definition of $\hat{b}_i$, one can let $\hat{b}_i(z_i)$ be any single unit vector for z_i in $\mathbf{Z}_{i0}$. Recall that $b_{ik}(z_i)$ is zero if v_k is not in $\phi_i(z_i)$; hence, by (3.6) and (3.5),

$$\mathcal{E}\hat{b}_i = \mathcal{E}b_i. \tag{3.7}$$

In addition, for almost every z_i, $b_i(z_i)$ is in $\phi_i(z_i)$, so that $\hat{b}_i$ is a pure, best response to b^{-i}.

If we define $\hat{b}_j$ in the corresponding way for each player j, then the strategy combination $\hat{b}$ has the properties:

$$\hat{b} \text{ is pure}; \tag{3.8a}$$

$$\text{for each } j,\ \hat{b}_j \text{ is a best response to } b^{-j}; \tag{3.8b}$$

$$\mathcal{E}\hat{b} = \mathcal{E}b. \tag{3.8c}$$

Recall that the expected utility to a player j depends on b^{-j} only through $\mathcal{E}b^{-j}$; it follows that, for each j, $\hat{b}_j$ is a best response to $\hat{b}^{-j}$. Hence $\hat{b}$ is a purification of b, which completes the proof of Theorem 1. Note, too, that (3.8c) is the same as (2.7c).

We now turn to our more general result. Suppose that each random variable Z_i has two components, say Z_i' and Z_i''; in other words, $\mathbf{Z}_i$ is itself a product of two measurable spaces, $\mathbf{Z}_i'$ and $\mathbf{Z}_i''$. Denote by Z'' the I-tuple $(Z_1'', \ldots, Z_I'')$.

THEOREM 2. *If for every player i,*
(a′) *the distribution of* Z_i' *is atomless,*
(a″) *the set* $\mathbf{Z}_i''$ *is finite,*
(b) *the random variables* $\{Z_j' : j \neq i\}$ *together with the random variable* $Y_i \equiv (Z_i', Z'', X_i)$ *form a mutually independent set, then every equilibrium has a purification.*

To illustrate the hypotheses of Theorem 2, consider an example like the one following the statement of Theorem 1, in which i's payoff is $u_i(a, X_0', X_i')$, and i's information is (Z_i', Z_i''), where

$$Z_i' = X_i' + E_i, \qquad Z_i'' = \zeta_i(X_0'),$$

and ζ_i is a function ("statistic") of the common payoff-relevant variable, X_0', with finitely many different values. (As before, the random variables $X_0', X_1', \ldots, X_I'$, $E_1, \ldots, E_I$, are mutually independent.)

PROOF OF THEOREM 2. The idea of the proof is to reformulate a new, equivalent game whose "moves" for each player i are functions from $\mathbf{Z}_i''$ to A_i, and then apply Theorem 1 to this new game. Thus let $\mathcal{A}_i$ be the set of all functions from $\mathbf{Z}_i''$ to A_i; since both $\mathbf{Z}_i$ and A_i are finite, so is $\mathcal{A}_i$. Let $\mathcal{S}_i$ denote the unit simplex in a Euclidean space of dimension equal to the number of elements in $\mathcal{A}_i$, and let $\mathcal{B}_i$ denote the set of all measurable functions from $\mathbf{Z}_i'$ to $\mathcal{S}_i$. We may interpret a function β_i in $\mathcal{B}_i$ as a behavior strategy in the new game. Thus, conditional on $Z_i' = z_i'$, the probability that i uses the decision function α_i (in $\mathcal{A}_i$) for responding to Z_i'' is $\beta_{i\alpha_i}(z_i')$.

To the behavior strategy β_i in the new game there corresponds in a natural way, a behavior strategy b_i in the old game, namely

$$b_{ia_i}(z_i', z_i'') = \sum_{\substack{\alpha_i \in \mathcal{Q}_i \\ \alpha_i(z_i'') = a_i}} \beta_{i\alpha_i}(z_i'). \tag{3.9}$$

On the other hand it is easy to show that to every b_i there corresponds at least one β_i that satisfies (3.9). Hence $\mathfrak{B}_i$ generates the set of all behavior strategies for i in the old game.

One can now define, in a straightforward way, the new game corresponding to the behavior strategies β_i, and apply Theorem 1; we omit the details.

L. S. Shapley has observed that the purifications of Theorems 1 and 2 possess a stronger feature than has been asserted. If any coalition is unable to gain by a coordinated switch from an equilibrium, then that same coalition cannot gain versus the purification. In particular, if the original equilibrium were a strong Nash equilibrium, the purification would be as well.

To assure the reader that our main result is not vacuous, we state, without proof:

THEOREM 3. *Under the hypotheses of Theorem* 2, *there exists an equilibrium.*

(One method of proof is to consider each player's set $\mathfrak{B}_i$ of behavior strategies as a subset of a space of essentially bounded measurable functions, endowed with the so-called weak* topology. With this topology, $\mathfrak{B}_i$ is a compact, convex set, and one can show that each player's expected utility is a continuous function of the I-tuple of all players' strategies. One can then apply the usual fixed-point method to demonstrate the existence of an equilibrium.[4])

4. Examples of nonexistence of equilibrium in pure strategies. In this section we give three examples of games that have no equilibria in pure strategies. In the first two examples, hypothesis (a) of Theorem 1 is maintained (atomless information), but the independence condition of hypothesis (b) is violated. In Example 3, the sets of moves are infinite, but the hypotheses of Theorem 1 are otherwise maintained.

EXAMPLE 1. This game, in which there are two players, brings out in a very simple context how statistical dependence between the information variables of the two players can prevent them from achieving an equilibrium in pure strategies, even if the information variables are irrelevant to the payoffs. Suppose that each player has two moves ($A_1 = A_2 = \{1,2\}$), and that the payoffs are those for the zero-sum game "matching pennies." The accompanying table shows the payoff matrix for player 1:

		Player 2's move	
		1	2
Player 1's move	1	1	0
	2	0	1

The players' information variables, Z_1 and Z_2, are distributed uniformly on the triangle of the unit square defined by $0 \leqslant z_1 \leqslant z_2 \leqslant 1$.

In order for a pair of behavior strategies to be an equilibrium, it is necessary and sufficient that, for each player i and almost every value z_i of his information variable Z_i, the conditional probabilities of the two moves of the other player (given that

[4]We are not aware of any theorem that would enable us to prove the existence of equilibrium in the general setup of our paper without adding some stronger hypotheses, such as those of Theorem 2. For a detailed proof of Theorem 3, see Radner and Rosenthal (1980). For further results on both existence and purification, see Milgrom and Weber (1980).

$Z_i = z_i$) should be equal. Suppose that there is an equilibrium in pure strategies, and let F denote the set of z_1 in the unit interval for which the first player uses move 1, i.e., for which $b_{11}(z_1) = 1$. The set F, which is measurable, characterizes the pure strategy of player 1. The equilibrium condition implies that, for almost every z_2,

$$\operatorname{Prob}\{Z_1 \in F | Z_2 = z_2\} = \frac{1}{2}. \tag{4.1}$$

Let $f(z_2)$ denote the conditional probability on the left-hand side of (4.1), and let λ denote Lebesgue measure on the line; then

$$f(z_2) = \frac{\lambda(F \cap [0, z_2])}{z_2}, \quad \text{a.e.} \tag{4.2}$$

We shall show that there does not exist a measurable set F such that $f(z_2) = 1/2$ for almost all z_2 in the unit interval.

Suppose that there were such a set F. Then, from (4.2), it would follow that, for almost every z,

$$\lambda(F \cap [0, z]) = z/2. \tag{4.3}$$

Note that $\lambda(F \cap [0, z])$ is continuous in z, so that (4.3) would hold for *all* z in the unit interval. Since F has positive measure, for any strictly positive number ϵ there is a nondegenerate interval, say (c, d), such that

$$\lambda(F \cap [c, d]) \geqslant (1 - \epsilon)(d - c) \tag{4.4}$$

(see, for example, Halmos (1950), p. 68, Theorem A). We shall take $\epsilon < 1/2$. The left-hand side of (4.4) equals

$$\lambda(F \cap [0, d]) - \lambda(F \cap [0, c]),$$

which by (4.3), equals $(d - c)/2$. Hence, from (4.4) we would have

$$(d - c)/2 \geqslant (1 - \epsilon)(d - c),$$

which is impossible.

Example 2. In this example there are again two players and the information variables are correlated, but the latter are conditionally independent given the payoff-relevant random variables. Suppose again that each player has two moves, and that the game is "matching pennies" as in Example 1, except that the diagonal entries equal to 1 in the payoff matrix are replaced by X, where X is distributed on the unit interval with probability density $3x^2$. Thus the payoff-relevant random variables for the players are $X_1 = X_2 = X$, but the realized value of X is irrelevant to the optimal strategies, so that, as in Example 1, for each player and almost every value of his information variable, the conditional probabilities of the two moves of the other player should be equal.

Suppose that, given that $X = x$, the two information variables, Z_1 and Z_2, are independent and are each distributed uniformly on the interval $[0, x]$. By a straightforward calculation, one finds that the marginal joint density of the information variables is

$$g(z_1, z_2) = \begin{cases} 3(1 - z_2), & z_1 \leqslant z_2, \\ 3(1 - z_1), & z_1 \geqslant z_2, \end{cases} \tag{4.5}$$

and that the conditional density of Z_1 given $Z_2 = z_2$ is

$$h(z_1|z_2) = \begin{cases} \dfrac{2}{1+z_2}, & z_1 \leqslant z_2, \\ \dfrac{2(1-z_1)}{1-z_2^2}, & z_1 \geqslant z_2. \end{cases} \tag{4.6}$$

Suppose that there were a pure-strategy equilibrium. Using the notation of Example 1, let F denote the set of values of Z_1 for which player 1 uses move 1. Again, (4.1) must hold in an equilibrium, for almost every z_2. Let I denote the indicator function for the set F, that is, $I(s) = 1$ for s in F, and 0 otherwise. It will be convenient to denote $\lambda(F \cap [0, t])$ by $L(t)$; thus

$$L(t) = \int_0^t I(s)\,ds.$$

The equilibrium condition (4.1) requires that, for almost every t,

$$\int_0^t I(s)\left(\frac{2}{1+t}\right)ds + \int_t^1 I(s)\frac{2(1-s)}{(1-t^2)}\,ds = \frac{1}{2},$$

or

$$\left(\frac{2}{1+t}\right)L(t) + \left(\frac{2}{1-t^2}\right)\int_t^1 (1-s)I(s)\,ds = \frac{1}{2}. \tag{4.7}$$

Integrating by parts, one can transform (4.7) into

$$\int_t^1 L(s)\,ds = \frac{1-t^2}{4}.$$

Differentiating this with respect to t, one gets $L(t) = t/2$, which is the same as (4.3), and hence impossible by the argument given in Example 1.

EXAMPLE 3. Again there are two players. Player 1 picks a number p in the unit interval and an element from the set $\{\alpha, \beta\}$, i.e., his move set is $[0, 1] \times \{\alpha, \beta\}$. Z_1 is uniform on $[0, 1]$. 2's move is a pair (C_α, C_β) where both C_α and C_β are Borel subsets of $[0, 1]$ the sum of the (Lebesgue) measures of which is one. Z_2 can be any random variable which is independent of Z_1. If player 1 picks (p, α), the payoff to player 2 is 1 if $p \in C_\alpha$ and (-1) if $p \notin C_\alpha$. Similarly, if 1 picks (p, β), 2's payoff is 1 if $p \in C_\beta$ and (-1) is $p \notin C_\beta$. The payoff to player 1 is the negative of the payoff to player 2 whenever $p = z_1$, the realization of Z_1. Otherwise 1's payoff is (-2).

Defining strategy spaces for this game raises technical issues which are not of central importance. We shall argue informally, therefore, that this game cannot possess a pure-strategy equilibrium. First notice that the situation as described is, in the notation of §3, one in which $X_1 = Z_1$, X_2 is trivial, and (X_1, Z_1) is independent of (X_2, Z_2). (A simpler version of this game would require $p = z_1$, but then hypotheses (b) of Theorem 1 would fail.)

At an equilibrium, player 1 must set $p = z_1$ for almost every z_1. To any pure strategy with this feature, however, player 2's set of pure best responses consists of all pairs (C_α, C_β) such that C_α (resp., C_β) agrees (disregarding sets of measure zero) with the set of z_1 values which player 1 assigns to α (resp. β). Thus any pure strategy equilibrium must result in payoffs of (-1) to player 1 and 1 to player 2. If player 1 sets $p = Z_1$ and then picks α or β with probability $1/2$ each, independently of Z_1, he guarantees himself an expected payoff of zero. This strategy combined with player 2 ignoring his information and setting $C_\alpha = C_\beta = [0, 1/2]$, for example, is evidently an equilibrium for any suitably defined spaces of strategies.

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