Econometrica, Vol. 51, No. 6 (November, 1983)

STATIONARY OPTIMAL POLICIES WITH DISCOUNTING IN A STOCHASTIC ACTIVITY ANALYSIS MODEL

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We consider optimal capital accumulation in a nonlinear activity analysis model in which production and primary resource supplies are affected by a stationary stochastic process of exogenous shocks; the optimality criterion is the sum of discounted expected future social utilities. Under various "neoclassical" conditions on technology and preferences, (i) there exists an optimal policy of investment and consumption expressible as a continuous time-invariant function of the capital stocks and the history of stochastic shocks, and (ii) there is a stationary stochastic process of capital stocks that is consistent with the optimal policy.

1. INTRODUCTION

THE DYNAMIC STABILITY of optimal growth in multisector models with uncertainty has been demonstrated under fairly general conditions for the case in which future expected social utility of consumption is not discounted, and the "overtaking" criterion of optimality is used.[2] The case in which future expected utility is discounted has been less tractable. Brock and Majumdar [4] have demonstrated stability in a multisector model that satisfies the following assumptions (among others): (i) the exogenous shocks to the economic system form a sequence of independent and identically distributed random variables, (ii) the optimal policy of investment and consumption can be expressed as a continuous time-invariant function of the state of the economic system, (iii) the random vectors of optimal capital stocks belong to a compact set, and (iv) the Hamiltonian system corresponding to the optimal process has suitable "curvature." One would like to be able to extend this result by allowing more general stochastic processes of exogenous shocks, and by demonstrating that properties (ii)–(iv) above are themselves consequences of natural assumptions about the technology and preferences of the economy.

The results in the present paper constitute partial progress in this program. In a fairly general nonlinear activity-analysis model, with a stationary stochastic process of exogenous shocks, we demonstrate that properties (ii) and (iii) above follow from "neoclassical" assumptions about the technology and preferences. We also demonstrate the existence of a stationary stochastic process of capital stocks that is consistent with the optimal policy function. Our method of analysis

[1]The research reported here was supported in part by the National Science Foundation with grants to Cornell University and the University of California, Berkeley, and by a fellowship from the John Simon Guggenheim Foundation to Mukul Majumdar. Much of the work was done while Roy Radner was a faculty member and Mukul Majumdar was a Ford Rotating Research Professor at the University of California, Berkeley in 1976. Thanks are due to Professor I. Zilcha for his comments on an earlier version. The views expressed here are those of the authors, and not necessarily those of Bell Laboratories or the Bell System.

[2]See Jeanjean [9], Dana [5], and Zilcha [20]. The literature on optimal accumulation in aggregative models is reviewed in Majumdar [12].

is an extension of known techniques of Markovian dynamic programming to the case in which the stochastic environment is stationary, but not necessarily Markovian; this extension may have some independent interest.

Recall that in a "neoclassical" model with constant technology and a constant supply of essential primary resources, an appropriate concept of dynamic stability is that of convergence towards constant consumption, investment, and capital stocks. In an economy in which the technology and supplies of primary resources are subject to random shocks, one cannot (in general) expect convergence to a *constant* state of the system, but only convergence to a stationary stochastic process of consumption, investment, and stocks (see **[16]**). For example, if the successive random shocks are independent and identically distributed, and a time-invariant policy function is used to determine consumption and investment, then the successive capital stock vectors will form a Markov process. A "steady state" of this process is characterized by an invariant probability distribution of the capital stocks, which together with the probability transition law of the system determines a stationary stochastic process of consumption, investment, and capital stocks. If the exogenous shocks are not independent, then one cannot expect the successive states of the economic system to be even Markovian, and one must be prepared to accept as a "steady state" a general stationary stochastic process. The reader should bear in mind that such a process can exhibit "stochastic cycles," i.e., fluctuations of varying magnitude and duration.

A second complication we wish our theory to be able to deal with is the multisector nature of the model. In one- or two-sector models, one is constrained to consider only the determination of aggregate investment. Thus, while the analysis of such a restricted model attempts to capture to some extent the choice between *present* consumption and *future* consumption (the possibilities of which depend on the present level of investment), it does not take into account the important question of the *distribution* of aggregate investment among various sectors and industries.[3] In our framework, a simple method of treating such problems is provided. In principle, the richer structure of our model can be helpful in any systematic study of many issues relevant for investment planning (e.g., the implications of alternative allocation policies for the distribution of employment among various sectors).

In our multisector model of capital accumulation, we allow both the technological process of production and the supply of primary resources (like "labor") to depend on the history of the stochastic environment. The evolution of the environment is assumed to be a *stationary* stochastic process, the probabilistic law of which does not depend on economic decisions. The production possibilities are described by a finite number of (possibly nonlinear) activities. The inputs needed and the outputs produced by these activities are influenced by the state

[3] In the literature on development planning, the importance of these problems has been recognized by Dobb, Sen, and others. See, e.g., Sen [**17**, p. 9].

of the environment. The stochastic model of production is specified in detail in Section 2. An interesting bound on the set of all feasible activity levels is computed (see (2.10)). It is also shown that the compact set of feasible activity levels varies in an upper semi-continuous manner with respect to changes in the history of the environment and the stocks of producible goods (Proposition 2.1). Some other continuity properties of the law of motion of the system are also derived (Lemma 2.2).

The optimal accumulation problem that we study can be viewed as a stochastic dynamic program with discounting.[4] In each period, the planner has to choose an "action," which in our context is simply a decision concerning the levels or intensities at which the activities are to be operated. Such a decision completely specifies the total consumption of all the commodities and the allocation of investment among the activities. A policy or a program consists of a sequence of decision rules, one for each period, which determines the action in every period corresponding to each evolution of the system up to that period. The optimality criterion is a discounted sum of expected (social) utilities generated by consumption in each period. We first prove the existence of an optimal policy that is stationary, i.e., that can be described as a "memoryless," time-independent optimal decision function (or, optimal policy function). Roughly speaking, certain continuity and boundedness conditions on the technology, the environment, and the utility function are sufficient to guarantee the existence of such an optimal policy function (Theorem 2.1). The optimal policy is also characterized in terms of the "functional equation" of dynamic programming (see (2.23)). We emphasize that for these results we do not require the standard neoclassical conditions on convexity of the technology and concavity of the utility function.

Under stronger conditions, including convexity of the technology and concavity of the utility function, the optimal policy function is continuous (Theorem 3.1).

The optimal policy function is a useful tool in investigating the qualitative behavior of the stochastic process of optimal accumulation.[5] It is shown, for example, that if the optimal policy function exists and is continuous, the stochastic process of optimal accumulation does have a steady state (Theorem 3.2). It is also noted that for a wide class of models, the optimal policy will actually require that some activities are operated at positive levels (so that aggregate investment is positive in each period).

The results that we use from the theory of dynamic programming are quite technical, and are not conveniently available in one or two references. Therefore, in the Appendix, we have provided a sketch of the mathematical background that we need, together with references to the relevant literaure.

[4]Radner [**15**] explored the dynamic programming approach in a deterministic context, and Jeanjean [**8**] used dynamic programming methods in a model with uncertainty and discounting.

[5]The stability analysis of Brock and Majumdar [**4**] relies on the assumption of the existence of such a policy function.

2. AN ACTIVITY ANALYSIS MODEL OF PRODUCTION UNDER UNCERTAINTY

2.1. *Notation*

In what follows, if S is a metric space, $\mathscr{B}(S)$ denotes the Borel σ-field of S (i.e., the smallest σ-field containing the open sets). $\mathscr{M}(S)$ is the set of all probability measures on $\mathscr{B}(S)$. On the notion of weak convergence in $\mathscr{M}(S)$, the reader is referred to Billingsley [**2**]. The set of all real-valued bounded $\mathscr{B}(S)$-measurable functions on S is denoted by $B(S)$ and the set of all real-valued bounded upper semi-continuous (u.s.c.) functions on S is denoted by $C_1(S)$. We use the *sup-norm* of $B(S)$ (see Dunford–Schwarz [**6**, p. 240]). For any f in $B(S)$, we let

$$\|f\|_B = \sup_{s \in S} |f(s)|. \tag{2.1}$$

For any f in $C_1(S)$, we let

$$\|f\|_{C_1} = \sup_{s \in S} |f(s)|. \tag{2.2}$$

We recall that, endowed with the topologies generated by the norms $\|\cdot\|_B$ and $\|\cdot\|_{C_1}$, the spaces $B(S)$ and $C_1(S)$ are *complete metric spaces* (see Maitra [**11**, Lemma 4.2]).

Finally, a vector $x = (x_i)$ in R^m is *nonnegative* (written $x \geqq 0$) if $x_i \geqq 0$ for all i; x is *semi-positive* (written $x > 0$) if $x \geqq 0$ *and* $x_i > 0$ for some i; x is *strictly positive* (written $x \gg 0$) if $x_i > 0$ for all i. For a vector $x = (x_i)$ in R^m, $\|x\| = \sum_{i=1}^{m} |x_i|$.

2.2. *The Environment*

The environment is described by the set Ω of all doubly infinite sequences $\omega = \langle \omega_t \rangle_{t=-\infty}^{\infty}$ where each ω_t belongs to a compact (nonempty) metric space W. For example, W can be taken as a closed interval $[a, b]$ of the real line, or more generally, a closed bounded set in a finite dimensional Euclidean space. A particular ω_t will be called the *environment at date t*; ω will be called a *complete history* of the environment (from the indefinite past into the indefinite future). Ω is endowed with the product topology. Hence, it is a compact metric space—in particular, it is a complete separable metric space (Dunford–Schwarz [**6**, I.6.15; p. 22]).

Let h_t be any *partial history* of the environment up to period t, i.e., $h_t = (\ldots, \omega_{-t}, \ldots \omega_0, \ldots, \omega_t)$. The partial history h_t is the most that can be observed about the environment up to date t, and decisions at date t can at most depend on h_t. We denote by H_t the set of all such partial histories of the evolution of the environment up to period t. Again, endowed with the product topology each H_t is a compact (hence, a complete, separable) metric space. Note that if we consider the shift transformation τ mapping any infinite sequence $\omega = (\omega_t)$ into an infinite sequence $\tau\omega$ defined as

$$(2.3) \qquad (\tau\omega)_t = \omega_{t-1},$$

we can verify that τ is one-to-one and continuous from H_t onto H_{t+1}. Thus, τ is a homeomorphism (Kelley [**10**, Ch. 5, no. 8, p. 141]). We shall use h (resp. H) to denote a particular partial history (resp. the set of all partial histories) up to some period when the terminal date is unimportant.

2.3. *Commodities, Activities, and the Technology*

There are m *producible* commodities at each date. In addition, there are primary factors, which cannot be produced but the supply of which is exogenously given. To simplify notation, we concentrate on the case in which there is a single primary factor (to be called "labor"), that is essential in production. To be sure, we do not use any result like the nonsubstitution theorem in which the fact that there is a *single* primary factor is of importance. What follows can be extended to allow for more than one primary factor, with each activity requiring at least one of these for operating at positive intensity.

The production possibilities at each date are described by J *activities* or techniques of production. At each date, the activity j $(= 1, 2, \ldots, J)$ is operated at an intensity or level z_j $(\geqq 0)$. To express the idea that the inputs and outputs corresponding to any given activity level may be stochastic we postulate that (i) the input requirements of these activities at any date t depend on the partial history h_t of the environment up to that date, and (ii) the corresponding outputs available in period $t+1$ depend on the history h_{t+1} of the environment up to date $t+1$. Thus, input requirements at date t are observable at date t, but the corresponding outputs may only be known at date $t+1$.

Let $l_j(h, z_j)$ be the labor requirement of activity j when operated at an intensity z_j if the partial history is given by h. The following assumption is a precise description of the essential role of labor in production.

ASSUMPTION A.1: For each $j = (1, 2, \ldots, J)$ the function $l_j(h, z_j)$ defined on $H \times R_+$ assumes values in R_+; moreover: (a) $l_j(h, 0) = 0$ for all h in H; (b) $l_j(h, z_j)$ is strictly increasing in z_j and as $z_j \to \infty$, $l_j(h, z_j) \to \infty$; (c) $l_j(\cdot, \cdot)$ is continous on $H \times R_+$.

Given a vector $z = (z_j)$ of activity levels, we denote by $l(h, z)$ the total labor requirement when the activities are operated at z, i.e.,

$$(2.4) \qquad l(h, z) = \sum_{j=1}^{J} l_j(h, z_j).$$

For any $z_j \geqq 0$, let $m_j(z_j) = \min_{h \in H} l_j(h, z_j)$, i.e., the minimum amount of labor needed to run the jth activity at intensity z_j, no matter what the partial history is. By the continuity and strong monotonicity of l_j and compactness of H, one easily gets

$$m_j(0) = 0, \qquad m_j(z_j^1) > m_j(z_j^2) \quad \text{if} \quad z_j^1 > z_j^2. \tag{2.5}$$

To express the idea that the supply of labor may also be a stationary stochastic process, we assume that it is exogeneously given by a function $\boldsymbol{M}$ on H satisfying a continuity property:

ASSUMPTION A.2: The function $\boldsymbol{M}$ from H into R_+ is continuous and strictly positive. Hence, there are numbers $\boldsymbol{M}_1, \boldsymbol{M}_2$ such that

$$\boldsymbol{M}_2 \equiv \max_{h \in H} \boldsymbol{M}(h) \geqq \min_{h \in H} \boldsymbol{M}(h) \equiv \boldsymbol{M}_1 > 0. \tag{2.6}$$

Turning now to the producible commodities, the requirement of the ith commodity in the jth activity operated at an intensity z_j at date t is specified by a function $R_{ij}(h_t, z_j)$ and the output of the ith commodity from the jth activity by a function $P_{ij}(h_{t+1}, z_j)$. Thus, if at date t the vector of activity levels is given by $z = (z_j)$ in R_+^J, the *total* input requirement for commodity i is given by

$$R_i(h_t, z) = \sum_{j=1}^{J} R_{ij}(h_t, z_j) \tag{2.7}$$

and the total output of the ith commodity generated by the activities at date $t+1$ is given by

$$P_i(h_{t+1}, z) = \sum_{j=1}^{J} P_{ij}(h_{t+1}, z_j). \tag{2.8}$$

We shall denote by $R(h_t, z)$ and $P(h_{t+1}, z)$ the m-vectors whose coordinates are $R_i(h_t, z)$ and $P_i(h_{t+1}, z)$, respectively. These are the vectors of inputs and outputs of the producible goods resulting from the operation of the activities at levels $z = (z_j)$ when the partial histories of the environment up to dates t and $t+1$ are given by h_t and h_{t+1}, respectively. The following assumptions are made about the nature of the functions R_{ij} and P_{ij}. Their interpretation (being standard) is not spelled out:

ASSUMPTION A.3: For each (i, j), $R_{ij}(h_t, z_j)$ is a continuous function on $H_t \times R_+$ with values in R_+. $R_{ij}(h_t, 0) = 0$ and $R_{ij}(h_t, z_j)$ is nondecreasing in z_j for each h_t in H_t.

ASSUMPTION A.4: For each (i, j), $P_{ij}(h_{t+1}, z_j)$ is a continuous function on $H_{t+1} \times R_+$ with values in R_+ and $P_{ij}(h_{t+1}, 0) = 0$ for all h_{t+1} in H_{t+1}.

The *feasibility requirements* on the possible choice of $z = (z_j)$ at any date t will now be specified. Given the stocks of producible commodities resulting from the activities in the previous period and the supply of labor, a vector z of activity levels is *feasible* if it satisfies the constraint that the input requirements cannot exceed the available supply. Some bounds on feasible activity levels will now be derived so that we can conveniently restrict our attention to a compact set of feasible activity levels.

For any feasible activity level $z = (z_j)$ one must have

$$l(h,z) \leqq M(h). \tag{2.9}$$

The left side of (2.9) is the total labor requirement for $z = (z_j)$ whereas the right side is the total supply of labor. The inequality (2.9) leads to the following bound:

Under Assumptions A.1 *and* A.2 *there is some* $\beta > 0$, *such that for all feasible activity levels* $z = (z_j)$,

$$\|z\| = \sum_{j=1}^{J} z_j \leqq \beta. \tag{2.10}$$

Let us now define

$$A = \{z \in R_+^J : z \geqq 0, \|z\| \leqq \beta\}. \tag{2.11}$$

Clearly the compact set A contains all the feasible activity levels, no matter what the environment at date t is. Let ϕ be the correspondence from $H \times R_+^m$ to A defined by:

$$\phi(h,k) = \{z \in A : l(h,z) \leqq M(h), R(h,z) \leqq k\}. \tag{2.12}$$

In other words, $\phi(h,k)$ is the set of all activity levels that are *feasible* given h and k, in the sense that the input requirements of capital and labor are no greater than the available supplies.

The following proposition is easily proved:

PROPOSITION 2.1: *Under Assumptions* A.1 *through* A.3, *the correspondence* $\phi(h,k)$ *from* $H \times R_+^m$ *into* A *is upper semi-continuous.*

In Section 3, we provide sufficient conditions for ϕ to be continuous. We note that for any producible commodity i one has (from 2.8)

$$y_i = P_i(h_{t+1}, z) = \sum_{j=1}^{J} P_{ij}(h_{t+1}, z_j). \tag{2.13}$$

It follows from the continuity of P_{ij} on the compact set $H_{t+1} \times A$ that there is some positive constant γ' such that for all h_t in H and z in A,

$$\|y\| \equiv \sum_{i=1}^{m} y_i \leqq \gamma'. \tag{2.14}$$

We denote the compact set of nonnegative m-vectors satisfying the bound (2.14) by K, i.e.,

$$K = \{k \in R^m : k \geqq 0, \|k\| \leqq \gamma'\}. \tag{2.15}$$

Writing $S_t = H_t \times K$ (the product being endowed with the product topology), we refer to S_t as the *state space* in period t. Thus, the generic element s_t of S_t is the pair (h_t, k_t) representing a partial history up to date t and the stocks of producible commodities at date t. It is clear that S_t is a compact (hence, a complete, separable) metric space. Note that for all t, S_t and S_{t+1} are homeomorphic to each other, and thus in any discussion in which only the topological structures are relevant, the subscript can be dropped, and we shall simply let $S_t = S$ whenever the time-period is unimportant. In what follows the domain of ϕ is restricted to S.

The optimal resource allocation problem that we study will be treated as a variant of the problem of discounted dynamic programming. In the dynamic programming terminology (see, e.g., Blackwell [3]), the compact set A is the set of all possible *actions* (see (2.11)) whereas $\phi(s)$ is the set of all *feasible* actions when the system is in the state s. Thus, taking an action in the dynamic programming means choosing activity levels in our context. Given the state in period t, say $s_t = s$ and an action chosen in that period, the system moves to a new state s_{t+1}. Let $q(\cdot \mid s, z)$ be the probability distribution of $s_{t+1} \equiv (h_{t+1}, k_{t+1})$ given $s_t = s$ and the action z. Thus, the family $q(\cdot \mid (s, z))$ for each (s, z) in $S \times A$ describes the *stochastic laws of evolution* of the system. We first note that the evolution of the environment is governed by a stochastic law that is exogenously specified and is independent of k_t or z_t. This stochastic law is formally specified by an initial distribution λ_1 on $\mathscr{B}(H_1)$ and the family $\lambda(\cdot \mid h_t)$. $\lambda(\cdot \mid h_t)$ is to be interpreted as the distribution of h_{t+1} given h_t, whereas λ_1 is the '*initial*' distribution of h_1. On the other hand, the distribution of k_{t+1} given (h_t, k_t, z_t) is determined according to

$$k_{t+1} = P(h_{t+1}, z_t). \tag{2.16}$$

Clearly, the distribution of k_{t+1} given h_t, k_t, and z_t is determined by $\lambda(\cdot \mid h_t)$ and the function P in (2.16). Thus, the family $q(\cdot \mid (s \cdot z))$ is well-defined for all s in S in z in A (irrespective of whether the feasibility requirements on z are met or not). At this stage, we introduce the following regularity condition on the stochastic law of the environment that will be assumed in the subsequent discussion.

ASSUMPTION A.5: If h_t^n converges to h_t, the sequence $\lambda(\cdot \mid h_t^n)$ of probability measures on $\mathscr{B}(H_{t+1})$ converges weakly to $\lambda(\cdot \mid h_t)$.

Given h_t, a partial history up to period t, the randomness of $h_{t+1} \equiv (h_t, \omega_{t+1})$ is solely due to the randomness of ω_{t+1}. Let $\lambda'(\cdot \mid h_t)$ be the distribution of ω_{t+1} given h_t, a probability measure on $\mathscr{B}(W)$. Clearly, if f is any continuous real valued function on H_{t+1}, one must have

$$(2.17) \quad \int_{H_{t+1}} f(h_{t+1})\lambda(dh_{t+1} \mid h_t) = \int_{H_{t+1}} f(h_{t,\omega_{t+1}})\lambda(dh_{t+1} \mid h_t) = \int_W f(h_t, \cdot)\lambda'(d\omega_{t+1} \mid h_t).$$

From (2.17) one can show that under Assumption A.5, *if h_t^n converges to h_t, the sequence $\lambda'(\cdot, h_t^n)$ of probability measures on $\mathscr{B}(W)$ converges weakly to $\lambda'(\cdot, h_t)$ on* $\mathscr{B}(W)$. To see this let g be any continuous real valued function on W. Define f on H_{t+1} as $f(h_t, \omega_{t+1}) = g(\omega_{t+1})$. Clearly f is continuous on H_{t+1}. And by A.5

$$(2.18) \quad \int_{H_{t+1}} f(h_{t+1})\lambda(dh_{t+1}, h_t^n) \quad \text{converges to} \quad \int_{H_{t+1}} f(h_{t+1})\lambda(dh_{t+1} \mid h_t).$$

But, by (2.18) we have

$$(2.19) \quad \int_W g(\cdot)\lambda'(d\omega_{t+1} \mid h_t^n) \quad \text{converges to} \quad \int_W g(\cdot)\lambda'(d\omega_{t+1} \mid h_t).$$

Two examples in which Assumption A.5 holds will be mentioned. If $\langle \omega_t \rangle$ is a sequence of independent and identically distributed random variables with a common distribution Θ on $\mathscr{B}(W)$, the conditional distribution $\lambda'(\cdot \mid h_t)$ is also Θ. One can use (2.17) and the Lebesgue dominated convergence theorem to verify Assumption A.5. Secondly, if ω_t is a Markov process with a stationary kernel $\Theta(\cdot, \omega)$ which is weakly continuous in ω, i.e., if ω^n converges to $\bar{\omega}$, the sequence $\Theta(\cdot, \omega^n)$ of probability measures converges weakly to $\Theta(\cdot, \bar{\omega})$, then Assumption A.5 can also be directly verified. In such a verification, and at various steps in what follows, the following mathematical lemma is needed (see Yushkevich [**19**, Lemma 2]):

LEMMA 2.1: *Let X, Y be compact metric spaces, and f be a continuous real valued function on $X \times Y$. Let μ^n be a sequence of probability measures on $\mathscr{B}(Y)$ converging weakly to a probability measure μ on $\mathscr{B}(Y)$. For any sequence x^n in X converging to x in X,*

$$(2.20) \quad \int_Y f(x^n, \cdot)\mu^n(dy) \quad \textit{converges to} \quad \int_Y f(x, \cdot)\mu(dy).$$

We shall now state and prove some useful continuity and measurability properties of the family of $q(\cdot \mid s, z))$ of probability distributions.

LEMMA 2.2: *Under Assumptions* A.1–A.5, *for any sequence (s^n, z^n) converging to (s, z), the corresponding sequence $q(\cdot \mid (s^n, z^n))$ of probability measures converges weakly to $q(\cdot \mid (s, z))$.*

PROOF: Use Lemma 2.1 (taking $Y = W$, $X = H_t \times A$, and noting the continuity of P and f) and the remarks following (2.17). *Q.E.D.*

LEMMA 2.3: *For any B in $\mathscr{B}(H_{t+1})$, $q(B \mid \cdot)$ is a $\mathscr{B}(S \times A)$-measurable function.*

PROOF: For any B in $\mathscr{B}(H_{t+1})$, $q(B \mid \cdot)$ is a function from $S \times A$ into $[0,1]$. We shall show that it is the composition of two measurable functions. Let $\psi_1 : S \times A$ into $\mathscr{M}(H_{t+1})$, be defined by: $\psi_1(s,z) = q(\cdot \mid (s,z))$. By Lemma 2.2, ψ_1 is a continuous, hence $\mathscr{B}(S \times A)$-measurable. Let $\psi_2 : \mathscr{M}(H_{t+1})$ into $[0,1]$ be defined as: $\psi_2(q) = q(B)$. Measurability of ψ_2 is proved in Varadarajan [**18**]. Then, for any fixed B in $\mathscr{B}(H_{t+1})$, $q(B \mid (s,z)) = \psi_2[\psi_1(s,z)]$. *Q.E.D.*

For each $t \geqq 1$, define the product set $\mathscr{E}_t$ as

$$\mathscr{E}_t \equiv \left[\prod_{\tau=1}^{t-1} (S_\tau \times A) \right] \times S_t .$$

The generic element of $\mathscr{E}_t$ is indicated by $e_t \equiv (s_1, z_1; s_2, z_2; \ldots s_{t-1}, z_{t-1}; s_t)$. $\mathscr{E}_t$ is endowed with the relevant product of the Borel σ-fields of the components. A *program π of resource allocation* (or, briefly, a *plan* or a *policy* according to the terminology of dynamic programming) is a sequence of functions $\pi = (f_1, \ldots, f_t, \ldots)$ where each f_t $(t \geqq 1)$ specifies the action selected in period t (i.e., the levels of the activities chosen in period t) as a (Borel-measurable) function of the evolution $e_t = (s_1, z_1; \ldots, s_{t-1}, z_{t-1}; s_t)$ of the system; moreover, the selection must be consistent with the feasibility requirements given by the correspondence ϕ; formally, each f_t $(t \geqq 1)$ is a measurable function from $\mathscr{E}_t$ into A, satisfying, for each $e_t \equiv (s_1, z_1; \ldots, s_{t-1}, z_{t-1}; s_t)$ in $\mathscr{E}_t$, $f_t(e_t) \in \phi(s_t)$.

We are especially interested in *stationary programs* or policies $\pi = (f, f, \ldots, f, \ldots) \equiv (f^\infty)$ defined by a *single* Borel-measurable function f from S into A, satisfying for each s in S, $f(s) \in \phi(s)$. Whenever the system is in s, the action chosen (i.e., the vector of activity levels selected) is given by $f(s)$, and this is true irrespective of how the system moved in the past and arrived at the state s: the policy is "memoryless."

A program of resource allocation π generates a *consumption program* $c = (c_t)$ in the following way: for each $t \geqq 1$, c_t is a function from H_t into R^m_+ defined as

$$c_t(h_t) = k_t - R(h_t, f_t(h_t, k_t)). \tag{2.21}$$

The utility derived from consumption is given by a function u defined on R^m_+. We assume that the *utility function* ("return" or "reward" function in the terminology of dynamic programming) has the following properties:

ASSUMPTION A.6: The function $u : R^m_+ \to R$ is bounded and continuous.

The relevant continuity properties imply that, if a sequence (h^n, k^n, z^n) converges to $(h, k; z)$ in $S \times A$, $u(k^n - R(k^n, z^n))$ converges to $u(h - R(h, z))$.

With a discount factor δ, satisfying $0 < \delta < 1$, the total discounted expected utility derived from any program π from the initial state $s = (h, k)$ is defined as

$$V_\pi(s) = \sum_{t=1}^{\infty} \delta^{t-1} u_t(\pi) \tag{2.22}$$

where u_t is the expected utility in period t generated by π. A program π is *optimal* if $V_\pi(s) \geqq V_{\pi'}(s)$ for all s in S, and for *every* program π'. Our first theorem asserts the existence of a *stationary* program $\pi = (f^\infty)$ that is optimal and characterizes the optimal program in terms of the "optimality equation" of dynamic programming. We refer to f as an *optimal policy function*. A continuity property of V_π is also established.

THEOREM 2.1: *Under Assumptions* A.1 *through* A.6, *there is a stationary program $\pi = (f^\infty)$ that is optimal; moreover, it satisfies*

$$V_\pi(h_t, k_t) = \max_{z \in \phi(h_t, k_t)} \left[u(k_t - R(h_t, z)) + \delta \int V_\pi(\cdot)\, dq(\cdot \mid k_t, h_t, z) \right]. \tag{2.23}$$

V_π is upper semi-continuous on $H \times K$.

REMARKS: (i) The proof of Theorem 2.1 is sketched in the Appendix. (ii) It should be emphasized that we have *not* made any convexity assumption on the technology or concavity assumption on the utility function to prove the existence of a stationary optimal program.[6]

3. THE OPTIMAL POLICY FUNCTION AND THE STOCHASTIC PROCESS OF OPTIMAL ACCUMULATION

In this section, we shall first establish some interesting properties of the optimal policy function. Conditions on the utility function and the input requirement functions under which the optimal policy function is continuous will be discussed. We shall also examine the stochastic processes of optimal capital stocks generated by a continuous optimal policy function. Conditions under which the process has a stochastic steady state are spelled out.

Going back to the "optimality equation" (2.23) which characterizes V_π, we note that the optimal policy function f is actually a selection from a correspon-

[6] Problems of intertemporal allocation when the technology is not convex have recently been studied in some detail by Majumdar and Mitra [13] in the framework of a deterministic, aggregative model. In particular, they have dealt with optimal growth when future utilities are discounted, and noted that the question of existence of a stationary optimal program gets more complicated when the production function has an initial phase of increasing returns. To our knowledge, extensions of the type of results reported in that paper to the case in which the nonconvex technology is subject to random shocks have not yet been achieved.

dence ψ (see the Appendix, especially L.a.1 and the arguments immediately preceeding (a.5)). By the well-known "maximum theorem" (Berge [1]), this correspondence ψ is upper semi-continuous if ϕ is continuous (given the other assumptions of our model). Thus, our first task is to strengthen Proposition 2.1 and to provide conditions under which ϕ is actually continuous. While upper semi-continuity of ϕ is rather trivially established, we have been able to prove lower semi-continuity only under a more restrictive condition (Assumption A.7 stated below) on the input-requirement functions. After deriving the continuity of ϕ, our next task is to give conditions under which ψ is itself a single-valued function rather than a correspondence. This, of course, means that ψ $(=f)$ is necessarily continuous.

To establish the continuity of ϕ, we introduce the following Assumption A.7 which is admittedly somewhat restrictive:

Assumption A.7: (i) For each j, $l_j(h, \cdot)$ is convex on R_+; for each (i, j), $R_{ij}(h, \cdot)$ is convex in z_j.

(ii) For any pair (i, j), if $R_{ij}(h, z_j) > 0$ for some $z_j > 0$ and some h in H, then $R_{ij}(h, z_j) > 0$ for all $z_j > 0$ and all h in H.

The convexity assumption (i) implies that ϕ is convex-valued: We can informally describe A.7(ii) as follows: if any capital good i is essential in operating activity j at positive intensity in *some* partial history of the environment, it remains so in *all* partial histories. We can now prove the following Proposition.

Proposition 3.1: *Under Assumptions* A.1, A.3, *and* A.7, *the correspondence* $\phi(h,k)$ *from* $H \times R_+^m$ *into* A *is continuous and convex-valued.*

Proof: In view of Proposition 2.1, we go to the proof of the lower semi-continuity of ϕ. Suppose that (h^n, k^n) is a sequence converging to (h, k). Let z be an arbitrary element of $\phi(h,k)$. This means that $z \geqq 0$ satisfies

$$(3.1) \qquad \begin{cases} R(h,z) \leqq k, \\ l(h,z) \leqq M(h). \end{cases}$$

We want to construct a sequence z^n converging to z such that for all n, z^n belongs to $\phi(h^n, k^n)$. Note that if $z = 0$, such a sequence can be constructed trivially by taking $z^n = 0$ for all n. Consider, therefore, the case when $z > 0$. The inequalities (3.1) are rewritten as follows: let I_1 be (the possibly empty) set of indices for commodities for which $k_i = 0$, and I_2 be the set for which $k_i > 0$. Thus $I_1 \cup I_2 = \{1, 2, \ldots, m\}$. Clearly one must have

$$(3.2) \qquad \begin{cases} R_i(h,z) = k_i = 0 & \text{for all} \quad i \in I_1, \\ R_i(h,z) \leqq k_i \ (>0) & \text{for all} \quad i \in I_2, \\ l(h,z) \leqq M(h). \end{cases}$$

Consider the sequence (h^n, k^n), and for each n, define

$$\Lambda_n = \{\lambda \in [0,1] : R(h^n, \lambda z) \leqq k^n;\ l(h^n, \lambda z) \leqq M(h^n)\}. \tag{3.3}$$

Observe that Λ_n is nonempty for each n; indeed it is necessarily a closed interval $[0, \lambda^n]$. Obviously, Λ_n is closed as the relevant functions are continuous, and given their monotonicity property, if $\lambda' \in \Lambda_n$ so does λ'' where $0 \leqq \lambda'' \leqq \lambda'$. Since by construction of Λ_n, $\lambda^n z$ is in $\phi(h^n, k^n)$, we shall prove that $\lambda^n z$ converges to z, i.e., λ^n converges to 1. Suppose that the sequence (λ^n) does not converge to 1. We can then find a subsequence $\lambda^{n'}$ converging to some $\lambda < 1$. By convexity,

$$R_i(h, \lambda z) = R_i(h, (1-\lambda)0 + \lambda z) \leqq \lambda R_i(h, z). \tag{3.4}$$

By strict monotonicity (since $z_j > 0$ for some j) of l_i (see A.1(b)),

$$l(h, \lambda z) < l(h, z) \leqq M(h). \tag{3.5}$$

Hence, we can assert

$$\begin{cases} R_i(h, \lambda z) = k_i = 0 & \text{for all } i \in I_1, \\ R_i(h, \lambda z) \leqq \lambda k_i < k_i & \text{for all } i \in I_2, \\ l(h, \lambda z) < M(h). \end{cases} \tag{3.6}$$

Using a continuity argument, one shows that there is some n'_0 such that $n' \geqq n'_0$ implies

$$\begin{cases} \text{(a)}\ \ R_i(h^{n'}, \lambda^{n'} z) = 0 \leqq k_i^{n'} & \text{for all } i \in I_1, \\ \text{(b)}\ \ R_i(h^{n'}, \lambda^{n'} z) < k_i^{n'} & \text{for all } i \in I_2, \\ \text{(c)}\ \ l(h^{n'}, \lambda^{n'} z) < M(h^{n'}). \end{cases} \tag{3.7}$$

From (3.6) it must be true that for all activities j such that $z_j > 0$, $R_{ij}(h, \cdot) = 0$ for all $i \in I_1$. By Assumption A.7, $R_{ij}(h^n, \cdot) = 0$ for all h^n. Hence (a) of (3.7) is satisfied. From (3.7) we get a contradiction to the maximality of $\lambda^{n'}$ defined following (3.3). *Q.E.D.*

In addition to Assumption A.6 (continuity and boundedness), we make the following assumption on the utility function:

Assumption A.8: $u : R_+^m \to R$ is nondecreasing and concave.

We say that a particular commodity i is *desirable* if for any consumption vector $c \geqq 0$, and any $\epsilon > 0$, $u(c') > u(c)$ where $c'_i = c_i + \epsilon$, $c'_k = c_k$ for all $k \neq i$. In other words, an increase in consumption of the ith commodity leads to an increment in utility. Going back to the technology, let us say that an activity j *uses* a commodity i, if for some h in H, $R_{ij}(h, z_j) > 0$ for some $z_j > 0$. In other words, if the jth activity is operated at some positive level given a partial history

h, a positive quantity of commodity i is required as an input. Indeed, by Assumption A.7, if the jth activity uses the commodity i, $R_{ij}(h, z_j) > 0$ for *all* h in H and all $z_j > 0$. Thus, by Assumption A.7 if the jth activity uses commodity i, a positive quantity of commodity i is *always* required in order to operate the jth activity at any positive intensity, no matter what the partial history is.

ASSUMPTION A.9: Each activity j uses at least one producible commodity i which is desirable; moreover, $R_{ij}(h, \cdot)$ is strictly convex for all h in H.

Given a pair (h, k) consider now two distinct feasible activity levels $z^1 \geqq 0$, $z^2 \geqq 0$. Without loss of generality, let us assume $z^2 \neq 0$, and that $z_j^2 > 0$ for some j. Let i be the desirable commodity that j uses. Thus, $R_{i,j}(h, z_j) > 0$ for all h in H, and the function $R_{i,j}(h, \cdot)$ is strictly convex. In other words,

$$R_{i,j}\left(h, \lambda z_j^1 + (1-\lambda) z_j^2\right) < \lambda R_{i,j}\left(h, z_j^1\right) + (1-\lambda) R_{i,j}\left(h, z_j^2\right)$$

for all λ in $(0, 1)$. By Assumption A.3, we have, for all i,

$$\sum_{j=1}^{J} R_{ij}\left(h, \lambda z_j^1 + (1-\lambda) z_j^2\right) \leqq \lambda \sum_{j=1}^{J} R_{ij}\left(h, z_j^1\right) + (1-\lambda) \sum_{j=1}^{J} R_{ij}\left(h, z_j^2\right)$$

and the inequality is strict for the commodity $\boldsymbol{i}$. Thus,

$$R\left(h, \lambda z^1 + (1-\lambda) z^2\right) < \lambda R(h, z^1) + (1-\lambda) R(h, z^2).$$

Note that $k - R(h, z^1) \geqq 0$, $k - R(h, z^2) \geqq 0$. Hence, $k - R(h, \lambda z^1 + (1-\lambda) z^2) > \lambda[k - R(h, z^1)] + (1-\lambda)[k - R(h, z^2)]$. Since u is nondecreasing and the commodity $\boldsymbol{i}$ is desirable, $u[k - R(h, \lambda z^1 + (1-\lambda) z^2)] > \lambda u[k - R(h, z^1)] + (1-\lambda) u[k - R(h, z^2)]$.

If we now go back to (2.23) and use the implications of Assumptions A.7–A.9 that have just been spelled out, we can verify that the set of maximizing activity levels is necessarily reduced to a single point. This leads to the following result:

THEOREM 3.1: *Under Assumptions* A.1 *through* A.9, *the optimal policy function f is continuous.*

In what follows, we shall study the stochastic process of optimal capital stocks generated by a continuous optimal policy function f. Strictly speaking, we examine the process $\langle \boldsymbol{h}, \boldsymbol{k}, \rangle = (h_t, k_t)$ which has two interesting properties. First, the conditional distributions $\lambda(\cdot \mid h_t)$ are exogenously given, and satisfy the continuity property A.5. Secondly, the sequence k_t must satisfy

$$k_{t+1} = P(h_{t+1}, f(h_t, k_t)), \tag{3.8}$$

where f is a continuous function from S into A. The stochastic law of the Markov process $\langle \boldsymbol{h},\boldsymbol{k}\rangle$ is determined by an initial distribution, say θ_1, of (h_1,k_1) and the *kernel* $\nu(B,(h,k))$ which specifies the "one step" transitional probabilities (which are stationary since the same function f is involved in (3.8) for every t). Formally, $\nu(B,\cdot)$ is a $\mathscr{B}(S)$-measurable function for every B in $\mathscr{B}(S)$; and for every (h,k), $\nu(\cdot,(h,k))$ is a probability measure on $\mathscr{B}(S)$.[7] Given θ_1, the initial distribution of (h_1,k_1), one determines θ_t, the distribution of (h_t,k_t) for all $t \geqq 2$ from the relationship

$$(3.9) \qquad \theta_t(B) = \int_S \nu(B,\cdot)\,d\theta_{t-1} \qquad \text{for all } B \text{ in } \mathscr{B}(S).$$

An invariant distribution θ^* of the Markov process $\langle \boldsymbol{h},\boldsymbol{k}\rangle$ has the property that if θ^* happens to be the distribution of (h_t,k_t), it is also the distribution of $(h_{t'},k_{t'})$ for all $t' \geq t$. Another way of describing the property is to use the relation (3.9). Let T^* be the operator from $\mathscr{M}(S)$ into $\mathscr{M}(S)$ defined as $T^*\theta(B) = \int_S \nu(B,\cdot)\,d\theta$ for all B in $\mathscr{B}(S)$. T^* is the operator mapping the distribution of (h_t,k_t) into the distribution of (h_{t+1},k_{t+1}). An invariant distribution or a steady state θ^* of the process $\langle \boldsymbol{h},\boldsymbol{k}\rangle$ is a *fixed point* of the mapping T^*, i.e., $\theta^* = T^*\theta^*$. A standard fixed point argument (see Brock and Majumdar [**4**, p. 240]) leads to the following theorem.

THEOREM 3.2: *Under Assumptions* A.1 *through* A.9, *there exists at least one invariant distribution* θ^* *of the process* $\langle \boldsymbol{h},\boldsymbol{k}\rangle$.

We have not yet ruled out the possibility that $f(h,k)=0$, i.e., the optimal policy is one of "inaction" and no activity is operated at positive intensity. It is not difficult to show that if there is an activity j' which produces a desirable commodity by using labor only, this situation cannot arise (note, however, that our A.9 used in deriving the continuity of f rules this condition out).

A second, and perhaps more interesting class of models in which inaction cannot be optimal can be characterized in terms of an appropriate *productivity* condition. To take the simplest example, suppose there is an activity $\boldsymbol{j}$ that uses one unit of a desirable commodity $\boldsymbol{i}$ and some labor to produce $\rho > 1$ units of $\boldsymbol{i}$ (with certainty). Consider the vector $\boldsymbol{c} = (0, \ldots, \rho, \ldots, 0)$ and suppose $\delta u(\boldsymbol{c}) > u(0, \ldots, 1, \ldots, 0)$. In this case it will clearly be nonoptimal to use no activity at all in all periods.

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Manuscript received November, 1980; final revision received January, 1983.

[7]Note that under Assumptions A.1 through A.9, convergence of (h_t^n,k_t^n) to (h_t,k_t) implies weak convergence of $\nu(\cdot,(h_t^n,k_t^n))$ to $\nu(\cdot\, h_t,k_t)$. This is proved by using Lemma 2.1. This continuity property can be used to verify that $\nu(B,\cdot)$ is $\mathscr{B}(S)$-measurable by following the arguments of Lemma 2.3.

APPENDIX

PROOF OF THEOREM 2.1: We shall sketch only the main steps leading to Theorem 2.1 since the proof relies on the ideas developed by Maitra [**11**] and Furukawa [7].

Our dynamic programming problem is specified by $\langle S, A, \phi, q, u\rangle$ defined and interpreted in Section 2. The following assumptions are made:

(p.1): *S is a complete separable metric space* (*recall* (2.15)).

(p.2): *A is a compact metric space* (*recall* (2.11)).

(p.3): *$\phi(s)$ is a nonempty subset of A, and ϕ is upper semi-continuous* (*recall Proposition* 2.1).

(p.4): *u is bounded and continuous on $S \times A$* (*recall* A.6).

(p.5): *$q(B \mid \cdot)$ is $\mathscr{B}(S \times A)$-measurable for each B in $\mathscr{B}(S)$* (*recall Lemma* 2.3) *and for each $(s,a) \in S \times A$, $q(\cdot \mid s,a)$ belongs to $\mathscr{M}(S)$.*

(p.6): *For any sequence (s_n, a_n) converging to (s,a), the sequence $q(\cdot \mid (s_n, a_n))$ of probability measures converges weakly to $q(\cdot \mid (s,a))$* (*recall Lemma* 2.2).

Define the map $Z: S \to R$ by

(a.1) $$Z(s) = \max_{a \in \phi(s)} u(s,a).$$

L.a.1. *Under* (p.1) *through* (p.6), *the function $Z: S \to R$ is u.s.c.; the correspondence $\psi: S \to A$ defined as $\psi(s) = \{a \in \phi(s) : u(s,a) = Z(s)\}$ is $\mathscr{B}(S)$-measurable; moreover, there is a $\mathscr{B}(S)$-measurable function $f: S \to A$ such that $f(s) \in \psi(s)$ for all s in S.*

PROOF: See Parthasarathy (**14**, Lemma 2.1, Theorem 2.1).

L.a.2. *Let $w: S \to R$ be in $C_1(S)$; then $g: S \times A \to R$ defined as*

(a.2) $$g(s,a) = \int w(\cdot)\, dq(\cdot \mid (s,a))$$

is also in $C_1(S \times A)$.

PROOF: See Maitra [**11**, Lemma 4.1].

For any w in $C_1(S)$, define the function $Fw: S \to R$ as

(a.3) $$Fw(s) = \max_{a \in \phi(s)} \left[u(s,a) + \delta \int w(\cdot)\, dq(\cdot \mid (s,a)) \right].$$

By (p.4) and L.a.2, the expression within the square bracket of the right side of (a.3) is u.s.c. in (s,a), and by (p.3), $\phi(s)$ is compact for all $s \in S$; hence, the maximum is attained for every s, and by L.a.1, Fw is u.s.c. and is clearly bounded. Hence, F maps $C_1(S)$ into $C_1(S)$. By using Maitra [**11**, Lemma 4.3], one gets the following result.

L.a.3. *With $0 < \delta < 1$, F is a contraction from $C_1(S)$ into $C_1(S)$; hence, F has a unique fixed point $w^* = Fw^*$.*

To complete the proof, let us define, for each g in $B(S)$ satisfying $g(s) \in \phi(s)$, the operator $\mathbb{L}_g$ on

$B(S)$ which maps w into $\mathbb{L}_g w$ defined as

$$\text{(a.4)} \qquad \mathbb{L}_g w(s) = u(s, g(s)) + \delta \int w(\cdot)\, dq(\cdot \mid (s, g(s)).$$

It is known that $V_{(g^{\infty})}$ is the unique fixed point of the contraction operator $\mathbb{L}_g$. Going back to L.a.3, and using L.a.1, there is some $\mathscr{B}(S)$-measurable $f\colon S \to A$, $f(s) \in \psi(s)$ such that $Fw^* = L_f w^* = V_{(f^{\infty})}$, i.e.,

$$\text{(a.5)} \qquad V_{(f^{\infty})}(s) = \max_{a \in \phi(s)} \left[u(s,a) + \delta \int V_{(f^{\infty})}(\cdot)\, dq(\cdot \mid (s,a)) \right].$$

Since w^* is in $C_1(S)$, the equality $w^* = V_{f(\infty)}$ also establishes that $V_{f(\infty)}$ is u.s.c. on S. Thus, the *stationary policy* $\pi \equiv (f^{(\infty)})$ *satisfies the "optimality equation"* (a.5), *and the total discounted expected return generated by* π *in u.s.c. on* S (*recall* (2.23)).

In order to complete the proof, it remains to show that π is optimal. One uses the steps leading to the basic result of Blackwell [3, Theorem 6(f) on p. 232]. The main difficulty is to extend the Lemma on p. 228 of Blackwell [3] to our framework, and this is overcome by appealing to Furukawa [7, Lemma 3.2 on p. 615].

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