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THE APPLICATION OF LINEAR PROGRAMMING TO TEAM DECISION PROBLEMS*[1]

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In a team decision problem there are two or more *decision variables*, and these different decisions can be made to depend upon *different* aspects of the environment, or *information variables*, the resulting payoff being a random variable. The choice of optimal rules for selecting information variables and for making decisions is the central problem of the economic theory of teams. This paper shows, by means of an example, how linear programming can be applied to obtain optimal team decision functions in the case in which the payoff to the team is a convex polyhedral function of the decision variables.

1. Introduction[2]

In a *team decision problem* there are two or more *decision variables*, and these different decisions can be made to depend upon different aspects of the environment, i.e., upon different *information variables*. The choice of optimal rules for selecting information variables and for making decisions is the central problem of the economic theory of teams. In a previous paper [1], Marschak has given an introduction to the main concepts of this theory. In the present paper I shall show how the technique of linear programming can be used to solve a typical class of team decision problems.

The "character" of a decision problem is determined by the form of the function that is to be maximized, which I shall call the *payoff function*. Much of the available data about business leads naturally to the formulation of decision problems in terms of what might be called *convex polyhedral* payoff functions; i.e., problems for which the space of decision variables can be divided into regions, whose boundaries are linear, such that within each region the payoff is a linear function of the decision variables. As is well known, such a problem is amenable to linear programming, and as I have shown in another paper [2], the introduction of probabilistic uncertainty, and of the further complications of a team situation, does not destroy the linear character of a programming problem, although it may result in a substantial increase in the "size" of the problem.

In this paper I will illustrate these ideas by means of an example; a general

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[2] I am indebted to A. Manne and J. Marschak for helpful comments on this paper.

formulation has been given in the paper just referred to, but the reader will probably have no trouble in providing such a generalization himself. The example to be used is about as simple as it can be without sacrificing any of the three features that I want to illustrate, namely, (1) uncertainty, (2) the fact that different decision variables can be made to depend upon different information variables, and (3) a nondegenerate convex polyhedral payoff function. (Therefore, the reader should not expect too much in the way of realism!)[3]

Within the framework of the example, I shall (1) show how to apply linear programming to compute optimal decision rules for any particular structure of information and communication; (2) compare different structures of information and communication; (3) discuss an effect of joint constraints on the decision variables, in a partly "decentralized" team; and (4) point out the relationship between team decision problems and sequential decision problems.

2. An Example

Consider a "firm" with two activities, which I will label "production" and "promotion," and suppose that the levels of expenditure on these two activities must be chosen for one period to come. Let a denote the amount of money alloted to production, and let xa be the resulting quantity produced (there is only one commodity concerned), where x can be interpreted as the "productivity" of the production activity. Similarly, let b denote the amount of money alloted to promotion, and let yb be the resulting demand generated. The quantity actually sold will therefore be the smaller of the two quantities, xa and yb; if both the product and the demand generated are "perishable," and if the units are chosen so that the price of the commodity is 1, then the profit resulting from the pair of expenditures (a, b) is

$$\min (xa, yb) - (a + b).$$

If the business were at all profitable, then the firm would of course expand its scale of operation indefinitely, were it not for the fact that its immediate supply of capital is limited. This limit is not absolute, but there is a substantial cost attached to obtaining more capital than is immediately available. Letting k denote the capital limit, and $(1 + f)$ denote the cost per dollar of additional capital, the firm's profit, as a function of the decision variables a and b, is given by the payoff function

$$u(a, b; x, y) = \min (xa, yb) - (a + b) - f \max (0, a + b - k). \tag{1}$$

If $0 < xy/x + y - 1 < f$, then the function u just defined is indeed convex and polyhedral, and its contours are shown in figure 1.

It is easy to see that, for given x and y, the function u attains its maximum when

$$ax = by, \qquad a + b = k, \tag{2}$$

[3] A. Manne has aptly described this type of example as "allegorical" rather than "realistic".

that is to say, when

$$a = \frac{yk}{x + y}, \qquad b = \frac{xk}{x + y}, \tag{3}$$

and the maximum value of u is

$$\max_{a,b} u = \left(\frac{xy}{x + y} - 1\right) k .$$

Suppose now that the firm is uncertain about the actual values of the "productivity" parameters, x and y, that will prevail during the period in question, and that these values can be predicted accurately only at some substantial cost. Two extreme alternatives suggest themselves. The firm could pay the cost and obtain the relevant information, and then make the appropriate decisions, as given by equation (3). This alternative will be called the case of *full information*. On the other hand, the firm could rely only upon its knowledge of the probability distribution of x and y, which is assumed to be known, and choose that pair of

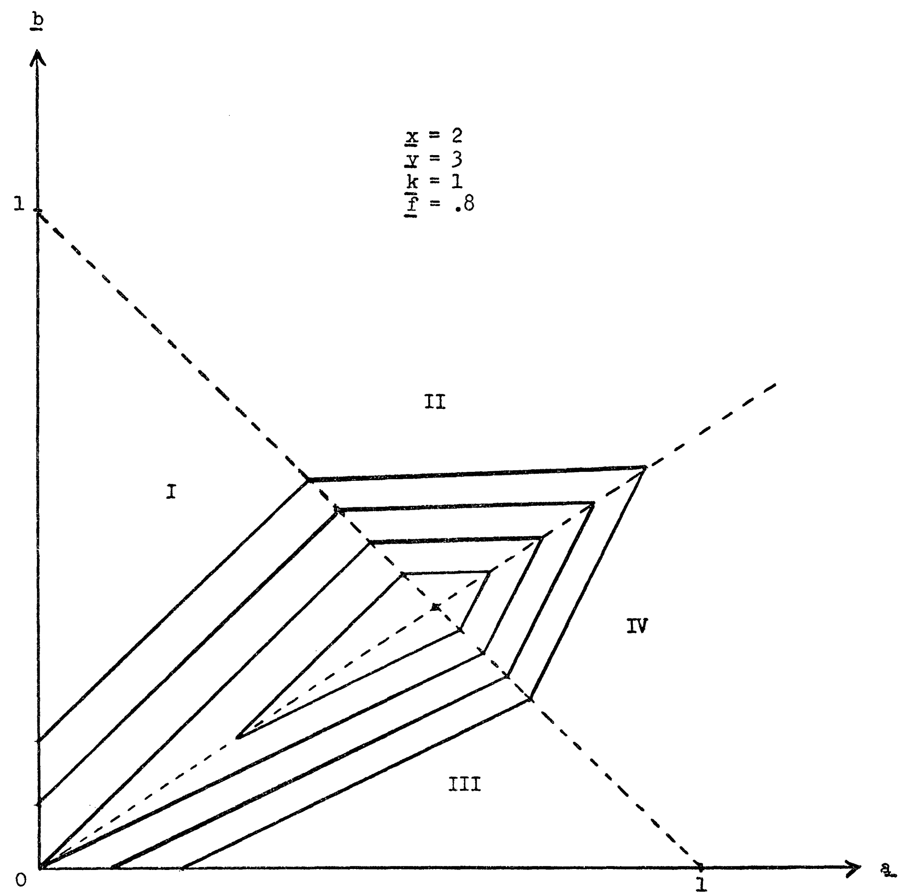

FIG. 1. Iso-profit curves for $u(a, b; x, y)$

expenditures that maximizes the expected, or average, profit. This will be called the case of *routine operation.* Each of these two alternatives involves a different structure of information. Which alternative is the better depends upon which one results in the higher expected profit, net of the cost of information.

A third, intermediate, alternative is suggested under a circumstance that has been described by Marschak as "cospecialization of action and information." In this case, it costs less for the person in charge of production to get the needed information about the parameter x than it does for the person in charge of promotion to do so, and the reverse holds for the parameter y. If, in addition, communication between these two persons is costly, it may be desirable to have the decision about the variable a made by the production manager only in the light of knowledge about x, and the decision about the variable b made by the promotion manager only in the light of knowledge about y, all however according to a decision rule agreed upon in advance. This last alternative will be called the case of *decentralization.*

The possibility of costly communication may seem far-fetched in the context of this simple example; however, if instead of a and b one thinks of two fairly complicated sequences of decision, with each person (or department) getting new information all the time, then it might indeed be costly to achieve a complete exchange of information between the two.

As a primary step toward solving the over-all problem of choosing both a best information structure and best decision rules, one must, at least in principle, solve the various "sub-optimizing" problems of choosing the best decision rules for given information structures, and this is the type of problem that will be considered in detail in the rest of this paper. Before doing so (in the next section), it may be helpful to look at the results for some given numerical values of the parameters.

Suppose that x and y are statistically independent, and can each take on one of two values, with equal probability, the values being given in table 1. Suppose, furthermore, that the amount of free capital (k) equals 1,000 dollars, and that the cost of additional capital $(1 + f)$ is 2.7 dollars per dollar. (It is clear from equations (2) and (3) that the value of k merely determines the scale of operation, and does not influence the relative expenditures.) The maximum possible expected profit for each of the three alternative information structures described above is given in Table 2.

In the "routine" case, the decision rules are, in a sense, degenerate; a best

TABLE 1

Joint Probability Distribution of x and y

x	y	
	2.8	3.6
3.0	1/4	1/4
3.4	1/4	1/4

single value of a and a best single value of b are to be chosen. In the "decentralized case," however, a *pair* of values (a_1, a_2) and a pair of values (b_1, b_2) are to be chosen, where a_1 denotes the expenditure that will be made by the production manager if he learns that x will have the value 3.0, a_2 is the expenditure corresponding to $x = 3.4$, etc. In the "full information" case, there are four values a_{ij} and four values b_{ij} to be chosen, where a_{ij} denotes the expenditure that will be made on production corresponding to the pair of parameter values (x_i, y_j), etc. Table 3 shows the best decision rules for each of the three information structures, and Table 4 shows the resulting allocations of resources.

An interesting feature of the "dencentralized" case (for this numerical example) is that, under the best decision rules, the capital limit is actually exceeded by a small amount whenever a and b both take on their largest values, an event that occurs with probability $\frac{1}{4}$. On the other hand, in the "routine" and "full information" cases one could as well, from the beginning, have imposed the constraint

$$a + b = k \tag{4}$$

and taken the payoff function to be

$$u(a, b; x, y) = \min(xa, yb) - (a + b).$$

However, imposing the constraint (4) on the decision functions in the "decentralized" case would be too stringent. In the present numerical example such a constraint would reduce the expected profit from 541 to 534, or by 7.5% of the difference in expected profit between the routine and full information cases. In general, if different decisions are based upon different information variables, as in the "decentralized case", a certain degree of lack of complete coordination will typically be introduced, and to require that a given constraint never be violated may turn out to be uneconomical when the true cost of a violation is actually weighted against the possible advantages.

Even in this simple example there are many other conceivable information structures besides the three already mentioned. Some of these will be mentioned

TABLE 2

Maximum Expected Profit

	Routine	Decentralized	Full Information
Maximum expected profit................	498	541	592

TABLE 3

Best Decision Rules

Routine	Decentralized	Full Information	
$a = 483$	$a_1 = 512$	$a_{11} = 483$	$b_{11} = 517$
$b = 517$	$a_2 = 452$	$a_{12} = 452$	$b_{12} = 548$
	$b_1 = 548$	$a_{21} = 545$	$b_{21} = 455$
	$b_2 = 426$	$a_{22} = 514$	$b_{22} = 486$

TABLE 4

Allocations of Resources Under Best Decision Rules

Parameter Values	Best Decision Rules		
	Routine	Decentralized	Full information
$x = 3.0$	$a = 483$	512*	483
$y = 2.8$	$b = 517$	548*	517
3.0	483	512	452
3.6	517	426	548
3.4	483	452	545
2.8	517	548	455
3.4	483	452	514
3.6	517	426	486

* Total expenditure exceeds immediate supply of capital.

in Section 4. In the next section I will take up the problem of computing best decision rules for any given information structure.

3. Computing the Optimal Decision Rules

The procedure for computing the optimal decision rules for a given information structure involves converting the team decision problem into an equivalent linear programming problem. The following discussion is in terms of two decision variables and two random parameters, in order to make more transparent its relation to the example of the previous section, but the generalization to any number of decision variables and random parameters is obvious.

Suppose that

$$u(a, b; x, y) = \min_n f_n(a, b; x, y), \qquad (n = 1, \cdots, N) \tag{5}$$

where, for every n, x, and y, f_n is linear in a and b. Suppose also that x and y take on only a finite number of values, with probabilities $p(x, y)$. Furthermore, let

$r = R(x, y)$ be the information on which action a is based,
$s = S(x, y)$ be the information on which action b is based,
A denote any function of r (a decision rule for a),
B denote any function of s (a decision rule for b),
Z denote any function of x and y.

Then the following two maximization problems are equivalent, in the sense that the maximum values are the same, and (A, B) is a solution of Problem I if and only if there is a Z such that (Z, A, B) is a solution of Problem II.

Problem I. Choose A and B so as to maximize

$$Eu(A[R(x, y)], B[S(x, y)]; x, y), \tag{6}$$

subject to $A(r)$, $B(s)$ nonnegative.

Problem II. Choose Z, A and B so as to maximize

$$EZ(x, y), \tag{7}$$

subject to $Z(x, y)$, $A(r)$, $B(s)$ nonnegative, and to the further constraints that

$$Z(x, y) \leqq f_n(A[R(x, y)], B[S(x, y)]; x, y) \tag{8}$$

for every n, x and y. (Note: the symbol E denotes mathematical expectation with respect to the random parameters x and y.)

Since $EZ(x, y) = \sum_{x,y} p(x, y)Z(x, y)$ is a linear function of the "variables" $Z(x, y)$, and since the constraints (8) are linear in $Z(x, y)$, $A(r)$ and $B(s)$, Problem II is a standard linear programming problem.

Returning to the example of the previous section, let the function u be given by equation (1); then it is easy to see that u can be expressed in the form (5) by taking

$$\begin{aligned} f_1 &= (x - 1)a - b, \\ f_2 &= (x - 1 - f)a - (1 + f)b + fc, \\ f_3 &= -a + (y - 1)b, \\ f_4 &= -(1 + f)a + (y - 1 - f)b + fc. \end{aligned} \tag{9}$$

(These 4 functions correspond to the regions I–IV, respectively, in Fig. 1.)

Consider the decentralization example; there one has the information structure

$$R(x, y) = x, \qquad S(x, y) = y. \tag{10}$$

Suppose, furthermore, that x and y can each take on one of two values, as in the numerical example; then A will take on one of two values, say a_1 and a_2, according as x equals x_1 or x_2; and likewise for B. $Z(x, y)$, however, will take on one of four values, say z_{ij}, corresponding to the four pairs (x_i, y_j). In this case Problem II takes the form:

Choose z_{ij}, a_i, b_j, so as to maximize $\sum_{ij} p_{ij}z_{ij}$, subject to z_{ij}, a_i, b_j non-

TABLE 5

Constraint Matrix

z_{11}	z_{12}	z_{21}	z_{22}	a_1	a_2	b_1	b_2	$\leqq$
E	0	0	0	$-G_1$	0	$-H_1$	0	F
0	E	0	0	$-G_1$	0	0	$-H_2$	F
0	0	E	0	0	$-G_2$	$-H_1$	0	F
0	0	0	E	0	$-G_2$	0	$-H_2$	F

where

$$E = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad G_i = \begin{bmatrix} x_i - 1 \\ x_i - 1 - f \\ -1 \\ -1 - f \end{bmatrix}, \quad H_j = \begin{bmatrix} -1 \\ -1 - f \\ y_j - 1 \\ y_j - 1 - f \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ fc \\ 0 \\ fc \end{bmatrix}. \quad \begin{array}{c} \text{Corresponds to region} \\ \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} \\ \text{on Fig. 1} \end{array}$$

negative, and the set of linear constraints presented in matrix ("detached coefficient") form in Table 5.

The fortunate pattern of 1's and 0's in the left half of the constraint matrix of Table 5 is characteristic of a linear problem derived from one with a polyhedral profit function; from a computational point of view, the addition of the variables z does not represent a significant increase in the number of variables. The scattering of 0's throughout the right half of the constraint matrix is typical of a team decision problem.

More generally, for the decentralized case in this example, if x can take on I values, and y can take on J values, then for Problem II there will be $IJ + I + J$ variables, and $4IJ$ constraints. Because of the special structure of this problem, the dual will always be considerably easier to solver than the primal form. In order to solve the dual, it should not take substantially more computing effort than a linear programming model with $I + J$ constraint equations.

4. Interpretation of Sequential Decision Problems as Team Problems

Thus far in this paper the different decision variables in a team decision problem have been interpreted as the decisions of different *persons*. Another class of problems with the same formal structure arises from sequential decision problems for even a single "person" (e.g., inventory and production scheduling problems). In this case the different decision variables correspond to decisions taken at different points of time. Thus, if there is a decision to be made in each of two successive time periods, and information about the parameter values also tends to become known sequentially, then, using the notation of the lemma of Section 3, either of the following information structures is likely to be relevant:

$$\begin{cases} R(x, y) = \text{constant} \\ S(x, y) = x \end{cases} \tag{11}$$

or

$$\begin{cases} R(x, y) = x \\ S(x, y) = (x, y) \end{cases} \tag{12}$$

The technique of Section 3 applies just as well, of course, to these information structures as it did to the ones considered there. However, the special "triangular" character of the information structures that arise in single-person sequential problems often leads to computational simplifications that do not apply to team problems in general. On the other hand, it is clear that sequential or "dynamic" elements can be incorporated into a team decision problem, without altering the basic mathematical framework.

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