Value and Capital Fifty Years Later

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15 Intertemporal General Equilibrium

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Nel mezzo del cammin di nostra vita
Mi ritrovai per una selva oscura,
Che la diritta via era smarrita.

Inferno, Canto I

I INTRODUCTION[1]

In the Preface to his book, *Causality in Economics* (1979), Hicks mentioned his participation in a conference on the 'Microfoundations of Macroeconomics', which was organised by the International Economic Association at S'Agaro in 1974. He reported that 'though some excellent papers were given, reviewers . . . rightly perceived that the conference as a whole was a failure. We did not get to grips with the question we were supposed to be discussing.' He went on to say

> One of the reasons . . . was that the question had been wrongly posed. It took for granted that 'micro' (the economics of the firm and of the individual) was a solid foundation, on which the more dubious 'macro' (economics of the whole economy, usually a national economy) was to be built. What were the grounds for holding that the one was more solid than the other? We were begging the question, but we should have faced it.

I hope that I shall face this question, at least indirectly, as I sketch the development of intertemporal general equilibrium theory since *Value and Capital*, and reflect a little on the significance of that development, and on where we are headed.

In the same Preface, Hicks lists five reasons for pursuing economic theory:

1. intellectual attraction;
2. ideological;
3. characterisation of optimal allocations (welfare economics);
4. descriptive – refinement and criticism of economic statistics;
5. analytical.

With regard to the last reason, he explains: 'I am thinking of analysis applied to facts. When theory is applied, it is being used as a means of explanation: we ask not merely what happened, but why it happened. That is causation; exhibiting the story, so far as we can, as a logical process.'

Thus one main purpose of a theory is to help us to explain some observed economic phenomena, or perhaps to predict some phenomena that have not yet been observed. However, it is sometimes said that, as a precondition for the success of an equilibrium theory, one must be able to demonstrate (in the context of the theory or model):

1. the existence of an equilibrium;
2. the uniqueness of equilibrium, or at least that there are only finitely many equilibria;
3. that the equilibria change continuously with the data (parameters) of the model, at least for most values of the data.

Of course, a theory might be interesting even if it did not satisfy preconditions 2 or 3 (I shall discuss this point below); nevertheless, it is generally accepted that it is crucial to understand whether or not these preconditions are satisfied.

It is interesting that since the publication of *Value and Capital* the bulk of general equilibrium theory has been devoted to the investigation of preconditions 1–3, rather than to the explanation of empirical regularities. (For the time being, I shall leave aside the 'welfare economics' part of general equilibrium theory.) A well-known economist, now dead, once remarked that he was disappointed in general equilibrium theory because it seemed so unambitious – for example, it had not tried to explain why demand curves usually slope downward (the law of demand), or at least had not yet succeeded in doing so. (He would have been pleased with more recent efforts; see Hildenbrand, 1983).

It might be said that general equilibrium theory explains the very existence of the economic institution of markets. Personally, I find this weak, because the theory assumes the presence of markets,

instead of demonstrating why markets, rather than some alternative form of economic organisation, occur.

It is also said that the theory explains prices and interest rates. I suppose this means that it explains the actual prices and interest rates that have been observed to exist during some particular historical period. An example of an activity coming close to this is the elaboration and estimation of econometric macroeconomic models. The success or failure of such a model then becomes a 'test' of the general equilibrium theory that lies behind it. However, this is not a direct test because, even if the underlying general equilibrium model were correct, the macroeconomic model might fail because the statistical aggregation of variables was incorrect, or even impossible to do correctly. In a more general and more ambitious enterprise, general equilibrium theory has been used to try to explain 'business cycles', if only in a qualitative fashion (see, for example, the last chapter of *Value and Capital*). The study of securities markets – 'finance', as the topic is usually called in business schools – is also an area where one might argue that general equilibrium theory has been empirically tested. I shall return to these topics later.

It is fascinating that, while general equilibrium theory has been subject to few direct empirical tests, the bulk of theoretical effort in this subject over the past fifty years has been directed towards making the theory more 'realistic'. By this I mean that the *assumptions* of the theory were made more realistic, no doubt in the hope of making the conclusions or predictions more realistic as well. I shall call this the process of *realism development*, and devote a part of this chapter to sketching this process. (I shall refrain from speculating about any possible casual connections between these two aspects of general equilibrium theory, namely, the paucity of direct testing and the process of realism development.)

A convenient starting-point from which to describe the process of realism development is the group of roughly contemporaneous papers by Arrow, Debreu, and McKenzie on the existence of general equilibrium, as summarised by G. Debreu in *Theory of Value* (1959). I shall refer to the model (or set of models) described so succinctly in the last work as the 'Arrow–Debreu–McKenzie' – or ADM – model. Very briefly, in the ADM model there are finitely many economic agents (consumers and producers), dates, and commodities at each date. In fact, to understand the beauty (and limitations) of the ADM model, it is best to think of the date of delivery or availability of a commodity as just one of its defining characteristics, along with its location and other physical characteristics. One can then imagine that before the beginning of time there is one large market for all of these commodities at once, in which the prices are determined by the equality of supply and demand. (Other stories are possible; see below). Although the 'method' is borrowed from a 'static' picture of an economy, the predictions can be 'dynamic' in the sense that the commodities are dated. Thus, if in the list of commodities one can identify something called 'wheat' at a number of successive dates, then an equilibrium of the model will predict the evolution of the quantities of wheat traded and the corresponding sequence of wheat prices.

It is possible to postulate a market structure other than the single market before the beginning of time that I have just described (see below for an example). However, if the set of trading opportunities is in a certain sense equivalent to that provided by the once-and-for-all, beginning-of-time market, then the market structure is called *complete*.

In the process of realism development, five features were incorporated into the ADM model:

1. *Uncertainty*: agents are uncertain about the environment of the economy, including the tastes and beliefs of the other agents.
2. *Incompleteness of markets*: the market structure is not such as to permit the full range of possible trades envisaged in the ADM model; in particular, there are not markets for bets about the outcomes of all possible future events.
3. *Heterogeneity of information*: not all agents come to the markets with the same information about the environment. This has three different consequences that can be taken account of in the theory:
   (a) an agent's information effectively constrains the set of strategies available to him;
   (b) agents may learn from equilibrium prices something about other agents' information, and hence something about the environment;
   (c) the contractual relations among agents are subject to *moral hazard* and *adverse selection*.
4. *Overlapping generations*: there is a long succession of agents in the economy, with many agents active at any one time, but with only partly overlapping active lifetimes.

5. *Departures from perfect competition*: this rubric covers a multitude of 'imperfections', and their analysis had made economic equilibrium theorists more and more dependent on the methods of game theory.

It is characteristic of the process of realism development that only one or two of the above features have been incorporated into any one elaboration of the ADM model. Nor is it clear that it should be a goal of equilibrium theory to incorporate all of these features into a single model, even though each one is clearly important and they probably interact in significant ways. I hope that the rest of this chapter will shed some light on this question.

Another side of general equilibrium theory is the study of 'economic welfare'. Recall that a feasible state of an economy is called *Pareto-optimal* if there is no other feasible state of that economy in which no consumer is worse off and at least one consumer is strictly better off. It is now well known (see Debreu, 1959; Arrow and Hahn, 1971), that under 'standard assumptions', (i) an equilibrium of an ADM model is Pareto Optimal, and (roughly speaking) (ii) every Pareto-optimal state is an equilibrium for some set of prices and some distribution among the consumers of the given total resources and shares of firms (the latter is sometimes called the 'distribution of initial endowments'). These will be referred to here as the two welfare theorems. Theorists have also studied the question whether one can tell from simple quantitative measures, like index numbers of output, if a change of state of an economy represents an improvement in economic welfare, and in what senses (see Chipman and Moore, 1980a, 1980b) though I shall have no space to discuss this interesting subject here.

The use of general equilibrium theory to assess the welfare effects of alternative economic policies is a worthy goal. But surely it is trite to observe that such an exercise is valid only to the extent that the theory is realistic in its description of the economy. Thus even the economist primarily interested in 'normative' economics cannot responsibly avoid the question of realism.

In the rest of this chapter, I shall give a brief and impressionistic survey of selected stages – or rather strands – in what I have called above the 'process of realism development'. For each strand I shall try to summarise in a stylised way what has been learned about preconditions 1–3, the two welfare theorems, and equilibrium dynamics. The concluding sections sketch an overall view of these developments, and offer some speculations about the course of future research.

2 TIME AND CERTAINTY

Recall that, under conditions of certainty, the ADM model accommodates 'dynamics' by dating commodities. More precisely, commodities that are supplied and/or consumed at different dates are distinguished as different, even if they otherwise have the same physical characteristics and location.

We may think of the market as taking place once and for all before the first date in the model. Contracts are made in that market specifying all of the deliveries and receipts that will ever take place. If prices are denominated in units of account, then payments in units of account are made at the beginning of time. All accounts are settled, all budgets must be balanced, and all plans must be verified as feasible at that time. Since there is complete certainty (and agreement!) about the future, there is no difficulty – other than computational – in doing all this.

With this interpretation, the prices of commodities corresponding to future dates are thus 'discounted'. However, there need be no single rate of interest; for each 'commodity' in ordinary parlance, the sequence of its prices determines a corresponding sequence of its 'own-rates' of interest.

Careful consideration of this last remark leads us to a deeper problem. Nothing about the ADM model requires that the list of today's commodities be the same as the list of tomorrow's commodities. For example, 'automobiles' next year are likely to be somewhat different from 'automobiles' today.[2] Furthermore, unless we make some specific assumptions about the structure of production and preferences, no conclusions specifically having to do with time will come out of the model.

By far the most common approach is to make some assumptions about the 'stationarity' of the data of the problem. Although I cannot go into much detail about this here, it is necessary to describe in a little detail the possible meanings of 'stationarity of technology'. First, I shall use the word 'commodity' in the ordinary sense, so that we can talk about the same 'commodity' at different dates. With this understanding, I shall suppose that there is the same list of commodities at all dates. Second, since many commodities are durable, but

their qualities change with age or use (e.g., wine, trees, machines), it will be useful to distinguish commodities at any one date by those relevant characteristics, including age or 'vintage' where those qualities are important.

Third, production – in the ADM model – transforms one sequence of available quantities of commodities into another such sequence; the difference is the sequence of *net outputs*. It is convenient, and conventional, to represent this sequence of net outputs as arising from a sequence of gross inputs and corresponding gross outputs. We suppose that at the beginning of each period there is a stock (possibly zero) of each commodity; from those stocks are subtracted the current consumption in that period, and the rest becomes the *gross input* into production. The stocks at the beginning of the next period are the sums of the *gross outputs* from the previous period and the exogenously determined supplies from outside the system ('endowments'). The *technology at a given date* for the economy as a whole is the set of feasible pairs of gross inputs and outputs corresponding to production during that period. The technology is *stationary* if the date-specific technologies are identical.

I shall not give here a correspondingly general definition of stationarity of preferences. At this point, it will suffice to assume that each consumer's preferences can be represented by a sum of discounted one-period utilities, with a constant rate of discount (impatience), and the same one-period utility function at all dates. (See below for remarks on the more general case.)

With regard to endowments, two special cases have been studied intensively:

1. There are two kinds of commodities, *primary* and *producible*. The primary commodities are not producible, but are necessary for production. The endowments of the primary commodities are constant in time, but the endowments of the producible commodities are zero except in the first period. All producible commodities are subject to physical deterioration. Given the constant supply of the primary commodities, there are sufficiently strong decreasing returns to scale in production (in terms of the producible commodities) that gross output is uniformly bounded over time.
2. All commodities are producible, and their endowments are zero except in the first period. There are constant returns to scale in production.[3]

Finally, we must say something about the numbers of consumers, producers, and dates. At this stage of my exposition, I shall assume that the numbers of consumers and producers are finite. If the number of dates is also finite, then the model fits quite nicely into the standard ADM framework, and the standard results about equilibrium and welfare are applicable, which from a technical point of view is quite satisfactory.

But from another point of view, the assumption of a finite number of dates is awkward, since it means that there is an end to the economy at a foreseeable date.[4] This is not merely a conceptual awkwardness; the finite horizon will typically induce an 'end-effect' on agents' plans that prevent equilibrium quantities and relative prices from being stationary. Of course, if the horizon is very distant, and the consumers are discounting the future in their preferences, one might hope that the end-effect would be small at the earlier dates. I shall return to this point below, but I hope that this remark makes it at least plausible that it may be mathematically convenient to approximate the case of a very distant but finite horizon with that of an infinite horizon.

With the assumptions of stationarity of the data, as in Cases 1 and 2 above, one can hope to get some fairly sharp predictions about the time path of equilibrium prices and quantities. In particular, in Case 1, with an infinite horizon, one can hope to show that equilibrium paths tend towards a *steady state* as time increases without limit, where a steady state is a path along which quantities and *relative* prices are constant in time, and all own-rates of interest equal a common, constant rate, which we are then justified in calling *the* rate of interest. In this situation, I shall say that the equilibrium path of prices and quantities is *dynamically stable*.

In Case 2, again with an infinite horizon, one can hope to show that the equilibrium path has the properties of dynamic stability except that the quantities – instead of approaching constants – tend towards a path of *proportional growth*, in which relative quantities (quantity ratios) are constant, and all quantities change at the same geometric rate of growth (positive or negative). In this situation, I shall say that the equilibrium path has the *turnpike property*, and that the limit path – the path of proportional growth and constant rate of interest towards which the equilibrium path tends – is the *turnpike*.

The early work on equilibria with the turnpike property, in which Professor Hicks played a stimulating role, concentrated on the special case in which there is only one consumer.[5] Because of the second welfare theorem (see Section 1) this case is not so special after all. Typically, a competitive equilibrium will be Pareto-optimal (the second welfare theorem), and therefore there will be a set of non-negative weights such that the Pareto-optimum in question also maximises a weighted linear combination of the utilities of the consumers. This linear combination – the social welfare function – could be thought of as the utility function of a single virtual consumer who represents society. This reduction to a 'representative consumer' or 'social welfare function' is not so innocuous as it may sound. The particular set of weights corresponding to a particular equilibrium will typically depend on the equilibrium in question. Thus a change in the production technology of the economy, leading to a new equilibrium, will also typically lead to a new social welfare function. Furthermore, if different consumers have different discount rates (impatience), then the weighted average of their utility functions will not represent stationary preferences.

In addition, the early turnpike theory also dealt with the case of a finite horizon. In this case, the turnpike property is rephrased: the path spends 'most' of its time close to the turnpike. Third, early turnpike theory dealt with the case in which the (single) consumer does not discount future utility, or assigns utility only to the final stock of commodities. (Because of the finite horizon, this does not lead to a poorly posed problem, as it would in the case of an infinite horizon.)

Early turnpike theory produced propositions of the following type: in Case 2, if society's preferences are homothetic and there is no impatience (no discounting of future utility), if the technology is in a certain sense indecomposable, and if the 'standard' ADM assumptions are otherwise satisfied, then for a sufficiently distant (but finite) horizon the equilibrium (optimal) path has the turnpike property.

The next step is to consider models with an infinite horizon. In what follows I shall limit myself (unless otherwise noted) to Case 1 above, in which gross output and all one-period utilities are bounded.

If one moves to a model with an infinite horizon, but tries to maintain the assumption of no impatience, one faces some definitional problems. For example, even if a sequence of one-period utilities is bounded (as it would be in Case 1), the *undiscounted* sum of utilities might not converge. One way to address this difficulty is to define the utility of the sequence to be the long-run-average of the one-period utilities.[6] Since this criterion is rather crude, more refined criteria for optimality without impatience have been studied. One such criterion was introduced by Ramsey (1928). Recall that in a steady state (we are in Case 1), the quantities produced and consumed are constant in time. As we vary the initial endowments of producible commodities, we generate a family of steady states.[7] With sufficient regularity assumptions, among these steady states there will be (at least) one that has maximum utility per period; call this last the *optimal steady-state utility*. For any given path and any given period, define the corresponding *one-period loss* to be the difference between the optimal steady-state utility and the utility for that path in that period. Ramsey's criterion for judging a path is the sum of the one-period losses, or the *total loss*. One has to show, of course, under what conditions this total loss is well-defined; in interesting cases it is either positive or infinite. In these cases, a path is *Ramsey-optimal* if, among all feasible paths with given initial endowments of producible commodities, it has minimum total loss. In a particular (one-consumer) model with a single producible commodity, Ramsey showed that a Ramsey-optimal path is dynamically stable, and moreover converges to an optimal steady-state path. Ramsey's analysis was significantly extended to the many-good case by Gale (1967), Brock (1970), and others, and conditions for the validity of the two welfare theorems have been rather thoroughly explored.[8]

I turn now to the case of impatience. In some sense, this case is mathematically better behaved. For any bounded sequence of one-period utilities, and any positive rate of discount, the discounted sum of utilities is well-defined, and is also well behaved as a function of the sequence of one-period utilities. Generally speaking, with regularity assumptions that are reasonable in the context of the model, the preconditions for equilibrium theory are met, and the two welfare theorems are valid.[9]

On the other hand, recent research has shown how, as the consumers' impatience increases, equilibrium paths may lose their dynamic stability. (An early example was provided by Sutherland, 1979.) Benhabib and Nishimura (1985) have studied a model with one consumption good and one capital good in which equilibrium paths tend towards cycles if the rate of discount of future utility lies in some interval, provided that the production of the consumption good is sufficiently more capital intensive than that of the capital good, and the capital good does not depreciate too rapidly. On the other hand, for discount rates below that interval, the equilibrium paths will be dynamically stable. In a continuous-time version of this model, Benhabib and Nishimura demonstrate similar phenomena, except

that they require the presence of at least two capital goods. Equilibrium cycles can also result from delays between investment and output; see (Rustichini, 1989).

Indeed, cyclic behaviour is not the only alternative to dynamic stability for equilibrium paths. Even more recent research has shown that the dynamics of equilibrium paths can be arbitrarily complex, that is, 'chaotic' (Boldrin and Montrucchio, 1986; Deneckere and Pelikan, 1986). In particular, Boldrin and Montrucchio have shown that, given a stationary technology and a positive discount rate, there is a one-period utility function such that the corresponding optimal path is chaotic.

Much remains to be done to explore the dynamics of equilibrium with many commodities and many agents. Nevertheless, a pattern seems to be emerging: for given one-period utility functions and stationary technology:

1. equilibrium paths will be dynamically stable if consumers are sufficiently patient, whereas
2. for discount rates away from zero, equilibrium paths may be asymptotically cyclic, or even chaotic.

All of this is possible with complete certainty and complete markets.

I close this section with some supplementary remarks about the case of certainty. The first remark concerns the concept of efficiency. A *programme* is a sequence of net outputs (totalled over all of the firms in the economy) that is feasible given the technology and total endowments of the economy. A programme is *efficient* if there is no other programme that has a larger net output of some commodity at some date, and no smaller net output of any commodity at any date. If all commodities are desirable, then efficiency is a necessary condition for Pareto-optimality, but is clearly not sufficient for any given set of preferences. Indeed, efficiency is a property of production alone. Corresponding to the two welfare theorems for competitive equilibrium, there is a close relationship between efficiency and the maximisation of profit by the units of production. Koopmans (1957, ch. 1, section 3, 4) provided an excellent exposition of this theory in both the static and intertemporal contexts, the latter based in part on the earlier contribution of Malinvaud (1953). More recent results are discussed by Cass and Majumdar (1979) and Majumdar (1988).

A second remark concerns the concept of stationary preferences. Koopmans was evidently the first to demonstrate that this concept could be generalised beyond the case of preferences represented by a sum of discounted identical one-period utility functions (with a constant discount rate); I shall call this latter case *stationary discounted utility*. Let C denote an infinite sequence of consumption vectors, let c denote the first vector in the sequence C, and let C^* denote a new infinite sequence formed by removing the first vector, c, from the original sequence, C. A general representation of stationary preferences is $U(C) = V[c, U(C^*)]$; this is sometimes called *recursive utility*. (I have omitted any discussion of mathematical regularity conditions; for a thorough treatment see Koopmans, 1986.) Although stationary discounted utility is a special case of recursive utility, it implies properties of intertemporal equilibrium that are strikingly different from (and probably less realistic than) the properties implied by 'most' recursive-utility preferences. For analyses of Pareto-optimal programmes with recursive utility see Lucas and Stokey (1984), Benhabib *et al* (1987), Dana and Le Van (1987), and the references cited there.

Finally, I remark that the conditions of non-increasing returns to scale and decreasing marginal productivity have typically played an important role in the analysis of ADM models; these conditions are summarised under the mathematical condition of 'convexity of production sets'. Nevertheless, some progress has been made in characterising optimal growth when non-convexities are present; see Majumdar *et al* (1988) and the previous research which they cite.

3 THE DEVELOPMENT OF REALISM

In the previous section, I reviewed models with complete markets, certainty, a finite number of agents, an infinite time horizon, and stationary preferences and technology. In that case, we see emerging a pattern in which:

1. the 'preconditions' for a satisfactory equilibrium theory are generally fulfilled, that is, equilibria exist, are locally unique, and are (generically) well-behaved as a function of the data of the model;
2. the two welfare theorems are valid (with some qualifications for the limiting case of no impatience);
3. equilibrium paths are dynamically stable if consumers are sufficiently patient, whereas
4. for discount rates away from zero, equilibrium paths may be asymptotically cyclic, or even chaotic.

In the present section I shall briefly sketch the different strands of research in the development of realism, as listed in Section 1.

3.1 Uncertainty with Complete Markets

The ADM model was not originally put forward for the case of uncertainty, but an ingenious device introduced by Arrow (1953), and further elaborated by Debreu (1953, 1959), enabled the theory to be reinterpreted to cover the case of uncertainty. The basic idea was to distinguish commodities, not only by their physical characteristics, location, and date, but also by the *environmental event* in which they are made available and/or used. Thus contracts could be made for delivery contingent on any event. For example, a 'bet' is a contract in which money is paid today in exchange for a promise to deliver money at a future date if a specified event has happened by that date, that is, a promise to deliver money at a specified *date–event pair*. In what I shall call the *Arrow–Debreu* (AD) model of markets with uncertainty, the market is complete with respect to both time and uncertainty.

If we introduce uncertainty into the ADM model in this way, but leave the other features unchanged (as in Section 2), then the results are similar to those for the case of certainty, except that the two dynamic behaviours, 'steady state' and 'cycling', which are distinct in the case of certainty, typically become merged in the behaviour of a stationary stochastic process.[10] Correspondingly, 'dynamic stability' is now interpreted as convergence to a stationary process. Note that a stationary process can produce sample paths that look something like 'business cycles', for appropriate patterns of serial correlation.[11]

On the other hand, these models, with or without uncertainty, do not justify any significant role for either (i) money-like assets and other purely financial securities, including stocks, or (ii) active commodity markets at every date (not just at the beginning of time). In this sense, these important institutional features of modern capitalist economies are not explained by these models.

3.2 Equilibria in Sequences of Incomplete Markets

Limitations on agents' capacities for observation, communication, and computation, and other 'transactions costs', make it intuitively clear why markets are not complete in the Arrow–Debreu sense, and hence why there are active, but incomplete, markets at every date.

The presence of incomplete markets, however, leads to some ambiguity about the best way to extend the Arrow–Debreu model, and the corresponding definition of equilibrium, to this case.[12]

A number of new issues arise. First, there is uncertainty about the prices that will hold in future markets, as well as uncertainty about the environment. Second, producers do not have a clear-cut natural way of comparing net revenues at different dates and states. Stockholders have an incentive to establish a stock exchange, since it enables them to change the way their future revenues depend on the states of the environment. As an alternative to selling his shares in a particular enterprise, a stockholder may try to influence the management of the enterprise in order to make the production plan conform better to his own subjective probabilities and attitude towards risk.

Third, consumers will typically not be able to discount all their 'wealth' at the beginning of time, because (i) their shares of producers' future (uncertain) net revenues cannot be so discounted and (ii) they cannot discount all their future resource endowments. *Consumers will be subject to a sequence of budget constraints*, one for each date (rather than to a single budget constraint relating the present cost of his consumption plan to his present net worth, as in the Arrow–Debreu economy).

Fourth, economic agents may have an incentive to speculate on the prices in future markets, by storing goods, hedging, etc. Instead of storing goods, an agent may be interested in saving part of one date's income, in units of account, for use on a subsequent date, if there is an institution that makes this possible. There will thus be a demand for 'money' in the form of demand deposits.

Fifth, agents will be interested in forecasting the prices in markets at future dates. These prices will be functions of both the state of the environment and the decisions of (in principle, all) economic agents up to the date in question.

Consider now a sequence of markets at successive dates. Suppose that no market at any one date is complete in the Arrow–Debreu sense: that is, at every date and for every commodity there will be some future dates and some events at those future dates for which it will not be possible to make current contracts for future delivery contingent on those events. In such a model, several types of 'equilibrium' concept suggest themselves.

First, we may think of a sequence of 'momentary' or 'temporary' equilibria in which the current market is cleared at each date. The prices at which the current market is cleared at any one date will

depend upon (among other things) the expectations that the agents hold concerning prices in future markets (to be distinguished from futures prices on the current market!). We can represent a given agent's expectations in a precise manner as a function (schedule) that indicates what the prices will be at a given future date in each elementary event at that date. This includes, in particular, the representation of future prices as random variables, if we admit that the uncertainty of the agent about future events can be scaled in terms of subjective probabilities (Savage, 1954).

In the evolution of a sequence of momentary equilibria, each agent's expectations will be successively revised in the light of new information about the environment and about current prices. Therefore, the evolution of the economy will depend upon the rules or processes of expectation formation and revision used by the agents. In particular, there might be interesting conditions under which such a sequence of momentary equilibria would converge, in some sense, to a (stochastic) steady state. This steady state (for example, a stationary probability distribution of prices) would constitute a second concept of equilibrium.[13]

A third concept of equilibrium emerges if we investigate the possibility of consistency among the expectations and plans of the various agents. I shall say that the agents have common expectations if they associate the same (future) prices to the same events. (Note that this does not necessarily imply that they agree on the joint probability distribution of future prices, since different agents might well assign different subjective probabilities to the same event). I shall say that the plans of the agents are consistent if (i) for each commodity, each date, and each event at that date the planned supply of that commodity at that date in that event equals the planned demand, and (ii) a corresponding condition holds for the stock markets. An *equilibrium of plans, prices, and price expectations* (PPPE) is a set of prices on the current market, a set of common expectations for the future, and a consistent set of individual plans, one for each agent, such that, given the current prices and price expectations, each individual agent's plan is optimal for him, subject to an appropriate sequence of budget constraints.

A fourth concept of equilibrium has been proposed to deal with the situation in which traders enter markets with different non-price information (information about the economic environment other than prices). This situation presents an opportunity for agents to learn about the environment from prices, since market prices reflect, in a possibly complicated manner, the non-price information signals received by the various agents. These inferences are derived, explicitly or implicitly, from individuals' own 'models' of the relationship between the non-price information received by market participants and the market prices. On the other hand, the true relationship is determined by the individual agent's behaviour, and hence (jointly) by their individual models. An equilibrium of this system, in which the individual's models are identical with the true model, is called a *rational expectations* (RE) equilibrium.

Of these four concepts of equilibrium, the third (PPPE) is perhaps the closest in spirit to the Arrow–Debreu theory. It is also, I believe, closest to the notion of 'equilibrium over time', described in *Value and Capital*. In chapter X of *Value and Capital*, Hicks distinguished two senses of equilibrium:

1. 'Temporary equilibrium', in which at a given date supply equals demand;
2. 'Equilibrium over time', which he defined by the additional 'condition that the prices realized on [each date] are the same as those which were previously *expected* to rule at that date' (*Value and Capital*, p. 132). With regard to uncertainty, he added that 'the expectations of entrepreneurs are in fact not precise expectations of particular prices, but partake more of the character of probability distributions . . .' (*Value and Capital*, p. 133).

I shall call this last the condition of *fulfilled expectations*.

In later writing, Hicks explicitly added a third condition, that no agent can improve his position by (unilaterally) changing his strategy. As he put it in *Capital and Growth* (p. 23): 'there is equilibrium when all "individuals" are choosing the quantities, to produce and consume, which they prefer. *To a conception of equilibrium that is of this type we must hold fast*' (my emphasis). This is echoed in *Causality in Economics* (p. 45): 'all opportunities for advantageous change that are presented within the model must be taken'. We might call this the *Hicks–Nash condition* of equilibrium.

All told, then, we have three conditions for equilibrium over time:

1. the Hicks-Nash condition;
2. fulfilled expectations;
3. market clearing.

I would classify the temporary equilibrium concept as a 'bounded rationality' approach (see Section 5). Although this is a very promising notion of equilibrium, and I leave it to Grandmont to discuss it in detail here.

3.3 Equilibrium of Plans, Prices, and Price Expectations

In the previous subsection I defined, in the context of a sequence of incomplete markets, an *equilibrium of plans, prices, and price expectations* (PPPE), which I argued was a natural elaboration of the Hicksian equilibrium over time. In a PPPE equilibrium, agents agree on the prices that will obtain at each date–event pair, but may have different beliefs about the likelihoods of those events, as well as different attitudes towards risk. In an equilibrium, they update their expectations and (individually optimal) plans like good Bayesians as they receive new information. This Bayesian updating is, of course, part of their individually optimal strategies.

In a PPPE equilibrium we begin to see a breakdown of the standard properties of the ADM model. First, equilibrium need not exist, even under 'standard' assumptions about preferences and technology. The difficulty stems from the fact that (because of the incompleteness of markets), agents face a sequence of budget constraints, instead of a single one at the beginning of time, and the positions in assets that agents may take – and may want to take – are unbounded. Even under strong regularity conditions, it appears that one can only guarantee the *generic* existence of a PPPE equilibrium.[14]

Second, equilibria need to be 'determinate', that is, need not be locally unique; in fact, there can be a continuum of equilibria. This can happen even in a two-period model, in which assets are denominated in 'inside money'.[15] It can also happen in so-called 'sunspot equilibria', in which eventual allocations depend on events that are not pay-off-relevant for anyone (extrinsic uncertainty). (In other words, in sunspot equilibria agents' actions may be functions of exogenous random variables that have no direct influence on preferences or technology.)[16] Taking this together with the previous paragraph, we see that the preconditions for a fully satisfactory equilibrium theory are no longer present in the case of incomplete markets.

Third, PPPE equilibria need not be optimal, not even if one defines 'optimality' to reflect the constrained set of resource allocations that can be attained with an incomplete set of markets. In fact, with a particular definition of 'constrained optimality', one can show in a model of pure exchange that PPPE equilibria are generically *not* optimal.[17] (I shall say nothing about the dynamics of PPPE equilibria, since the topic has so far received little attention.[18])

3.4 Rational Expectations

The preceding discussion has concerned models in which agents are assumed to have *homogeneous information*, that is, all agents have the same (incomplete) information at each date event pair. The more realistic case of *heterogeneous information* across agents leads to further problems in both the definition and characterisation of equilibria. In any reasonable definition of equilibrium, prices will reveal something of the information held by individual agents (in the language of statisticians, a price will be a 'statistic'), and hence reveal something about the state of the economic environment that affects agents' utilities directly (not just through their budget constraints). In a *rational expectations* (RE) equilibrium agents know enough about the relationship between states of the environment and equilibrium prices at a given date-event pair to be able to condition their expectations about future events on those prices.

Even with a finite time horizon (two periods!), RE equilibria need not exist, but it has been shown that (roughly speaking) in pure-exchange economies RE equilibria exist generically. RE equilibria will typically not be fully optimal, except in the special case in which the equilibrium prices fully reveal all the agents' information. In fact, the latter will happen generically if there are only finitely many alternative states of information for all of the agents taken together.[19]

I think it can be argued that the theory of equilibrium with transactions costs and incomplete markets makes a significant step towards 'explaining' – at least qualitatively – such facts of economic life as speculation and the trading of money-like assets and stocks. On the other hand, both the PPPE and (even more so) the RE equilibria place unrealistically great demands on agents for sophistication, computation, and understanding of the economy.

3.5 Overlapping Generations

The *overlapping generations* (OLG) model takes a further step towards realism by recognising that the lives of economic agents (especially consumers) are short compared with the life of the economy. Formally, in OLG models agents have finite lives, whereas the economy

lasts for infinitely many periods, so there are infinitely many agents. One implication of the OLG assumption is that equilibrium may be indeterminate under conditions of certainty, provided there is a suitable complementarity between present and future consumption. OLG models can also have sunspot equilibria. Finally, OLG equilibria need not be optimal. On the positive side, with the OLG assumption one can construct models with equilibria in which agents hold fiat money (although there may also be other equilibria in which they do not).[20]

A curious feature of an OLG equilibrium is that, in a sense, the equilibrium prices have to be 'known' or 'foreseen' from the beginning of time, even though only a finite number of the (infinitely many) agents are alive at the beginning of time. This would not be totally absurd if the equilibrium were unique, so that all agents, once they began to make economic decisions, could calculate and foresee the same sequence of equilibrium prices. With indeterminacy of equilibrium, the agents who are active at the beginning of time must (jointly) pick an equilibrium, and then make sure that all succeeding generations are informed and convinced about the details of the one that they have chosen. To tell a different story, if the equilibrium is picked by some 'Walrasian *tâtonnement*', or other algorithm, then all the excess demand functions of all the agents (unborn as well as born) must be represented in the equilibrating process at the beginning of time! Note that this problem arises even if there are only finitely many equilibria, but more than one.

3.6 Moral Hazard and Adverse Selection

Heterogeneity of information across agents is accommodated in some extent in the extended Arrow–Debreu model and the concept of rational expectations equilibrium. But beyond these implications of heterogeneous information, one must add the phenomena of *moral hazard* and *adverse selection*. In a way, we can think of these phenomena as introducing externalities into the relations among agents. These externalities are not only present with large numbers of competing agents, but they have a tendency to lead to long-term relationships among agents, involving bargaining and other strategic behaviour inconsistent with the assumptions of pure competition and price-taking.

Recent years have seen an explosion of research on moral hazard and adverse selection.[21] However, it would be an exaggeration to say that this research has had as yet more than a minimal impact on intertemporal general equilibrium theory.

3.7 Relationships with Game Theory

The concept of general equilibrium is close to that of the Nash equilibrium of a non-cooperative game, except that:

1. the ADM model – as usually formulated – does not specify the outcomes to economic agents out of equilibrium;
2. agents do not take account of the potential effects on prices of changes in their own actions, that is, agents are assumed to be 'price takers'.

Recent developments in game theory, especially the theory of Harsanyi equilibrium with incomplete information, have revealed problems analogous to those we have found in intertemporal general equilibrium theory: indeterminateness of equilibria, and 'unsatisfactory' nature of many equilibria, unreasonable demands on the agents' powers, etc. The role of expectations seems to be key in generating these problems in both areas of research.[22]

A more substantive connection between the two fields is developing as economic theorists attempt to incorporate elements of imperfect competition into theories of market equilibrium, all the way back to the study of bargaining.[23] This development is too vast for me to summarise here, but I speculate that the substantive connection will lead to a convergence of methodology as well.

4 TAKING STOCK

The part of the theory of intertemporal general equilibrium that is closest to *Value and Capital* is embodied in ADM models with certainty, a finite or infinite time horizon, and possibly stationary preferences and technology (see Section 2). Although some of the most interesting results in this area are quite recent, and the process of realism development started quite early, there is a certain logic in taking the ADM model with time and certainty as the point of departure for the process of realism development. As we look back on this process, several features stand out (although each is subject to various qualifications, or has been only imperfectly explored):

1. The *assumptions* of the model were progressively modified to make them more realistic.
2. Certain important features or institutions of economic life – like active markets at every date, stock markets, the holding of fiat money, etc. – which were not compatible with the ADM model with certainty and time, became compatible with the new models.
3. With more realistic assumptions, the preconditions for a fully satisfactory equilibrium theory did not emerge as clearly, or were no longer present. Equilibria could be shown to exist only generically. When they existed, they might be indeterminate.
4. The range of validity of the two welfare theorems was diminished; in particular, equilibria were sometimes – or even 'most of the time' (generically) – not optimal.
5. Although the range of possible equilibrium dynamics was already quite large in the ADM model with certainty, with the introduction of uncertainty the meaning of 'dynamic stability' became less sharp, and the predictions of the theory became less crisp. (Indeed, little research seems to have been done on the dynamics of markets with incomplete markets.)
6. While the assumptions became more realistic, *the definitions of equilibrium became less so*. Greater and greater demands were placed on the abilities of the agents to process information, compute optimal strategies, understand the 'laws' governing the economic system, and select one of the possibly many equilibria. In the case of the OLG model, the definition of equilibrium raised the further question of whether any social process or algorithm could *even in principle* produce an equilibrium set of prices and plans.
7. We noted a parallel between these developments and similar developments in the general theory of non-cooperative games with incomplete information, especially sequential games.

What have been the positive contributions of the development of intertemporal general equilibrium theory?

1. First, in a general, qualitative way, it has improved our 'understanding of how markets work', of financial institutions, capital asset pricing, the 'invisible hand' and its limitations, and some possible sources of dynamic stability and instability.
2. Even the indeterminacy of equilibrium has its positive side. It suggests that, even if we were currently on an equilibrium path,

we would have to look to 'historical accident' to account for the particular selection of that path from among the many equilibrium candidates. (In other words, 'history matters', which is not implausible.) It also points to the possibility of shifts from one equilibrium path to another, brought about by concerted social action. (See Geanakoplos and Polemarchakis, 1986a.)

3. One might hope for more detailed and even quantitative confirmation of intertemporal general equilibrium theory. One natural place to look for such confirmation is the field of macroeconomics. I hesitate to make any generalisations here, because I am definitely out of my field of expertise. However, my impression is that there is not an abundance of 'empirical regularities' waiting to be explained (see, for example, Tobin, 1985). On the other hand, it may be too much to expect equilibrium theories to explain annual (not to mention quarterly) changes in macroeconomic magnitudes. Tobin, writing about the 'older neoclassicals' (including Hicks and Samuelson), described them as finding 'the neoclassical paradigm useful for long-run trends but saw nothing problematic in departures from those trends for a variety of reasons, for example, market imperfections, adjustment costs, information lags. These departures need not imply any irrationality or any permanent failure of markets to clear; the properties of full general equilibrium should not be expected to hold every day or every year' (Tobin, 1985, p. 114).[24]

One might therefore turn to the field of economic development for a potentially fruitful field of application. Here again I am out of my depth, but it is my impression that empirical regularities in this field seem more elusive today than they did two or three decades ago.[25]

5 LOOKING AHEAD

Much can still be done to further the process of realism development in models of intertemporal equilibrium; I mention a few obvious topics:

1. survival and failure, entry and exit;
2. research and development, 'Schumpeterian competition';
3. increasing returns to scale, the behaviour of large firms.

But unless the development of the realism of assumptions is matched

by a corresponding development of the realism of the *equilibrium concept*, I fear that overall progress will be limited.

Recall the definition of equilibrium given by Hicks in *Capital and Growth* (p. 23): 'there is equilibrium when all "individuals" are choosing the quantities, to produce and consume, which they prefer. To a conception of equilibrium that is of this type we must hold fast.' This is echoed in *Causality in Economics* (p. 45): 'all opportunities for advantageous change that are presented within the model must be taken'. I have called this the Hicks–Nash condition of equilibrium. There are, of course, two other conditions that are required by a full-fledged Walrasian–Hicksian market equilibrium (as in the equilibrium of plans, prices, and price expectations). The first is that expectations should be fulfilled (at least probabilistically, or state-by-state). The second is that markets should clear. So all in all we have three equilibrium conditions, or more generally, three types of equilibrium condition:

1. the Hicks-Nash condition;
2. fulfilled expectations;
3. market clearing.

It is the first two conditions that I shall concentrate on here.

We have been told that – as economic theorists – we should pay more attention to (i) bounded rationality and (ii) out-of-equilibrium behaviour. It seems to me that, from the perspective of the equilibrium conditions 1 and 2 above, these lead to the same inquiry.[26] By definition, if the Hicks–Nash condition is not satisfied, some agent is not exploiting an 'opportunity for advantageous change', something we would usually characterise as 'economically irrational', or 'boundedly rational'. If expectations are not fulfilled, that is, if an agent observes an event to which he had previously assigned a zero probability, then one might also say that his expectations were in some sense irrational or boundedly rational, although the case is not so compelling.[27]

Earlier I alluded to the possibility that a multiplicity of equilibria would give a greater importance to history in determining the current equilibrium path. Hahn (1987) has shown how this could actually happen through the mechanism of out-of-equilibrium behaviour. As he put it, 'the state of the economy that we single out as an equilibrium depends on our theory of the behavior of agents out of that equilibrium' (p. 321).

What are the obstacles to economic rationality that economic theorists should consider? One of the best-known recent writers on this subject is Herbert Simon, who has emphasised the limitations on observation, reasoning, and general information processing to which humans are subject. However, despite the writing and urging of a few colleagues,[28] these ideas have made hardly a dent in intertemporal equilibrium theory. The recent interest in bounded rationality in game theory has concentrated on the limits that finite memory imposes on the complexity of strategies that players can pursue in sequential games. One must applaud this interest, although I suspect that limitations of memory will not turn out to be the most crucial factor that limits the complexity of strategies.

Hicks and others before him were aware of the problem of complexity. In *Capital and Growth* (p. 102), after discussing a problem in stock adjustment, he wrote: 'It is hardly a discovery . . . to find that we are unable to "simulate" the behaviour of intelligent business management by any simple rule.' Later in the same book (p. 197), after a discussion of the problems that arise in a 'traverse' from one dynamic equilibrium to another, he wrote: 'In an actual economic situation, all these problems arise at once, while (because of the advance in technology) the equilibrium at which the economy is aiming is continually shifting. No wonder there is a problem of business management!'

Another well-known obstacle to rationality is 'passion'. A. O. Hirschman has discussed this matter at length in a stimulating and delightful book, *The Passions and the Interests* (Hirschman, 1977), and I cannot resist reproducing here some passages cited by him. The background is the emergence in the seventeenth and eighteenth centuries in Europe of an 'infatuation with interest as a key to the understanding of human action'.[29] Thus Helvetius is quoted as proclaiming: 'As the physical world is ruled by the laws of movement so is the moral universe ruled by laws of interest' (p. 43). But there were dissenters from this view. For example, Spinoza: 'All men certainly seek their advantage, but seldom as sound reason dictates; in most cases appetite is their only guide, and in their desires and judgements they are carried away by their passions, which take no account of the future or of anything else' (p. 44).

Hirschman also quotes 'this ironical remark of the Marquis of Halifax: "If men must be supposed always to follow their true interest, it must be meant of a new manufactory by God Almighty; there must be some new clay, the old stuff never yet made any such infallible creature"' (p. 45).

As one might expect from the French, La Bruyère summed it up most dramatically and concisely: 'Nothing is easier for passion than to defeat reason: Its great triumph is to gain the upper hand over interest' (p. 46).

I doubt that I have to cite more recent sources to convince you that passion plays an important role in economic life. Unfortunately, I am not aware of any research on this topic by general equilibrium theorists (even intertemporal!), nor do I intend to pursue this topic further here.

On the other hand, J. M. Clark described the psychological model of 'economic man' used by economists of his day as follows:

> If one wished to be unfair to economists in general, he might select, for purposes of comparison with these psychological principles, a certain well-known though fictitious character whose idiosyncrasies furnish alternate joy and irritation to modern readers of economics. He is a somewhat inhuman individual who, inconsistently enough, carries the critical weighing of hedonistic values to the point of mania. So completely is he absorbed in his irrationally rational passion for impassionate calculation that he often remains a day laborer at pitifully low wages from sheer devotion to the fine art of making the most out of his scanty income and getting the highest returns from his employers for such mediocre skill as he chooses to devote to their service. Yet he cannot fail to be aware that the actuarial talent he lavishes outside of working hours would suffice to earn him a relatively princely salary in the office of any life insurance company. So intricate are the calculations he delights in that even trained economists occasionally blunder into errors in recording them (Clark, 1918, p. 24).

Returning to bounded rationality and out-of-equilibrium behaviour, one comes across the following terms: (i) learning, (ii) adjustment, (iii) adaptation, (iv) evolution. Such ideas have been discussed by economists for a long time (but also infrequently, compared to the entire literature on economic theory).[30] These terms describe behaviours that are responses – of individuals and groups – to the boundedness of rationality. It to be expected, therefore, that the successful analysis of such behaviours would require an understanding in some depth of the boundedness of rationality. This is not the place for an essay on this topic, but I think that a few words are in order here to remind you of some of the difficult issues that have to be faced.[31]

Although the hypothesis of economic rationality might be quite appropriate for some simple, repetitive decision problems under conditions of certainty, it becomes less and less appropriate as we consider decision problems involving more uncertainty and longer time horizons. In such cases, the theorist may make use of the hypothesis in several ways:

1. The theorist calculates the actions that would be optimal for the economic agent,[32] and uses this optimal action as a first approximation to what the agent actually does; for this research strategy to work, the hypothesised decision problem must not be too complicated for the theorist to solve!
2. The theorist replaces the actual decision problem by a simpler one, or hypothesises that the agent does this.
3. The theorist hypothesises that the agent's actual actions will 'tend towards' the optimal action (in time), provided the agent has repeated opportunities to improve his action in an unchanging environment.

In his book, *The Foundations of Statistics*, L. J. Savage emphasised the dilemma of decision theory that is reflected in the two proverbs, 'Look before you leap', and 'You can cross that bridge when you come to it.' In his chapter on 'Preliminary Considerations on Decision in the Face of Uncertainty' he comments:

> Carried to its logical extreme, the 'Look before you leap' principle demands that one envisage every conceivable policy for the government of his whole life (at least from now on) in its most minute details, in the light of the vast number of unknown states of the world, and decide here and now on one policy. This is utterly ridiculous, not – as some might think – because there might later be cause for regret, if things did not turn out as had been anticipated, but because the task implied in making such a decision is not even remotely resembled by human possibility. It is even utterly beyond our power to plan a picnic or to play a game of chess in accordance with the principle, even when the world of states and the set of available acts to be envisaged are artificially reduced to the narrowest reasonable limits (Savage, 1954, p. 16).

If we are not to discard entirely the rationality postulate in economic theory, then we must elaborate more sophisticated and empirically relevant concepts of rational behaviour – call it boundedly rational behaviour if you like – concepts that nevertheless retain the important insights provided by the notion of 'economic man'.

I shall not attempt here to give a precise definition of bounded rationality. However, three aspects of bounded rationality do seem important for decision theory: (i) existence of goals, (ii) search for improvement, and (iii) long-run success.

It is no doubt useful to explain much of economic behaviour in terms of 'goals' or 'motives', and normative economics would appear to be meaningless without reference to goals. On the other hand, an individual economic agent may have 'conflicting' goals, and it may be bad psychology in many instances to assume that these conflicts are resolved in terms of a single transitive preference ordering. Such conflicts may be 'resolved' in a dynamic way by various mechanisms for switching attention and effort, with results that do not appear to be transitive. (There are, perhaps, useful analogies between individuals with conflicting goals and groups of individuals with conflicting interests.) Also, the set of goals may be endogenous, so that, through time, some goals may be dropped and others added to the list.

Even if the theorist draws back from assuming that economic agents behave according to optimal lifetime strategies, it is no doubt useful to postulate that they search for improvements, at least from time to time, and that they take advantage of perceived improvements. How, and under what circumstances, agents search for improvements, and how these improvements are perceived, is, of course, an important subject of study. If repeated improvements can be made in the solution of the same problem, then we have a situation of 'expanding rationality'. On the other hand, an environment that changes at unpredictable times and in unpredictable directions may make past improvements obsolete, so that the individual is engaged in a race between improvement and obsolescence.

A strategy of search may itself be the object of an improvement effort (as in the planning of research and development), but this leads to a 'regression' in the model of decision-making: one eventually reaches a level of behaviour at which it is no longer fruitful to assume that the search for improvement is itself being conducted 'optimally'.

The notion of 'adjustment', as it has commonly been used in economic theory, is in the spirit of bounded rationality in the following sense. At a given date the economic agent adopts a particular action (or strategy) that is optimal with respect to the agent's formulation of the decision problem and the agent's 'expectations'. At the next date, the agent receives new information, which causes him to revise his expectations in *a way that was not anticipated at the previous date*, or even causes him to revise his formulation of the decision problem. This revision of expectations or of problem formulation is to be distinguished from the behaviour of a Bayesian statistician with an optimal sequential decision rule, who periodically revises his *a posteriori* probability distribution on the states of the environment in response to new information, according to a well-defined and completely anticipated (optimal) transformation.

My final point is that, once we have been even moderately successful in developing theories of the boundedness of rationality, then *we shall probably no longer be so interested in equilibrium*. Here I should distinguish between two senses of equilibrium that we have been using:

1. a dynamic path that satisfies the Hicks–Nash condition;
2. a stationary stochastic process.

It seems plausible to me that our knowledge of our social, economic, and even physical environment, although expanding, is going to remain extremely limited in the foreseeable future. But if our knowledge is increasing – or even changing – then our economic environment will not be stationary, at least to the extent that we act on our knowledge. (This will be so, even if the underlying exogenous physical environment is approximately stationary on the relevant time-scale.) Notice that this not simply a case of the economy aiming at a continually shifting equilibrium (see the quote from Hicks, at the beginning of Section 5). What I am describing is a thoroughly non-stationary process, but not necessarily one that is without any 'laws of motion' at all.[33] In addition, because of the boundedness of rationality, the Hicks–Nash condition is never satisfied, either.[34]

These ideas are not new; they are not even new among mathematical economists (for example, Koopmans, 1974). Indeed, they are repeated in one form or other every so often, and usually meet with approval, and then are largely ignored. What explains this curious behaviour of economic theorists?

In fact, some recent research is more encouraging. I cite, for example, the work of Nelson and Winter (1982), Selten (1988), and Binmore (1985), to name a few. (However, the two last-named papers are on game theory.) Recently, a whole conference at Cornell University was devoted to the topic, 'Learning from Endogenous Data'.[35] Some research in experimental economics may also be relevant to these issues.[36]

So I would like to close on an optimistic note. I have concluded that the long development of intertemporal general equilibrium

theory that was launched in *Value and Capital* ultimately revealed the theory's serious limitations, but probably that exploration was necessary. I hope that this chapter is not just another of those premature calls for serious research on intertemporal *non-equilibrium* theory, but rather a report that this new stage of research interest has already begun.

Notes

1. The views expressed here are those of the author, and are not necessarily those of AT&T Bell Laboratories. I would like to acknowledge the help I received from J. Benhabib, D. Duffie, F. H. Hahn, P. Hillebrandt, M. K. Majumdar, E. Malinvaud, A. Mas-Colell, L. W. McKenzie, H. Polemarchakis, and K. Shell while I was preparing this chapter. The responsibility for errors and omissions remains mine, of course. Citations in the text are by author and date. Full citations are gathered at the end of the chapter. Although the list is already very long, I make no pretence that it is complete. I have attempted to include enough recent references for the reader to have an entrée to the relevant literature. I have also cited some works of historical interest that came up naturally in the text, *but not all of them!* Nobody should feel insulted if his or her important contributions have not been adequately cited.
2. However, since the present model assumes complete certainty, it requires us to be able to know next year's automobiles perfectly!
3. Alternatively, there are primary commodities that are necessary for production, as in Case 1, but the endowments of these are large enough (and growing fast enough) for them to be effectively unlimited (unconstraining).
4. This is an oversimplification. The date at which the economy ends could be a random variable taking on only a finite set of possible values, but then we would be in the case of uncertainty; see Section 3.1 of this chapter.
5. See (Hicks, 1961). For recent comprehensive accounts of 'turnpike theory' see (McKenzie, 1986, 1987). I owe my own introduction to the subject to a talk by Professor Hicks at the University of California in the spring of 1960, in which he expressed doubts about the validity of a proposition put forward by Dorfman, Samuelson, and Solow (1958), who first used the term 'turnpike' in this context. (See articles by Hicks, Morishima, and Radner, in the *Review of Economic Studies*, 1961, for the sequel.) Independently, McKenzie was pursuing a similar line of research (1963).
6. Even the long-run-average (Cesaro sum) of a bounded sequence may fail to exist (Hardy, 1949).
7. Note that not every initial endowment need be consistent with a steady state.
8. See Zilcha (1978), Cass and Majumdar (1979).
9. See Kehoe and Levine (1985), Kehoe, Levine, and Romer (1986), McFadden *et al* (1980).

10. To be precise, a stochastic process can exhibit purely cyclic behaviour; for example, see Feller (1968), pp. 404–5, for a discussion of periodic Markov chains. More typically, however, the addition of 'noise' to a cyclical dynamic system will result in a stationary process, as in an irreducible aperiodic Markov chain. For such a stochastic process, the invariant probability distribution corresponds, in a sense, to the 'steady state' of a deterministic process. A stochastic process may also converge to a stationary process, in various senses (some stronger that others), but this topic is too technical to discuss here (see, e.g., Doob, 1953, chs. V and VI).
11. Brock and Mirman (1973) provided an early stability result for a one-good optimal growth model. Further contributions on dynamics and on the two welfare theorems were provided by Radner (1973), Dana (1976), Zilcha (1976ab, 1978), Mirman and Zilcha (1977), Föllmer and Majumdar (1978), Majumdar and Radner (1983), and the references cited in those papers. A summary of the literature on one-good models can be found in Majumdar, Mitra, and Nyarko (1988), as well as an analysis of the implications of certain non-convexities. In a preliminary way, Radner (1968) explored the consequences and problems of extending the AD model to the case in which different agents have different information.
12. Radner (1968) argued that heterogeneity of information among agents would lead to incomplete markets, and hence to a sequence of markets. Hahn (1971) studied equilibrium in markets with transactions costs; see also the references cited there.
13. See notes 10 and 11 above.
14. Hart (1975) gave an example of the non-existence of equilibrium. On the generic existence of equilibrium, see Duffie and Shafer (1985, 1987). For a different approach, see Younès (1986, 1990).
15. See Cass (1984), Geanakoplos and Mas-Colell (1987), Balasko and Cass (1989), and the references cited there.
16. See Cass and Shell (1983).
17. See Geanakoplos and Polemarchakis (1986b) for the case of pure exchange, and Geanakoplos *et al* (1987) for a model with production. However, Repullo (1988) defines a new (and weaker) concept of constrained optimality ('D-efficiency') for which (roughly speaking) the two welfare theorems hold for incomplete markets.
18. See, however, Duffie *et al* (1988) on the existence of stationary Markov equilibria, and Dutta and Polemarchakis (1989) on serial dependence in output and prices.
19. RE equilibria were introduced formally in Radner (1967), and independently by Lucas (1972) and Green (1973). Examples of non-existence were given by Kreps (1977) and Green (1977). For a particular definition, existence has been shown to be generic except when the dimensions of the price and information spaces are equal; see Jordan and Radner (1982) and Allen (1986) for summaries of the relevant literature.
20. The OLG model was introduced by Allais (1947) and Samuelson (1958); other early contributions were by Malinvaud (1953), Lerner (1959), Cass and Yaari (1966), Shell (1971), and Gale (1973). (The little-noticed contribution by Allais is described by Malinvaud (1987).) A later treatment and bibliography are in Balasko and Shell (1980, 1981). On existence of equilibrium, see Balasko *et al* (1980) and Wilson (1981). On indeterminacy, see Geanakoplos and Polemarchakis (1986a). On 'sunspot equilibria', see Peck (1988) and the references cited there. On cyclic and chaotic dynamics, see Grandmont (1985) and Benhabib and Laroque (1988). On the core of an OLG model see Younès (1990).

21. See Radner (1987) for an introduction and bibliography. The papers in Groves *et al* (1988) provide a sampling of recent research, and an entry to the literature.
22. See van Damme (1987, 1988) and Fudenberg *et al* (1988) for accounts of recent developments.
23. See, for example, Linhart *et al* (1989) on bargaining with incomplete information.
24. See Kydland and Prescott (1982) for an attempt to explain aspects of post-war US macroeconomic magnitudes with an equilibrium growth model. This attempt is relatively modest in that it looks at 'the auto-co-variances of real output and the co-variances of cyclical output with other aggregate economic time series . . .' See also Mannelli (1986), and the other papers in the same issue of the *Quarterly Review* of the Federal Reserve Bank of Minneapolis.
25. For some recent attempts to use intertemporal general equilibrium theory to explain the dynamics of economic development, see Lucas (1985) and Romer (1987).
26. 'Out-of-equilibrium behaviour' could also refer, of course, to behaviour when markets do not clear, as in the models of 'non-Walrasian' equilibrium. I have left that topic to Professor Malinvaud.
27. This problem also arises in the theory of equilibria of sequential games. See Savage (1954) p. 39, for a brief discussion of this problem in the context of single-person decision problems.
28. See, for example, Simon (1972), Nelson and Winter (1982), Day and Groves (1975); in self-defence, I should cite some of my own attempts (1975, 1980, 1986).
29. P. 43. 'Interest' here refers to 'self-interest', as opposed to 'passion' and 'reason'.
30. See next note for references.
31. See, for example, Day and Groves (1975), Simon (1972), Nelson and Winter (1982), Radner (1975), and the references cited there. For a discussion of 'obstacles' to the attainment of equilibrium, see Hicks (1979) pp. 44 ff.
32. It is to be understood that the word 'action' may, where appropriate, be interpreted as 'decision rule' or 'strategy'.
33. For a few simple examples of this, see Radner (1975).
34. It might be argued that the Hicks–Nash condition could be satisfied if agents were 'doing their best', given their bounds on rationality. I question whether this is a meaningful statement, but cannot elaborate this point here; cf. Savage (1954) p. 7, fn, and p. 16.

35. Organised by D. Easley and N. M. Kiefer. For learning in a rational expectations equilibrium context, see Blume and Easley (1982), Bray (1982), Frydman (1982, 1987), and Frydman and Phelps (1983).
36. See Alger (1986), Selten (1988), Binmore (1985). Professor A. Schotter and I have begun to explore these issues experimentally at New York University; for preliminary reports, see Radner and Schotter (1989) and Linhart *et al* (1990).

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Comment*

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1 INTRODUCTION

Roy Radner's (1972) notion of 'equilibrium of plans, prices, and price expectations', an extension of Arrow's (1953) model of general equilibrium under uncertainty, is now the closest thing to a standard paradigm for intertemporal general equilibrium. I have heard a rumour that Radner recently voiced scepticism over his own notion of equilibrium. There is certainly some hint of that in his (1988) review of post-war developments of 'Intertemporal General Equilibrium'. On the whole, however, his review has a balance of optimism and scepticism concerning developments of the theory since Hicks's *Value and Capital* appeared. Radner's perspective is well-informed and pragmatic. In this discussion of Chapter 15, I will highlight a few of his observations that I find most novel, and also add a few comments of my own.

2 THE DRIVE FOR REALISM – THE LACK OF PRACTICAL AMBITION

A major portion of Chapter 15 outlines developments of the theory that are devoted to more realistic assumptions. The incorporation of uncertainty, multi-period trading with incomplete markets, heterogeneous information with rational expectations (learning from prices), overlapping generations, imperfect competition, and so on – these make for a model closer and closer to our perception of markets. As Radner points out, and this seems to be the principal thesis of his chapter, 'realism development' in the theory has also created greater and greater tension along three fronts.

First, it has become more and more challenging (both in terms of the economist's technical competence as well as what the model itself can produce) to obtain what Radner calls the preconditions of an equilibrium theory: '(1) the existence of an equilibrium; (2) the uniqueness of equilibrium, or at least that there are only finitely many equilibria; and (3) that the equilibria change continuously with the data (parameters) of the model, at least for most values of the data.' Shafer (1988) is an excellent survey of how Radner's three preconditions can be drawn from an equilibrium model. Even given Radner's preconditions, the classical welfare properties of general equilibrium are easily left by the wayside as realism is added to the model.

Second, each bit of additional realism seems to come at a cost to the ability of the model to explain empirical regularities in market data. My own perspective from the finance faculty of a business school leaves me forever in wonder at the mountains of security price data that are hopefully fitted to equilibrium asset-pricing models that contain little of the above listed touches of realism.[1] The pity is not that we have yet to make enough theoretical or econometric progress to enable us to fit the data to the more realistic models. It is, rather, that the more realistic the models become, the fewer restrictions they place on equilibrium behaviour. This is related mainly to modelling precondition (2). Incomplete markets with monetary assets, for example, deliver a continuum of possible equilibria, as shown by Cass (1985), Geanakoplos and Mas-Colell (1985), and Balasko and Cass (1986). McAllister's (1988) model of equilibrium with rational expectations and heterogeneous expectations has the same property.

Third, the drive for realism places an incredible strain on the notion of a rational economic agent. Not only are the agents populating the newest and most realistic models expected to solve infinite-dimensional stochastic dynamic programming problems for which no solution algorithms can be contemplated, they are also expected to have at their disposal a complete list of all possible states of the future (each state corresponding to a complete description of all relevant economic information). Kreps (1988) offers one possible escape route, a model of preferences with unforeseen contingencies. Bewley (1986, 1987) suggests a 'Knightian' alternative based on incomplete preference orders. Neither of these preference models has been incorporated into the intertemporal general equilibrium theory. Finally, some of the more realistic models have game-theoretic features demanding common knowledge of rationality (I'm rational, you're rational, I know you're rational, you know I'm rational, I know that you know that I'm rational, *ad infinitum*).[2] Common knowledge of rationality effectively demands each agent's

ability to solve every agent's decision problem simultaneously in common knowledge. Radner's comments on the bounds of rationality and the role of passion in place of rationality make for enjoyable reading.

3 INTERACTION WITH FINANCE

As Radner points out, one may argue that finance, the study of security markets, has been a proving ground for intertemporal general equilibrium theory. Security prices can only be explained by paying attention to the temporal resolution of uncertainty, a focus of developments in intertemporal general equilibrium models. Security price data are readily available in large volume. What is more, the application and testing of general equilibrium models in the setting of security markets are both motivated by obvious market incentives to measure financial risks and explain the determination of security prices.

Since Arrow's (1953) paper, 'The Role of Securities in the Allocation of Risk Bearing', and the subsequent development of uncertainty in the general equilibrium model by Debreu (1953; 1959, ch. 7), some major milestones in finance that have had a strong influence on general equilibrium theory are:[3]

(i) the capital asset pricing model (CAPM) of Sharpe (1964) and Lintner (1965), which gave a substantial practical focus to the theory;[4]
(ii) the Black and Scholes (1973) option-pricing formula and its elaboration by Merton (1973b); the formula itself had little to do with general equilibrium theory but its possibility suggested the importance of dynamics in extending the scope of Arrow's (1953) results;
(iii) Breeden's (1979) consumption-based capital asset pricing model, an extension of the CAPM to multiperiod settings using Merton's (1969, 1973a) continuous-time dynamic programming approach;
(iv) the martingale pricing theory of Harrison and Kreps (1979), a dramatic conceptual simplification of the problem of pricing securities in multiperiod markets that set off an avalanche of further developments to the Black and Scholes model as well as a new examination of general equilibrium theory for infinite-dimensional consumption spaces.[5]

Most recently, finance theorists have devoted special attention to equilibria with asymmetric information and to the 'microstructure' of security markets; examples are the work of Glosten and Milgrom (1985) or Admati and Pfleiderer (1988). This literature shows that finance has the practical good sense to leave the invisible hand behind when the process of price-formation is itself to be explained.

It is from this focus on price formation that I expect finance theory will continue to be a force sustaining the development of intertemporal general equilibrium theory.

4 SOME GAPS TO BE FILLED

Radner reviews some of the gaps in intertemporal general equilibrium theory. Let me supply a few of the most important from my own menu.

First, the endogenous innovation of security markets seems to me one of the most important processes to be explained. Radner (1972) himself, along with Hart (1975), led us to focus on the incomplete structure of security markets and what that means for the preconditions of the theory and the welfare properties of equilibria. (Geanakoplos, 1988, surveys the many developments of the past few years; 'incomplete markets' has recently been one of the most active topics in general equilibrium theory.) Unfortunately, however, there has been precious little in the way of explaining the formation of security markets and where it stops. (Grossman and Hart, 1987; Harris and Raviv, 1987; Duffie and Jackson, 1989; and Allen and Gale, 1987, are early attempts.) What markets will be set up, when, and by whom? Is the innovation process, presumably guided by entrepreneurial incentives, efficiently trading off the market set-up and transactions costs against gains from the availability of insurance?

Second, I would say we know too little about the role of the firm in intertemporal equilibrium. I speak of the firm, and not merely the set of technologically feasible production plans.

(a) What is the goal of the firm? Market-value maximisation is unanimously supported by shareholders, is Pareto-optimal, and is a well-posed problem, provided there is a complete set of security markets. Otherwise, each of these propositions can be overturned. (See, for example, Duffie and Shafer, 1986.)

(b) How is the financial policy of the firm relevant? Even with incomplete markets (but otherwise standard assumptions), extensions of the Modigliani–Miller (1958) theorem suggest that financial policy is irrelevant for both the value of the firm (Duffie and Shafer, 1986) as well as the utilities of shareholders (DeMarzo, 1988). Some work obviously remains to be done to justify the emphasis of modern corporations on their financial strategies. Aghion and Bolton (1987) is a good example of how the theory may develop.

Third, and this seems one of the most demanding basic questions, what is an appropriate model for the expectations of agents with asymmetric information? Surely we cannot take the ideal case of fully rational and simultaneous learning from prices in a model of completely foreseen contingencies all that seriously! In particular, it seems questionable to me that agents in actual markets treat current and future prices as functions of the underlying state of the world, drawn from some mysterious all encompassing set Ω (common to all agents), despite my own everyday use of this assumption for modelling purposes. Beyond the incredible demands of this assumption, it has failed to generate much of a theory of intertemporal equilibrium with asymmetric information. Neglecting results implying completely revealing prices, there is little to build on in the way of general models of equilibrium, by which I mean models whose properties can be characterised without referring to explicit solutions based on parametric forms for preferences and probability distributions. A key aspect of the problem is that there is no reason to believe in the continuity of the function mapping the revealed price to the conditional distribution of future states. To see this in a one-state one-price model, simply plot a continuous function mapping the state to the price. Unless the price function is strictly monotonic, the inverse image correspondence (which determines the conditional state distribution) is not lower hemi-continuous, but that form of continuity is needed, in general, for the continuity of the conditional distribution of the state as a function of the price. There are generic arguments that avoid discontinuities when the dimensions of the state and price spaces are appropriate for completely revealing prices (see Allen, 1981), but completely revealing prices seem too much to hope for or to rely on. McAllister (1988) offers an alternative formulation of the model satisfying Radner's precondition (1) (existence), but not satisfying preconditions (2) and (3).

I believe these three problems, the endogenous formation of markets, the theory of the firm, and the treatment of expectations, are central to the satisfactory future development of the intertemporal general equilibrium model.

Notes

* The support of Batterymarch Financial Management and the comments of Lionel McKenzie are much appreciated.

1. Lucas's (1978) model is a prime example.
2. See, again, McAllister (1988).
3. When presenting this discussion it occurred to me how much my perspective may be influenced by my surroundings at Stanford, with which all these four models have had some connection by way of one or more of their authors.
4. The later contributions of Borch (1968) to the development of this model, particularly its welfare properties, have been largely overlooked; I am grateful to Dieter Sondermann for bringing them to my attention.
5. Mas-Colell (1986) had the first results treating the class of continuous-time consumption spaces envisioned by Harrison and Kreps. Many more papers quickly followed, most of these citing applications in finance as an impetus.

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Postscript