Economic Analysis of Markets and Games

Essays in Honor of Frank Hahn

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Transfer Payments and the Core of a Profit-Center Game

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If a firm is organized as a collection of profit centers, and if some profit centers use goods, services, or resources provided by other centers, then the issue arises of how the supplier centers will be compensated in accounting dollars. This is sometimes called the problem of "transfer pricing," although the term may be misleading since it suggests fixing a price per unit of the commodity that is provided.

If the profit centers—or *divisions*, as I shall call them here—are allowed to bargain freely among themselves regarding the terms of such transfers of commodities (goods, services, resources) and the corresponding transfers of accounting dollars, one might predict the outcome to be in the core of a suitably defined game, should the core be nonempty. Roughly speaking, an element of the core is a feasible production plan for the firm, together with an allocation of accounting profits among the divisions, such that no subset of divisions obtains a total profit smaller than it could get by operating "on its on." (These concepts will be defined more precisely in section 2.) Thus, an element of the core is in some sense "stable" against recontracting by groups of divisions. (See, however, the remarks at the end of this introduction.)

The nonemptiness of the core has been established for models of pure-exchange economies, and for economies with production in which every subset of consumers has available to it—in principle—the same set of production possibilities.[1] The essence of the typical firm is that different divisions have available *different* production possibilities; each division specializes in a particular set of activities (lines of business, research and development, production, marketing, etc.). On the other hand, a firm is simpler than an economy as a whole in that there are no "consumers" in the usual sense. Finally, a realistic model of the internal economics of the firm should allow for "nonconvexities," i.e., possibilities of increasing returns to scale, fixed costs, externalities, etc.

In this paper I define a "profit-center game" for a firm organized into divisions, and I make some progress toward characterizing the core for some simple models with nonconvexities. The results for these models suggest the following general conclusions:

- For the case of a single "supplier" division and several "customer" divisions, under quite general conditions, the following imputation of profit

is in the core: the supplier gets all of the "surplus" that it generates within the firm. In particular, if a customer can get from the outside market inputs that are comparable to those it gets from the supplier, then the transfer price corresponding to this core imputation may be as high as the market price. In general, there will typically be other core imputations in which the customers share some of the surplus. For some special cases I examine the upper bounds on the shares of the surplus that the customers can receive, or, correspondingly, lower bounds on the transfer prices that they can pay the supplier. For example, if the supplier's cost function is linar, with a fixed cost and a constant marginal cost for each commodity, then (for an imputation in the core) the customers must pay transfer prices that are at least as large as the corresponding marginal costs, and some of them must pay strictly larger transfer prices (unless, of course, the supplier is subsidized by the headquarters, a possibility that is not envisioned in the present model).

- The core will be nonempty under a variety of non-neoclassical technological conditions that occur naturally within a firm, provided there is no competition among internal suppliers who produce close substitutes. In particular, I give some conditions under which the suppliers of strictly complementary inputs can (in the core) jointly extract all of the surplus generated by the use of those inputs by their internal "customers."
- The core may be empty if there are alternative internal suppliers of close substitutes, and their average costs are declining and then increasing in the relevant ranges of output.

This last result suggests that, in setting up profit centers, the firm will try to avoid creating "internal competition" among internal suppliers of close substitutes. However, further research is needed to determine the range of models for which these conclusions are valid.

For completeness, I also give the usual argument that establishes the nonemptiness of the core under general conditions of convexity. This argument uses the standard method of duality to impute "shadow profits" to the divisions, calculated according to the "shadow prices" corresponding to an optimal production plan.

A more fundamental extension of the model would be needed in order to study how an element of the core might be selected. Such an extension would require a more explicit consideration of the information available to the players. Although such an extension is not attempted here, a few informal remarks may help put the present paper in better perspective. At one extreme, suppose that all the data of the problem (cost and revenue functions, etc.) were known to the headquarters of the firm, and that the headquarters had the capacity to compute a profit-maximizing plan and the power to enforce its instructions to the divisions. In this case, the imputation of profits to individual divisions would be unnecessary, as would be the exercise of finding an element of the core. At the other extreme, suppose that the data about each division were initially known only to that division. If—as suggested at the beginning of this introduction—one were to rely on bargaining among the divisions (or division heads) to produce a plan and an imputation of profit, then, depending on the bargaining mechanism and the initial data, the outcome might or might not be optimal for the firm. For example, even in the case of one customer and one supplier, if the bargaining mechanism has the property that the divisions are neither "taxed" nor "subsidized" by the headquarters, then the outcome cannot be in the core for all configurations of the data.[2] On the other hand, if taxes and subsidies are allowed, then an appropriate "Groves mechanism" will produce outcomes in the core for all configurations of the data.[3]

In most firms, the typical situation will be somewhere between the two extremes sketched above. In a continuing relationship among the divisions, if a plan and a profit imputation are not in the core, this fact may be discovered by the interested parties, and they will have an incentive to change the arrangement by some means, even if this requires an evasion of bureaucratic rules. Thus, an arrangement not in the core may not be viable in the long run. On the other hand, there may be no mechanism that is guaranteed to lead to a core arrangement.

1 The Economy of the Firm

The firm is divided into *divisions*, indexed by i in I. The goods, services, and resources that are used, produced, bought, or sold by the divisions are called *commodities*. Those commodities that can be bought or sold on markets that are external to the firm are called *market commodities*; those that are owned or produced by the firm and used internally are called *nonmarket commodities*. It will be convenient to make the convention that a commodity must be either market or nonmarket, but not both. A simple device will show that this can be done without loss of generality (see below). Commodities may also be distinguished by date and location, as is normally done in the theory of production (see, for example, Debreu 1959, Chapter 2).

The *net output* of division i will be denoted by $y_i = (m_i, n_i)$, a vector with dimension equal to the total number of commodities. The subvectors m_i and n_i correspond to the market and nonmarket commodities, respectively. A coordinate of y_i is positive or negative according as the corresponding commodity is a net output or a net input of that division. The total net output of the firm is the sum of the net outputs of the individual divisions.

A division is characterized by its *production set*, Y_i, and its initial resources, r_i. The production set of the division is its set of technologically feasible net output vectors. The vector r_i of initial resources describes the quantities of nonmarket commodities that are assigned to (or owned by) division i before the process of production or before any buying or selling takes place. A *plan* for the firm is a combination of net output vectors, one for each division. A plan (y_i) is *feasible* if each division's net output vector is technologically feasible and for each nonmarket commodity the firm's total net input does not exceed its total initial resources. This can be expressed formally as follows:

$$\text{for each } i \text{ in } I,\ y_i \text{ is in } Y_i; \tag{1.1}$$

$$\sum_{i \in I} (n_i + r_i) \geq 0. \tag{1.2}$$

Suppose that the market commodities are bought and/or sold at fixed prices; denote the vector of these prices by p. (A more general treatment would allow the market prices to depend on the firm's net output vector.) The *profit* of the firm is the value of the net output of market commodities, evaluated at the market prices. Thus, the profit is the inner product pm, where m is the (vector) sum of the market-commodity net-output vectors of the divisions. Since the commodities may be dated, this formulation allows one to interpret the profit as the present discounted value of the firm's net outputs of market commodities.

If a particular good or service can serve as either a market or a nonmarket commodity, the two versions of the good or service will be distinguished formally but will be treated as perfect substitutes in production. This perfect substitutability will, of course, be expressed in the production sets of the divisions.

In modeling the initial resources of the firm, it might seem unnecessary, or even inappropriate, to assign them to the individual divisions at the beginning, before the choice of the production plan of the firm. This is, however, only a convention, and it can be interpreted in several ways. First, one of the "divisions" might be the headquarters of the firm, whose "production" consists in the assignment of centrally controlled resources to the several divisions. Second, resources that are initially assigned to one division can be reassigned to others as part of the production plan, without cost. Finally, one wants to be able to model the situation in which certain resources or "capacities" are naturally associated with certain divisions, for organizational or informational reasons that are not explicitly treated in the model in more detail.

In course of the paper, various assumptions will be made about the productions sets of the divisions. Two assumptions, however, will be maintained throughout the paper, unless otherwise noted. The first asserts that any division can do nothing, producing no outputs and using no inputs. The second asserts that commodities can be unsold or unused without using up other commodities.

A1 (inaction) For each i, Y_i contains the zero vector.

A2 ("free" disposal) For each i and each y_i in Y_i, if $y_i' \leq y_i$ then y_i' is also in Y_i.

Furthermore, it is natural to suppose that the prices of all market commodities are strictly positive. It should be pointed out, however, that in conjunction with assumption A2 this is not merely a convention. For example, if one of the "outputs" of the firm is a toxic waste, the firm may have to pay a positive price to have the waste treated and taken away. On the other hand, once one has ruled out negative prices, there is no essential further loss of generality in assuming that all prices are strictly positive.[4]

A3 The prices of all market commodities are strictly positive.

A feasible production plan for the firm is called *optimal* if it has maximum profit in the set of all feasible plans.

2 The Profit-Center Game and the Core

In order to model the organization of the firm as a collection of profit centers, I shall define a "cooperative game" whose outcome is not only a production plan but also an imputation of an "accounting profit" to each

division. I shall suppose that the sum of the accounting profits of the divisions equals the profit of the firm, pm, as defined in section 1. The theory used to predict the outcome of the game is that the outcome will be in the core of the game (defined below). The basic analytical problems addressed in this paper are (1) the determination of conditions under which the core is nonempty and (2) as complete a description of the core as is possible under such conditions.

In what follows I shall consider nonempty subsets of the set of all divisions—called *coalitions*—and the maximum profit that each coalition could make if it acted as a "firm" on its own. Roughly speaking, an imputation of accounting profits to the several divisions of the firm is in the core if the total accounting profit imputed to each coalition is not less than the actual profit it could earn if it acted as a firm on its own.

Thus, for any coalition S, a nonempty subset of I, define $v(S)$ by

$$v(S) = \max p\left(\sum_{i \in S} m_i\right) \quad \text{subject to} \tag{2.1}$$

1. for each i in S, y_i is in Y_i,
2. $\sum_{i \in S} (n_i + r_i) \geq 0$.

In particular, $v(I)$ is the maximum profit of the entire firm.[5]

As a consequence of assumption A1 (inaction), the zero vector is a feasible net output for any coalition, and so the maximum in (2.1) is nonnegative. Thus,

$$v(S) \geq 0 \quad \text{for each coalition } S. \tag{2.2}$$

On the other hand, it would be quite usual for coalitions other than I (the entire firm) not to be able to make strictly positive profits on their own, so that $v(S)$ might be zero for many—or even all —coalitions other than I.

A *profit imputation* $x = (x_i)$ is a vector whose ith coordinate is to be interpreted as the accounting profit attributed to division i. A profit imputation is *feasible* if there is a feasible net output vector for the firm such that the total accounting profit of the divisions equals the total profit of the firm.

A profit imputation $x = (x_i)$ is said to be in the *core* if

$$\text{for each coalition } S, \sum_{i \in S} x_i \geq v(S) \tag{2.3}$$

and

$$\sum_{i \in I} x_i = v(I). \tag{2.4}$$

Condition (2.4) implies in particular that a profit imputation in the core is feasible. The core is thus the set of feasible profit imputations that (1) give each coalition other than I at least as much profit as it could get on its own and (2) have a total profit equal to the maximum profit of the firm as a whole.

There is nothing in the model described thus far that guarantees that the core is nonempty.

3 The Core of the Profit-Center Game is Nonempty for Convex Production

If the production set of the firm is convex, and the usual duality theory is applicable, then the core of the profit-center game is nonempty. A profit imputation in the core can be obtained as follows. Recall that a production plan for the firm is optimal if it maximizes the total value of the net output of market commodities, subject to its production and resource constraints. Corresponding to an optimal plan there will be a "shadow price" for each nonmarket commodity. Impute to each division an accounting profit equal to the sum of (1) the value of its net output of market commodities, (2) the "shadow value" of its net output of nonmarket commodities, and (3) the "shadow value" of its initial resources.

Although the above construction is well known, I shall sketch it here more formally for the sake of completeness. (The reader can skip to section 4 without loss of continuity.) I make the following assumption, in addition to A1–A3:

A4 Each production set Y_i is closed and convex.

Let $(\hat{y}_i) = ((\hat{m}_i, \hat{n}_i))$ be an optimal production plan for the firm. By standard duality theory,[6] there exists a non-negative, nonzero vector q such that

for each division i, and each (m_i, n_i) in Y_i,

$$pm_i + qn_i \leq p\hat{m}_i + q\hat{n}_i \tag{3.1}$$

and

$$q \sum_{i \in I} (\hat{n}_i + r_i) = 0. \tag{3.2}$$

An interpretation of (3.1) is that, at the optimal production plan, each division is maximizing its "shadow profit." Since q is non-negative, (3.2) implies that if the resource constraint of a nonmarket commodity is not binding, then the corresponding shadow price is zero.

Define the profit imputation $\hat{x} = (\hat{x}_i)$ as follows: For each i in I,

$$\hat{x}_i = p\hat{m}_i + q(\hat{n}_i + r_i). \tag{3.3}$$

Note that, by (3.2),

$$\begin{aligned} \sum_{i \in I} \hat{x}_i &= p \sum_{i \in I} \hat{m}_i + q \sum_{i \in I} (\hat{n}_i + r_i) \\ &= \sum_{i \in I} \hat{m}_i \\ &= v(I). \end{aligned}$$

I shall now show that $\hat{x}$ is in the core, as defined in section 2. Consider any coalition S smaller than the entire firm, and let $(y_i)_{i \in S}$ be a production plan that is feasible for S on its own, i.e., that satisfies conditions 1 and 2 of (2.1). By those conditions, together with (3.1), (3.3), and condition 2 of (2.1),

$$\begin{aligned} \sum_{i \in S} pm_i &\leq \sum_{i \in S} (p\hat{m}_i + q\hat{n}_i - qn_i) \\ &= \sum_{i \in S} (p\hat{m}_i + q\hat{n}_i + qr_i - qn_i - qr_i) \\ &= \sum_{i \in S} \hat{x}_i - q \sum_{i \in S} (n_i + r_i) \\ &\leq \sum_{i \in S} \hat{x}_i. \end{aligned} \tag{3.4}$$

Hence, $v(S)$ cannot exceed the right-hand side of (3.5).

This argument can be extended, under appropriate conditions, to cases in which the net revenue from the net output of market commodities is a concave function of the net outputs of those commodities.

4 Four Elementary Propositions about the Core

From (2.3) and (2.4) it is clear that the core is defined entirely in terms of the function v, which assigns a real number to each nonempty subset of players. The function v is usually called the characteristic function of the game. Thus, two firms that have different "economies" (as in section 1) but the same characteristic function will have the same core.

In this section I present four elementary but useful propositions about the core, which will be applied in subsequent sections. Throughout this section I make no assumption about the characteristic function, v, other than those made explicit in the statements of the propositions.

4.1 Strategically Equivalent Games

PROPOSITION 4.1 If $c = (c_i)$ is a vector with coordinates corresponding to the players i in I, and if $\tilde{v}$ and v are two characteristic functions such that, for each coalition S (a nonempty set of players),

$$\tilde{v}(S) = v(S) + \sum_{i \in S} c_i,$$

then x is in the core of v if and only if $\tilde{x} = x + c$ is in the core of $\tilde{v}$.

4.2 A Bound on Individual Imputations in the Core

PROPOSITION 4.2 If a vector x is in the core, then for each player i

$$x_i \leq v(I) - v(I \backslash \{i\}).$$

(Recall that I denotes the set of all players.)

4.3 Necessary Players

In light of proposition 4.1, there is no loss of generality in making the convention that, for each player i, $v(i) = 0$, where I use the abbreviation $v(i)$ for $v(\{i\})$. Suppose that, in addition, each set of players can get at least zero, i.e., the function v is non-negative. A player j will be called *necessary* if no set of players that does not contain j can get more than zero. The next proposition states that, under these conditions, an imputation that gives everything to a necessary player is in the core.

PROPOSITION 4.3 With the convention that, for every player i, $v(i) = 0$, suppose that, for each coalition S,

1. $0 \leq v(S) \leq v(I)$,
2. if player j is not in S, then $v(S) = 0$.

Then the core contains the vector x^* defined by

$$x_j^* = v(I),$$

$$x_i^* = 0 \quad \text{if } i \neq j.$$

Proof If j is in S, then

$$\sum_{i \in S} x_i^* = x_j^* = v(I) \geq v(S).$$

If j is not in S, then

$$\sum_{i \in S} x_i^* = 0 = v(S),$$

which completes the proof.

Note that there can be other imputations in the core, and in particular there can be more than one necessary player.

4.4 Superfluous Players

A player is called *superfluous* if deleting him from the set of all players, I, does not reduce the value of the characteristic function. Again, make the convention that $v(i) = 0$ for all i.

PROPOSITION 4.4 If $v(I) = v(I \backslash \{j\})$, then for any x in the core, $x_j = 0$.

Proof If $x_j > 0$, then

$$\sum_{i \neq j} x_i < \sum_{i \in I} x_j = v(I) = v(I \backslash \{j\}),$$

so that x cannot be in the core. On the other hand, $v(i) = 0$ implies that $x_i \geq 0$ if x is in the core.

5 One Supplier and Several Customers

In this section I consider a firm with an economy as described in section 1, but with the following special structure: one division, the "supplier," produces nonmarket commodities for the other divisions, the "customers." The customers may get inputs from the market, too, but they produce only market commodities. The supplier gets its inputs from the market, produces nonmarket commodities for customers in the firm, and may also produce market commodities. Although this structure is special, I make no special assumptions (such as convexity) about the production possibilities.

Using the elementary propositions of section 4, I show that, under very general conditions, the following imputation of profit is in the core: the supplier gets all of the "surplus" that it generates within the firm. In particular, if a customer can get from the outside market inputs that are comparable to those it gets from the supplier, then the transfer price corresponding to this core imputation may be as high as the market price.

In general, the above imputation will not be the only one in the core; there will typically be others in which the customers share some of the surplus. For some special cases I examine the upper bounds on the shares of the surplus that the customers can receive, or correspondingly, lower bounds on the transfer prices that they can pay the supplier. For example, if the supplier's cost function is linear, with a fixed cost, and a constant marginal cost for each commodity, then (for an imputation in the core) the customers must pay transfer prices that are at least as large as the corresponding marginal costs, and some of them must pay strictly larger transfer prices (unless, of course, the supplier is subsidized by the headquarters, a possibility that is not envisioned in the present model).

Let player 0 denote the supplier, and let the customers be indexed by $i \geq 1$. As before, I denotes the set of all the players. Denote by z_i the vector of nonmarket commodities supplied by 0 to i, and let z denote the vector of the total quantities of nonmarket commodities supplied by 0 to the customers, i.e.,

$$z = \sum_{i \geq 1} z_i.$$

The supply vector z is constrained to be in some set, say Z, that represents what is technically feasible for the supplier (without regard to cost). Thus far, nothing is assumed about Z except that it is some subset of the nonnegative part of the vector space whose dimension equals the number of nonmarket commodities. Let $F_0(z)$ denote the profit of the supplier (before transfers from other divisions) if it provides the vector z to the customers. The function F_0 reflects any opportunities the supplier has to produce market goods, as well as the cost of producing z. If the supplier has any capacity limitations, these will also be reflected in F_0.

For example, let $R_0(z')$ be the supplier's revenue from selling z' to the outside market, and let $C_0(z + z')$ be the supplier's cost of production.

Then

$$F_0(z) = \max_{z'} [R_0(z') - C_0(z + z')]. \tag{5.1}$$

In particular, if the supplier cannot (or may not) sell its output to the market, the $z' = 0$ and $F_0(z) = -C_0(z)$.

I now consider the customers. For $i \geq 1$, let $F_i(z_i)$ denote customer i's "gross" profit if it gets input z_i from the supplier. Here "gross" means before making any transfer payments to the supplier. F_i reflects any opportunities the customer has for buying inputs on outside markets. For example, suppose that i buys s_i on the market,[7] at cost $C_i(s_i)$, and suppose that from its total inputs $(z_i + s_i)$ it produces outputs that earn it a total revenue of $R_i(z_i + s_i)$. Then

$$F_i(z_i) = \max_{s_i} [R_i(z_i + s_i) - C_i(s_i)]. \tag{5.2}$$

Note that $F_i(0)$ is i's maximum profit if it buys all its inputs on the outside market.

Incidentally, in the notation of section 1,

$$m_0 = F_0(z),$$

$$n_0 = z,$$

$$Y_0 = \{(m_0, n_0) | m_0 = F_0(n_0), n_0 \geq 0\}.$$

For $i \geq 1$,

$$m_i = F_i(z_i),$$

$$n_i = -z_i,$$

$$Y_i = \{(m_i, n_i) | m_i = F_i(-n_i), n_i \geq 0\}.$$

A (feasible) *plan* for the firm is a specification

$$\zeta = (z_i)_{i \geq 1}$$

of the commodity z_i $(i \geq 1)$ supplied by 0 to the several customers i, such that

$$z = \sum_{i \geq 1} z_i \text{ is in } Z.$$

The profit to the firm from a plan ζ is

$$\pi(\zeta) = F_0(z) + \sum_{i \geq 1} F_i(z_i).$$

Similarly, if S is a coalition, and the supplier is in S, then a *plan*

$$\zeta_S = (z_i), \quad i \geq 1, \quad i \text{ in } S$$

is feasible for S if

$$\sum_{\substack{i \in S \\ i \geq 1}} z_i \text{ is in } Z.$$

On the other hand, if the supplier is *not* in S then all the customers in S must buy their inputs on the outside market.

I can now specify the characteristic function for this game. If the supplier is in S, then

$$v(S) = \max \left\{ F_0(z) + \sum_{\substack{i \geq 1 \\ i \in S}} F_i(z_i) | z_j = 0 \text{ for } j \notin S, z \in Z \right\}, \tag{5.3}$$

where z denotes, as above, the sum of the vectors z_i. If the supplier is not in S, then

$$v(S) = \sum_{\substack{i \geq 1 \\ i \in S}} F_i(0). \tag{5.4}$$

I assume that the maxima in (5.1) and (5.2) are attained. In particular, $v(I)$ is the maximum profit of the firm. Let $\hat{\zeta} = (\hat{z}_i)$ denote a plan that attains $v(I)$, and let $\hat{z}$ denote the corresponding vector of the total commodities supplied by 0 to the customers.

The first result of this section states that there is an imputation in the core such that the supplier gets all of the surplus due to its presence in the firm!

THEOREM 5.1 If Z contains the vector 0, then the core contains the profit imputation $x^* = (x_i^*)$ defined by

$$x_0^* = v(I) - \sum_{i \geq 1} F_i(0),$$

$$x_i^* = F_i(0), \quad i \geq 1.$$

Proof Use propositions 4.1 and 4.3.

Corresponding to the core imputation x^* in theorem 5.1 is a set of transfer payments from the customers to the supplier:

$$t_i^* = F_i(z_i) - F_i(0), \quad i \geq 1. \tag{6.5}$$

In general, there will be imputations in the core other than x^*. Bounds on what a player can obtain in a core imputation are provided by proposition 4.2.

EXAMPLE 5.1 This example,[8] although not particularly realistic, will be useful in illustrating the preceding ideas. It will also be used in section 6 to construct an example in which the core is empty.

Suppose that there are three customers and one supplier. Each customer can obtain either 1 unit or 0 units from the supplier, and nothing from the outside. The supplier's cost is c_k if it supplies a total of k units ($k = 0, 1, 2$, or 3). Thus, in the preceding notation,

$$F_0(k) = -c_k.$$

The profit to a customer is b if he gets one unit of the input, and 0 otherwise. Thus, for $i \geq 1$,

$$F_i(1) = b > 0,$$

$$F_i(0) = 0.$$

Assume that

$$0 = c_0 < c_1 < c_2 < c_3,$$

$$c_1 > b, \quad c_2 < 2b, \quad c_3 < 3b,$$

$$c_3 - c_2 > b.$$

Then

$$v(0,i,j) = 2b - c_2, \quad \text{if } i \neq j, i \text{ and } j \geq 1,$$

$$v(0,1,2,3) = 3b - c_3,$$

$$v(S) = 0, \quad \text{otherwise.}$$

If x is in the core, then

$$\sum_0^3 x_i = 3b - c_3 = v(I),$$

so that

$$x_3 = 3b - c_3 - x_0 - x_1 - x_2$$
$$\leq 3b - c_3 - 2b + c_2$$
$$= b - (c_3 - c_2).$$

The same inequality is satisfied by x_1 and x_2, so that

$$\sum_1^3 x_i \leq 3b - 3(c_3 - c_2).$$

Also,

$$\sum_1^3 x_i = 3b - c_3 - x_0.$$

Hence, given $x_0 \geq 0$, there is an x in the core giving x_0 to 0 if and only if

$$3b - 3(c_3 - c_2) \geq 3b - c_3 - x_0,$$

or

$$\frac{c_2}{2} \geq \frac{c_3}{3} - \frac{x_0}{6}. \tag{5.6}$$

In particular, if the above inequality is satisfied, then $(x_0, b - t, b - t, b - t)$ is in the core with the transfer price t,

$$t = \frac{c_3 + x_0}{3}, \tag{5.7}$$

provided that $(b - t)$ is non-negative, i.e., that

$$x_0 \leq 3b - c_3. \tag{5.8}$$

Hence there is no x in the core with $x_0 = 0$ if

$$\frac{c_2}{2} < \frac{c_3}{3}. \tag{5.9}$$

EXAMPLE 5.2 Suppose that there is no outside market for the supplier's output, and that the supplier's cost function is linear, with a positive fixed cost, a, and a vector c of constant, positive marginal costs. In other words, in the notation of (5.1),

$$F_0(z) = -(a + cz),$$

where cz denotes the inner product ("dot product") of c and z. Suppose further that each customer can obtain corresponding inputs on the outside at constant external prices e (a vector), with each coordinate of e strictly greater than the corresponding coordinate of c, or[9] $e \gg c$. Let $\hat{z}_i$ maximize $[R(z_i) - cz_i]$, and assume that $\hat{z}_i > 0$. Also, let $\hat{z}$ (as usual) be the sum of the vectors $\hat{z}_i$, and assume that, with the plan $(\hat{z}_i)$ the firm's profit is positive, i.e.,

$$\hat{\pi} = \sum_{i \geq 1} R_i(\hat{z}_i) - a - c\hat{z} > 0. \tag{5.10}$$

It follows that (z_i) is an optimal plan for the firm, and so

$$v(I) = \hat{\pi}. \tag{5.11}$$

On the other hand, let $\tilde{s}_i$ maximize $[R_i(s_i) - es_i]$, so that in the notation of (5.2)

$$F_i(0) = R_i(\tilde{s}_i) - e\tilde{s}_i. \tag{5.12}$$

Under the firm's optimal plan, customer i's gross profit (before transfers) is

$$F_i(\hat{z}_i) = R_i(\hat{z}_i). \tag{5.13}$$

Hence, the core imputation x^* of theorem 5.1 corresponds to the following transfers from customers to the supplier:

$$t_i^* = R_i(\hat{z}_i) - R_i(\tilde{s}_i) + e\tilde{s}_i. \tag{5.14}$$

Suppose that R_i is strictly increasing, *strictly concave* (at least in the relevant region), and differentiable, and that $\tilde{s}_i$ is an interior point of the feasible set. It follows that

$$\tilde{s}_i < \hat{z}_i,$$
$$R_i(\tilde{s}_i) > R_i(\hat{z}_i),$$
$$\nabla R_i(\tilde{s}_i) = e,$$

where ∇R_i denotes the gradient of R_i. Hence,

$$R_i(\hat{z}_i) - R_i(\hat{s}_i) < \nabla R_i(\tilde{s}_i)(\hat{z}_i - \tilde{s}_i),$$
$$= e(\hat{z}_i - \tilde{s}_i),$$

so that

$$t_i^* < e(\hat{z}_i - \tilde{s}_i) + e\tilde{s}_i = e\hat{z}_i. \tag{5.15}$$

In other words, the transfer price under x^* is strictly less than the external market price. (It is, of course, trivial that the transfer price corresponding to a core imputation cannot exceed the external-market price in this example.)

The transfer t_i^* is, of course, an upper bound on what a customer would pay the supplier in a core imputation. To get a lower bound, one uses proposition 4.2. I consider here only the case in which, without any one customer i, it is still profitable for the firm to supply internally all of the remaining customers' demands; i.e., for each i,

$$\sum_{j \neq i} [R_j(\hat{z}_j) - c\hat{z}_j] - a > 0. \tag{5.16}$$

In this case, proposition 4.2 implies that for an imputation in the core, for $i \geq 1$,

$$x_i \leq v(I) - v(I \backslash \{i\})$$
$$= R_i(\hat{z}_i) - c\hat{z}_i.$$

The corresponding bound on the transfer is

$$t_i \geq c\hat{z}_i. \tag{5.17}$$

This corresponds to a lower bound on the transfer price (vector) of c. Notice, however, that not every customer can pay this lower bound in a core imputation, since then the supplier would make a loss. In other words, (5.17) cannot be simultaneously binding for all customers. On the other hand, for every customer i there is a core imputation for which (5.17) is binding.

These conclusions must be modified if condition (5.16) is not satisfied. In such cases there is at least one customer j such that it is not profitable for the supplier to meet the demands of the remaining customers. Such a customer is a "necessary" player in the language of section 4, and hence there is a core imputation that gives him all of the surplus. In this case the supplier is still a necessary player, too, so it is still true that there is also a core imputation that gives the supplier all of the surplus.

6 Several Suppliers and Customers

6.1 Competition among Suppliers

An extreme case of "competition" among internal suppliers is the one in which two or more suppliers produce identical nonmarket commodities for the use of customers in the firm. If the average costs of these suppliers is declining and then increasing in the relevant range of output, but the optimal plan requires that all of the output (of this nonmarket commodity) be produced by a single supplier, then the core may be empty. The following example illustrates this phenomenon.

EXAMPLE 6.1 Consider the situation of example 5.1, but suppose that there are two suppliers instead of one, with identical cost functions. Recall that there are three potential customers, each of which can use either one unit or none. If a customer uses one unit, his benefit is $b > 0$; otherwise it is 0. The cost for a single supplier of producing k units ($k = 0, \ldots, 3$) is $C(k)$, where

$$\begin{aligned} &0 = C(0) < C(1) < C(2) < C(3), \\ &C(1) > b, \\ &C(k) < kb \text{ for } k \geq 2, \\ &kb - C(k) \text{ is increasing in } k \text{ for } 0 \leq k \leq 3. \end{aligned} \tag{6.1}$$

Note that the last line of (6.1) is equivalent to

$$C(k+1) - C(k) < b \text{ for } 0 \leq k \leq 2. \tag{6.2}$$

It follows from (6.1) that each customer should get one unit, and that one supplier should produce all three units and the other none. The characteristic function is

$$v(S) = \begin{cases} 2b - C(2) & \text{if } S \text{ has 2 customers and at least 1 supplier;} \\ 3b - C(3) & \text{if } S \text{ has 3 customers and at least 1 supplier;} \\ 0 & \text{otherwise.} \end{cases}$$

Each supplier is a "superfluous player" in the sense of subsection 4.4;

hence, for any core imputation, each supplier receives 0. Let x_k denote the imputation of profit to customer k ($k = 1, 2, 3$). For a core imputation,

$$\begin{aligned} x_1 + x_2 &\geq 2b - C(2),\\ x_1 + x_3 &\geq 2b - C(2),\\ x_2 + x_3 &\geq 2b - C(2),\\ x_1 + x_2 + x_3 &= 3b - C(3). \end{aligned} \tag{6.4}$$

Hence, for each customer k,

$$\begin{aligned} x_i &\leq [3b - C(3)] - [2b - C(2)]\\ &= b - [C(3) - C(2)]. \end{aligned}$$

Hence,

$$\sum_1^3 x_k \leq 3b - 3[C(3) - C(2)]. \tag{6.5}$$

Comparing (6.5) with the last line of (6.4), we see that the core will be empty if the right-hand side of (6.5) is strictly less than $3b - C(3)$, or if

$$\frac{C(2)}{2} < \frac{C(3)}{3}. \tag{6.6}$$

On the other hand, if the inequality in (6.6) is reversed (weakly), then the imputation

$$x_k = b - \frac{C(3)}{3}, \quad k = 1, 2, 3 \tag{6.7}$$

in which the three customers share the surplus equally, is in the core. (This is essentially the same analysis as was used in example 5.1 to determine when there was an imputation in the core that gave the single supplier 0.)

It is clear that examples of this kind can be constructed with any number of suppliers and customers, and that the commodities produced by the several suppliers need not be perfect substitutes.

On the other hand, it is well known that if average cost is declining throughout the relevant range, and other conditions are satisfied, then the core will not be empty, as the following example illustrates.

EXAMPLE 6.2 Let there again be two identical suppliers, each with cost function C. Let K denote the set of customers, and suppose that customer k has a linear profit function

$$F_k(q) = b_k q, \quad 0 \leq q \leq Q_k$$

where q is a real number and $b_k > 0$. Let $Q \equiv \sum Q_k$, and make the following assumptions:

A1 For $0 < q \leq Q$, $C(q)$ is positive, differentiable, and increasing; average cost $C(q)/q$ is minimized at $q = Q$; $C(0) = 0$.

A2 For every customer k, and for $0 < q \leq Q$, $C'(q) < b_k$.

A3 $\sum_k b_k Q_k - C(Q) > 0$.

These assumptions imply that the total profit is maximized when a single supplier produces the total input Q and each customer k gets input Q_k; the maximum profit is

$$\hat{\pi} = \sum_k b_k Q_k - C(Q).$$

Let a denote the average cost $C(Q)/Q$. It is easy to check that the following imputation is in the core: each supplier gets 0, and each customer k is charged the unit price equal to a, i.e., k receives the imputation $(b_k - a)Q_k$. In fact, we need only check that, for every subset S of customers in K,

$$\sum_S (b_k - a)Q_k \geq \left(\sum_S b_k Q_k\right) - C\left(\sum_S Q_k\right),$$

since the assumptions imply that the right-hand side is the value of the characteristic function for a set consisting of S together with one or both suppliers. But the last inequality is equivalent to

$$a = C(Q)/Q \leq C\left(\sum_S Q_k\right) \Big/ \sum_S Q_k,$$

which is true by assumption A1.

6.2 Complementary Suppliers

At the opposite pole from the case of competing suppliers is the case in which the several suppliers produce nonmarket commodities that are "strict complements." As we shall see, this case is very much like the case of a single supplier; in fact, in a very precise sense it will correspond to the case of a single supplier resulting from a "merger" of the several complementary suppliers.

I shall use a model that is close to that of section 5, but with several suppliers. Let J denote the set of suppliers and K the set of customers; J and K are disjoint, and I is the union of J and K. Let $z_{jk} \geq 0$ denote the vector of nonmarket commodities supplied by j to k, and define

$$z_{j.} \equiv \sum_k z_{jk},$$

the total supply of supplier j. The latter is constrained to be in some set Z_j; the product

$$Z \equiv \prod_j Z_j$$

corresponds to the set Z in section 5. I assume that each set Z_j contains the vector 0.

Denote by z_k the vector of nonmarket inputs that customer k receives;

$$z_k = (z_{jk})_{j \in J}.$$

We can define the profit function, F_k, for customer k, as in section 5 (e.g., as in (5.2)). We can also define the profit function, say G_j, for supplier j, in the same way that F_0 is defined in section 5 (e.g., as in (5.1)). Let ξ denote a plan, $((z_{jk}))$.

The *assumption of strict complementarity* (SC) states that, if the nonmarket input from any supplier to any customer is 0, then the maximum profit for that customer is the same as it would be if the nonmarket inputs from all the suppliers were 0. Formally,

(1) For every j in J and k in K, if $z_{jk} = 0$ then $F_k(z_k) = F_k(0)$.

I also add to SC the following conditions:

(2) For all j in J and k in K, all $z_k \geq 0$ and $z_{j.} \geq 0$ in Z_j,

$$F_k(z_k) \geq F_k(0),$$

$$G_j(z_{j.}) \leq G_j(0),$$

$$G_j(0) \geq 0.$$

For any situation defined as above, define the corresponding *one-supplier game* by replacing the set of suppliers with a single supplier whose production possibility set is Z and whose profit function is

$$F_0(z) = \sum_{j \in J} G_j(z_{j.}). \tag{6.8}$$

Thus, the corresponding one-supplier game fits the model of section 5.

The main result for complementary suppliers is as follows.

THEOREM Under the assumption of strict complementarity (SC), if $[x_0^*, (x_k^*)_{k \in K}]$ is in the core of the one-supplier game, and if

$$\sum_{j \in J} x_j^* = x_0^*,$$

where

$$x_j^* \geq G_j(0) \text{ for all } j \text{ in } J,$$

then $(x_i^*)_{i \in I}$ is in the core of the several-supplier game.

Proof First, observe that if $S \leq I$ and S does not contain every supplier, then

$$v(S) = \sum_{j \in S \cap J} G_j(0) + \sum_{k \in S \cap K} F_k(0). \tag{6.9}$$

To see this, observe that if S does not contain every supplier, then by SC the inputs of all other suppliers are worthless, so the maximum profit of customer k is $F_k(0)$. Since no customer needs any positive nonmarket inputs, the maximum profit of any supplier j in S is $G_j(0)$. These two observations imply (6.9). Second, observe that if $[x_0^*, (x_k^*)_{k \in K}]$ is in the core of the one-supplier game, then for all customers k in K

$$x_k^* \geq F_k(0) = v(\{k\}). \tag{6.10}$$

We must now consider two cases. In the first case, not every supplier is in S; then

$$v(S) = \sum_{\substack{j \in S \\ j \in J}} G_j(0) + \sum_{\substack{k \in S \\ k \in K}} F_k(0).$$

Hence,

$$\sum_{i \in S} x_i^* \geq v(S). \tag{6.11}$$

In the second case, every supplier is in S ($J \subset S$). Let $\bar{v}$ denote the characteristic function of the one-supplier game. Then

$$v(S) = \bar{v}(\{0\} \cup (K \cap S)).$$

According to section 5, $\bar{v}(T)$ is the maximum of

$$F_0(z) + \sum_{k \in T} F_k(z_k) \tag{6.12}$$

subject to the constraint that, for some feasible plan ξ with $z_k = 0$ for all k not in T,

$$\sum_k z_k = z.$$

By (6.8), the maximum of (6.12) is also $v(S)$ where $S = J \cup T$; hence (6.11) is satisfied in the second case, too, which completes the proof.

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Notes

1. See Scarf 1986 and Sharkey 1982. Moulin (1987) gives a very brief survey and additional references. The topic is closely related to "cost allocation." I do not here consider issues of fairness or justice; see, e.g., Zajac 1978.

2. A precise statement of this result is beyond the scope of the present paper; "outcomes" of a bargaining mechanism are assumed to be determined by Bayes-Nash equilibria of the incomplete-information game that is induced by the mechanism (see Myerson and Satterthwaite 1983).

3. See Radner 1986 for a precise statement. The Groves mechanism in question will specify a noncooperative "revelation game" in which, for each player, it is a dominant strategy to tell the truth. The corresponding outcome maximizes the firm's profit and is individually rational for both players. To achieve the latter, the headquarters may have to "subsidize" the profit centers.

4. Recall that in the model the prices of market commodities are exogenous and fixed. If the price of a market commodity were 0, then it would not enter the profit calculations, although the availability of the commodity would be reflected in the description of the production possibility sets.

5. In all the particular models investigated in this paper, the maxima in (2.1) will be attainable.

6. In fact, to obtain the desired results, some further regularity conditions must be imposed; see any standard treatment of duality theory in nonlinear programming.

7. Note that s_i and z_i are vectors in the same space. Thus, if a particular input is "computing services," part may be purchased on the market and part obtained internally.

8. This example is adapted from Zajac 1978. However, it is probably older.

9. For vectors u and w, I shall write $u \geq w$ if each coordinate of u is as large as the corresponding coordinate of w, $u > w$ if $u \geq w$ but $u \neq w$, and $u \gg w$ if each coordinate of u is strictly greater than the corresponding coordinate of w.

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