Capital, Investment and Development

Essays in Memory of Sukhamoy Chakravarty

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A note on the theory of cost–benefit analysis in the small

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1 Introduction and summary

Cost–benefit analysis is a familiar technique for determining whether a proposed project is economically desirable. In this chapter I shall provide a theoretical justification of this technique in the case in which the project is "small" in a precise sense that will be defined below and certain "regularity" conditions are satisfied. Roughly speaking, the latter conditions will require that market equilibria vary "smoothly" with the data of the problem.

For the purposes of this chapter, a *project* determines a change in the total supply of goods and services – commodities – available for consumption, currently and/or in the future. A project is a *potential Pareto improvement* if the new supply can be allocated among the consumers in such a way as to make everyone better off – or at least someone better off and no one worse off.

Let z^h denote the change in the supply of commodity h that is determined by the project, and let $z = (z^h)$ be the vector with coordinates z^h. Positive coordinates correspond to net outputs, and negative ones to net inputs. The vector z is sometimes called the "net-output vector" for the project, and for our purposes we may identify z with the project.

A common test to determine whether a project z is a potential improvement involves the following steps.

1 Determine a "correct" set of prices p^h for those commodities for which the corresponding coordinates of z are not zero, i.e. for those commodities that actually appear as inputs and/or outputs of the project. For those inputs or outputs that occur in the future, the corresponding prices are to be interpreted as *discounted* prices, with the discounting done according to some "correct" rates of interest.
2 Calculate the total "net benefit," $p \cdot z = \Sigma_h p^h z^h$, for the project z. Accept the project if the total net benefit is positive. If the project involves future inputs and/or outputs, then the total net benefit is also called the *present value*. The positive terms $p^h z^h$ are the (discounted) *benefits* of the project, and the negative terms are the (discounted) *costs*. (In the formula for total net benefit, the prices and quantities of *all* the commodities appear, but for those commodities that do not appear as inputs or outputs in the project, i.e. those commodities h for which z^h is zero, it clearly does not matter how the prices are determined.)

In this chapter I shall consider the situation in which the initial supply of commodities has been allocated according to a market equilibrium that is competitive on the consumers' side, and where (p^h) are the equilibrium prices. (In other words, the initial supply is given by some mechanism, the consumers are price-takers, and the prices are market-clearing.) I shall also assume that the consumers have "textbook preferences" (this will be made precise in subsequent sections).

It is well known that, even in this situation, a project that passes the net-benefit test need not be a potential Pareto improvement.[1] This problem is illustrated in figure 6.1. There are two commodities, corresponding to the two axes in the figure. The point x represents the initial total supply. The curve C represents the *community indifference curve*,[2] namely, the set of total supplies that can be efficiently allocated in such a way as to make every consumer exactly as well off as in the initial equilibrium allocation. The relative prices corresponding to the initial equilibrium are determined by the slope of the straight line L that is tangent to C at the point x. Points that are above and to the right of C represent total supplies that are potentially Pareto superior to x. On the other hand, points that are above and to the right of L have a higher value than x does at the initial equilibrium prices (p^h). In the northwest quadrant of the

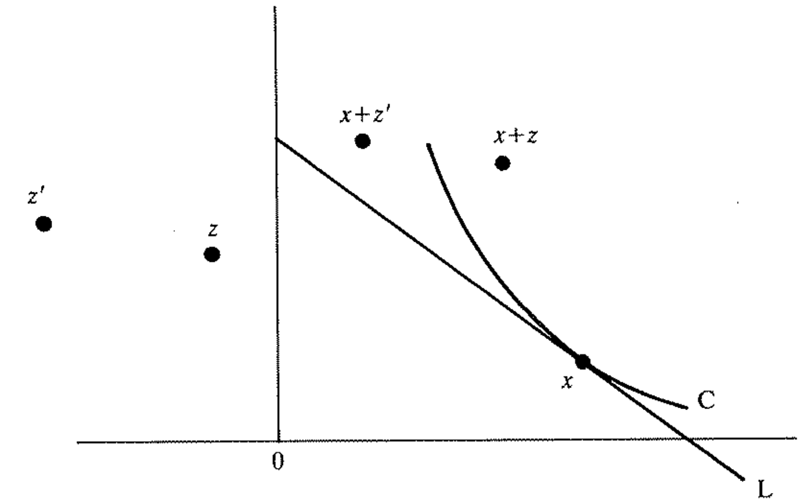

Figure 6.1 The net-benefit test in the large does not always work.

figure, the points z and z' represent two different projects, both of which use commodity 1 as a net input and produce commodity 2 as a net output. Corresponding to z, we have the new total supply $x + z$ in the northeast quadrant, and corresponding to z' is the new total supply $x + z'$. We see that both z and z' have a positive net benefit, whereas only $x + z$ is potentially Pareto superior to x.

To deal with this problem, let us expand the concept of a project. A "project" usually can be built at different scales. Let $\zeta(s)$ denote the corresponding net-output vector if the "project" is built at scale s, where $0 \leq s \leq S$; of course, $\zeta(0) = 0$. In figure 6.2, the broken curve in the northwest quadrant is traced out by $\zeta(s)$ as s varies from 0 to S, whereas the broken curve in the northeast quadrant is traced out by $x + \zeta(s)$. Suppose that the coordinates of $\zeta(s)$ vary "smoothly" with s, and let $\zeta'(0)$ denote the vector of derivatives of the coordinates of $\zeta(s)$ with respect to s, evaluated at $s = 0$. In figure 6.2, $\zeta'(0)$ is represented by the arrow emanating from the point x and tangent to the curve $x + \zeta(s)$ at x. Since $\zeta'(0)$ points upward and to the right of the line L, we see that $p\zeta'(0) > 0$. Furthermore, we see that for some *entire interval of positive scale values s, the new supply $x + \zeta(s)$ is potentially superior to the initial supply x.*

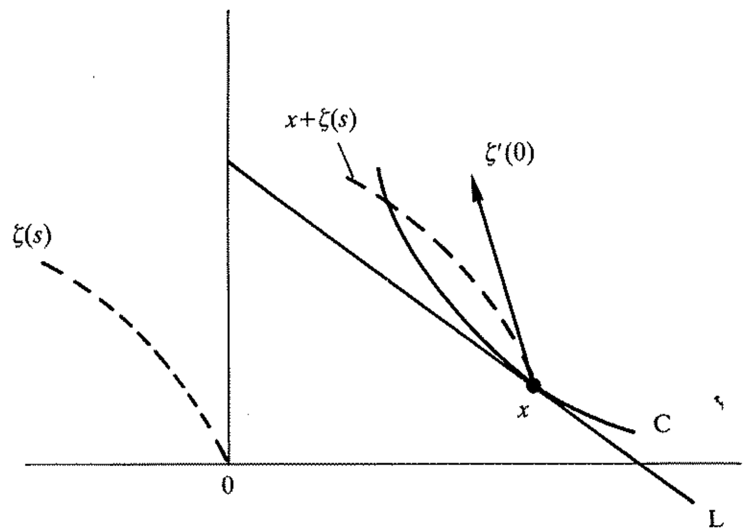

Figure 6.2 The net-benefit test in the small.

Figure 6.2 suggests the following reformulation of the net-benefit test. A *project* is a function ζ from an interval $[0, S]$ to the commodity space; $\zeta(s)$ represents the net-output vector of the project if it is adopted at scale s. We shall say that the project passes the net-benefit test if $p\zeta'(0) > 0$. The main result of this chapter is that, under certain regularity conditions that will be described below, if a project passes the net-benefit test, then at some positive scale s^* the net-output vector $\zeta(s)$ is a potential Pareto improvement for all s such that $0 < s \leq s^*$. The actual argument will be analytical, rather than geometrical as above. This will enable me to elucidate conditions under which an allocation of the new supply is an actual – not just potential – improvement, and also to make precise the related idea of "consumer surplus in the small."

Sections 2 and 3 set out the model and show how to calculate the changes in consumers' welfares that result from a new allocation. Section 4 states and proves the main result. Section 5 shows how changes in wealth (income) can be used to realize a potentially superior improvement, and elucidates the concept of "consumer surplus in the small."

The main result makes use of quite strong "smoothness" assumptions. No attempt has been made here to justify these assumptions in terms of more primitive assumptions about the consumers' preferences and consumption sets. The tools for such a justification would be found in the book by Mas-Colell (1985). Nor have I attempted to relate this chapter to the (now vast) literature on cost–benefit analysis, beyond providing a few citations in the notes.

In fact, the immediate origins of this chapter were lecture notes for a course in microeconomic theory that I gave at the University of California, Berkeley, during the 1960s and 1970s. They were part of my attempts to illustrate how the abstract (and, to the students, often boring) constructs of utility and demand theory might be applied to practical problems, a goal which seems particularly appropriate to pursue in a volume dedicated to the memory of Sukhamoy Chakravarty.

2 Allocations and projects

Consider an economy with I consumers, and a vector x of quantities of H commodities allocated among the m consumers. Call x the *total supply*. An *allocation* of x is an I-tuple (x_i) such that $\Sigma_i x_i = x$. Given an allocation (x_i) of x, suppose that it is proposed to replace x by x', another vector of total supplies. We are interested in determining whether or not the proposed total supply x' represents an "improvement." The difference $z \equiv x' - x$ may be interpreted as a *project*; typically, some of the coordinates of z will be positive and some will be negative.[3]

The proposed total supply x' will be called *potentially superior* to the allocation (x_i) if there is an allocation of x' that is Pareto superior to the allocation (x_i); equivalently, in this case we shall call the project z a *potential improvement* with respect to the allocation (x_i). (Recall that an allocation (x_i') is Pareto superior to an allocation (x_i) if, for every i, x_i' is at least as good as x^i for consumer i and, for at least one i, consumer i strictly prefers x_i' to x_i.)

As noted in section 1, instead of a single project we shall consider a one-parameter family of vectors $\zeta(s)$, where s is interpreted as the "scale" of the project and $x' = x + \zeta(s)$ is the new total supply if the project is adopted at scale s. Here s is a nonnegative real number; I shall assume that $\zeta(s)$ is defined for s in some interval $[0, S]$, that the coordinates ζ^h of ζ are continuously differentiable functions on that interval, and that $\zeta(0) = 0$. Let $\zeta'(s)$ denote the vector of derivatives of the coordinates of ζ with respect to s; then $\zeta'(0)$ will be called the *direction of change of the total supply* at $s = 0$.

In order to formulate the net-benefit test precisely, I introduce the concept of a valuation equilibrium. An allocation is a valuation equilibrium with respect to a price vector p if, for each consumer i, any consumption vector that i strictly prefers to his own allocation would cost more at the given prices. Suppose now that the initial allocation (x_i) of the initial supply x is a valuation equilibrium with respect to p. For any two vectors $y = (y^h)$ and $z = (z^h)$ let $y \cdot z$ denote, as usual, the inner product

$$\sum_h y^h z^h$$

I shall say that the project $\zeta(.)$ passes the net benefit test if

$$p \cdot \zeta'(0) > 0 \tag{6.1}$$

In what follows I shall show that, under certain "regularity" conditions, *if ζ passes the net-benefit test, then there is some positive scale s^* such that $\zeta(s)$ is a potential Pareto improvement for all s such that* $0 < s < s^*$.

3 Changes in consumer utility

Let us represent the preferences of consumer i by a numerical utility function, say u_i. It is to be understood that all consumption vectors are nonnegative. I shall assume that each consumer's utility function is concave and continuously differentiable. (Further regularity assumptions will be made as appropriate.) I shall also make the assumption of nonsatiation for each consumer i, i.e. for each consumer i and each consumption vector x_i, there is another consumption vector that consumer i strictly prefers to x_i.

Using the notation that has just been introduced, an allocation (x_i^*) is a valuation equilibrium with respect to a price vector p if, for every consumer i and every nonnegative consumption vector x_i, $p \cdot x_i \leqslant p \cdot x_i^*$ implies $u_i(x_i) \leqslant u_i(x_i^*)$.

Suppose that (x_i) is an allocation of the initial supply x and is a valuation equilibrium with respect to the price vector p. For each i,

let $W_i = p \cdot x_i$ denote the *wealth* of consumer i. The consumption x_i can be characterized as maximizing consumer i's utility subject to the "budget constraint" that the cost of consumption not exceed his wealth. I shall assume that the valuation equilibrium satisfies, for each i, the familiar first order ("marginal") conditions for such a constrained maximum: there is a number m_i such that[4]

$$D_h u_i(x_i) = m_i p^h \qquad h = 1, \ldots, H \tag{6.2}$$

$$\sum_h p^h x_i^h = W_i \tag{6.3}$$

The assumption of nonsatiation implies that, for each i, $m_i > 0$.

As a consequence of the prevailing economic and social policy, if the project were adopted at scale s, then the new supply $x + \zeta(s)$ would be allocated in some way which we may write (without loss of generality) in the form $[x_i + \zeta_i(s)]$, where

$$\sum_i \zeta_i(s) = \zeta(s) \tag{6.4}$$

Consumer i's utility at this new allocation would then be

$$V_i(s) \equiv u_i[x_i + \zeta_i(s)] \tag{6.5}$$

Assume now that, for each i, ζ_i is continuously differentiable, and that $\zeta_i(0) = 0$. Then the derivative of consumer i's utility with respect to the scale of the project is, from (6.5),

$$V_i'(s) = \sum_h D_h u_i[x_i + \zeta_i(s)] D\zeta_i^h(s) \tag{6.6}$$

Setting $s = 0$ in (6.6), and using (6.2), we get

$$\begin{aligned} V_i'(0) &= \sum_h D_h u_i(x_i) D\zeta_i^h(0) \\ &= m_i \sum_h p^h D\zeta_i^h(0) \\ &= m_i p \cdot \zeta_i'(0). \end{aligned} \tag{6.7}$$

If we use the suggestive notation

$$\mathrm{d}u_i \equiv V_i'(0) \qquad \mathrm{d}x_i \equiv \zeta_i'(0) \tag{6.8}$$

then (6.7) can be rewritten as

$$\mathrm{d}u_i = m_i p \cdot \mathrm{d}x_i \tag{6.9}$$

4 The net-benefit test

I shall now prove the main result concerning the validity of the net-benefit test. Recall the definition of a *competitive equilibrium* in the case of pure exchange. For each i, let ε_i be i's *endowment vector* (in $\mathbb{R}^H$), and let $\varepsilon \equiv \Sigma_i \varepsilon_i$. A competitive equilibrium (CE) relative to (ε_i) is an $(I + 1)$-tuple $[(\alpha_i), \varphi]$ such that

1 (α_i) is an allocation of ε;
2 (α_i) is a valuation equilibrium with respect to φ;
3 for each i, $\varphi \cdot \alpha_i \leqslant \varphi \cdot \varepsilon_i$.

Let (k_i) be H strictly positive numbers summing to 1, and for each scale s define (with a slight abuse of the previous notation)

$$\varepsilon_i(s) = x_i + k_i \zeta(s) \tag{6.10}$$

where – as in sections 2 and 3 – (x_i) is the initial allocation and $\zeta(s)$ is the project at scale s. For each s let $[\alpha_i(s), \varphi(s)]$ be a CE relative to $[\varepsilon_i(s)]$. Assume that φ and (α_i) are continuously differentiable, with $\varphi(0) = p$ and $\alpha_i(0) = x_i$ for all i. I shall show that, *if* $p \cdot \zeta'(0) > 0$, *then there is an* $s^* > 0$ *such that the allocation* $[\alpha_i(s)]$ *is strictly Pareto superior to* (x_i) *for all* s *such that* $0 < s < s^*$.

Define $\zeta_i(s) \equiv \alpha_i(s) - x_i$. By (6.7) it is sufficient to show that, for each i, $p \cdot \zeta_i'(0) > 0$. By the assumption of nonsatiation, for each i and s,

$$\varphi(s) \cdot \alpha_i(s) = \varphi(s) \cdot \varepsilon_i(s)$$

$$\varphi(s)[x_i + \zeta_i(s)] = \varphi(s) \cdot [x_i + k_i \zeta(s)]$$

$$\varphi(s) \cdot \zeta_i(s) = k_i \varphi(s) \cdot \zeta(s)$$

Differentiating both sides of this last equation with respect to s and setting $s = 0$ we get

$$\varphi'(0) \cdot \zeta_i(0) + \varphi(0) \cdot \zeta_i'(0) = k_i[\varphi'(0)\zeta(0) + \varphi(0)\zeta'(0)]$$

But $\zeta_i(0) = \zeta(0) = 0$ and $\varphi(0) = p$, so that

$$p \cdot \zeta_i'(0) = k_i p \cdot \zeta'(0) > 0$$

5 Wealth, prices, and consumer surplus in the small

In section 4, the achievement of a Pareto-superior allocation was accomplished by giving each consumer his initial allocation plus a share of the *physical* net output of the project, and then letting the consumers trade in a corresponding competitive pure exchange equilibrium. In many (if not most) applications, it will be more natural to suppose that the net benefits of the project are distributed in the form of *monetary wealth*, followed by a corresponding change in equilibrium prices and consumptions (allocations). In this section, using standard demand theory, I shall characterize the directions of change in the wealth distribution that are consistent with both a Pareto improvement and an equilibrium of supply and demand.[5]

As in section 2, we start with an allocation (x_i) of an initial supply x that is a valuation equilibrium with respect to a price vector p. Consumer i's initial "monetary wealth" is $W_i \equiv p \cdot x_i$. We suppose further that if the project is adopted at scale s, then consumer i would dispose of a monetary wealth $\omega_i(s)$, where $\omega_i(0) = W_i$. Let ξ_i denote consumer i's demand function, i.e. if his wealth were $\tilde{W}_i$ and prices were $\tilde{p} = (\tilde{p}^h)$, then his demand (vector) would be $\xi_i(\tilde{W}_i, \tilde{p})$. Accordingly, if the project were adopted at scale s, the equilibrium price vector $\varphi(s)$ would be determined by the equation

$$\sum_i \xi_i[\omega_i(s), \varphi(s)] = x + \zeta(s) \tag{6.11}$$

This actually represents H equations in the H unknowns $\varphi^h(s)$.

If the individual demand functions are sufficiently "well-behaved," then we can change the wealth functions $\omega_i(.)$ arbitrarily – at least in some neighborhood of the original functions – and still get solutions to (6.11), thus obtaining a variety of different allocations of the net output of the project.

Returning to the case in which the wealth functions are fixed, at scale s consumer i will receive, in equilibrium, the allocation

$$\alpha_i(s) \equiv \xi_i[\omega_i(s), \varphi(s)] \tag{6.12}$$

and his resulting utility will be

$$V_i(s) \equiv u_i[\alpha_i(s)] \tag{6.13}$$

We wish to study the first derivatives of the functions V_i in the neighborhood of $s = 0$. To this end, we write down the first-order conditions that determine the demand functions at each scale s, corresponding to (6.2): for each i there is a positive number $\mu_i[\omega_i(s), \varphi(s)]$ such that

$$D_h u_i[\alpha_i(s)] = \mu_i[\omega_i(s), \varphi(s)]\varphi^h(s) \qquad h = 1, \ldots, H \tag{6.14}$$

Also, corresponding to the budget constraint (6.3) we have

$$\varphi(s) \cdot \alpha_i(s) = \omega_i(s) \tag{6.15}$$

Assume that the functions ζ_i and μ_i are continuously differentiable, as well as the functions ζ, ω_i, and φ (and hence α_i); in particular,

$$\begin{aligned} \zeta(0) &= 0 \qquad \omega_i(0) = W_i \\ \alpha_i(0) &= x_i \qquad \varphi(0) = p \\ \mu_i(W_i, p) &= m_i \end{aligned} \tag{6.16}$$

Differentiating (6.13) with respect to s:

$$V_i'(s) = \sum_h D_h u_i[\alpha_i(s)] D\alpha_i^h(s)$$

By (6.14),

$$\begin{aligned} V_i'(s) &= \mu_i[\omega_i(s), \varphi(s)] \sum_h \varphi^h(s) D\alpha_i^h(s) \\ &= \mu_i[\omega_i(s), \varphi(s)]\varphi(s)\cdot\alpha_i'(s) \end{aligned} \tag{6.17}$$

By (6.15),

$$\begin{aligned} \omega_i'(s) &= \frac{d}{ds}[\varphi(s)\cdot\alpha_i(s)] \\ &= \varphi'(s)\cdot\alpha_i(s) + \varphi(s)\cdot\alpha_i'(s) \end{aligned}$$

$$\varphi(s)\cdot\alpha_i'(s) = \omega_i'(s) - \varphi'(s)\cdot\alpha_i(s) \tag{6.18}$$

Hence, from (6.17),

$$V_i'(s) = \mu_i[\omega_i(s), \varphi(s)][\omega_i'(s) - \varphi'(s)\cdot\alpha_i(s)] \tag{6.19}$$

Setting $s = 0$ and recalling (6.16) we have

$$V_i'(0) = m_i[\omega_i'(0) - \phi'(0)\cdot x_i] \tag{6.20}$$

If we use the suggestive notation

$$du_i \equiv V_i'(0) \qquad dW_i \equiv \omega_i'(0) \qquad dp \equiv \varphi'(0) \tag{6.21}$$

then (6.20) can be rewritten as

$$du_i = m_i(dW_i - x_i\cdot dp) \tag{6.22}$$

Recall the standard result of demand theory that m_i is the "marginal utility of wealth" for consumer i at his original allocation. That is, define

$$U_i(W_i, p) = u_i[\xi_i(W_i, p)]$$

to be the maximum utility in the optimization problem corresponding to (6.2)–(6.3); then

$$m_i = \frac{\partial}{\partial W_i} U_i(W_i, p) \tag{6.23}$$

(The function U_i is the consumer's *indirect utility function*.) It follows from (6.22) that, for small s, consumer i's change in utility will be approximately $sm_i\, d\tilde{W}_i$, where

$$d\tilde{W}_i \equiv dW_i - x_i\cdot dp \tag{6.24}$$

Thus $sd\tilde{W}_i$ is the change in the amount of i's wealth that would lead approximately to a change sdu_i in utility, provided there were no concomitant change in prices. We are therefore justified in calling $d\tilde{W}_i$ consumer i's consumer surplus in the small corresponding to the project ζ.

Let $dx_i \equiv \alpha_i'(0)$; from (6.22) we see that

$$\frac{du_i}{m_i} = d\tilde{W}_i \equiv dW_i - x_i dp = p \cdot dx_i \tag{6.25}$$

Hence

$$\sum_i \tilde{W}_i = \sum_i \frac{du_i}{m_i} = p \cdot dx \qquad dx \equiv \zeta'(0) \tag{6.26}$$

We see from (6.26) that, if we can achieve a Pareto improvement at small scale s with some new wealth distribution (and the corresponding equilibrium allocation), then the total net benefit $p \cdot dx$, which is the same as the total surplus, must be strictly positive. Conversely, if the total net benefit is strictly positive, then as was shown indirectly in section 4 there is some distribution of wealth that makes every consumer's surplus $d\tilde{W}_i$ positive, i.e. results in a Pareto improvement. There will, in general, be many such wealth distributions.

Equation (6.24) tells us what information will enable us to estimate a given consumer's surplus generated by a small project: his preproject consumption, and the changes in his wealth and in prices induced by the adoption of the project. The changes in prices can possibly be estimated from knowledge of aggregate demand functions, but the other data are about individual consumers. One typically tries to overcome this last problem by dividing the consumers into a small number of groups (e.g. by wealth, location, etc.), in the hope that wealth changes and demands will be approximately constant within the groups.

Notes

I am grateful for comments by M. K. Majumdar and A. Mas-Colell on an earlier draft. The views expressed here are those of the author, and not necessarily those of AT&T Bell Laboratories.

1 See Chipman and Moore (1973). Material on cost–benefit analysis can be found in Little and Mirrlees (1969) and in Drèze and Stern (1987).

2 See Debreu (1959, p. 97, note 1). This concept of the community indifference curve is apparently due to Scitovsky (1942).

3 The present paper deals only with projects whose inputs and outputs are consumer goods and services (including labor). Many projects (such as an irrigation system or a steel mill) produce only outputs that are inputs into further production. The present analysis can be extended to such situations but considerable additional care is needed to ensure that the prices used reflect the actual rates of transformation in production.

4 I shall use the following notation for partial derivatives. If f is a function of H variables, say $z_1, \ldots, z_H$, then $D_h f(z)$ will denote the partial derivative of f with respect to the hth variable, evaluated at the particular point $z = (z_h)$. If f is a function of one variable, then the derivative will be denoted by Df or f'.

5 For a related discussion, see Milleron (1970).

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