GAMES AND ECONOMIC BEHAVIOR **12,** 245–265 (1996)
ARTICLE NO. 0016

The Design and Performance of Sharing Rules for a Partnership in Continuous Time*

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Received June 8, 1989

We study repeated partnerships with imperfect monitoring and risk neutrality. The interval between the partners' decisions, the delay, is given but can be arbitrarily small. Each stage-game's output is Gaussian, with mean and variance depending on the partners' actions, making the sequence of outcomes a discretization of a diffusion. A sharing rule is efficient if there is an equilibrium of the corresponding game whose outcomes are Pareto efficient; it is stable if these equilibria approach a limit as the delay approaches zero. We characterize partnerships for which there exist stable, efficient sharing rules, and describe the corresponding equilibria. *Journal of Economic Literature* Classification Numbers: C73, D2, D82. © 1996 Academic Press, Inc.

1. INTRODUCTION AND SUMMARY

A *partnership* is an agreement between a group of players (partners) to jointly produce some output and to distribute the outcome according to some mutually accepted and enforceable sharing rule. Each partner derives utility only from the share of outcome he receives and disutility from his effort. If each player cannot observe the strategies used by the other players,

* The views presented here are those of the authors, and are not necessarily those of AT&T Bell Laboratories. The authors are indebted to an anonymous referee for useful suggestions.

0899-8256/96 $12.00

but only some signal (like the output) correlated with the actions of the partners, then we have *imperfect monitoring*. In general, the output itself will not be given in a deterministic way by the actions of the partners; rather the actions of the partners determine the distribution of the random variable output. A description of a one-shot partnership game is given in Radner (1986). Here we shall be concerned with a repeated partnership game. In particular, the game will be set in continuous time, but defined as the limit of discrete time games, as the length of the intervals tends to zero.

Partnerships differ, among other things, by the length of time that is necessary for each partner to adjust his effort according to the flow of information he is receiving. This length of time may vary because of technical conditions (it may not be possible to change the level of effort instantaneously) or because the flow of information is not continuous. The importance of such differences in the reaction time and information delays is discussed and analyzed in the work of Abreu *et al.* (1991), which is relevant for many of the points discussed below. Such differences may of course be expected to influence the set of Nash equilibria of the repeated partnership, and in particular the efficiency of equilibrium outcomes.

It is known from the work of Radner *et al.* (1986) that if players discount future utility then, under certain conditions, every equilibrium of the repeated partnership game has an outcome in the utility space that is bounded away from efficiency, uniformly in the discount rate, even with general sharing rules. This contrasts with ''Folk-theorem'' results for games with perfect monitoring, and with the result of Radner (1986) that there exist efficient equilibria if players do not discount future utility (see Fudenberg *et al.*, 1989).

On the other hand, the efficiency of equilibria can be improved in partnership games if players are risk neutral and are allowed to design sharing rules in which the fractional shares vary with the outcome. The role of such ''flexible'' sharing rules is discussed in Williams and Radner (1994).

This paper will formulate the partnership game in a framework that allows the analysis of the role played by intervals between decisions of different length, including the limiting case of instantaneous adjustment. The outcome is a stochastic process of Wiener type: as the reaction time tends to zero this process tends to the solution of a stochastic differential equation. It may be objected that in real-world partnerships the reaction time and the flow of information are always, for practical purposes, different from zero, so partners cannot adjust instantaneously all the time. In fact, the same objection may be raised against any model of a dynamic game in continuous time. On the other hand, a fixed lower bound on the interval between decisions seems difficult to justify, and is particularly unsatisfactory if the equilibrium set depends critically on its value. The question therefore arises of characterizing the stability of the properties of the set of equilibria and of the strategies when the decision interval becomes arbitrarily small. In other words, the analysis of a continuous time model may be considered as a way of testing the robustness of the results for a discrete time model; the study of the limit situation should clarify which properties of a discrete time model depend critically on the fixed delay in the reaction of the players when that delay is small.

The first question we analyze is the characterization of *efficient sharing rules,* i.e., sharing rules that have the efficient outcome as a (Nash) equilibrium. This question was first examined in Williams and Radner (1994), where the generic existence of efficient sharing rules was demonstrated in a class of partnership models. Like Williams and Radner, we assume that the partners are risk-neutral. For our model, we provide a complete characterization of the partnerships for which the design of efficient sharing rules is possible, and a characterization of such rules. This characterization is particularly simple to state in the case where the outcome is a random variable with a normal distribution. In such case, the condition requires that at least two of the partners are different enough, in the sense that the variance and the mean of the outcome respond differently to the efforts of these two partners. It is interesting to note that the preferences of the players (in our case, the disutility of the effort) play no role. We then present a general procedure to design such sharing rules. From this very construction, it will be apparent that the set of efficient sharing rules is very large.

We then examine the problem of the existence of a limit for the optimization problem of each partner, as the interval between decisions tends to zero. The existence of such a limit is important from the point of view of the robustness of the equilibrium, as the repeated games become (in the limit) a continuous time game. In fact, the existence of such a limit is a necessary condition for the concept of equilibrium to be well defined. We prove that it is always possible to construct sharing rules that are both efficient and stable (with respect to this limit process). Indeed, very simple sharing rules can be formed; quadratic functions are enough.

1.1. *A Formal Description of the Model*

As we mentioned in the Introduction, we propose to study the partnership as a continuous time game. But we shall construct and analyze the continuous time model as the limit of a sequence of discrete time games, as the length of the time interval h tends to zero.

Most of the work will therefore be concentrated on the study of a family of one-shot games indexed by h. We construct the sharing rules for each discrete game; but the main technical difficulty will be to show that the sharing rules can be defined also in the limit as h tends to zero, so that rules for the continuous time game are defined.

There are m partners, indexed by an integer j, producing only one output, denoted y. An *action profile* is a vector of actions $(a_1, \ldots, a_m)$ of the partners. We denote as usual $a_{-j} = (a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_m)$ and $(a_j, a_{-j}) = (a_1, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_m)$. The *action space* for each partner, i.e., the set of possible effects, is the real line.

We first describe the one-shot game. At the beginning of each period each partner chooses a rate of effort, a_j, that he keeps constant for that interval. The disutility for such effort is $Q_j(a_j)h$. The disutility function Q_j is a strictly convex continuously twice-differentiable function defined on the real line.

Given an action profile, a, the total output at the end of the period is a random variable Y, normally distributed with mean $\alpha a h$ and variance $\tau(a)^2 h$. Here αa denotes the inner product of the two vectors α and a. α is a vector in $\mathbb{R}^m$, and τ is assumed to be a (continuously twice-differentiable) function of the action profile. Note that the mean is linear in the action profile. This is mostly used to simplify the computation.

We denote the density of the distribution of Y, given a, as

$$g^h(y \mid a) \equiv \frac{1}{\sqrt{2\pi}\tau h^{1/2}} \exp\left[-\frac{1}{2}\left(\frac{y - \alpha a h}{\tau h^{1/2}}\right)^2\right]. \tag{1.1}$$

We refer to the pair (α, τ) as the *output function* parameters.

The choice of a normal distribution as describing the randomness of the system reduces the set of possible different assumptions about the output function; the partners' actions can influence the mean and the variance of the process. Such simplification may be justified in light of one of the results of the paper to the effect that if partners can affect the variance then any action profile that is individually rational can be sustained as an equilibrium: so that the effect on moments higher than the second can be considered redundant.

The realized final output y for the period is common knowledge for every partner. Given y, each partner j receives a share $s_j^h(y)$ of the output.

The partnership works in isolation with no free disposal, so the sharing functions $\{s_j^h : j = 1, \ldots, m\}$ must satisfy the budget constraint:

$$\sum_{j}{}_1^m s_j^h(y) = y \qquad \text{for every } y \in \mathbb{R}. \tag{1.2}$$

We denote the expectation of the random variable s_j^h, given the action profile a, by $E(s_j^h \mid a)$. The utility for each partner for the period is the difference between the share of the output he receives and the disutility of his effort; the partners are therefore risk neutral, i.e.,

$$s_j^h(y) - Q_j(a_j)h.$$

The infinite horizon game is now defined as usual. The total final utility for each partner is the discounted sum of the utilities for each period, with discount factor $\rho e^{-\rho h}$. The information each player (partner) receives at the end of each period is the realization of the output. As usual, a strategy for each player j is a sequence of measurable functions $\{\sigma_j^h(k)\}_{k=1}^{\infty}$, where

$$\sigma_j^h(1)\colon \mathbb{R} \to \mathbb{R}; \qquad \text{and} \qquad \sigma_j^h(k+1)\colon \mathbb{R}^k \to \mathbb{R}$$

for $j = 1, \ldots, m$.

A Nash equilibrium is an m-tuple of strategies such that no partner can increase his expected payoff by unilaterally changing his strategy. The activity of the partnership lasts for an infinite sequence of periods.

The m-tuple of strategies for each game with $h > 0$ determines a solution of the stochastic differential equation

$$dY(t) = \alpha a\, dt + \tau(a)\, dW(t), \qquad Y(0) = 0 \text{ a.s.},$$

where W is a standard Brownian motion (see Fleming and Rishel (1975), in particular pages 106–127, for details).

When each of the partners different from j chooses for each period the same action profile a_{-j}, then the jth partner faces the simple maximization problem

$$\max\left\{\sum_{k}{}_0^{\infty} [E(s_j^h \,|\, (a_j^k, a_{-j})) - Q_j(a_j^k)h]\rho e_j^{-\rho kh} : \alpha^k, 1 \le k < \infty\right\}, \tag{1.3}$$

where a_j^k is the action taken by the jth partner in the kth period. A particular solution of the above problem consists of the infinite repetition of the solution to the one-period problem:

$$\max_{a_j \in \mathbb{R}} E(s_j^h \,|\, (a_j, a_{-j})) - Q_j(a_j)h. \tag{1.4}$$

Consider now any $h > 0$. A sequence of actions $\{(a^k)_{k=1}^{\infty}\}$, $a^k = (a_1^k, \ldots, a_m^k)$ will be called efficient for the partnership if it is Pareto optimal. A particular efficient sequence is easily found. Consider the problem

$$\max_{a=(a_1,\ldots,a_m)} \left\{ E(Y|a) - \sum_j{}_1^m Q_j(a_j)h \right\}. \tag{1.5}$$

We assume that it has a solution, $\hat{a} = (\hat{a}_1, \ldots, \hat{a}_m)$, characterized as the solution of the system:

$$\frac{\partial Q_j}{\partial a_j}(\hat{a}_j) = \alpha_j, j = 1, \ldots, m. \tag{1.6}$$

Clearly the sequence $\{(\hat{a}^k)\}_{k=1}^{\infty}$, $\hat{a}^k \equiv \hat{a}$ for every k, is efficient.

We now proceed to discuss the questions and answers presented in this paper.

First, in the next section we consider the general case with no a priori restrictions on the sharing functions and on the dependence of the variance on the actions of the partners. In particular, we consider the following question. Let $\hat{a} = (\hat{a}_1, \ldots, \hat{a}_m)$ be a given vector of efforts. Can we devise a family of sharing functions $\{(s_1^h, \ldots, s_m^h); h > 0\}$ with the following properties:

1. For each h and for each partner j, if the other partners produce in each period the effort vector $\hat{a}_{-j}$, then the best choice for player j is exactly $\hat{a}_j$; that is,

$$\arg\max_{a_j} E(s_j^h | (a_j, \hat{a}_{-j})) - Q_j(a_j)h = \hat{a}_j. \tag{1.7}$$

2. For such a family of sharing functions, as h tends to zero, an "instantaneous expected utility" for each player is defined; that is,

$$\lim_{h \downarrow 0} \frac{1}{h}(E(s_j^h | a) - Q_j(a)h) \tag{1.8}$$

exists and is finite.

The condition (1.8) above is necessary if we want to define a game and an equilibrium in the continuous time case as a limit of the discrete time games with time unit h. If the limit described above exists, then it can be interpreted as the expected instantaneous payoff, in the continuous time game, for any action chosen by the players.

The vector of efforts $\hat{a}$ in which we are mostly interested is the vector of Pareto optimal efforts, but the arguments we present extend with minor modifications to any other vector of efforts. The first conclusion of this section is that a necessary and sufficient condition for the existence of a family of sharing rules which satisfy the first-order conditions and the existence of a finite limit is that the partners are different enough, in a sense that is precisely defined below. We then examine conditions under which such conditions are also sufficient.

We remark that in the case of constant variance the condition that the partners are different is not satisfied.

2. The Design of the Sharing Rules

2.1. *Design of the Sharing Rules: The First Order Conditions*

From the assumptions on Q_j and the distribution of Y we have that the function of a_j given by $\int_{\mathbb{R}} s_j^h(y) g_j^h(y \mid (a_j, a_{-j}))\, dy - Q_j(a_j)h$ is twice continuously differentiable, for every measurable function s_j^h, and that it has an interior maximum.

Therefore the following are necessary conditions for Nash equilibrium:

$$(F^h)\begin{cases} \dfrac{\partial}{\partial a_j}\left(\int_{\mathbb{R}} s_j^h(y) g^h(y \mid (a_j, \hat{a}_{-j}))\, dy - Q_j(a_j)h\right)\Bigg|_{a_j=\hat{a}_j} = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for every } j = 1, \ldots, m, \\ \sum_j{}_1^m s_j^h(y) = y. \end{cases}$$

We shall say that a family of functions $\{(s_j^h): h > 0, j = 1, \ldots, m\}$ determines a *limit sharing rule* if

(L) $\lim_{h \downarrow 0} \dfrac{1}{h} E(s_j^h(y \mid a))$ exists and is finite, for every action profile a of the partners.

A solution of the system F^h is a family of sharing rules: $\{(s_j^h): h > 0, j = 1, \ldots, m\}$; we refer to such a family as a *solution of the first order problem.*

We shall also simplify our notation as follows:

$$\hat{q}_j \equiv \frac{dQ_j}{da_j}(\hat{a}_j), j = 1, \ldots, m, \qquad \text{and} \qquad \hat{q} \equiv (\hat{q}_1, \ldots, \hat{q}_m),$$

$$g_j^h \equiv \frac{\partial}{\partial a_j} g^h(\cdot|(a_j, \hat{a}_{-j}))\Big|_{a_j=\hat{a}_j}, \qquad g^h = (g_1^h, \ldots, g_m^h),$$

$$f^\perp \equiv \{g : (f, g) = 0\}, \qquad \text{where } (f, g) \equiv \int_{\mathbb{R}} f(t)g(t)\, dt,$$

and

$$\|f\| \equiv (f, f)^{1/2}.$$

The system F^h can now be compactly written:

$$(F^h) \begin{cases} (s_j^h, g_j^h) = q_j h & j = 1, \ldots, m, \\ \sum_j{}_1^m s_j^h(y) = y, & \text{for all } y. \end{cases}$$

We assume $q \neq 0$ and $g^h \neq 0$. At this point we temporarily suppress the superscript h (up to the statement of Theorem 2.2).

For a finite family of continuous functions defined on the real line, $\{g_j\}_{j=1}^m$, we say they are *pairwise linearly dependent* if for every pair (i, k) there exists a real number c_{ik} such that $g_i = c_{ik} g_k$. We define now a condition on the family $\{g_j\}_{i=1}^m$ defined above.

(I) The functions $\{g_j\}_{j=1}^m$ are not pairwise linearly dependent.

Note that such a condition is satisfied if at least one of the functions is not a constant multiple of the others. Note that

$$\frac{\partial}{\partial a_j} g(y \,|\, a)|_{a=\hat{a}} = g(y \,|\, \hat{a}) p_j(y \,|\, \hat{a}) \tag{2.1}$$

with

$$p_j(y \,|\, \hat{a}) \equiv \left(\frac{y - \alpha \hat{a} h}{\hat{\tau} h^{1/2}}\right) \frac{\alpha_j h^{1/2}}{\hat{\tau}} + \frac{\hat{\tau}_j}{\hat{\tau}} \left(\left(\frac{y - \alpha \cdot \hat{a} h}{\hat{\tau} h^{1/2}}\right)^2 - 1\right), \qquad j = 1, 2, \tag{2.2}$$

where

$$\hat{\tau} = \tau(\hat{a}); \qquad \hat{\tau}_j \equiv \frac{\partial \tau}{\partial a_j}(\hat{a}).$$

Now two functions g_i and g_j are linearly dependent if and only if their ratio is a constant function, and so if and only if the ratio $[x\alpha_i h^{1/2} + \hat{\tau}_i(x^2 - 1)]/[x\alpha_j h^{1/2} + \hat{\tau}_j(x^2 - 1)]$ is a constant independent of x, which is in turn equivalent to the condition $\hat{\sigma}_i\alpha_j - \hat{\sigma}_j\alpha_i = 0$ for every i and j; so that we can state a condition I' equivalent to I:

$$(I') \quad \text{The set } \left\{\frac{\hat{\tau}_j}{\alpha_j}\right\}_{j=1}^{m} \text{ has at least two different numbers.}$$

Note that if I holds then at least one of the $\hat{\tau}_j$'s is nonzero. Whenever in the following we assume that I holds, we also agree that $\hat{\tau}_1 \neq 0$. Unless otherwise noted, all of the functions we deal with have as argument the output variable, which we have denoted above by y. To avoid repeating the symbol y, we use the notation "id" for the identity function. Thus, for example: $(\text{id}, u) \equiv \int_{\mathbb{R}} yu(y)\, dy$.

LEMMA 2.1. *The first-order system has a solution if condition I is satisfied, for any vector q. Conversely, if the vector is the efficiency vector $\hat{q}$ and F can be solved, then condition I is satisfied.*

Proof. The conclusion follows easily from the classical theorem of the alternative (Aubin and Ekeland, 1984): F has a solution if and only if the dual system F^* in the variables $(u_1, \ldots, u_m; u_{m+1})$, where $u_1, \ldots, u_m$ are scalers, $u_{m+1}: \mathbb{R} \to \mathbb{R}$,

$$(F^*) \quad \begin{cases} g_j u_j = u_{m+1}, & j = 1, \ldots, m, \\ \sum_{j}{}_1^m \hat{q}_j u_j h + (\text{id}, u_{m+1}) \neq 0, \end{cases}$$

does not have a solution. Clearly then if the condition I is satisfied then the system F^* does not have a solution, and so F has a solution. Conversely, assume now that condition I is not satisfied, and assume w.l.o.g. that, say, $q_1 \neq 0$. then define $u_{m+1} \equiv g_1$ and let u_j be the constants c_{1j} given by the definition of pairwise linear dependence. The first condition of F^* is satisfied. Also,

$$\sum_{j}{}_1^m h\hat{q}_j u_j - (\text{id}, u_{m+1}) = \sum_{j}{}_1^m (\text{id}, g_1) - (\text{id}, g_1) = (m-1)\hat{q}_1 h \neq 0$$

and so the second condition of F^* is satisfied, too. ■

For simplicity of exposition we now concentrate our attention on the case $m = 2$. The reader will note that the results of this section extend to the case $m \geq 2$. From Lemma 2.1 above we already know that condition I is sufficient to ensure the existence of a solution to F. Now we want to characterize such solution. We remark that in this section we do *not* use the fact that the density is Gaussian. The first step in solving the system F is to choose a special sharing rule for partner 1 (say).

Specifically, we let

$$s_1 \equiv \frac{\hat{q}_1}{\|g_1\|^2} g_1 h \qquad \text{for every } h > 0. \tag{2.3}$$

According to this sharing function, the first partner is paid a constant proportion of his marginal contribution. Note that

$$\frac{\partial}{\partial a_1} E(s_1|a) = (s_1, g_1)\hat{q}_1 h. \tag{2.4}$$

Now observe that for any $z \in g_1^{\perp}$ we also have $(s_1 + z, g_1) = \hat{q}_1 h$. Denote $s_1 + g_1^{\perp} \equiv \{f : f = s_1 + y, y \in g_1^{\perp}\}$. Then the problem F can be reformulated as the problem of finding a function s_2 such that:

$$(F_1) \begin{cases} s_2 \in \mathrm{id} - (s_2 + g_1^{\perp}). \\ (s_2, g_2) = \hat{q}_2 h. \end{cases}$$

Now from condition I it follows that $g_2^{\perp} \neq g_1^{\perp}$; choose therefore a z in $g_1^{\perp}$ such that $(z, g_2) = (\mathrm{id} - s_1, g_2) - \hat{q}_2$, and conclude that the F problem has the solution pair $(\tilde{s}_1, \tilde{s}_2)$, where

$$\begin{cases} \tilde{s}_1 \equiv g_1 c(h) + z, \\ \tilde{s}_2 \equiv \mathrm{id} - \tilde{s}_1, \end{cases} \tag{2.5}$$

with $c(h) \equiv \hat{q}_1 h / \|g_1\|^2$.

Conversely, it is clear that any solution of the F^h problem can be written in the form above. We have proved

THEOREM 2.2. *The set of solutions of the system F^h is described by the family* (2.5).

The choice of a family $\{s_j\}$ of sharing functions has been reduced to the choice of a family $\{z\}$, or (since we now return to use the superscript h) a family $\{z^h\}$. Recall that we do not have a complete freedom of choice for the functions $\{z^h\}$. In fact, we still have to show that we can choose functions

$\{z^h\}$ so that the limit condition (L) is also satisfied. In the form described above, the system of conditions (F^h) and (L) is difficult to treat. The next step is to reduce such a system to an equivalent one, but one of a much simpler form. This is done in Appendix 1.

We summarize here, for the convenience of the reader, what parts of the appendix are needed in the following. To ease the calculations the following change of variable is introduced:

$$w^h\left(\frac{y-\alpha\hat{a}h}{\hat{\tau}h^{1/2}}\right) \equiv z^h(y),$$

so that the functions we are looking for are the functions $\{w^h\}_{h\geq 0}$. The change of variable is a simple normalization.

The next step is to prove that the infinite-dimensional system $\{F^h\}$ is in reality a two-dimensional system (as one might expect, since the assumption of normality of the random variable output reduces the degrees of freedom to only two). Indeed, if we introduce the two quantities

$$m(w^h) \equiv \frac{\hat{\tau}^{-1}}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-x^2/2}(x^2-1)w^h(x)\,dx,$$

$$n(w^h) \equiv \frac{\hat{\tau}^{-1}}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-x^2/2}xw^h(x)\,dx,$$

then it is proved in Theorem A.3 that the original system (F^h) and (L) has a solution if and only if the following system in the unknown functions $\{w^h\}_{h\geq 0}$ has a solution

$$(S_3)\quad \begin{cases} m(w^h) = -\alpha_1\dfrac{q_1}{\Delta}\left(\dfrac{\hat{\tau}_2}{\hat{\tau}_1}+\omega(h)\right)h, \\[2ex] n(w^h) = \dfrac{q_1\hat{\tau}_2}{\Delta}h^{1/2}, \\[2ex] \lim\limits_{h\downarrow 0} hG(w^h,a). \end{cases}$$

Here Δ is a real number, G is a function of w^h given by Eq. (A.7) in the Appendix, and ω is a continuous function of h, with $\omega(0) = 0$, described in Lemma A2. The reader will find the precise definition in the appendix.

After the transformation summarized above, the task of finding a solution of the first-order problem is easy. This is done in the following theorem.

THEOREM 2.3. *Assume the condition I. A family of solutions to the first-order problem can be found by solving a system of linear equations* (*see* (2.8), (i) *and* (ii), *below*) *in two variables.*

Proof. We adopt the convention $-1! = 0! = 1$, and we recall that

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-x^2/2}\, x^k\, dx = 0 \text{ if } k \text{ is odd,} \quad \text{and} \quad = (k-1)! \text{ if } k \text{ is even.} \tag{2.6}$$

We must find functions w^h that satisfy the system S_3. We shall construct a solution in the form of a power series. There may, of course, be other solutions; in fact, we shall see that there are many solutions of the type

$$w^h(x) = h^{1/2} \sum_{j\,\text{odd}} c_j\, x^j + h \sum_{j\,\text{even}} c_j(h)\, x^j, \tag{2.7}$$

with $c_j \in \mathbb{R}$, for j odd, and c_j continuous functions, $c_j(0) = 0$, for j even.

For the function w^h in (2.7) one easily computes

$$\begin{aligned} m(w^h) &\equiv \frac{1}{\sqrt{2\pi}\,\hat{\tau}}\int_{\mathbb{R}} e^{-x^2/2}\,(x^2-1)\, w^h(x)\, dx \\ &= \sum_{j\,\text{even}} c_j(h)\,\frac{h}{\hat{\tau}}\{(j+1)\,! - (j-1)!\} \end{aligned}$$

(note that c_0 is indeterminate), and

$$n(w^h) = \frac{1}{\sqrt{2\tau}\,\hat{\tau}}\int_{\mathbb{R}} e^{-x^2/2}\, x w^h(x)\, dx = \sum_{j\,\text{odd}} c_j\,\frac{h^{1/2}}{\hat{\tau}}\, j!.$$

We have reduced the problem F^h to the search for coefficients c_j satisfying:

$$\sum_{j\,\text{even}} c_j(h)\,\frac{(j+1)! - (j-1)!}{\hat{\tau}} = -\alpha_1\,\frac{q_1}{\Delta}\left(\frac{\hat{\tau}_2}{\hat{\tau}_1} + \omega(h)\right), \tag{2.8.i}$$

$$\sum_{j\,\text{odd}} c_j\,\frac{j!}{\hat{\tau}} = \frac{q_1\,\hat{\tau}_2}{\Delta}. \tag{2.8.ii}$$

It is enough to choose the family of coefficients $\{c_j(\cdot) : j \text{ even}\}$ in the form $c_j(h) = c_j + \omega_j(h)$, with ω_j continuous and $\omega_j(0) = 0$. The series will converge if we impose the condition that the coefficients c_j are eventually zero.

For such a choice of c_j one easily checks that the condition L (existence of a limit) is also satisfied. ■

2.2. *Sufficient Conditions*

In the previous section we have characterized the solutions of the first-order problem. We now discuss if, and under which conditions, the efficient profile is also a solution of the maximization problem of each partner:

$$\max_{a_j \in \mathbb{R}} \left\{ \lim_{h \downarrow 0} \frac{1}{h} E(s_j^h \,|(a_j, \hat{a}_{-j})) - Q_j(a_j) \right\} \equiv \max_{a_j \in \mathbb{R}} V((a_j, \hat{a}_{-j}); \{s_j^h\}). \quad (2.9)$$

One can easily compute

$$\frac{\partial V}{\partial a_j}(a) = \frac{\tau_j}{\tau^2} \frac{\hat{\tau}_1}{2\pi \hat{\tau}} \frac{q_1}{C_1} \int_{\mathbb{R}} e^{-(x^2/2)((1+\hat{\tau}^2/\tau^2))} (x^2 - 1) \left(x^2 \frac{\hat{\tau}^2}{\tau^2} - 1 \right) dx \quad (2.10)$$

and

$$\frac{\partial^2 V}{\partial a_j^2}(a) = \left(\tau_{jj} - 2 \frac{\tau_j^2}{\tau} \right) f_1(\tau) + \tau_j^2 f_2(\gamma), \quad (2.11)$$

where f_1, f_2 are continuous functions that satisfy

$$f_1(\hat{\tau}) = \frac{q_1}{\hat{\tau}_1}$$

$$f_2(\hat{\tau}) = \frac{q_1}{\hat{\tau}_1} \frac{1}{2\hat{\tau}},$$

so that

$$\frac{\partial^2 V}{\partial a_j^2}(\hat{a}) = \frac{q_1}{\hat{\tau}_1} \left(\hat{\tau}_{jj} - \frac{3}{2\hat{\tau}} \tau_j^2 \right). \quad (2.12)$$

Obviously,

$$\frac{\partial^2 G}{\partial a_j^2} = -\tau_{jj} \frac{\alpha_1 \hat{\tau}_2}{\hat{\tau}_1} \frac{q_1}{\Delta}. \quad (2.13)$$

EXAMPLE. Here we assume that the variance only depends on the effort of the first partner, i.e., $\tau = \tau(a_1)$, with $\hat{\tau}_1 \neq 0$. Then condition (I) is satisfied, and we have $m(w^h) = n(w^h) = 0$. Therefore

$$E(s_1^0(y|a)) = V(a),$$
$$e(s_2^0(y|a)) = \alpha \cdot a - V(a).$$

The second-order conditions for the second partner are clearly satisfied, since

$$\frac{\partial^2 V}{\partial a_2}(\hat{a}_1, \hat{a}_2) = -Q''(\hat{a}_2) < 0. \tag{2.14}$$

For the first partner we have

$$\frac{\partial^2 V}{\partial a_1}(\hat{a}_1, \hat{a}_2) = \frac{\partial^2 V}{\partial a_1^2}(\hat{a}) - Q_1''(\hat{a}). \tag{2.15}$$

It is sufficient (for a local maximum, but for some q function this may be a global maximum) that

$$\frac{\partial^2 V}{\partial a_1}(\hat{a}) \leq 0. \tag{2.16}$$

This happens for instance if

$$\hat{\tau}_1 > 0; \qquad \hat{\tau}_{11} < 0. \tag{a}$$

The first partner, by increasing his effort rate, increases both mean and variance (this last at a decreasing rate). Note that $V(\hat{a}) < 0$, so a lump sum transfer may be necessary to convince 1 to stay in the partnership with this sharing rule. As another example, suppose

$$\hat{\tau}_1 < 0, \qquad \hat{\tau}_{11} > 0. \tag{b}$$

The first partner performs "quality control," with decreasing returns. The efficient profile of action gives a (best) maximum if the variance term $\hat{\tau}$ is large or $\hat{\tau}_1$ is small.

3. Conclusion

The previous example shows a natural case in which condition (I) is easily verified. At the same time, the first-order conditions are also sufficient to insure that, for sharing rules of the type described by Eq. (2.5), the efficient action is also the optimal choice for each partner.

Our main theorem shows that this is indeed a general situation: sharing rules that support the efficient solution as an equilibrium, and at the same time can be extended to hold in the limit game in continuous time, exist if and only if the partners are different enough. The meaning of the condition *different enough* is made precise by the two equivalent conditions (I) or (I').

In the literature on partnerships special attention has been given to a particular class of sharing rules: the ones defined by a constant proportion of the output. In an extended version of this paper we examine the performance of this class of sharing rules in the context of a partnership game as defined in the present paper, as the length of the reaciton time tends to zero. In particular, we consider the case of two identical partners, and equal splitting of the outcome. The main result in this case is that when the reaction time becomes smaller than a *fixed positive quantity* the only equilibrium of the repeated game is the equilibrium of the one-shot game. This result reveals an interesting way in which the equilibria depend on the length of the parameter h. If we formally set $h = \infty$ (i.e., the partners have to choose their level of effort once and for all at the beginning of the game), then clearly the only equilibrium is the (inefficient) equilibrium of the one-shot game. Also, we have seen that, for h small enough, again the only equilibrium is the one-shot Nash. On the other hand, for intermediate values of h, the equilibrium set is larger. When properly defined, therefore, the efficiency of the equilibria expressed as a function of the parameter h has a maximum at some finite, nonzero h.

Appendix

In this section we provide the details of the transformation of the first-order problem into the simpler form summarized before Theorem 2.3.

We shall first need some easy computational results. Note that here we are using the fact that g^h is Gaussian.

Lemma A.1. 1. *The equality*

$$\left(\mathrm{id} - \frac{\hat{q}_1 h g_1^h}{\|g_1^h\|^2} - z^h, g_2^h\right) = \hat{q}_2 h$$

holds if and only if

$$(g_1^h, g_2^h)\,\frac{\hat{q}_1 h}{\|g_1^h\|^2} = (z^h, g_2^h).$$

2. *One has*

$$\|g_1^h\|^2 = h^{-1/2}C_1 + h^{1/2}C_2$$

where C_1, C_2 *are positive real numbers.*

3. *The limit*

$$\lim_{h\downarrow 0}\frac{1}{h}\int_{\mathbb{R}} g_1^h(y\,|\,\hat{a})g^h(y\,|\,a)\,dy$$

exists and is finite, *for every choice* $a = (a_1, a_2)$.

Proof. (i) Simply note that

$$\begin{aligned}(\mathrm{id}, g_j^h) &= \frac{1}{\sqrt{2\tau}}\int_{\mathbb{R}} e^{-x^2/2}\left(x\,\frac{\alpha_j h^{1/2}}{\hat{\tau}} + (x^2-1)\,\frac{\hat{\tau}_j}{\hat{\tau}}\right)\\ &\quad\times(x\hat{\tau}h^{1/2} + \alpha\cdot\hat{a}h)\,dx\\ &= \left(-\frac{\hat{\tau}_j}{\hat{\tau}}\alpha\cdot\hat{a} + \alpha_{jj} + \frac{\hat{\tau}_j}{\hat{\tau}}\alpha\cdot\hat{\tau}\right)h = \hat{q}_j h.\end{aligned}$$

(ii) Note that

$$\begin{aligned}\|g_1^h\|^2 &= \int_{\mathbb{R}} g^h(y\,|\,\hat{a})p_1(y\,|\,\hat{\mathrm{a}})^2\,dy\\ &= \frac{1}{2\pi\hat{\tau}h^{1/2}}\int_{\mathbb{R}} e^{-x^2}\\ &\quad\times\left(\frac{\hat{\tau}_1^2}{\hat{\tau}^2}(x^2-1)^2 + 2\,\frac{\hat{\tau}_1}{\hat{\tau}_2}x\alpha_1 h^{1/2} + \frac{x^2}{\hat{\tau}^2}\alpha_1^2 h\right)dx\\ &= h^{-1/2}C_1 + h^{1/2}C_2,\end{aligned}$$

where

$$C_1 \equiv \frac{\hat{\tau}_1^2}{\hat{\tau}^3}\frac{1}{2\pi}\int_{\mathbb{R}} e^{-x^2}(x^2-1)^2\,dx = \frac{\hat{\tau}_1^2}{\hat{\tau}_3}\frac{3}{8\sqrt{\pi}}$$

$$C_2 \equiv \frac{\alpha_1^2}{\hat{\tau}^3}\frac{1}{2\pi}\int_{\mathbb{R}^3} e^{-x^3}x^2\,dx = \frac{\alpha_1^2}{\hat{\tau}^3}\frac{1}{4\sqrt{\pi}}.$$

Finally, for (iii) we note that

$$\begin{aligned}
&(1/h)\int_{\mathbb{R}} g_1^h(y\,|\,\hat{a})g^h(y\,|\,a)\,dy \\
&= \frac{1}{2\pi\tau h^{1/2}}\int_{\mathbb{R}} \exp\left\{-\frac{1}{2}\left(x^2\left(1+\frac{\hat{\tau}^2}{\tau^2}\right)\right)\right. \\
&\qquad \left. + 2\alpha(\hat{a}-a)h^{1/2}x\,\frac{\hat{\tau}}{\tau^2} + \frac{[\alpha\cdot(\hat{a}-a)]^2}{\tau^2}h\right\} \\
&\qquad \times\left(\frac{\hat{\tau}_1}{\hat{\tau}}(x^2-1) + x\,\frac{\alpha_1 h^{1/2}}{\hat{\tau}}\right)\frac{q_1}{\|g_1^h\|^2}\,dx \\
&= \frac{1}{2\pi\tau}\int_{\mathbb{R}} e^{-(x^2/2)(1+\hat{\tau}^2/\tau^2)} e^{h^{1/2}(-\alpha(\grave{a}-a)x(\hat{\tau}/\tau^2)-([\alpha\cdot(\hat{a}-a)]^2 h)/2\tau^2)} \\
&\qquad \times\frac{\hat{\tau}_1}{\hat{\tau}}(x^2-1)\frac{q_1}{C_1+C_2h}\,dx
\end{aligned}$$

(the last equality uses part (i) above). One easily sees that, as h tends to 0, this last expression tends to

$$V(a) = \frac{\hat{\tau}_1}{2\pi\tau\hat{\tau}}\frac{q_1}{C_1}\int_{\mathbb{R}} e^{-(x^2/2)(1+(\hat{\tau}^2/\tau^2))}(x^2-1)\,dx \tag{A.1}$$

(C_1 is defined above), so that

$$\lim_{h\downarrow 0}\frac{1}{h}E(s_j^h(y\,|\,a)) = V(a). \tag{A.2}$$

This concludes the proof of Lemma A.1. ■

It follows immediately from the previous lemma that

$$\lim_{h\downarrow 0}\frac{1}{h}E(\tilde{s}_1^h(y\,|\,a)) = \lim_{h\downarrow 0}\frac{1}{h}\int_{\mathbb{R}}[g_1^h(y\,|\,\hat{a})c(h) + z_1^h](g^h(y\,|\,a))\,dy$$
$$= V(a) + \lim_{h\downarrow 0}\frac{1}{h}(z_1^h, g^h(\cdot;a))$$

and

$$\lim_{h\downarrow 0}\frac{1}{h}E(\tilde{s}_2^h(y\,|\,a)) = \lim_{h\downarrow 0}\frac{1}{h}\int_{\mathbb{R}}(\mathrm{id} - g_1^h(y\,|\,\hat{a})\,c(h) - z_1^h)(g^h(y\,|\,a))\,dy$$
$$\alpha\,a - V(a) - \lim_{h\downarrow 0}\frac{1}{h}(z_1^h, g^h(y\,|\,a)),$$

whenever the limits exist. It is easily checked with a direct computation that the derivative of the limit function $V(a)$ satisfies

$$\frac{\partial V}{\partial a_1}(\hat{a}) = \hat{q}_1. \tag{A.3}$$

Also, one has

$$V(\hat{a}) = -\frac{1}{4\sqrt{\pi}}\frac{\hat{q}_1}{C_1}\frac{\hat{\tau}_1}{\hat{\tau}^2}. \tag{A.4}$$

The second lemma we need is

LEMMA A.2.

$$(g_1^h, g_2^h)\,\|g_1^h\|^{-2} = \frac{\hat{\tau}_2}{\hat{\tau}_1} + \omega(h),$$

where ω is a continuous function of h and $\omega(0) = 0$.

Proof.

$$
\begin{aligned}
(g_1^h, g_2^h) &= \frac{1}{2\pi\hat{\tau}^2 h}\int_{\mathbb{R}} e^{-((y-\alpha\cdot\hat{a}h)/\hat{\tau}h^{1/2})^2} p_1^h(y\,|\,\hat{a}) p_2(y\,|\,\hat{a})\, dy \\
&= h^{-1/2}\frac{\hat{\tau}_1\hat{\tau}_2}{\hat{\tau}^3}\frac{1}{2\pi}\int_{\mathbb{R}} e^{-x^2}(x^2-1)^2\, dx \\
&\quad + h^{1/2}\frac{\alpha_1\alpha_2}{\hat{\tau}^3}\frac{1}{2\pi}\int_{\mathbb{R}} e^{-x^2}x^2\, dx \\
&= h^{-1/2}\frac{\hat{\tau}_1\hat{\tau}_2}{\hat{\tau}^3}\frac{3}{8\sqrt{\pi}} + h^{1/2}\frac{\alpha_1\alpha_2}{\hat{\tau}^3}\frac{1}{4\sqrt{\pi}}.
\end{aligned}
$$

The claim now follows using Lemma A.1.ii. This concludes the proof of Lemma A.2. ■

THEOREM A.3. *Assume condition I. Then the set of solutions of the first-order problem is the family of functions* w^h, *where* $z^h(y) = w^h\,((y - \alpha\cdot\hat{a}h)/\hat{\tau}h^{1/2})$ *and* w^h *solves the system* S_3 *below.*

Proof. Recall the functions $\{z^h\}_{h\geq 0}$ have to satisfy the system:

$$
S_1 \begin{cases}
(z^h, g_2^h) = hq_1\left(\dfrac{\hat{\tau}_2}{\hat{\tau}_1} + \omega(h)\right) & \text{for } h > 0 \\
(z^h, g_1^h) = 0 & \text{for } h > 0 \\
\lim\limits_{h\downarrow 0}\dfrac{1}{h}(z^h, g^h(\cdot\,|\,a)) \text{ exists and is finite.}
\end{cases}
$$

It is convenient to introduce a change of variable

$$
w^h\left(\frac{y - \alpha\cdot\hat{a}h}{\hat{\tau}h^{1/2}}\right) \equiv z^h(y), \tag{A.5}
$$

so that we can rewrite

$$
\begin{aligned}
(z^h, g_j^h) &= \frac{\hat{\tau}_j}{\hat{\tau}}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-x^2/2}(x^2-1)w^h(x)\, dx \\
&\quad + \frac{\alpha_j}{\hat{\tau}}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-x^2/2}xw^h(x)\, dx\, h^{1/2},
\end{aligned} \tag{A.6}
$$

and

$$
\begin{aligned}
(z^h, g^h(\cdot \mid a)) &= \frac{\hat{\tau}}{\tau}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-(1/2)(x+(\alpha\cdot(\hat{a}-a)/\hat{\tau}))h^{1/2})^2}(\hat{\tau}^2/\tau^2)w^h(x)\,dx \\
&\equiv G(w^h, a).
\end{aligned}
\tag{A.7}
$$

If we now define

$$
\frac{\hat{\tau}^{-1}}{\sqrt{2\pi}}\int_{\mathbb{R}}^{-x^2/2}(x^2-1)w^h(x)\,dx \equiv m(w^h)
$$

$$
\frac{\hat{\tau}^{-1}}{\sqrt{2\pi}}\int_{\mathbb{R}}^{-x^2/2}xw^h(x)\,dx \equiv n(w^h)
$$

then we can rewrite S_1 as

$$
(S_2)\quad \begin{cases}
m(w^h)\hat{\tau}_1 + n(w^h)h^{1/2}\alpha_1 = 0 & \text{for } h > 0 \\
m(w^h)\hat{\tau}_2 + n(w^n)h^{1/2}\alpha_2 = q_1\left(\dfrac{\hat{\tau}_2}{\hat{\tau}_1} + \omega(h)\right)h & \text{for } h > 0 \\
\lim\limits_{h\to 0}\dfrac{1}{h}G(w_h, a) \text{ exists and is finite.}
\end{cases}
$$

Now, thanks to the (I) condition, we can solve the linear part of the above system in the two variables $m(w^h)$ and $n(w^h)$, and obtain therefore the following equivalent formulation:

$$
(S_3)\quad \begin{cases}
m(w^h) = -\alpha_1\dfrac{q_1}{\Delta}\left(\dfrac{\hat{\tau}_2}{\hat{\tau}} + \omega(h)\right)h & \text{for } h > 0 \\
n(w^h)h^{1/2} = \dfrac{q_1\hat{\tau}_2}{\Delta}h & \text{for } h > 0 \\
\lim\limits_{h\downarrow 0}\dfrac{1}{h}G(w_h, a) \text{ exists and is finite.}
\end{cases}
$$

We have reduced our original problem S_1 to the (equivalent) one of determining a suitable family of functions $\{(w^h)\colon h > 0\}$, which satisfies S_3 above.

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