III. Paths of Economic Growth that are Optimal with Regard only to Final States: A Turnpike Theorem[1]

1. *Introduction*[2]

In this paper I consider the problem of determining best paths of economic growth, when the citerion of preference among paths focuses entirely upon the final state. This problem is considered within the framework of a model of a closed economy with constant returns to scale, of the type proposed by von Neumann in his paper on balanced growth equilibrium [5]. The main result (section 5) is that under certain conditions all best growth paths must be " close " to the von Neumann path of balanced growth, except possibly for a finite number of periods, which number is independent of the length of the path. Within the framework of this model of the closed economy, the two most restrictive conditions are (1) that the preference function on the final states be homogeneous, and (2) that the von Neumann balanced growth path be the unique profit maximizing direction of growth under the von Neumann equilibrium prices.

One source of interest in such results may be the general problem of planning economic growth. In particular, they point to the possibility of planning the direction of growth over a fairly long period without precise specification of a social preference ordering of final states.

Although in the optimization problems discussed here the criterion of preference is focused upon the final states, this does not imply that social wants during intermediate states are necessarily ignored. Human services constitute an important group of " commodities," and consumption needs can be expressed in a limited way in terms of the technology of the production of services.

The problem and results considered in this paper were suggested by certain problems of efficient capital accumulation discussed by Dorfman, Samuelson and Solow in [1]. The problem I pose is, however, somewhat more general than theirs in certain directions, and the methods of proof are different.

I believe that this problem area has come to be known under the general heading of " The Turnpike Theorem," the " turnpike " in this case being the von Neumann path.[3]

[1] This research was supported by the Office of Naval Research under contract Nonr-222(77), project NR 049 029 with the University of California. Reproduction in whole or in part is permitted for any purpose of the United States government.

[2] I wish to thank G. Debreu, F. Hahn, and R. Solow for helpful comments and criticisms.

[3] I am indebted to a recent talk by Professor J. R. Hicks for my own introduction to this topic.

No attempt will be made here, however, to trace the history of this topic, nor of the general topic of balanced growth. The model of production and balanced growth used here is essentially due to Gale [2]. I have followed the excellent treatment of Karlin in chapter 9 of [3], to which the reader is also referred for references to other work. A recent paper by Malinvaud [4] should also be mentioned.

Section 2 below sets out the model of production used here, section 3 the class of preference functions for final states, and section 4 the concepts of prices, interest, profit, and balanced growth equilibrium. The " turnpike theorem " is presented and proved in the final section.

2. *Production*

I will deal with the so-called " closed model of production," which is somewhat of a generalization of the model used by von Neumann in his well-known paper [5] on balanced growth (for a discussion of the general closed model, with references, see, for example, Karlin [3], section 9.10).

There is a fixed list of commodities, numbered from 1 to M. Production takes place during each of a sequence of periods. At the beginning of each period, some vector x of nonnegative quantities of the various commodities is used up as an *input*, and at the end of the period a vector y is produced as an *output*, this output being available for use as input for production in the next period. The vectors x and y are of course each in M-dimensional Euclidean space, the nonnegative part of which will be called the commodity space. The set of technologically possible pairs (x, y) is assumed to be the same for every period, and is denoted by T.

I will assume throughout this paper:

*A*1. *T is a closed cone in the nonnegative orthant of* 2*M-dimensional Euclidean space.*

*A*2. *If* $(0, y)$ *is in T, then* $y = 0$.

The interpretation of $A1$ is that the technology exhibits continuity and constant returns to scale. The interpretation of $A2$ is that it is impossible to produce something from nothing.

The following definitions will be useful:

*D*1. x is balanced if there is a real number $\rho > 0$ such that $(x, \rho x)$ is in T. The number ρ is called an associated *growth factor.* (The associated growth rate is $(\rho - 1)$.)

*D*2. The *coefficient of expansion* of any pair (x, y) is[1]

$$\lambda(x, y) = \max \{c \mid y \geqq cx\}.$$

*D*3. A sequence $\{x_n\}_{n=0}^{N}$ of commodity vectors is called *feasible,* given x_0, if (x_n, x_{n+1}) is in T for $n = 0, 1, \ldots, N-1$.

[1] For vectors y and z, $y \geqq z$ denotes $y_i \geqq z_i$, $i = 1, \ldots, M$; $y \geqslant z$ denotes $y \geqq z$ and $y \neq z$; and $y > z$ denotes $y_i > z_i$, $i = 1, \ldots, M$.

Example 2.1. $M = 2$, and T is the set of all (x, y) such that

$$y_1^2 + y_2^2 \leqq k\ x_1x_2,$$

$$x_1,\ x_2,\ y_1,\ y_2 \text{ all } \geqq 0,$$

where k is some positive number.

Example 2.2 (von Neumann).

$$T = \{(x,y) \mid y = Ba,\ x = Aa,\ a \geqq 0\}$$

where A and B are matrices with nonnegative elements. The co-ordinates of the vector a are to be interpreted as the " levels " or " intensities " of a certain finite number of productive processes. In this case T is a polyhedral cone.

3. *Terminal Objectives*

This paper is concerned with feasible sequences that, given N and x_0, maximize $u(x_N)$ in the set of all feasible sequences $\{x_n\}\underset{n=0}{N}$, where u is some given function on the commodity space. Such a sequence will be called *u-optimal.*

The interpretation of the above is that the planner's (or " society's ") preferences among feasible sequences depend only upon comparisons among the *terminal* states of such sequences, the function u representing those preferences.

The following assumptions are made about the preference function u.

A3. u is nonnegative and continuous on the commodity space, and there exists x such that $u(x) > 0$.

A4. u is " quasi-homogeneous," i.e., for all nonnegative M-dimensional vectors x′ and x″, and all numbers $k > 0$, $u(x') \geqq u(x'')$ *if and only if* $u(kx') \geqq u(kx'')$. Without loss of generality one may *take u to be homogeneous of degree* 1.

Example 3.1. $u(x) = \Pi_{i=1}^{M} x_i^{\alpha_i}$, where the α_i are fixed nonnegative numbers.

Example 3.2. $u(x) = \max\{c \mid x_i \geqq c\alpha_i,\ i = 1, \ldots, M\}$, where the α_i are fixed nonnegative numbers with $\Sigma\alpha_i = 1$. In this case the numbers $\alpha_i, \ldots, \alpha_M$ represent the " desired proportions " of the several commodities.

Example 3.3. $u(x) = \Sigma_{i=1}^{M} w_i x_i$, where $w \geqq 0$. In this example, w_i could be the price of commodity i, so that $u(x)$ would be the value of x under the set of prices $w_1, \ldots, w_M$.

4. *Prices, Interest, Profit, and Balanced Growth Equilibrium*

The proof of the turnpike theorem of this paper makes use of certain " shadow prices," by comparing the growth of a corresponding " shadow profit " along alternative paths of growth. Thus, for any nonegative M-dimensional vector p, any positive real number μ, and any pair (x,y) of commodity vectors, the quantity

$$p{\cdot}(y - \mu x)$$

may be interpreted as *profit*, if p is thought of as a *price vector* and μ an *interest factor* (where $p{\cdot}z$ denotes the scalar product $\Sigma_{i=1}^{M} p_iz_i$), although these terms are only suggestive.

D4. (p, μ) is called an *equilibrium price-interest* pair if

(*a*) $p \cdot (y - \mu x) \leqq 0$ for all (x,y) in T;

(*b*) $p \geqslant 0$, $\mu > 0$.

Von Neumann, and others following him, showed that under certain conditions on the production possibility set T, there exist $\hat{x}$, p and ρ such that

(1) $\hat{x}$ is balanced, with growth factor ρ,

(2) (p,ρ) is an equilibrium price-interest pair.

A triple $(\hat{x},p,\rho)$ satisfying (1) and (2) will here be called a *von Neumann balanced growth equilibrium.*

For example, it can be shown that conditions (1) and (2) follow from assumption *A*2, and the following assumptions (see Karlin [3], vol. 1, sec. 9.10, and the references given there):

*A*1′. *T is a closed convex cone in the nonnegative orthant of* 2*M-dimensional Euclidean space.*

*A*5. *If* (x,y) *is in* T, *and* $x' \geqq x$, $y' \leqq y$, *then* (x', y') *is in* T.

*A*6. *For every* $i = 1, \ldots, M$, *there is an* (x,y) *in* T *for which the* ith *co-ordinate of* y *is strictly positive.*

The interpretation of *A*1′ is that, in addition to *A*1, when some variables are fixed, one has nonincreasing marginal productivity in all other variables. *A*5 can be interpreted as stating that disposal activity is costless; *A*6, that every commodity can be produced.

The ray through $\hat{x}$ will be called the von Neumann ray.

An additional consequence of the above assumptions is

$$\rho = \max_{(x,\,y)\,\in\,T} \lambda(x,y), \tag{3}$$

i.e., that x, when used as an input, yields the greatest possible coefficient of expansion. Indeed, it is this last fact that leads one to suspect that there is some intimate connection between optimal growth and von Neumann equilibrium.

5. *A "Turnpike Theorem": The Relation between u-Optimal Sequences and von Neumann Balanced Growth Equilibrium*

In this section I show that under certain conditions, including a uniqueness property of the von Neumann ray, all u-optimal sequences must be "close" to the von Neumann ray except possibly for a finite number of periods, which number is independent of the length of the sequence.

First, I define a "distance" between two vectors that is essentially equivalent to the angle between them.

*D*5. $$d(x,z) \equiv \left| \frac{x}{\|x\|} - \frac{z}{\|z\|} \right|,$$

(where, as usual, $\|x\| = (x \cdot x)^{1/2}$).

THEOREM. *Under assumptions* $A1$ *to* $A4$, *if*

(i) $(\hat{x}, p, \rho)$ *is a von Neumann equilibrium,*

(ii) $p \cdot (y - \rho x) < 0$ for all (x, y) in T that are not proportional to $(\hat{x}, \rho\hat{x})$,

(iii) there is a number $K > 0$ such that for all commodity vectors x, $u(x) \leqq Kp \cdot x$,

(iv) an initial commodity vector x_0 is given such that for some number $L > 0$, $(x_0, L\hat{x})$ is in T,

(v) $u(\hat{x}) > 0$;

then, for any $\varepsilon > 0$ there is a number S such that for any N and any u-optimal sequence $\{x_n\}_{n=0}^{N}$, the number of periods in which $d(x_n, \hat{x}) \geqq \varepsilon$ cannot exceed S.

It should be noted that the number S is independent of the length N of the sequence considered.

A number of remarks on conditions (i) to (v) follow the proof of the theorem.

LEMMA. *Under assumptions $A1$ and $A2$, and conditions* (i) *and* (ii) *of the theorem, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any (x, y) in T for which $d(x,\hat{x}) \geqq \varepsilon$, it follows that*

$$p \cdot y \leqq (\rho - \delta)\, p \cdot x.$$

Proof of Lemma. It follows from $A1$ and $A2$ that the set

$$T_1 \equiv \{y \mid (x,y) \text{ in } T.\ \ \|x\| = 1\}$$

is bounded. For if not, there is a sequence (x_k,y_k) such that $\|x_k\| = 1$, and $\|y_k\| \longrightarrow \infty$; but then the sequence

$$\left(\frac{x_k}{\|y_k\|}, \frac{y_k}{\|y_k\|}\right), \qquad k = 1, 2, \text{etc.},$$

being bounded and in T, has a limit point $(0,\bar{y})$ in T, with $\|\bar{y}\| = 1$, which contradicts $A2$.

I now proceed to prove the lemma.

Suppose to the contrary that there is an $\varepsilon > 0$ and a sequence (x_k, y_k) in T such that $p \cdot x_k > 0$, $d(x_k, \hat{x}) \geqq \varepsilon$ and

$$\frac{p \cdot y_k}{p \cdot x_k} \longrightarrow \rho. \tag{4}$$

By normalization, one may take

$$\|x_k\| = 1, \qquad k = 1, 2, \text{etc.}$$

Hence the sequence (x_k, y_k) is bounded, and so has a limit point $(\bar{x}, \bar{y})$. Since the sequence $p \cdot x_k$ is also bounded, it follows from (4) that $p \cdot (\bar{y} - \rho\bar{x}) = 0$; but this last, together with the fact that $(\bar{x}, \bar{y})$ cannot be proportional to $(\hat{x}, \rho\hat{x})$, contradicts condition (ii). Thus the lemma is proved.

Proof of Theorem. First define a feasible sequence $(\tilde{x}_n)_{n=0}^{N}$ by

$$\left\{\begin{aligned} \tilde{x}_0 &= x_0 \\ \tilde{x}_1 &= L\hat{x} \\ \tilde{x}_n &= L\rho_{n-1}\hat{x}, \qquad n = 1, \ldots, N. \end{aligned}\right. \tag{5}$$

The idea of the proof is to show that the sequence $\{\tilde{x}_n\}$ is better than any sequence that departs too far for too long from the von Neumann ray (even though $\{\tilde{x}_n\}$ may not itself be u-optimal). Consider any $\varepsilon > 0$ and any sequence $\{x_n\}_0^N$ that is feasible given x_0. For any period n for which $d(x_n, \hat{x}) \geqq \varepsilon$, it follows from the lemma that

$$p \cdot x_{n+1} \leqq (\rho - \delta) p \cdot x_n, \tag{6}$$

where $\delta > 0$ is as in the lemma. On the other hand, it follows from the definition of a von Neumann equilibrium that for all n one has at least

$$p \cdot x_{n+1} \leqq \rho p \cdot x_n. \tag{7}$$

Suppose that $d(x_n, \hat{x}) \geqq \varepsilon$ for P periods. Then from (6) and (7) one has

$$p \cdot x_N \leqq (\rho - \delta)^P \rho^{N-P} p \cdot x_0. \tag{8}$$

Hence, by condition (iii),

$$u(x_N) \geqq K(\rho - \delta)^P \rho^{N-P} p \cdot x_0. \tag{9}$$

On the other hand, by (5) and the homogeneity of u,

$$u(\hat{x}_N) = L\rho^{N-1} u(\hat{x}) > 0. \tag{10}$$

Combining (9) and (10) gives

$$\frac{u(x_N)}{u(\hat{x}_N)} \leqq C\left(\frac{\rho - \delta}{\rho}\right)^P$$

where

$$C \equiv \frac{K\rho p \cdot x_0}{Lu(\hat{x})}$$

Hence, for $\{x_n\}_0^N$ to be optimal, surely

$$C\left(\frac{\rho - \delta}{\rho}\right)^P \geqq 1,$$

or

$$P \leqq \frac{\log C}{\log \dfrac{\rho}{\rho - \delta}}.$$

Hence the proof is completed by taking

$$S = \max \left[1, \frac{\log C}{\log (\rho/\rho - \delta)}\right]. \tag{11}$$

Remark 1. Condition (ii) of Theorem 1 is fulfilled if, in particular, T is a " strictly-convex cone " with nonempty interior.[1] To see this, suppose to the contrary that (x', y') in T is not proportional to $(\hat{x}, \hat{y})$ (where $\hat{y} = \rho\hat{x}$), and that $p \cdot (y' - \rho x') = 0$. Let $(x,y) = \frac{1}{2}(\hat{x},\hat{y}) + \frac{1}{2}(x',y')$; then it follows that $p \cdot (y - \rho x) = 0$. But (x,y) is in the interior of T; hence there exist sufficiently small commodity vectors a and b such that $(x + a, y + b)$ is in T, and such that $p \cdot (b - \rho a) > 0$. This last implies, however, that $p \cdot [(y + b) - \rho(x + a)] > 0$, which contradicts the assumption that $(\hat{x}, p, \rho)$ is a von Neumann equilibrium.

[1] A convex cone C will be said to be a " strictly-convex cone " if for any and b in C that are not proportional, and any α for which $0 < \alpha < 1$, the point $\alpha a + (1 - \alpha)b$ is in the interior of C. Note that a " strictly-convex cone " is not a strictly convex set!

It should be added that the converse of this remark is not true. It would be interesting to have a simple characterization of those cones T that do satisfy (ii). It should also be added that (ii) is typically not satisfied when T is polyhedral (the case treated by von Neumann).

Remark 2. Condition (iii) is fulfilled if, for example, all the co-ordinates of p are positive.

Remark 3. Condition (iv) is fulfilled if, for example, there is free disposal ($A5$) and x_0 provides positive amounts of all those commodities for which $\hat{x}$ is positive. On the other hand, Condition (iv) can be weakened to the following: an initial vector x_0 is given such that for some number $L > 0$ and some integer $N_0 \geqq 1$, there is a feasible sequence from x_0 to $L\hat{x}$ in N_0 periods.

Remark 4. Condition (v) can be weakened to the following: for some integer $N_1 \geqq 0$ and some commodity vector y for which $u(y) > 0$, there is a feasible sequence from $\hat{x}$ to y in N_1 periods.

With Conditions (iv) and (v) modified as indicated in Remarks 3 and 4, equation (11) would have to be modified to read

$$S = \max\left[N_0 + N_1, \frac{\log C}{\log (\rho/\rho - \delta)}\right]. \tag{11'}$$

Remark 5. Consider the special case of Example 3.3 in which $u(x) = w \cdot x$, where w is some fixed vector $\geqslant 0$. A necessary and sufficient condition that (iii) be satisfied in this case is that $w_i = 0$ for all i for which $p_i = 0$. (In other words, the preference function must give no weight to goods that are " free " under the von Neumann equilibrium prices.) Under these circumstances the smallest value of K that can be used in (iii) is

$$K = \max_{p_i > 0} \frac{w_i}{p_i},$$

and, correspondingly, this gives a value of C in the theorem of

$$C = \frac{\rho p \cdot x_0}{Lw \cdot \hat{x}} \max_{p_i > 0} \frac{w_i}{p_i}.$$

Note that this last value for C is not affected by multiplying either p, w, or $\hat{x}$ by a positive constant, as is to be expected.

Berkeley, California. ROY RADNER.

REFERENCES

[1] Dorfman, R., P. A. Samuelson, and R. M. Solow, *Linear Programming and Economic Analysis*, New York: McGraw-Hill, 1958.

[2] Gale, D., " The closed linear model of production," in Kuhn and Tucker (eds.), *Linear Inequalities and Related Systems*, Annals of Math. Study No. 38, Princeton, N. J.: Princeton University Press, 1956, pp. 285-303.

[3] Karlin, S., *Mathematical Methods and Theory in Games, Programming and Economics* Vol. 1, Reading, Mass.: Addison-Wesley, 1959.

[4] Malinvaud, E., " Programmes d'expansion et taux d'interet," *Econometrica*, Vol. 27, No. 2 (April, 1959), pp. 1-9.

[5] von Neumann, J., "A model of general economic equilibrium," *Review of Economic Studies*, Vol. 13 (1945-6), pp. 1-9.

LINKED CITATIONS

- Page 1 of 1 -

You have printed the following article:

Paths of Economic Growth that are Optimal with Regard only to Final States: A Turnpike Theorem

Roy Radner

The Review of Economic Studies, Vol. 28, No. 2. (Feb., 1961), pp. 98-104.

Stable URL:

http://links.jstor.org/sici?sici=0034-6527%28196102%2928%3A2%3C98%3APOEGTA%3E2.0.CO%3B2-M

This article references the following linked citations. If you are trying to access articles from an off-campus location, you may be required to first logon via your library web site to access JSTOR. Please visit your library's website or contact a librarian to learn about options for remote access to JSTOR.

References

[4] **Programmes d'Expansion et Taux d'Interet**

Edmond Malinvaud

Econometrica, Vol. 27, No. 2. (Apr., 1959), pp. 215-227.

Stable URL:

http://links.jstor.org/sici?sici=0012-9682%28195904%2927%3A2%3C215%3APDETD%3E2.0.CO%3B2-N

[5] **A Model of General Economic Equilibrium**

J. V. Neumann

The Review of Economic Studies, Vol. 13, No. 1. (1945 - 1946), pp. 1-9.

Stable URL:

http://links.jstor.org/sici?sici=0034-6527%281945%2F1946%2913%3A1%3C1%3AAMOGEE%3E2.0.CO%3B2-H

NOTE: *The reference numbering from the original has been maintained in this citation list.*