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On the Core of a Cartel

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Abstract. Consider an industry with a given set of firms producing a homogeneous product, in which all the firms have the same constant cost per unit output, but possibly different capacities. Each firm chooses a level of output, and the market price is determined by the total industry output and the industry demand function (the "Cournot model"). I show that the core of the corresponding game, in which the firms are the players, is not empty, and includes the allocation in which the firms' outputs and profits are proportional to their respective capacities. The core is characterized by a convex "Lorenz curve" that limits the inequality in the distribution among firms of profit per unit capacity; this curve can be calculated explicitly, given the cost parameter and the industry demand function. I also consider a two-stage game in which, in the first stage, the firms choose their capacities noncooperatively, at a constant cost per unit capacity, and in the second stage they receive the proportional allocation described above. In the limit, as the number of firms increases without bound, total industry capacity and output are positive, but net profit is zero; price and output are in fact what would be optimal for the cartel if capacity were costless.

1 Introduction

Suppose that the firms in an industry form a cartel to fix total industry output so as to maximize total industry profit. What is a "rational" way to divide industry output and profit among the several firms? One answer to this question is provided by the game-theoretic concept of the *core*. An allocation of output and profit is said to be in the core if it yields to each subset of firms at least as much profit as that subset could guarantee itself by acting on its own. Thus the core is a *set* of allocations, possibly empty (the definition of a subset's guaranteed profit will be made precise in Section 2).

In his influential monograph, *Core and Equilibria of a Large Economy* (1974), Werner Hildenbrand provided a thorough study of the core of an exchange economy, in which the players were the consumers. He also introduced the idea of the core of an economy with production, again with the consumers as the players, who could also form coalitions for the purpose of production. The present paper complements Hildenbrand's study with a preliminary investigation of the core of an industry in which the active players are the *firms*, and the consumers are passive "demanders."

In this note I shall consider a very simple model of an industry in which the firms produce a homogeneous good, have the same unit cost, but have possibly different capacities. The guaranteed profit of a subset of firms can then be reasonably defined as the maximum profit that the subset can earn if all the other firms produce at capacity (Sec. 2). I first consider the case in which the number of firms and their capacities are fixed. Under some regularity conditions on demand, I show that the core is not empty, and in particular contains the allocation in which each firm produces a share of the total cartel output proportional to its capacity, and receives the corresponding profit; call this the *proportional allocation* (Sec. 3). In other words, the proportional allocation equalizes profit per unit of capacity across firms. Note that if capital investment is proportional to capacity, then this core allocation also equalizes the rate of return on investment across firms.

Under some further regularity conditions, I can fully characterize the core (Sec. 4). If the total industry capacity does not exceed the optimal cartel output (namely, the output that would maximize the cartel profit in the absence of any capacity constraint), then the proportional allocation is the only one in the core. On the other hand, if industry capacity exceeds the optimal cartel output, then there will be many allocations in the core. In fact, the core will consist of all allocations such that the Lorenz curve of the distribution across firms of profit-per-unit-capacity lies on or above a given function, which I denote by G. Thus the function G bounds the inequality in the distribution of profit-per-unit-capacity for allocations in the core. The function G can, in principle, be calculated explicitly from the demand function, the unit cost, and the industry capacity, and does not depend on the number of firms. More detailed information about G is also derived, depending on whether industry capacity is less than or greater than the industry output at which price equals unit cost. Section 5 describes an example.

I next consider the noncooperative game in which the firms simultaneously choose their respective capacities in the expectation that thereafter they will receive the proportional allocation (Sec. 6). In this model, I assume that the cost of capacity is proportional to capacity, and that this unit cost is the same for all firms. The Nash equilibrium of this game is easy to calculate, and has the property that the industry output exceeds the output that would be optimal for the cartel if capacity were costless (i.e., the optimal cartel output of Section 2). In fact, as the number of firms increases without bound (which is a way of describing free entry), then in equilibrium,

1. individual firm capacity tends to zero;
2. total industry capacity approaches a positive limit;
3. total industry profit (net of the cost of capacity) tends to zero, and hence so does individual firm profit;
4. total industry output equals that which would be optimal for the cartel if capacity were costless.

I conclude the discussion with some remarks on (1) an alternative definition of the core, (2) an extension of the analysis (by A. Postlewaite) to the case in which firms have different unit costs, and (3) possible extensions to models of sequential actions by the firms (Sec. 7).

Acknowledgments

Andrew Postlewaite worked out an extension of Section3 to the case in which firms may differ in their unit costs (Section 7.2). Martin Osborne, Richard Steinberg, Wilhelm Neuefeind, and Walter Trockel provided useful comments on earlier drafts. Martin Osborne also called my attention to two papers (Osborne and Pitchik, 1983, 1987), in which the Nash Bargaining Solution is used to predict the behavior of a cartel.

2 The Model with fixed capacity constraints

Consider an industry producing a single product, with a demand function

$$P = \phi(Q), \tag{1}$$

where P is the price at which the total quantity Q will be demanded. There are n firms, all with the same linear cost function, so that if firm j produces Q_j, its corresponding cost is γQ_j. Therefore, if the firms' respective outputs are $Q_1, ..., Q_n$, then the total output is $Q = \sum_j Q_j$ and firm j's profit is

$$\pi_j = Q_j\phi(Q) - \gamma Q_j = Q_j[\phi(Q) - \gamma]. \tag{2}$$

Finally, each firm j has a capacity K_j, so that

$$0 \leq Q_j \leq K_j, \ j = 1, ..., n. \tag{3}$$

By an *allocation* I shall mean an n-tuple of feasible outputs, $Q_1, ..., Q_n$, together with the n-tuple or corresponding profits, $\pi_1, ..., \pi_n$. The *core* is the set of all the allocations such that no subset of firms receives less total profit than it could *guarantee* to itself, acting on its own. To make the notion of guaranteed profit precise, let J denote any (nonempty) subset of firms, let x denote its total output, and let y denote the total output of all firms not in J. The total profit of subset J is, from (2),

$$f(x, y) \equiv x[\phi(x + y) - \gamma]. \tag{4}$$

Define the *guaranteed profit* of subset J by

$$g_J \equiv \max_x \min_y f(x, y), \tag{5}$$

subject to

$$0 \leq x \leq \sum_{j \in J} K_j \equiv K', \text{ and } 0 \leq y \leq \sum_{i \notin J} K_i \equiv K''. \tag{6}$$

The core is defined as the set of allocations $(\pi_1, \cdots, \pi_n)$ such that

$$\sum_{j \in J} \pi_j \geq g_J, \text{ for all } J. \tag{7}$$

In particular, taking J to be the set of all firms, g_J is the monopoly (cartel) profit, and the inequality in (7) must be an equality.

I shall assume in what follows that ϕ is continuous and strictly decreasing except where $\phi(Q) = 0$, that $\phi(Q)$ converges to zero as Q increases without limit, and that $\phi(0) > \gamma$.

First observe that, for any group of firms, the least favorable situation is the one in which all other firms produce at capacity; this follows from the assumption that the demand function is decreasing, and therefore the profit function (4) is decreasing in y or every x. From this it follows that the guaranteed profit levels g_J are given by:

$$g_J = \max_x f(x, K'') \text{ subject to } 0 \leq x \leq K'. \tag{8}$$

Let K denote the total industry capacity, i.e.,

$$K \equiv \sum_{j=1}^{n} K_j,$$

and for any subset J let r_J denote the ratio of J's capacity to total industry capacity, i.e.,

$$r_J \equiv \frac{1}{K} \sum_{j \in J} K_j. \tag{9}$$

If J's total output is x and all other firms produce at capacity, then J's total profit is

$$f(x, (1 - r_J)K);$$

this motivates the following definition:

$$G_K(r) \equiv \max_x f(x, (1-r)K) \text{ subject to } 0 \leq x \leq rK. \tag{10}$$

We can now rewrite (8) as

$$g_J = G_K(r_J). \tag{11}$$

The study of the core thus reduces to the study of the function G_K.

3 The proportional outcome is in the core

Let $\hat{\pi}$ denote the maximum total industry profit and let $\hat{Q}$ be the corresponding optimal cartel output. Define the *proportional allocation* as the one for which each firm produces a share of the optimal cartel output proportional to its capacity, and receives a corresponding share of the total industry profit, i.e.,

$$Q_j = (K_j/K)\,\hat{Q}, \text{ and } \pi_j = (K_j/K)\,\hat{\pi}, \quad j = 1, ..., n. \tag{12}$$

Proposition 1. *The proportional allocation is in the core.*

Proof. Recall that, for any subset J, r_J is the ratio of J's capacity to the total capacity. Let

$$K' \equiv r_J K, \;\; K'' \equiv (1 - r_J)K.$$

If J produces a total output x $(0 \leq x \leq K')$ and all other firms produce at capacity, then J's profit is, from (4)

$$\begin{aligned} f(x, K'') &= x[\phi(x + K'') - \gamma] \\ &= \frac{x}{x + K''}(x + K'')[\phi(x + K'') - \gamma]. \end{aligned} \tag{13}$$

However,

$$\frac{x}{x + K''} \leq \frac{K'}{K' + K''} = r_J, \text{ and } (x + K'')[\phi(x + K'') - \gamma] \leq \hat{\pi}.$$

Hence, from (13),

$$f(x, K'') \leq r_J \hat{\pi}. \tag{14}$$

Since this is true for every x $(0 \leq x \leq K')$,

$$g_J = G_K(r_J) \leq r_J \hat{\pi},$$

which completes the proof.

4 Further characterization of the core

As noted at the end of Section 2, the study of the core reduces to the study of the properties of the function G_K. Recall that

$$f(x, y) \equiv x[\phi(x + y) - \gamma], \text{ and} \tag{15}$$

$$G_K(r) \equiv \max_x f(x, (1-r)K) \text{ subject to } 0 \leq x \leq rK, \tag{16}$$

where $K > 0$ and $0 \leq r \leq 1$. Recall also that $f(x, y)$ is the profit of a subset producing output x when the remaining firms produce a total of y, and $G_K(r)$ is the guaranteed profit of a subset that has a fraction r of the total capacity K. It is immediate that

$$G_K(r) \geq 0, \quad G_K(0) = 0, \quad G_K(1) = \hat{\pi}, \tag{17}$$

where $\hat{\pi}$ is the maximum total industry profit. I shall show below that, with an additional assumption about demand, for each K the function G_K is nondecreasing and convex in r. Further details about G_K depend upon the magnitude of total industry capacity, K.

Before stating and proving these results in detail, I want first to show how the function G_K provides a bound for core allocations on the distribution across firms of profit per unit of capacity. To lighten the notation, I shall henceforth suppress the subscript K, i.e., I shall write $G(r)$ for $G_K(r)$.

For a given core allocation, let π_j be the profit of firm j. Let the firms be ranked in increasing order of the ratio of profit to capacity, with the firm with the lowest ratio numbered 1, etc. For any number r, define $J(r)$ to be the set of firms for which $(\pi_j/K_j) \leq r$, let $\pi(r)$ be the sum of their profits, and let $\rho(r)$ be the ratio of their total capacity to industry capacity, i.e.,

$$\pi(r) \equiv \sum_{j \in J(r)} \pi_j, \text{ and } \rho(r) \equiv r_{J(r)}. \tag{18}$$

Since the allocation is in the core,

$$\pi(r) \geq G(\rho(r)), \quad \text{for all } r. \tag{19}$$

There are, of course, only n different (nonempty) sets $J(r)$, where n is the number of firms. If we plot the n points with coordinates $(\rho(r), \pi(r))$, corresponding to the n different sets $J(r)$, in the size-profit plane, all of these points must lie on or above the graph of G, by (19). In particular the last point will have the coordinates $(1, \hat{\pi})$.

Let L be the "curve" obtained by connecting successive points by straight-line segments, and by connecting as well the origin and the first point. (See Fig. 1.) The curve L is in a sense the "Lorenz curve" for the *distribution of profits per unit capacity.* It is easy to see that: (1) the curve is convex, and lies on or below the line joining the origin with $(1, \hat{\pi})$; (2) the curve L lies above (or touches) the graph of G; (3) from (17) and the conclusion that the function G is convex and nondecreasing (see below), for any ordering of the firms there is a corresponding allocation in the core for which all of the

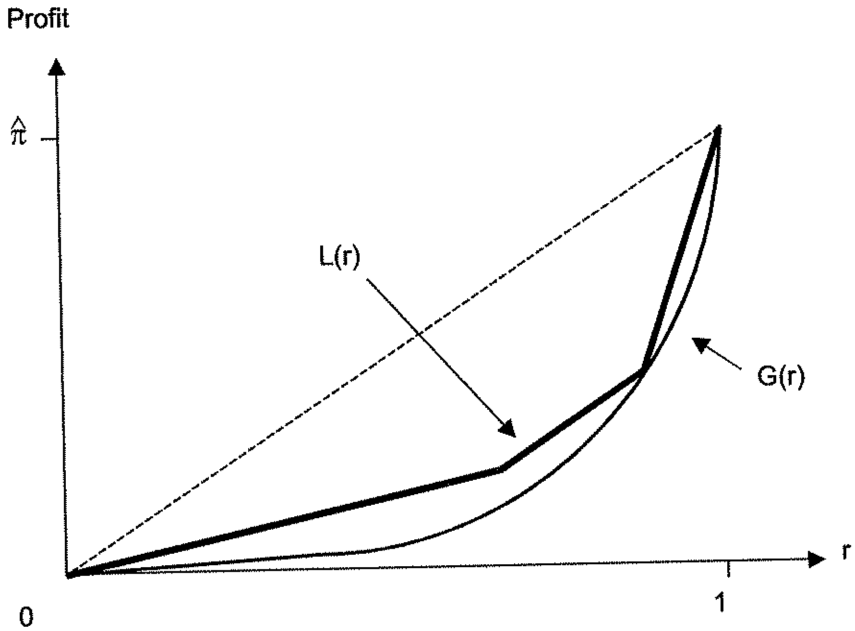

Fig.1. A Lorenz curve for the distribution of profits per unit capacity for a cartel with 3 firms.

points $(\rho(r), \pi(r))$ actually lie on the graph of G (maximal inequality); (4) as the number of firms increases, and if the distribution of shares of capacity approaches a continuous distribution on the interval [0,1], then the lower envelope of all L-curves approaches the graph of G.

I turn now to the properties of G. Henceforth, I shall assume that the following condition is satisfied.

Condition U: *For each* $y \geq 0$, *the function* $f(x, y)$ *has a unique global maximum in* $x \geq 0$.

For Condition U to be satisfied it is sufficient, but not at all necessary, that the demand function ϕ be concave where $\phi(x)$ is positive.

Two output levels are important for determining the properties of G, and hence of the core. Let Q^M denote the smallest industry output that would maximize total industry profit *in the absence of any capacity constraint*, and let Q^* denote the total (industry) output for which price equals marginal cost, i.e., $\phi(Q^*) = \gamma$. As is well known,

$$0 < Q^M < Q^*.$$

(For this, one uses only the assumptions about demand stated in Section 2.) It is useful to distinguish three cases, according as (1) K does not exceed Q^M, (2) K lies between Q^M and Q^*, or (3) K exceeds Q^*. Propositions 2-4 characterize the function G, and hence the core, in each of the three cases, respectively. Figures 2-4 illustrate the propositions.

Proposition 2. *If* $K \leq Q^M$, *then* $G(r) = r\hat{\pi} = rK[\phi(K) - \gamma]$. *(See Fig. 2.)*

Corollary 1. *If* $K \leq Q^M$, *then the proportional allocation is the only one in the core.*

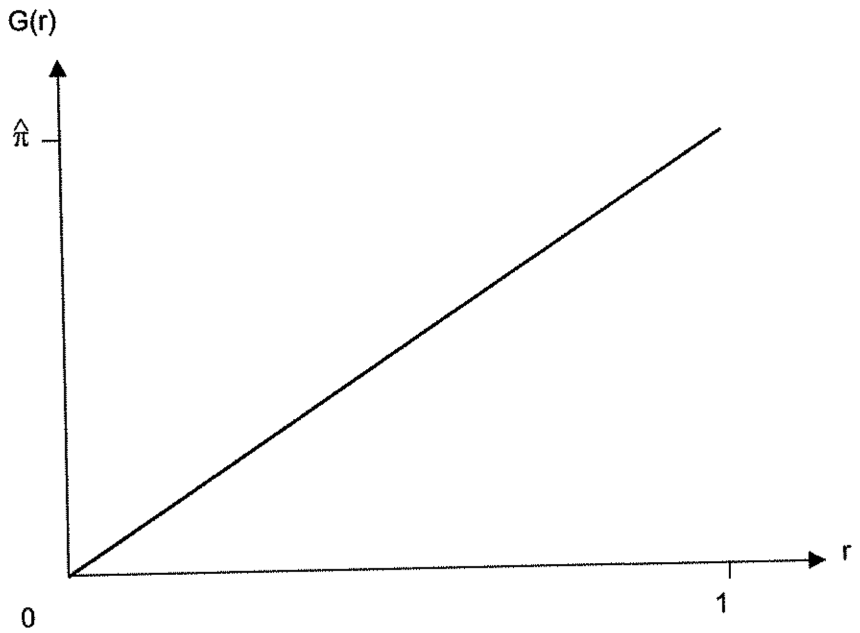

Fig. 2. G(r) for K < QM .

Proof. From Condition U it follows that total industry profit is maximized at capacity, so that,

$$\hat{Q} = K, \quad \text{and } \hat{\pi} = K[\phi(K) - \gamma].$$

From (14), for $0 \leq r \leq 1$ and $0 \leq x \leq rK$,

$$f(x,(1-r)K) \leq r\hat{\pi} = f(rK,(1-r)K),$$

which proves the proposition. On the other hand, in the proportional allocation each subset J receives exactly $G(r_J)$, and since it is producing at capacity it cannot make a higher profit in the core, which proves the corollary.

Proposition 3. *If $Q^M < K < Q^*$, then G is convex and strictly increasing. Furthermore, there exists $r^M > 0$ (depending on K) such that, for $0 \leq r \leq r^M$, the function G is linear in r,*

$$G(r) = rK[\phi(K) - \gamma], \tag{20}$$

with positive slope less than $\hat{\pi}$, and for a subset of firms of total capacity r its guaranteed profit is attained by producing at capacity. (See Fig. 3.)

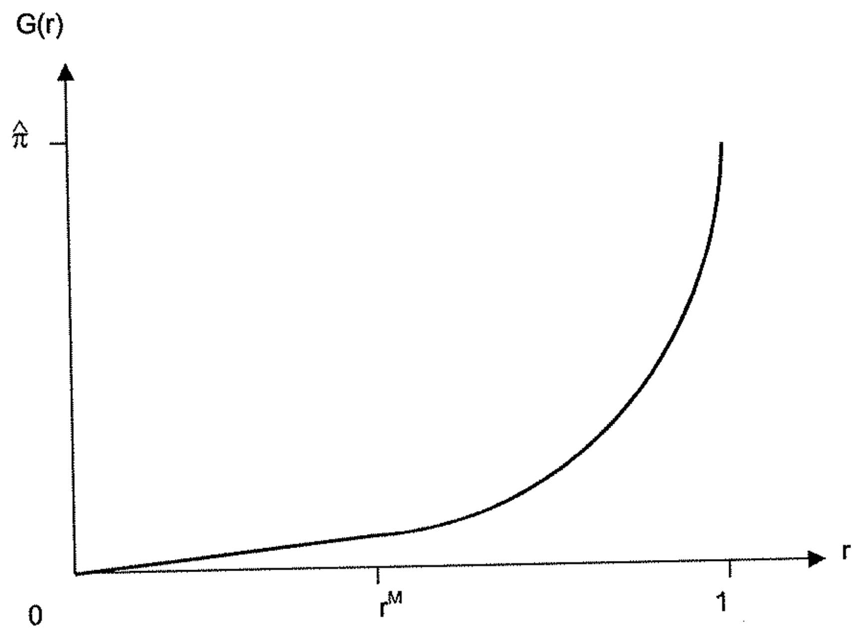

Fig. 3. G(r) for QM < K < Q*.

The proof is based on the following lemma, which is proved at the end of this section. For any r $(0 \leq r \leq 1)$ let $\xi(r)$ denote the unique value of x that maximizes $f(x,(1-r)K)$ subject to $0 \leq x \leq rK$.

Lemma 1. *For any r and r_0 in* $[0,1]$,

$$G(r) \geq G(r_0) + (r - r_0)K[\phi(q_0) - \gamma], \tag{21}$$

where $q_0 \equiv \xi(r_0) + (1 - r_0)K$.

To see that the lemma implies that G is strictly increasing, note that $K < Q^*$ implies that $\phi(q_0) - \gamma > 0$, so that, from (21), if $r > r_0$ then $G(r) > G(r_0)$.

The convexity of G also follows immediately from the lemma, since if $r_1 < r_0 < r_2$ then

$$\frac{G(r_2) - G(r_0)}{r_2 - r_0} \geq K[\phi(q_0) - \gamma] \geq \frac{G(r_0) - G(r_1)}{r_0 - r_1}.$$

To prove the rest of the proposition's conclusion, for each r in [0,1] let $\xi_0(r)$ denote the unique x that maximizes $f(x, (1-r)K)$ subject only to $x \geq 0$. By the Theorem of Maximum, ξ_0 is continuous. Furthermore, $f(x, K)$ is zero at $x = 0$ but strictly positive for $x > 0$ and sufficiently small; hence $\xi_0(0) > 0$. Therefore, for some $r' > 0$ and sufficiently small,

$$\xi_0(r) \geq rK \text{ for } 0 \leq r \leq r';$$

Let r^M be the largest such r'. Condition U now implies that

$$G(r) = rK[\phi(K) - \gamma] \; for \; 0 \leq r \leq r^M,$$

which is (20), since (21) implies that $f(x, (1-r)K)$ is nondecreasing on the interval $[0, rK]$. Finally, $K > Q^M$ implies that $K[\phi(K) - \gamma] < \hat{\pi}$, the maximum cartel profit. Hence, from (20) the slope of G on the interval $(0, r^M)$ is strictly less than $\hat{\pi}$.

Proposition 4. *Let $r^* \equiv 1 - (Q^*/K)$. If $K \geq Q^*$,then, for $0 \leq r \leq r^*$, $G(r) = 0$, and G is convex and strictly increasing for $r > r^*$. (See Fig. 4.)*

Proof. For $0 \leq r \leq r^*$ we have $(1 - r)K \geq Q^*$, and hence for $x \geq 0$,

$$\begin{aligned} \phi[x + (1-r)K] &\leq \gamma, \\ f(x, (1-r)K) &\leq 0, \\ G(r) &= 0. \end{aligned}$$

For $r > r^*$ we are in a situation like that of Proposition 3, and a corresponding analysis yields the desired conclusion that G is convex and strictly increasing. I omit the details.

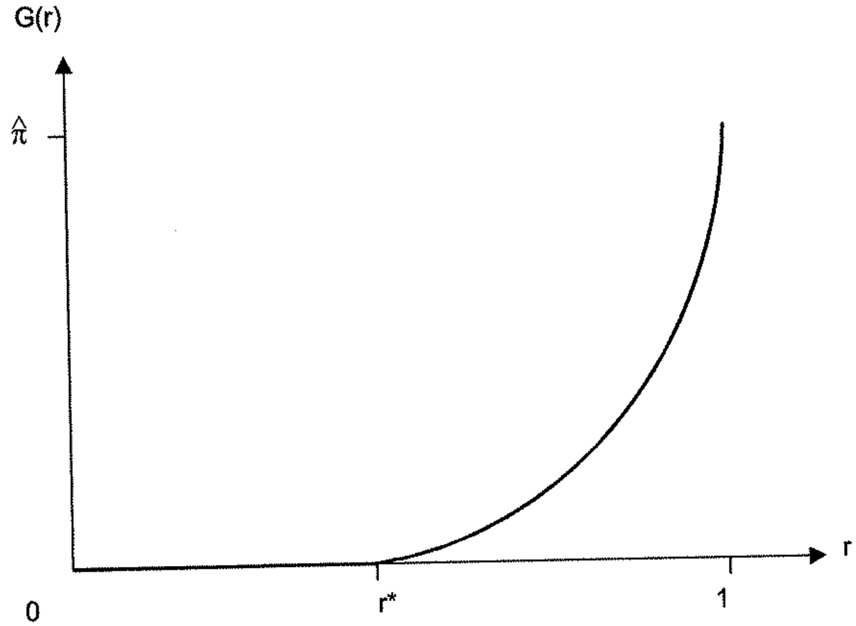

Fig. 4. G(r) for Q* < K.

To prove the lemma, define

$$\begin{aligned} x_0 &\equiv \xi(r_0), \\ x &\equiv x_0 + (r - r_0)K. \end{aligned} \tag{22}$$

Let H denote the right-hand side of (21). Then

$$\begin{aligned} H &= x_0[\phi(q_0) - \gamma] + (r - r_0)K[\phi(q_0) - \gamma] \\ &= x[\phi(q_0) - \gamma]. \end{aligned}$$

Further

$$\begin{aligned} x + (1 - r)K &= q_0, \text{ and} \\ x - rK &= x_0 - r_0 K \leq 0. \end{aligned}$$

If $x \geq 0$, then the last inequality implies that x is feasible for r, and $H = f(x, (1 - r)K) \leq G(r)$. If $x < 0$, then $H < 0$, and again $H \leq G(r)$, which completes the proof.

5 An example

Suppose that the demand function, ϕ, is linear where positive, so that

$$\phi(x) = \begin{cases} \alpha - \beta x & \text{for} \quad 0 \le x \le \frac{\alpha}{\beta}, \\ 0 & \text{for} \quad \ge \frac{\alpha}{\beta}, \end{cases} \tag{23}$$

where α and β are strictly positive parameters. To avoid trivialities, assume that $\phi(0) = \alpha > \gamma > 0$; we can then make the convention that

$$\alpha = 1, \ 0 < \gamma < 1. \tag{24}$$

If the cartel's total output is Q, its total profit is

$$(1-\gamma)Q - \beta Q^2. \tag{25}$$

One immediately verifies that

$$Q^M = \frac{1-\gamma}{2\beta}, \ Q^* = \frac{1-\gamma}{\beta} \tag{26}$$

(recall that Q^M maximizes (25) and that $\phi(Q^*) = \gamma$; see Sec. 4). One easily verifies that

$$\begin{aligned} \xi_0(r) &= \frac{1 - \beta K - \gamma + \beta r K}{2\beta} \\ &= Q^M - \frac{(1-r)K}{2} \\ &= \frac{1}{2}(Q^* - (1-r)K), \end{aligned} \tag{27}$$

so that

$$0 < \xi_0(r) < Q^M, \text{ and} \tag{28}$$

$$\xi_0'(r) = \frac{K}{2}. \tag{29}$$

Hence

$$\begin{aligned} f(\xi_0(r), (1-r)K) &= \frac{1 - \gamma - \beta(1-r)K^2}{4\beta} \\ &\equiv \pi_0(r), \end{aligned}$$

$$\begin{aligned} \hat{\pi} = \pi_0(1) &= \frac{(1-\gamma)^2}{2\beta}, \\ \pi_0'(r) &= \xi_0(r)\beta K > 0, \\ \pi_0''(r) &= \frac{\beta K^2}{2} > 0. \end{aligned}$$

If $Q^M < K < Q^*$, then

$$r^M = \frac{2Q^M}{K} - 1 = \frac{Q^*}{K} - 1,$$

$$0 < r^M < 1, \text{ and}$$

$$G(r) = \begin{cases} rK[1 - \beta K - \gamma] & \text{for} \quad 0 \le r \le r^M, \\ \pi_0(r) & \text{for} \quad r^M \le r \le 1. \end{cases}$$

It is straightforward to verify that G is differentiable at r^M.

On the other hand, if $K \ge Q^*$, then

$$r^* = 1 - \frac{Q^*}{K}, \text{ and}$$

$$G(r) = \begin{cases} 0 & \text{for} \quad 0 \le r \le r^*, \\ \pi_0(r) & \text{for} \quad r^* \le r \le 1. \end{cases}$$

Again, one can verify that G is differentiable at r^*.

6 Noncooperative choice of capacities

Imagine now that each of the n firms chooses its capacity at time zero, anticipating that thereafter it will obtain an allocation in the core of the "output game." Of course, if the core allocation is not unique, then each firm's anticipation will not be well-determined. Suppose then that each firm anticipates that it will obtain the corresponding proportional allocation in the core (see Sec.3). This assumption is, of course, somewhat *ad hoc*, but the proportional allocation has in its favor a certain fairness property, namely, that it equalizes across firms the ratio of profit per unit capacity, and hence, in the model described below, it *equalizes the rate of return on capital.*

Suppose, for the moment, that the number, n, of firms is given exogenously, and that at time zero the firms choose their respective capacities, K_i, simultaneously and noncooperatively. After that; each firm receives in every period its corresponding proportional allocation in the core. I assume that all firms discount future profits with the same discount factor. If Γ is the unit cost of capital for each firm, K is the total capacity of the industry, $\hat{\pi}(K)$ is the industry cartel profit, and δ is the (common) discount factor, then firm i's total discounted profit, $\Pi_i(K_1, ..., K_n)$, will be

$$\begin{aligned} \Pi_i(K_1, ..., K_n) &= -\Gamma K_i + \sum_{t=1}^{\infty} \delta^t \frac{K_i}{K} \hat{\pi}(K) \\ &= \frac{\delta}{1-\delta} \frac{K_i}{K} \hat{\pi}(K) - \Gamma K_i, \end{aligned} \tag{30}$$

where $K \equiv K_1 + ... + K_n$. Recall from Sec. 4 that

$$\hat{\pi}(K) = \begin{cases} K[\phi(K) - \gamma] & \text{for} \quad K \leq Q^M \\ Q^M[\phi(Q^M) - \gamma] & \text{for} \quad K \geq Q^M. \end{cases} \tag{31}$$

where Q^M denotes the optimal cartel output without a capacity constraint. Equations (30) and (31) define an n-person simultaneous-move game in which player (firm) i's action is $K_i (\geq 0)$.

Let $\pi^M \equiv \hat{\pi}(Q^M)$. I shall show that if

$$\frac{\delta \pi^M}{1-\delta} - \Gamma Q^M > 0, \tag{32}$$

then for sufficiently large n the Nash equilibrium of this game is given by

$$K_i = k(n) \equiv \frac{(n-1)\delta\pi^M}{n^2(1-\delta)\Gamma}, \tag{33}$$

$$K = nk(n) = \frac{(n-1)\delta\pi^m}{n(1-\delta)\Gamma} > Q^M, \tag{34}$$

and the corresponding individual firm and industry profits are

$$p(n) = \frac{\delta\pi^M}{n^2(1-\delta)}, \tag{35}$$

$$np(n) = \frac{\delta\pi^n}{n(1-\delta)}. \tag{36}$$

Note that these profits are independent of Γ!

If free entry were possible at time zero, then – since the individual firm profit (35) is strictly positive – the number of participating firms would be unbounded. I shall not attempt a serious analysis of entry here, but (35) and (36) suggest that unlimited entry would drive both individual firm and industry profits to zero. This would come about, of course, entirely through the acquisition of excess capacity, since the market price and output would remain at their monopoly levels.

To derive (33)-(36), first suppose that $K < Q^M$. Firm i's profit function is

$$\Pi_i(K_1, ..., K_n) = \frac{\delta}{1-\delta} K_i[\phi(K) - \gamma] - K_i \Gamma,$$

and its partial derivative with respect to its own capacity is

$$\frac{\partial \Pi_i}{\partial K_i} = \frac{\delta}{1-\delta}[\phi(K) - \gamma] - \Gamma + \frac{\delta}{1-\delta} K_i \phi'(K).$$

One can show that an equilibrium must be symmetric, so that in equilibrium, $K_i = K/n$, and

$$\begin{aligned}\frac{\partial \Pi_i}{\partial K_i} &= \frac{\delta}{1-\delta}[\phi(K)-\gamma] - \Gamma + \frac{1}{n}\frac{\delta}{1-\delta}K\phi'(K)\\ &= \frac{n-1}{n}\frac{\delta}{1-\delta}[\phi(K)-\gamma] - \Gamma\\ &\quad+\frac{1}{n}\frac{\delta}{1-\delta}[\phi(K)-\gamma+K\phi'(K)]\\ &= \frac{n-1}{n}\frac{\delta}{1-\delta}[\phi(K)-\gamma] - \Gamma\\ &\quad+\frac{1}{n}\frac{\delta}{1-\delta}\hat{\pi}'(K).\end{aligned}$$

Since $K < Q^M$, it follows that $\hat{\pi}'(K) > 0$. Furthermore, assumption (32) implies that

$$\frac{\delta}{1-\delta}[\phi(K)-\gamma] - \Gamma > 0.$$

Hence, for n sufficiently large,

$$\frac{n-1}{n}\frac{\delta}{1-\delta}[\phi(K)-\gamma] - \Gamma > 0,$$

which, together with $\pi'(K) > 0$, implies that

$$\frac{\partial \Pi_i}{\partial K_i} > 0 \text{ for } K_i = \frac{K}{n}, K < Q^M.$$

Hence there is no equilibrium with $K < Q^M$.

To complete the calculation, suppose that $K \geq Q^M$; then

$$\Pi_i(K_1, ..., K_n) = \frac{\delta}{1-\delta}\frac{K_i}{K}\pi^M - \Gamma K_i,$$

and a routine calculation leads to (33)-(36).

7 Further remarks

7.1 An alternative definition of guaranteed profit

In the definition of the core (Sec. 2), the guaranteed profit of a subset of firms was defined to be the maximum profit the subset can earn if all of the other firms produce at capacity. Recall that Q^* is the industry output at which price equals unit cost. Consider a subset J, and let J' denote the complementary subset. If the industry capacity exceeds Q^*, then if J is small enough, the capacity of J' will exceed Q^*; in this case the firms in J' would make a loss if they produced at capacity. However, if we changed the definition of the core so that J' produces at its capacity or Q^*, whichever is smaller, then the guaranteed profit of J would be unchanged, since the maximum profit of J would be zero in either case. Thus the core would be unchanged, as well.

7.2 An extension

Andrew Postlewaite (in a private communication) has extended the analysis of Section 3 to cover the case in which different firms may have different unit costs. He shows that the following analogue of the proportional allocation is in the core. Given the capacities of the firms, there will be an optimal cartel output; to attain this output some firms may not produce anything. Let I denote the set of all the firms that actually participate in producing the optimal cartel output, and let L denote the largest unit cost of firms in I. Now consider a hypothetical industry in which all the firms have the unit cost L, but the same capacities as before, and let P be the maximum industry profit for this second (hypothetical) industry. One can show that there is a core allocation with the following allocation of profit: (1) a firm not in I receives a share of P proportional to its capacity; (2) a firm in I receives its proportionate share of P, plus a premium. For a firm in I, its premium equals its capacity times the difference between L and its own (true) unit cost. *Note that a firm not in I receives a positive share of the cartel profit, although it produces no output.*

7.3 Sequential games

A more thorough theoretical treatment of collusion would consider a model in which the firms make a sequence (possibly infinite) of decisions regarding their *capacities* and *outputs*. In addition to characterizing the core of such a game, one might also like to know under what conditions (if any) a collusive allocation can be sustained as a sequential noncooperative equilibrium. I know of no such treatment in the literature. Note that this situation is not a repeated game, since the capacities of the firms may change.

One could envisage a more limited sequential model in which the capacities of the firms are given in advance, but the firms are free to change their outputs from one period to the next. This would be a repeated game in the strict sense of the term. According to a theorem of Aumann, in a repeated game with no discounting (the payoff to a player is the long-run average of his one-period payoffs) the set of strong-Nash equilibrium payoffs of the repeated game is the same as the set of core payoffs of the one-period game. (Recall that a strong-Nash equilibrium is a profile of strategies such that no subset of players can increase its total payoff by jointly changing the members' strategies.) Of course, strong Nash-equilibrium is not exactly a noncooperative game concept.

Elsewhere, I studied the (noncooperative) epsilon-equilibria in such a model (Radner, 1980). Recall that in an epsilon-equilibrium each firm is satisfied to be close (within epsilon in profit, where epsilon is a "small" number) to its optimal response to the other firms' sequential strategies. I considered a game with a finite number of repetitions, and a fixed number of firms, and focused on the proportional allocation as the target for collusive behavior.

I showed that: (1) If the lifetime of the industry is large compared to the number of firms, then collusion can be sustained in equilibrium for "most" of the lifetime of the industry; (2) if the number of firms is large compared to the industry's lifetime, then all equilibria will be close (in some sense) to the competitive equilibrium.

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