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ALLOCATION OF RESOURCES IN LARGE TEAMS

By K. J. Arrow¹ and R. Radner²

We study a team with many processes; the output process depends on the resources allocated to it by the resource manager, on a local decision by the process manager, and on a (random) parameter of the process. We compare two communication patterns: (1) resource allocations are based on full information, but local decisions are based only on corresponding local information; (2) all decisions are based on full information. We show that, if the criterion is expected average output per process, and if the process parameters are independent and identically distributed, then (under certain regularity assumptions) for “large” teams the additional communication among process managers in (2) over (1) has approximately no value.

1. INTRODUCTION

One of the oldest themes in economics is the use of the market in the coordination of widespread and diverse but interdependent activities. The underlying situation may be taken to be one of optimal resource allocation, that is, the maximum achievement of some objective subject to constraints on the resources used. Under suitable hypotheses of convexity and differentiability, it is well known that optimality conditions can be stated in terms of what the economist would call a price system and the mathematician a set of Lagrange parameters.

The constraint system, which represents the technology, may be very complicated indeed, so much that it is unreasonable to suppose it known to any single individual or available in the memory of even the largest computer. On the other hand, each part of the technology is known to someone, the individual or individuals who have to use that part. Hence, the system as a whole has, in a certain sense, more knowledge than is possessed by any single member.

In a world of dispersed knowledge, obviously the constrained maximization cannot be carried out in the same way as if all the necessary information were concentrated. But the economics of socialism has thrived on the argument that, when production processes are additively separable, then the price system provides a means of arriving at the optimum without any process manager having to transmit a description of his entire production structure. In this model, the decisions of the different processes are interdependent only because they compete for the same primary resources directly or indirectly.

There is a large literature on the validity and limits of this proposition, and we do not wish to enter into that here. We wish rather to emphasize the somewhat implicit calculation of informational costs in the discussion. The price system appears as a process of successive approximations. A tentative price is announced, and the individual processes are supposed to respond with demand for inputs and offers of outputs. Hence at each stage, information is being supplied.

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Now it is held to be a great advantage of the price system that the process manager need only transmit a vector instead of a description of the entire production structure. This argument presupposes that transmission of a vector is cheaper than transmission of a production possibility set or of a production function. This is clearly true if nothing is known a priori about the production structure; but as modern communication theory makes clear, the costs of transmission can be greatly reduced if advantage is taken of a priori knowledge. Thus, if a production function is known to be of the Cobb-Douglas form, then it is completely specified by a vector of parameters. Hence, the comparison of complete transmission of production structures with transmission of demands and supplies becomes a comparison of two finite-dimensional vectors.

A third and most important point is that the process of price adjustment requires in principle an infinite number of iterations. Therefore, the total transmission of information may be very large indeed, and it would require a more delicate measurement of communication costs to make clear whether decentralization is indeed superior to centralization.

Team theory [5] introduces a new way of looking at the centralization-decentralization choice. On the one hand, it assumes fixed information and communication structures. In the simplest models, there is an initial information pattern which may be followed by one step in which some of this information is transmitted to some (possibly all) agents in the team. On the other hand, team theory relies more heavily on the a priori structure of the information concerning productive possibilities. Following the standard Bayesian approach, it assumes that there is a prior probability distribution over the production structures of all the processes. In the specific examples that have been worked out, a priori assumptions take the form of drawing the production structures from a finite-parameter family, with a probability distribution over the parameters. This prior distribution is known to the team when it is deciding on its decision and communication functions. At any given realization, each process manager knows the parameters which determine his production structure, but not anyone else’s. A communication structure then determines that each firm manager transmits some of the parameters to some of the other agents. Each agent now has certain information, and takes the variables he does not observe to be distributed according to the conditional distribution obtained from the prior by conditioning on the information he has.

Suppose the production parameters for the processes are regarded as random drawings from the same distribution. If all information were conveyed to all agents (the equivalent of centralization), the optimal allocation to any agent would be determined by his parameters and by the realized (empirical) distribution of the parameters among the other processes (under suitable assumptions of convexity and symmetry). But if the number of processes is large, then the empirical distribution is essentially the distribution from which the random drawings were made and therefore is known, to a high degree of accuracy, a priori. It is the purpose of this paper to examine this situation and deduce conditions under which limited communication yields almost as high a return as full communication.
The application of team theory to resource allocation has been developed in Radner [6] and in Groves and Radner [3]. To represent joint resource constraints, we follow Radner in assuming that there is a resource manager who allocates the scarce resources. For our purposes, we confine ourselves to communication structures in which the process managers convey all their information to the resource manager. If the allocation of scarce resources completely determined the outputs of the individual processes, all such communication structures would be equivalent to complete centralization. However, it is assumed that each process manager must make another decision not subject to joint resource constraints; for vividness, we call the second decision “labor” as contrasted with “resources.” The two decisions are interrelated, in the sense that the production function for each process is not additively separable in the two variables. Then the optimal labor decision for each process depends on resources allocated to it, and the latter in turn depends on the production parameters of all processes. Hence, it makes a difference whether or not each firm knows the production parameters of all other firms. Accordingly, we compare the payoffs to the following two communication structures; full exchange of information (FEI) in which every agent (all process managers and the resource manager) are supplied with the production parameters of every process, and complete exchange with the center (CEC) in which each process manager supplies the resource manager with his production parameters but there is no exchange of information among the process managers. (Radner and Groves also considered two further communication structures, one in which no communication took place and one which turned out, in their context, to be equivalent to CEC. They also considered the possibility that the total resources might be initially uncertain, known in any realization to the resource manager but not to the process managers. In this paper, we do not consider these questions; in particular, total resource supply is assumed known to all a priori.)

Radner assumed that the output of each process was a concave quadratic function of the resource and labor decisions. Only the linear coefficients were uncertain. Among other results, he showed that as the number of firms became large, the difference in payoff between the FEI and CEC communication structures was bounded, but both payoffs were approaching infinity. Hence, he concluded, the value of the additional information in FEI becomes negligible as the number of firms becomes large. A careful look at the payoff formulas shows that these results hold even if total capital is held constant.\(^3\) Now this result is clearly surprising; with limited resources one should get only finite output. (To be sure, labor is not limited in this model; but if one subtracts the output obtainable from labor in the absence of capital, the difference still approaches infinity with fixed total capital.) The difficulty is that the resource allocations were not

\(^3\) See Radner [6, equations (5.3), (6.6), (6.7), and (7.1), pp. 227, 229, and 231]. The formulas have to be read carefully because of some notational assumptions; in effect, Radner’s formulas have to be read as the difference in payoff between that with the actual distributions of the parameters and that which would hold if each firm received, with certainty, the mean values of the parameters. Hence, the dispersion of the production parameters is, by itself, producing an unboundedly increasing team output.
constrained to be nonnegative. Large differences in the productivity-of-capital parameter between processes could be exploited by giving a negative allocation to the inefficient process and a corresponding positive allocation to the efficient process.

Hence, the specific results of Radner's paper cannot be accepted in a resource allocation context. But, as already seen, the idea that the additional information in FEI over CEC may be unimportant for large numbers of firms is intuitively reasonable. It is the purpose of this paper to establish this conclusion rigorously.

In Section 2, we illustrate the problem by a simple case, that in which each process is characterized by fixed coefficients. In Section 3, the model and its assumptions are presented, and the main result stated. The proof of the main result appears in Section 4.

In Section 5 we consider the case in which the number of firms is large compared to the total quantities of resources to be centrally allocated. We formalize this by letting the number of firms increase without limit while the vector of total resources remains fixed. In this case, total output will remain bounded; we shall show that the loss in expected total output due to using a complete exchange of information with the center, instead of full exchange of information, tends to zero as the number of firms increases.

In Section 6 we briefly consider the case in which the production parameters of the firms are statistically dependent. In general, full information will remain definitely superior, as measured by expected output per firm, even as the number of firms increases without limit. This is explored by using the concept of an exchangeable sequence of random variables.

2. AN EXAMPLE

A simple example will serve to illustrate both the formulation of the problem and some of the technical difficulties. Suppose that for each firm there is a single output and two inputs. The first input, the "resource," is centrally allocated by the resource manager, and the second input, "labor," is determined locally by the firm manager. For each firm there is a fixed-coefficients production function. There is also a fixed wage rate for labor, the same for all firms. The "net output" of the firm is

\[ F(k, l, t) = \min\{tk, l\} - wl, \]

where \( k \) is the input of the resource, \( l \) is the input of labor, \( t \) is a parameter that may vary from firm to firm, and \( w \), the wage rate, is a fixed parameter that is the same for all firms. Thus \( t \) is the output-resource ratio; it is also the labor-resource ratio if labor resources are used efficiently. Assume that \( 0 < w < 1 \), and that \( t \) varies in the unit interval. The inputs \( k \) and \( l \) must be nonnegative numbers.

Let there be \( I \) such firms, with a total quantity \( Ik \) of the resource to be allocated among them. For simplicity, we take \( \kappa = 1 \). Let \( \theta_i \) denote the particular value of the parameter \( t \) for firm \( i \). The numbers \( \theta_1, \ldots, \theta_I \) are assumed to be independent and identically distributed according to some probability distribution \( P \) on the
unit interval. The probability distribution $P$ is known to all the managers. We denote by $\theta^I$ the $I$-tuple $(\theta_i)$, i.e., the “random sample” of firm parameters.

The decision variable of firm manager $i$ is the quantity $l_i$ of his local input. The decision variable of the resource manager is the allocation $(k_1, \ldots, k_I)$ of the resource to the $I$ firms. This allocation must be nonnegative, and the total quantity of the resource allocated must not exceed $I$, i.e.,

$$\sum_i k_i \leq I. \tag{2.2}$$

The corresponding average net output per firm for the $I$ firms is

$$\frac{1}{I} \sum_i F(k_i, l_i, \theta_i). \tag{2.3}$$

In the case of full exchange of information (FEI), every decision variable is allowed to depend on the complete $I$-tuple $\theta^I$. If the goal is to maximize the total or average net output, then the optimal team decision function is easy to describe. First, for each firm, the quantity of local input should be adjusted optimally to its resource allocation, i.e., $l_i = \theta_i k_i$. The resulting net output for firm $i$ is

$$G(k_i, \theta_i) = \theta_i k_i (1 - w). \tag{2.4}$$

Thus $G(k, t)$ is the maximum output of a firm with resource $k$ and parameter $t$. Second, the total quantity, $I$, of the resource should be allocated among the firms so as to maximize

$$\frac{1}{I} \sum_i G(k_i, \theta_i) = \frac{1}{I} \sum_i \theta_i k_i (1 - w). \tag{2.5}$$

This can be accomplished, for example, by allocating all of the resource equally among all of the firms with the largest parameter value; the resulting average net output per firm is

$$(1 - w) \max_i \theta_i. \tag{2.6}$$

The expected value of (2.6) is the maximum expected average net output that is attainable under the full exchange of information; we denote this by

$$\omega_{FEI}(I) = (1 - w)E_{\theta^I} \max \{\theta_i : i = 1, \ldots, I\}. \tag{2.7}$$

Note that $\omega_{FEI}(I)$ depends on $I$, the number of firms.

In the case of complete exchange with the center (CEC), the resource allocation is allowed to depend on the complete $I$-tuple, $\theta^I$, but each firm’s quantity of local input may depend only on its own parameter, $\theta_i$. This represents the situation in which each firm reports its own parameter value to the resource manager before the latter determines his allocation, but each firm manager must decide on the quantity of his own local input before he knows how much of the resource he will get. The expected average net output under such an information structure may be
represented as

\begin{align}
(2.8) \quad E_\theta \frac{1}{I} \sum_i F[K_i(\theta^I), L_i(\theta_i), \theta_i],
\end{align}

where the functions \( K_i \) and \( L_i \) are constrained to be nonnegative, and

\begin{align}
(2.9) \quad \frac{1}{I} \sum_i K_i(\theta^I) \leq 1,
\end{align}

for every \( \theta^I \) (or with probability 1). The maximum (or supremum) of (2.8), subject to these constraints, will be denoted by \( \omega_{CEC}(I) \). If the probability distribution \( P \) is concentrated on finitely many \( t \) values, then there are only finitely many unknowns in (2.8), and the supremum \( \omega_{CEC} \) will be attained; the problem of maximizing (2.8) can be formulated as a linear programming problem. (We do not know for what more general class of probability distributions \( P \) on the unit interval the supremum \( \omega_{CEC} \) can actually be attained.)

Since the class of decision functions available to the team under CEC is essentially contained in the class of decision functions available under FEI, it is clear that \( \omega_{CEC}(I) \) cannot exceed \( \omega_{FEI}(I) \), and it will typically be strictly smaller. However, an implication of the main result of this paper is that, as \( I \) increases without limit, \( \omega_{CEC}(I) \) and \( \omega_{FEI}(I) \) both approach the same limit. In other words, for a sufficiently large number of firms, the loss in expected average net output per firm due to the incompleteness of information inherent in CEC can be made negligible by a suitable choice of CEC team decision functions.

We now sketch how this result can be proved in the context of the present example. We shall consider two special cases regarding the probability distribution \( P \) of the parameter \( t \); (1) the probability distribution \( P \) is concentrated on a finite number of \( t \) values; (2) \( P \) is the uniform distribution on the unit interval.

For the first case suppose that \( P \) is concentrated on the (distinct) points \( t_1, \ldots, t_N \) in the unit interval, with corresponding (strictly positive) probabilities \( p_1, \ldots, p_N \). For simplicity, we assume that \( t_N = 1 \). As \( I \) increases without limit,

\begin{align}
E_\theta \max \{ \theta_i : i = 1, \ldots, I \}
\end{align}

converges to 1; hence, from (2.7),

\begin{align}
(2.10) \quad \lim_{I \to \infty} \omega_{FEI}(I) = 1 - w.
\end{align}

We may interpret \((1 - w)\) as the maximum possible average output per firm for an infinite population of firms in which the parameter \( t \) has the relative frequency function \((p_n)\), and the average resource per firm is constrained not to exceed unity.

We shall now consider a particular sequence of CEC team decision functions, and shall show that the expected average output per firm for this sequence of decision functions also approaches \( (1 - w) \) as \( I \) increases without limit. To motivate this particular choice of decision functions, we note that in the case of the FEI information structure, if the relative frequency of firms for which \( \theta_i = t_N = 1 \) is \( q^I_N \), and \( q^I_N \) is strictly positive, then each such firm will receive an allocation of the
resource equal to $(1/q^t_N)$, and this will also be equal to the quantity of the local input. For all firms with $\theta_i < 1$, the resource and local inputs will both be zero (provided $q^t_N > 0$). By the Law of Large Numbers, $q^t_N$ converges to $p_N$ as $I$ increases without limit, so that "in the limit" each firm for which $t = 1$ will have a local input equal to $(1/p_N)$. This suggests the following sequence of CEC team decision functions: For any fixed $I$,

$$L_i(\theta_i)= \begin{cases} 1/p_N & \text{if } \theta_i = 1, \\ 0 & \text{if } \theta_i < 1; \end{cases}$$

(2.11a)

the total resource is divided equally among all those firms, if any, for which $\theta_i = 1$; if there are no such firms, the resource is thrown away.

With this decision function, the average output per firm is

$$q^t_i \left[ \min \left( \frac{1}{q^t_i}, \frac{1}{p_N} \right) - \frac{w}{p_N} \right] = \min \left( 1, \frac{q^t_N}{p_N} \right) - \frac{q^t_N w}{p_N},$$

where it is understood that this is zero if $q^t_N$ is zero. Appealing again to the Law of Large Numbers, we see that (2.12) converges (almost surely) to $(1-w)$ as $I$ increases without limit; hence so does its expected value (since it is bounded). Thus the expected average output per firm for the decision functions (2.11) converges to $(1-w)$. But $\omega_{CEC}(I)$ is at least as large as the expected value of (2.12), and not greater than $\omega_{FEI}(I)$, so that $\omega_{CEC}(I)$ also converges to $(1-w)$.

We turn now to the second case, in which $P$ is the uniform distribution on the unit interval. Equation (2.7) is still valid for $\omega_{FEI}(I)$. Note however that, for any $I$, there will be only one firm with the maximum parameter value in the sample (with probability 1), and the probability that this maximum is 1 is zero. Nevertheless, for the uniform distribution,

$$E_{\theta} \max \{\theta_i : i = 1, \ldots, I\} = \frac{I}{I+1},$$

(2.13)

so that (2.10) is valid for the uniform distribution, too. On the other hand, the interpretation of the limit in terms of an optimal allocation for an infinite population of firms is not so straightforward as in the case of a $P$ with finite support. We might pose the problem for the infinite population as follows: Find a function $K(\cdot)$ on the unit interval that maximizes the integral

$$\int_0^1 tK(t)(1-w) \, dt,$$

subject to the constraints

$$\int_0^1 K(t) \, dt = 1, \quad K(\cdot) \geq 0.$$

(2.14)

Unfortunately, although the supremum of (2.14) subject to (2.15) exists and equals $(1-w)$, this supremum is not attained by any function satisfying (2.15). We
might say that, in the limit, “the firm with the maximum value of \( t \) (i.e., \( t = 1 \)) should get an infinite quantity of the resource.”

Nevertheless, in this case, too, we can establish that

\[
\lim_{I \to \infty} \omega_{\text{CEC}}(I) = 1 - w.
\]

The idea is similar to the one used in the case of \( P \) with finite support, but a suitable approximation must be made. Let \( c \) be a positive number less than 1, and let \( S \) denote the interval \([1 - c, 1]\). For any sample of \( I \) firms, let \( r^I \) denote the relative frequency of firms for which \( \theta_i \) is in \( S \). Consider the following sequence of team decision functions for the CEC information structure: For any fixed \( I \),

\[
\begin{align*}
(2.17a) \quad L_i(\theta_i) = & \begin{cases} 
1/c & \text{if } \theta_i \text{ is in } S, \\
0 & \text{otherwise}; 
\end{cases} \\
(2.17b) \quad K_i(\theta_i) = & \begin{cases} 
1/r^I & \text{if } \theta_i \text{ is in } S \text{ and } r^I > 0, \\
0 & \text{otherwise}. 
\end{cases}
\end{align*}
\]

The corresponding average output per firm, given \( \theta^I = (\theta_1, \ldots, \theta_I) \), is

\[
A^c_I = \frac{1}{I} \sum_{\theta_i \in S} \left[ \min \left( \frac{\theta_i}{r^I}, \frac{1}{c} \right) - \frac{w}{c} \right] 
\]

\[
\geq \frac{1}{I} \sum_{\theta_i \in S} \left[ \min \left( \frac{1-c}{r^I}, \frac{1}{c} \right) - \frac{w}{c} \right] 
\]

\[
= r^I \left[ \min \left( \frac{1-c}{r^I}, \frac{1}{c} \right) - \frac{w}{c} \right].
\]

By the Law of Large Numbers, \( r^I \) converges to \( c \) (almost surely), so that the last expression converges to

\[
c \left[ \min \left( \frac{1-c}{c}, \frac{1}{c} \right) - \frac{w}{c} \right] = 1 - c - w.
\]

In other words

\[
\lim \inf_{I \to \infty} A^c_I \geq 1 - c - w, \quad \text{almost surely.}
\]

Since \( A^c_I \) is bounded, it is also true that

\[
\lim \inf_{I \to \infty} E A^c_I \geq 1 - c - w.
\]

But \( \omega_{\text{CEC}}(I) \) is at least as large as \( E A^c_I \), so we have shown that, for every \( c > 0 \),

\[
\lim \inf_{I \to \infty} \omega_{\text{CEC}}(I) \geq 1 - c - w,
\]

or simply,

\[
\lim \inf_{I \to \infty} \omega_{\text{CEC}}(I) \geq 1 - w.
\]
Therefore, since $\omega_{CEC}(I)$ cannot exceed $\omega_{FEI}(I)$, we have again that

\begin{equation}
\lim_{I \to \infty} \omega_{CEC}(I) = \lim_{I \to \infty} \omega_{FEI}(I).
\end{equation}

Notice that the expected value of $A^e_t$ could have been calculated exactly, but the inequality in (2.18) permitted us to avoid such an exact calculation. Another approximation serves the same purpose, and is more suggestive of how to proceed in the more general case. Suppose that in the expression for $A^e_t$ we replace every $\theta_i$ (in $S$) by 1; this will result in an error of at most $c$. For any $t$ in $S$ and any positive number $r$,

$$0 \leq \min\left(\frac{1}{r}, \frac{1}{c}\right) - \min\left(\frac{t}{r}, \frac{1}{c}\right) \leq \frac{c}{r}.$$ 

Hence

\begin{equation}
A^e_t \geq \frac{1}{I} \sum_{\theta_i \in S} \left[ \min\left(\frac{1}{r}, \frac{1}{c}\right) - \frac{w}{c} - \frac{c}{r} \right] 
= \min\left(1, \frac{r^I}{c}\right) - \frac{r^I w}{c} - c,
\end{equation}

which converges to the same limit as does (2.18). In fact, this same approximation could have been obtained by replacing each $\theta_i$ in $S$ by any other point in $S$.

3. GENERAL STATEMENT OF THE MAIN RESULT

The output of an individual firm is $F(k, l, t)$, where:

- $k$ is the vector of resources centrally allocated to the firm, a vector in the nonnegative part, $K$, of a Euclidean space, $\mathbb{R}^P$;
- $l$ is the local decision of the firm manager, a point in a closed convex subset, $L$, of a Euclidean space; $0 \in L$ (a convention);
- $t$ is the parameter that characterizes the individual firm’s production function; $t$ is a point in a compact metric space $T$;
- $F$ is the firm’s production function, a real-valued function on $K \times L \times T$.

Let $P$ be a probability measure on the Borel sets of $T$, such that $T$ is the support of $P$. (All functions on $T$ are to be understood to be Borel-measurable, and all probability measures on $T$ to be on the Borel sets of $T$.)

For any vector, $x$, we define $\|x\| = \sum |x_i|.$

We shall make the following assumptions about $F$:

**Assumption 1:** $F$ is continuous on $K \times L \times T$.

**Assumption 2:** For every $l$ in $L$ and $t$ in $T$, $F(\cdot, l, t)$ is nondecreasing on $K$, and $F(\cdot, \cdot, t)$ is concave on $K \times L$.

**Assumption 3:** For every $k$ in $K$ and $t$ in $T$

$$G(k, t) = \max_{l \in L} F(k, l, t)$$
exists; furthermore, this maximum is achieved at a point \( \tilde{L}(k, t) \) in \( L \) such that 
\[ \|\tilde{L}(k, t)\| \leq \lambda \|k\|, \]
where \( \lambda \) is a positive number independent of \( k \) and \( t \).

**Assumption 4**: If \( \|k_n\| \) diverges to \( +\infty \), \( k_n/\|k_n\| \) converges to \( \bar{k} \), and \( t_n \) converges to \( \bar{t} \), then \( G(k_n, t_n)/\|k_n\| \) converges to a number that depends only on \( \bar{k} \) and \( \bar{t} \).

**Full Exchange of Information (FEI)**

There are \( I \) firms, labeled \( 1, \ldots, I \), with respective parameters \( \theta_1, \ldots, \theta_I \), where the latter are independently and identically distributed on \( T \) with common probability distribution \( P \). There is a vector \( \kappa \) of total resources to be allocated among the \( I \) firms (\( \kappa \), a known vector in \( K \), represents the average resource per firm). A team decision function for FEI is a \( 2I \)-tuple of functions \( (K_1, \ldots, K_I, L_1, \ldots, L_I) \) on \( T^I \) such that each \( K_i \) takes values in \( K \), each \( L_i \) takes values in \( L \), and for every \( \theta^I = (\theta_1, \ldots, \theta_I) \) in \( T^I \),
\[
\sum_i K_i(\theta^I) \leq \kappa.
\]
The expected average output per firm for such a team decision function is
\[
E_{\theta^I}(\frac{1}{I}) \sum_i F[K_i(\theta^I), L_i(\theta^I), \theta_i],
\]
where the expectation is with respect to \( \theta^I \), i.e., with respect to the product measure on \( T^I \) derived from \( P \). Let \( \omega_{FEI}(I) \) denote the maximum expected average output *per firm* that can be achieved using team decision functions for FEI.

**Complete Exchange with the Center (CEC)**

The situation is the same as in the case of FEI, except that each \( L_i \) is further constrained to be a function of \( \theta_i \) only, i.e., each \( L_i \) is a function from \( T \) to \( L \), and the expected average output is
\[
E_{\theta^I}(\frac{1}{I}) \sum_i F[K_i(\theta^I), L_i(\theta_i), \theta_i].
\]
Let \( \omega_{CEC}(I) \) denote the supremum of the expected average output per firm that can be achieved using only team decision functions for CEC.

Clearly, for every \( I \),
\[ \omega_{CEC}(I) \leq \omega_{FEI}(I), \]
since the set of team decision functions for CEC is equivalent to a subset of team decision functions for FEI.

**Our main result is:**
\[
\lim_{I \to \infty} \omega_{CEC}(I) = \lim_{I \to \infty} \omega_{FEI}(I).
\]
In other words, for large $I$, the information structure CEC is almost as good, per firm, as the complete information structure FEI.\footnote{Although (3.4) is stated in terms of expected values, we actually show that the average outputs under FEI and CEL converge to a common value almost surely.}

The main result will be demonstrated by showing that each of the limits in (3.4) is equal to the solution of an auxiliary maximization problem, which can be interpreted as the maximum output per firm for FEI with "infinitely many" firms.

Recall that

\[(3.5) \quad G(k, t) = \max_{l \in L} F(k, l, t).\]

The function $G$ gives the output of a firm when the local decision is optimally adjusted to a given resource vector and parameter. We shall show that $G$ is continuous on $K \times T$, that for every $t$, $G(\cdot, t)$ is concave and nondecreasing on $K$, and that there exist positive $\bar{a}$ and $\bar{b}$ such that

\[(3.6) \quad G(k, t) \leq \bar{a} + \bar{b} \cdot k.\]

We may write $\omega_{FEI}(I)$ in the form

\[(3.7) \quad \omega_{FEI}(I) = E_{\theta^I} \max \left\{ \frac{1}{I} \sum G(k_i, \theta_i) : \frac{1}{I} \sum k_i \leq \kappa, k_i \geq 0 \right\}.\]

By the symmetry among firms and the concavity of $G$ in $k$, the maximum in (3.7) can be achieved by allowing $k_i$ to depend only on $\theta_i$ and the empirical distribution of $\theta_1, \ldots, \theta_I$, but not on $i$. That is to say, for any subset $S$ of $T$, let $P_I(S; \theta^I)$ denote the relative frequency of firms $i$ for which $\theta_i$ is in $S$; also, let $\mathcal{P}$ denote the set of all probability measures on $T$; then there is a function $\hat{K}_I$ from $T \times \mathcal{P}$ to $K$ such that

\[(3.8) \quad \omega_{FEI}(I) = E_{\theta^I} \left( \frac{1}{I} \sum G(\hat{K}_I(\theta, P_I[\cdot; \theta^I]), \theta_i) \right).\]

This last can be rewritten formally as

\[(3.9) \quad \omega_{FEI}(I) = E_{\theta^I} \int_T G[\hat{K}_I(t, P_I[\cdot; \theta^I]), t]P_I(dt; \theta^I).\]

Note that the expression,

\[(3.10) \quad \int_T G[\hat{K}_I(t, P_I[\cdot; \theta^I]), t]P_I(dt; \theta^I),\]

is the maximum average output per firm, given $I$, the number of firms, and $P_I(\cdot; \theta^I)$, the empirical probability measure corresponding to the "random sample" $\theta^I = (\theta_1, \ldots, \theta_I)$. The expression (3.10) is itself a random variable, since it depends on the random sample $\theta^I$, and its expected value is equal to $\omega_{FEI}(I)$.

By analogy, let $\mathcal{K}$ denote the set of all nonnegative functions $K$ from $T$ to $K$
such that

\[ (3.11) \quad \int_T K(t)P(dt) \leq \kappa, \]

and define

\[ (3.12) \quad V(K) = \int_T G[K(t), t]P(dt). \]

Equation (3.12) may be interpreted as representing the average output per firm for an infinite population of firms in which, for any measurable subset \( S \) of \( T \), \( P(S) \) is the relative frequency of firms whose parameter \( t \) lies in \( S \), and \( \int_S K(t)P(dt) \) is the average allocation of resources per firm for those firms.

By (3.6), for any \( K \) in \( \mathcal{K} \),

\[ (3.13) \quad V(K) \leq \bar{a} + \bar{b} \cdot \kappa. \]

Hence \( V(K) \) is bounded on \( \mathcal{K} \); let

\[ (3.14) \quad \omega = \sup \{ V(K): K \in \mathcal{K} \}. \]

We shall prove the main result, (3.4), by showing that each side of (3.4) is equal to \( \omega \).

4. PROOF OF THE MAIN RESULT

**Lemma 4.1:** (i) \( G \) is continuous on \( K \times T \); (ii) for every \( t \), \( G(\cdot, t) \) is concave and nondecreasing on \( K \); (iii) there are positive \( \bar{a} \) and \( \bar{b} \) such that, for all \( k \) and \( t \),

\[ (4.1) \quad G(k, t) \leq \bar{a} + \bar{b} \cdot k. \]

**Proof:** By Assumption 3, for every \( k \) and \( t \),

\[ G(k, t) = \max \{ F(k, l, t): l \in L \}. \]

Hence, by one form of the "Theorem of the Maximum" (e.g., Berge [1, p. 123]), it follows from the continuity of \( F \) that \( G \) is continuous. Part (ii) of the lemma is now immediate from the continuity of \( G \) and Assumption 2.

We now prove part (iii). Let \( 1 \) denote the vector in \( \mathbb{R}^D \) all of whose coordinates are unity. For every \( t \), \( G(\cdot, t) \) is concave and continuous; hence one can show that there exist \( \alpha(t) \) in \( \mathbb{R} \) and \( \beta(t) \) in \( \mathbb{R}^D \) such that

\[ (4.2) \quad G(k, t) \leq \alpha(t) + \beta(t) \cdot k, \quad \text{for all } k \in K, \]

\[ (4.3) \quad G(1, t) = \alpha(t) + \beta(t) \cdot 1. \]

The pair \([ \alpha(t), \beta(t) ]\) corresponds to a hyperplane in \( \mathbb{R}^{D+1} \) that is tangent to the graph of \( G(\cdot, t) \) at the point \([1, G(1, t)]\). Since \( G(\cdot, t) \) is nondecreasing, \( \beta(t) \) is nonnegative. Since the vector 0 is in \( K \), (4.2) implies that \( \alpha(t) \geq G(0, t) \). But \( G(0, \cdot) \) is continuous on \( T \), which is compact, and hence attains a minimum, say \( g_0 \),
on $T$; hence

(4.4) $\alpha(t) \geq g_0$, for all $t$ in $T$.

Similarly, $G(1, t)$ attains a maximum, say $g_1$, on $T$, so that from (4.3),

(4.5) $\alpha(t) + \beta(t) \cdot 1 \leq g_1$, all $t$ in $T$.

Combining (4.4) and (4.5) with the nonnegativity of $\beta(t)$, we see that the set of pairs $[\alpha(t), \beta(t)]$ is bounded. The proof of part (iii) of the lemma is completed by taking

$$\tilde{a} = \sup_{t} \alpha(t),$$

$$\tilde{b}_i = \sup_{t} \beta_i(t),$$

$$\tilde{b} = (\tilde{b}_1, \ldots, \tilde{b}_D).$$

**Lemma 4.2:** For every $\varepsilon > 0$ there is a $\delta_\varepsilon > 0$ such that, for every $k$ in $K$, and every $t$ and $t'$ in $T$ whose distance does not exceed $\delta_\varepsilon$,

(4.6) $|G(k, t) - G(k, t')| \leq \varepsilon \max (1, \|k\|)$.

**Proof:** Suppose, to the contrary, that there exists a sequence $(t_n, t'_n, k_n)$ such that the distance between $t_n$ and $t'_n$ converges to zero, but

(4.7) $|G(k_n, t_n) - G(k_n, t'_n)| > \varepsilon \max (1, \|k_n\|)$.

Since $T$ is compact, there is a subsequence for which both $(t_n)$ and $(t'_n)$ converge; thus, without loss of generality, we may assume that the original sequences $(t_n)$ and $(t'_n)$ both converge to, say, $\tilde{t}$.

**Case 1:** $\|k_n\|$ is bounded. There is a subsequence for which $k_n$ converges; then, for $n$ in this subsequence, (4.7) contradicts the continuity of $G$ on $K \times T$.

**Case 2:** $\|k_n\|$ is unbounded. Let $N$ be a subsequence for which $\|k_n\|$ diverges to $+\infty$ but $k_n/\|k_n\|$ converges. By Assumption 4,

$$\lim_{n \in N} \frac{G(k_n, t_n)}{\|k_n\|} = \lim_{n \in N} \frac{G(k_n, t'_n)}{\|k_n\|},$$

which contradicts (4.7).

**Lemma 4.3:** $\limsup_{I \to \infty} \omega_{FEI}(I) \leq \omega$.

**Proof:** By Lemma 2, for every $\varepsilon > 0$ there exists a finite partition $\{T^n_\varepsilon\}$ on $T$ such that, for all $n$, and for all $t$ and $t'$ in $T^n_\varepsilon$,

$$|G(k, t) - G(k, t')| \leq \varepsilon \max (1, \|k\|), \quad \text{all } k \text{ in } K.$$
For each \( n \), take a point \( t^e_n \) in \( T^e_n \). For \( I \) firms with parameters \( \theta_1, \ldots, \theta_I \), define
\[
\tilde{\theta}_i = t^e_n \quad \text{if } \theta_i \text{ is in } T^e_n.
\]

Hence, for any \( k_1, \ldots, k_I \) in \( K \) for which \( \sum_i k_i = I\kappa \),
\[
\frac{1}{I} \sum_i [G(k_i, \tilde{\theta}_i) - G(k_i, \tilde{\theta}_i)] \leq \frac{\varepsilon}{I} \sum_i \max(1, \|k_i\|)
\]
\[
\leq \frac{\varepsilon}{I} \left[ I + \sum_i \|k_i\| \right]
\]
\[
= \varepsilon (1 + \|\kappa\|),
\]
which implies that
\[
(4.8) \quad \frac{1}{I} \sum_i G(k_i, \theta_i) - \frac{1}{I} \sum_i G(k_i, \tilde{\theta}_i) + \varepsilon (1 + \|\kappa\|).
\]

As in Section 3, let \( P_I(S; \theta) \) denote the relative frequency of firms \( i \) for which \( \theta_i \) is in \( S \), given \( \theta^I = (\theta_1, \ldots, \theta_I) \), for any subset \( S \) of \( T \). Define
\[
q^I_n = P_I(T^e_n; \theta^I),
\]
\[
q^I = (q^I_1, \ldots, q^I_{N_\varepsilon}),
\]
where \( N_\varepsilon \) is the number of elements in the partition \( \{T^e_n\} \). Thus \( q^I \) is a random probability vector, whose probability distribution depends on \( I \), the number of firms.

For any \( N_\varepsilon \)-dimensional probability vector \( q = (q_n) \) define
\[
W(q) = \max \left\{ \sum_n q_n G(k_n, t^e_n): \sum_n q_n k_n \leq \kappa, \ k_n \geq 0 \right\}.
\]

The first term on the right side of (4.8) does not exceed \( W(q^I) \); hence, from (3.7),
\[
(4.11) \quad \omega_{FEI}(I) \leq EW(q^I) + \varepsilon (1 + \|\kappa\|).
\]

Let \( P^*_n \) denote \( P(T^e_n) \), and \( p^e = (p^e_n) \). By the Strong Law of Large Numbers, \( q^I \) converges to \( p^e \) almost surely. The function \( W(\cdot) \) is continuous on a compact set, and therefore, by the Lebesgue Dominated Convergence Theorem,
\[
(4.12) \quad \lim_{I \to \infty} EW(q^I) = W(p^e).
\]

Suppose that \( W(p^e) \) is achieved by the \( N \)-tuple \( (k^e_n) \), i.e.,
\[
W(p^e) = \sum_n p^*_n G(k^e_n, t^e_n).
\]

Define the simple function \( K^e \) in \( \mathcal{H} \) by
\[
K^e(t) = k^e_n \quad \text{if } t \text{ is in } T^e_n.
\]
By an argument similar to the one leading to (4.8), one can show that

$$W(p^e) \leq \int G[K^e(t), t]P(dt) + \varepsilon (1 + \|\kappa\|),$$

and the first term on the right side of this last expression does not exceed $\omega$. Therefore

$$W(p^e) \leq \omega + \varepsilon (1 + \|\kappa\|).$$

Combining (4.11), (4.12), and (4.13), we get the result that, for every $\varepsilon > 0$,

$$\limsup_{I \to \infty} \omega_{FEI}(I) \leq \omega + 2\varepsilon (1 + \|\kappa\|),$$

from which the conclusion of Lemma 3 follows immediately.

Recall that

$$V(K) = \int G[K(t), t]P(dt),$$

$$\omega = \sup \{V(K): K \text{ in } \mathcal{H}\}.$$

**Lemma 4.4:** For every positive $\varepsilon$ there is a bounded function $K_\varepsilon$ in $\mathcal{H}$ such that $V(K_\varepsilon) \geq \omega - \varepsilon$.

**Proof:** There is a sequence $(K'_m)$ of functions in $\mathcal{H}$ such that $V(K'_m)$ converges to $\omega$. For every pair of positive integers $m$ and $n$ define

$$K''_{mn}(t) = \min \{K'_m(t), n\kappa\},$$

where the minimum is taken coordinatewise. As $n$ increases without limit, with $m$ fixed, $K''_{mn}$ converges to $K'_m$ pointwise; also, since $G(\cdot, t)$ is nondecreasing on $K$,

$$G[K''_{mn}(t), t] \leq G[K'_m(t), t].$$

Hence, by the continuity of $G(\cdot, t)$ on $K$, and the Lebesgue Dominated Convergence Theorem, for every $m$,

$$\lim_{n \to \infty} V(K''_{mn}) = V(K'_m).$$

For every positive number $\varepsilon$ there is an $m_\varepsilon$ for which

$$V(K'_m) \geq \omega - \frac{\varepsilon}{2},$$

and by (4.14) there is an $n_\varepsilon$ such that

$$V(K''_{mn_\varepsilon}) \geq V(K'_m) - \frac{\varepsilon}{2}.$$

Take $K_\varepsilon = K_{m_\varepsilon n_\varepsilon}$. Then

$$V(K_\varepsilon) \geq \omega - \varepsilon,$$
which completes the proof of Lemma 4.4. Note that \( K_e \) is bounded above by \( n_e \).

**Lemma 4.5:** \( \liminf_{I \to \infty} \omega_{CEC}(I) \geq \omega. \)

**Proof:** Consider the sequence \((K_e)\) in Lemma 4.4, and fix \( \varepsilon \). By Assumption 3 there is a function \( L_e \) from \( T \) to \( L \) such that, for all \( t \),

\[
G[K_e(t), t] = F[K_e(t), L_e(t), t],
\]

\[
\|L_e(t)\| \leq \lambda \|K_e(t)\| \leq \lambda n_e \|\kappa\|.
\]

Let \( K_e \) be the set of \( k \) in \( K \) with \( \|k\| \leq n_e \|\kappa\| \), and let \( L_e \) be the set of \( l \) in \( L \) with \( \|l\| \leq \lambda n_e \|\kappa\| \). The function \( F \) is continuous on \( K_e = L_e \times T \), and hence uniformly continuous. Hence, there is an \( \eta_e > 0 \) such that, for all \( k \) in \( K_e \) and \( l \) in \( L_e \), and all \( t \) and \( t' \) in \( T \) whose distance does not exceed \( \eta_e \),

\[
|F(k, l, t) - F(k, l, t')| \leq \varepsilon.
\]

Let \( \{S_m\} \) be a finite partition of \( T \) such that no \( S_m \) has diameter greater than \( \eta_e \); also, let \( s_m \) be any point in \( S_m \). (Note that \( \{S_m\} \) and \( \{s_m\} \) depend on \( \varepsilon \).)

For any \( m \) for which \( P(S_m) > 0 \), define

\[
k_m = \frac{1}{P(S_m)} \int_{S_m} K_e(t)P(dt),
\]

\[
l_m = \frac{1}{P(S_m)} \int_{S_m} L_e(t)P(dt);
\]

for other \( m \) choose \((k_m, l_m)\) arbitrarily in \( K_e \times L_e \). By the construction of \( \{S_m\} \) and \( \{s_m\} \), for every \( m \) and every \( t \) in \( S_m \),

\[
F[K_e(t), L_e(t), s_m] \geq F[K_e(t), L_e(t), t] - \varepsilon.
\]

By the concavity of \( F(\cdot, \cdot, s_m) \),

\[
P(S_m)F(k_m, l_m, s_m) \geq \int_{S_m} F[K_e(t), L_e(t), s_m]P(dt).
\]

Hence, combining (4.15), (4.17), and (4.18), we get

\[
\sum_{m} P(S_m)F(k_m, l_m, s_m) \geq V(K_e) - \varepsilon.
\]

We now proceed to construct a particular team decision function for the CEC information structure whose expected average output per firm will converge to something at least as large as the left side of (4.19) as the number of firms increases without limit. Let \( M \) be the number of elements in the partition \( \{S_m\} \). For any \( M \)-dimensional probability vector \( r = (r_1, \ldots, r_M) \), define \( Z(r) \) to be an \( M \)-tuple \( z = (z_m) \) that maximizes

\[
\sum_{m} r_m F(z_m, l_m, s_m)
\]
subject to the constraints:

\[ z_m \text{ in } K_\varepsilon \quad (m = 1, \ldots, M); \]

\[ \sum_m r_m z_m \leq \kappa; \]

let \( U(r) \) denote the corresponding maximum. For any \( I \)-fold sample \( \theta^I = (\theta_1, \ldots, \theta_I) \), let \( r^I_m \) be the relative frequency of firms for which \( \theta_i \) is in \( S_m \), i.e.,

\[ r^I_m = P(S_m; \theta^I), \]

and let \( r^I = (r^I_1, \ldots, r^I_M) \). (Note that \( r^I \) depends on \( \theta^I \).)

Consider the following team decision function: for all \( \theta_i \) in \( S_m \),

\[ K_i(\theta^I) = Z_m(r^I), \]

\[ L_i(\theta_i) = l_m. \]

The corresponding average output per firm is

\[ \frac{1}{I} \sum_{m \theta_i \in S_m} F[Z_m(r^I), l_m, \theta_i]. \]

By the construction of \( \{S_m\} \) and \( \{s_m\} \), (4.21) is at least as large as

\[ \frac{1}{I} \sum_{m \theta_i \in S_m} \{F[Z_m(r^I), l_m, s_m] - \varepsilon\}, \]

i.e., at least as large as \( U(r^I) - \varepsilon \). Hence

\[ \omega_{CEC}(I) \geq E U(r^I) - \varepsilon. \]

Let \( p_m = P(S_m) \) and \( p = (p_m) \). By the Strong Law of Large Numbers, \( r^I \) converges to \( p \) almost surely as \( I \) increases without limit. By the “Theorem of the Maximum” (again, see Berge [1, p. 123]), \( U \) is continuous (on the compact set of \( M \)-dimensional probability vectors); appealing to Lebesgue once again, we have

\[ \lim_{I \to \infty} E U(r^I) = U(p). \]

But \( U(p) \) is at least as large as the left side of (4.19). Hence, combining this fact with (4.19), (4.22), and (4.23), we have proved that, for every positive \( \varepsilon \),

\[ \liminf_{I \to \infty} \omega_{CEC}(I) \geq U(p) - \varepsilon \]

\[ \geq V(K_\varepsilon) - 2\varepsilon. \]

By letting \( \varepsilon \to 0 \) in (4.24) one completes the proof of the lemma.

Putting together Lemmas 4.3 and 4.5, we have

\[ \limsup_{I \to \infty} \omega_{FEI}(I) \leq \omega \leq \liminf_{I \to \infty} \omega_{CEC}(I). \]
On the other hand, for every $I$,

$$\omega_{CEC}(I) \leq \omega_{FEI}(I).$$

It follows that

$$\lim_{I \to \infty} \omega_{CEC}(I) = \lim_{I \to \infty} \omega_{FEI} = \omega,$$

which is our main result, (3.4).

5. THE CASE IN WHICH THE NUMBER OF FIRMS INCREASES WITH TOTAL RESOURCES FIXED

We now consider the case in which the number of firms is large compared to the total quantities of resources to be centrally allocated. We formalize this by letting the number of firms increase without limit while the vector of total resources remains fixed. In this case, total output will remain bounded; we shall show that the loss in expected total output due to using a complete exchange of information with the center (CEC), instead of full exchange of information (FEI), tends to zero as the number of firms increases.

To simplify the argument, we restrict ourselves in this section to the case in which the set $T$ of possible parameter values $t$ is finite. Let $p(t)$ denote $\text{Prob}(\theta_i = t)$, and take $p(t) > 0$ for all $t$. As the number $I$ of firms increases without limit, the number of firms of each type $t$, say, $N_I(t)$, will also increase without limit, by the Law of Large Numbers. Since all firms of the same type will get the same vector of resources,\(^5\) this implies that, as $I$ increases, the resource vectors allocated to individual firms will tend towards zero. Thus, when the number of firms is large, the total output for each type $t$ will be determined by (1) the total number $N_I(t)$ of firms of that type, (2) the marginal productivity of the allocated resources near zero input (i.e., at very small scale), and (3) the local decisions. In a sense, if the number of firms is large, each type will behave approximately as an industry with constant returns to scale.

We retain the model of Section 3, except that the resource constraint (3.1) is replaced by

$$\sum_{i=1}^{I} K_i \leq \kappa,$$

where $\kappa$ is the same for all $I$, and the performance criterion is expected total output.

Full Exchange of Information

Let $\theta$ denote the infinite sequence $(\theta_1, \theta_2, \ldots)$ of random variables $\theta_i$, independent and identically distributed on $T$ (finite), with

$$p(t) = \text{Prob}(\theta_i = t).$$

\(^5\) Under assumptions of symmetry and concavity.
ALLOCATION OF RESOURCES

Define

\[(5.3) \quad \Omega_I(\theta) = \max \sum_{i=1}^{I} F(k_i, l_i, \theta_i),\]

subject to

\[k_i \in K, \quad l_i \in L, \quad \sum_{i=1}^{I} k_i = \kappa;\]

\[(5.4) \quad \omega_{FEI}(I) = E\Omega_I(\theta).\]

We maintain assumptions (1)–(3) of Section 3. Hence, \(\Omega g I(\theta)\) is well defined and is uniformly bounded, so that \(\omega_{FEI}(I) < \infty\). Furthermore, for every \(\theta\), \(\Omega_I(\theta)\) is nondecreasing in \(I\). Hence,

\[(5.5) \quad \lim_{I \to \infty} \Omega_I(\theta) = \Omega(\theta)\]

exists for every \(\theta\), and is uniformly bounded, and hence,

\[(5.6) \quad \lim_{I \to \infty} \omega_{FEI}(I) = E\Omega(\theta).\]

Indeed, we shall show that, almost surely, \(\Omega(\theta) = E\Omega(\theta)\), i.e., \(\Omega(\theta)\) has the same value for almost all \(\theta\).

Let \(N_I(t)\) denote the total number of firms among the first \(I\) for which \(\theta_i = t\). Recall that we may take the values of \(k_i\) and \(l_i\) that achieve the maximum (5.3) to be the same, respectively, for all firms of the same type \(t\). Denote such optimal values by \(\hat{k}_I(t)\) and \(\hat{l}_I(t)\), respectively. Keep in mind that \(N_I(t), \hat{k}_I(t),\) and \(\hat{l}_I(t)\) are all random variables, i.e., depend on \(\theta\). Define

\[(5.7) \quad k_I(t) = N_I(t)\hat{k}_I(t), \quad l_I(t) = N_I(t)\hat{l}_I(t), \quad y_I(t) = [k_I(t), l_I(t)].\]

(We may interpret \(k_I(t)\) as the total resource vector allocated to all firms of type \(t\).) With this notation we may write

\[(5.8) \quad \Omega_I(\theta) = \sum_{t} N_I(t) F[y_I(t)/N_I(t), t].\]

By the Law of Large Numbers,

\[(5.9) \quad \lim_{I \to \infty} \frac{N_I(t)}{I} = p(t) > 0,\]

almost surely, so that, in particular,

\[(5.10) \quad \lim_{I \to \infty} N_I(t) = +\infty.\]

The vectors \(k_I(t)\) are uniformly bounded, and therefore, by Assumption 3 of Section 3, we may also take the vectors \(l_I(t)\) to be uniformly bounded. It follows
that, for every \( t \), \( y_I(t)/N_I(t) \) will converge to zero almost surely. Rewrite (5.8) as

\[
\Omega_I(\theta) = \sum_t \|y_I(t)\| \frac{F[y_I(t)/N_I(t), t]}{\|y_I(t)/N_I(t)\|}.
\]

The sequence of vectors \( y_I(t) \) will have a convergent subsequence (depending on \( \theta \)) for which the corresponding vectors

\[
y_I(t) \quad \text{and} \quad \frac{y_I(t)/N_I(t)}{\|y_I(t)/N_I(t)\|}
\]

will also converge. This motivates the following assumption, which replaces Assumption 4 of Section 3.

Before stating the assumption we first point out that, for any \( y \) in \( K \times L \) and \( t \) in \( T \), the function \( f \) defined by

\[
f(x) \equiv F(xy, t)
\]

is concave for \( x \) in the closed interval \([0, 1]\). Therefore, if \( y \) is not 0, the limit

\[
H(y, t) = \lim_{\substack{x \to 0 \\ x > 0}} \frac{F(xy, t)}{x\|y\|}
\]

exists (with \( +\infty \) a possible value). For each \( t \), the function \( H(\cdot, t) \) is homogeneous of degree 0 (note that \( H(0, t) \) is not defined).

\textbf{Assumption 4$: If \{y_n\} is a sequence in } K \times L \text{ such that:}

(i) \( y_n \to 0 \),

(ii) \( \frac{y_n}{\|y_n\|} \to \bar{y} \),

then, for every \( t \) in \( T \),

\[
\frac{(y_n, t)}{\|y_n\|} \to H(\bar{y}, t) < \infty.
\]

It follows from Assumption 4$ that \( H(\cdot, t) \) is continuous on \( K \times L \) at every point \( y \neq 0 \); it further follows then that \( H(\cdot, t) \) is uniformly bounded. If we make the convention that

\[
\|y\|H(y, t) = 0 \quad \text{if} \quad y = 0,
\]

then the function

\[
\sum_t \|y(t)\|H[y(t), t]
\]
is continuous in the variables \( y(t) \). Define

\begin{equation}
\omega = \max \sum_t \|y(t)\|H[y(t), t],
\end{equation}

subject to:

(i) for all \( t \), \( y(t) \in K \times L \),

(ii) \( \sum_t k(t) \leq \kappa \).

For every \( \theta \) there is a subsequence \( J(\theta) \) of \( I \)'s such that the limits

\begin{equation}
\tilde{y}(t) = \lim_{I \in J(\theta)} y_I(t),
\end{equation}

\begin{equation}
\bar{y}(t) = \lim_{I \in J(\theta)} \frac{y_I(t)}{\|y_I(t)\|},
\end{equation}

exist for every \( t \). By Assumption 4', and the definitions (5.5), (5.16), and (5.18),

\begin{equation}
\Omega(\theta) = \lim_{I \to \infty} \Omega_I(\theta),
\end{equation}

\begin{equation}
= \lim_{I \in J(\theta)} \Omega_I(\theta),
\end{equation}

\begin{equation}
= \sum_t \|\tilde{y}(t)\|H[\tilde{y}(t), t],
\end{equation}

\begin{equation}
= \sum_t \|\bar{y}(t)\|H[\bar{y}(t), t],
\end{equation}

\begin{equation}
\leq \omega.
\end{equation}

We have thus proved:

**Lemma 5.1:** For every \( \theta \),

\begin{equation}
\Omega(\theta) \leq \omega.
\end{equation}

**Complete Exchange with the Center**

Suppose that the maximum, \( \omega \), in (5.18) is achieved at \( \tilde{y}(t) = [\tilde{k}(t), \tilde{l}(t)] \). Consider the following CEC decision functions: for every \( \theta \) and \( I \), and every \( t \) for which \( N_I(t) > 0 \), let

\begin{equation}
K_i = \frac{\tilde{k}(t)}{N_I(t)},
\end{equation}

\begin{equation}
L_i = \frac{\tilde{l}(t)}{I_p(t)}.
\end{equation}
The corresponding total output is

\begin{equation}
W_t(\theta) = \sum_i N_t(t) F \left[ \frac{\hat{k}(t)}{N_t(t)}, \frac{\hat{l}(t)}{Ip(t)} \right]
\end{equation}

By the Strong Law of Large Numbers,

\begin{equation}
\left[ \frac{\hat{k}(t)}{N_t(t)}, \frac{\hat{l}(t)}{Ip(t)} \right] \to \left[ \hat{k}(t), \hat{l}(t) \right] = \hat{y}(t) \quad \text{a.s.}
\end{equation}

We may, without loss of generality, take \( \hat{k}(t) \) to be nonzero, and thus \( \hat{y}(t) \) nonzero. Therefore, a.s.,

\begin{equation}
\left[ \frac{\hat{k}(t)}{N_t(t)}, \frac{\hat{l}(t)}{Ip(t)} \right] \to \hat{y}(t)
\end{equation}

so that by Assumption 4' and the homogeneity of \( H(\cdot, t) \), the second line of (5.23) converges for almost every \( \theta \) to

\begin{equation}
\sum_i \| \hat{y}(t) \| H(\hat{y}(t) / \| \hat{y}(t) \|, t) = \sum_i \| \hat{y}(t) \| H(\hat{y}(t), t),
\end{equation}

i.e.,

\begin{equation}
\lim_{t \to \infty} W_t(\theta) = \omega, \quad \text{a.s.}
\end{equation}

Furthermore, \( W_t(\theta) \) is uniformly bounded, so

\begin{equation}
\lim_{t \to \infty} E_\theta W_t(\theta) = \omega.
\end{equation}

But, since \( E_\theta W_t(\theta) \) is the expected total output for a particular CEC decision function,

\begin{equation}
E_\theta W_t(\theta) \leq \omega_{CEC}(I).
\end{equation}

Therefore, we have proved:

**Lemma 5.2:**

\begin{equation}
\lim_{t \to \infty} \omega_{CEC}(I) \geq \omega.
\end{equation}
Combining Lemmas 5.1 and 5.2 with (5.5) and (5.6), we have

\[ \lim_{I \to \infty} \omega_{FEI}(I) \leq \omega \leq \liminf_{I \to \infty} \omega_{CEC}(I). \]

On the other hand, \( \omega_{CEC}(I) \leq \omega_{FEI}(I) \),

\[ \limsup_{I \to \infty} \omega_{CEC}(I) \leq \lim_{I \to \infty} \omega_{FEI}(I). \]

Therefore,

\[ \lim_{I \to \infty} \omega_{CEC}(I) = \lim_{I \to \infty} \omega_{FEI}(I) = \omega. \]

6. FULL INFORMATION IS SUPERIOR WHEN PRODUCTION PARAMETERS ARE CORRELATED

If the production parameters are independently distributed, then, for large teams, the empirical distribution is effectively known a priori, and the firm's knowledge of its own parameter, \( \theta_i \), defines its place in the general distribution, hence supplies all the information that could possibly be obtained. However, if the \( \theta_i \)'s are correlated but not perfectly, then the inferences of any one firm as to the empirical distribution of the others' parameters is less certain; it knows that they are drawn from the conditional joint distribution given its own parameter values, but since the latter is a random variable, so is the empirical distribution.

Retain the presupposition that production parameters of the firms are a priori indistinguishable from the viewpoint of other firms or of the resource managers. In a terminology introduced by de Finetti [2, Chapter 4], the production parameters of the different firms are taken to be exchangeable random variables: the (marginal) joint distribution of any finite subset of them is symmetric. Then, as shown by de Finetti (see also Hewitt and Savage [4]), the joint distribution of an infinite sequence of exchangeable random variables can be represented as the distribution of an infinite random (independently and identically distributed) sequence from a distribution \( P \) which is itself a random variable. Formally, suppose \( \hat{T} \) is the infinite Cartesian power of \( T \) (the domain of the production parameter), \( \Pi \) the space of probability distributions of \( T \), and \( \sigma \) is the probability measure on \( \hat{T} \) for a sequence of exchangeable random variables. Then there exists a probability measure \( \pi \) on \( \Pi \) such that, for any sequence \( \{A_i\} \), where \( A_i \) is a measurable subset of \( T \),

\[ \sigma(A_1xA_2x\ldots) = \int_{\Pi} [P(A_1)P(A_2)\ldots] d\pi(P). \]

We may interpret this as meaning that the actual \( \theta \) for any firm is determined by the general conditions applicable to all firms, which are summed up in the parameters determining the distribution of \( P \), and idiosyncratic conditions, represented by taking \( \theta \) to be a random drawing from \( P \).
For a large team, therefore, the empirical distribution can be taken to be (approximately) $P$.

In the FEI case, $P$ can, in the limit, be observed by all participants; hence, an optimizing policy will have the form of a pair of functions, $K_i(\theta, P)$, $L_i(\theta, P)$, not actually depending on $i$. If the optimal policy exists, then clearly $L_i(\theta, P)$ will in general depend genuinely on $P$. Even if there is no optimal policy, the sequence of approximately optimal policies will depend on $P$.

In the CEC case, $P$ is observed by the center but not by the firms. Hence $k$ can be chosen to depend on $\theta$ and $P$, but $l$ must be chosen to be a function of $\theta$ only. This is a genuine restriction on the class of optimal policies. If the optimal policy for the FEI case really depends on $P$, then no policy in which the local decisions are independent of $P$ can achieve the same level of aggregate output. Hence, the CEC case is not equivalent to the FEI case even in the limit.

We can illustrate this conclusion in the example studied in Section 2. We assume that the team is large, effectively being at the limit. Take the case where the probability distribution of the parameter $t$ is concentrated on a finite set. In the present context, we mean that there is a finite set of parameter values such that each $P$ is concentrated on that set (or a subset thereof). As before, the largest possible value of $t$ is $t_N = 1$. Assume that $P_N = P(t_N) > 0$ for all distributions $P$ and that $P_N$ takes on at least two different values as $P$ varies. We can for simplicity assume that the probability distribution of $P$ is concentrated on a finite set, each possible $P$ having a positive probability.

If $P$ is known to all managers, as it is under FEI, then as Section 2 shows, the optimal policy for large teams is to set $L_i(\theta_i) = 1/P_N$ if $\theta_i = 1$, and $=0$ otherwise, the total resource being divided equally among the firms with $\theta = 1$. This yields a conditional expected average output of $1 - w$ for each $P$, and therefore an expected average output of $1 - w$ when $P$ is considered as a random variable. If, for any $P$, any other policy were adopted the expected average output would be less than $1 - w$ for that $P$.

This policy depends on $P$, through $P_N$, and is therefore not a CEC policy. A CEC policy must determine $L_i$ as a function of $\theta_i$, independent of $P$. It is impossible that $L_i(1) = 1/P_N$ for all $P$, since $P_N$ takes on at least two different values. Therefore, for at least one $P$, the expected average output for the optimal CEC policy conditional on $P$ must be less than the maximum attainable, $1 - w$. Since each possible value of $P$ has a positive probability, the unconditional expected average output under the optimal CEC policy is less than $1 - w$, and so less than the expected average output under the optimal FEI policy.

It might be asked if a modicum of information can be added to the CEC case to realize the same average output level as the FEI case. Clearly, it is sufficient to identify $P$ in the class of admissible $P$-values, and any statistic which will accomplish that is enough. If, for example, $E(\theta|P)$ varies with $P$, the sample arithmetic mean of the $\theta$'s (or even of a random sample of the actual $\theta$'s, but one which goes to infinity in size with the number of firms) is sufficient. In particular cases, less information would suffice, since all that is really needed is to identify the function $L_i(\theta, P)$ which gives the optimal FEI local decision. Thus, in the
preceding example, it is enough to have a consistent estimate of $P_n$, since the other aspects of $P$ distributions are irrelevant to the asymptotically optimal FEI policy.

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