Equilibrium Risk Premia for Risk Seekers

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Abstract

Several landmark papers (Friedman and Savage 1948, Markowitz 1952, Kahneman and Tversky 1979, Tversky and Kahneman 1992) have suggested that utility functions of individual investors have convex regions corresponding to risk seeking behavior. At the same time, the degree of individual risk aversion required to explain aggregate equity returns must be high (the “equity premium puzzle” of Mehra and Prescott 1985). Can both be true? Can individual investors exhibit risk seeking behavior and at the same time, in the aggregate, demand a high premium for holding risky assets? We study aggregation properties in an economy where all agents are risk seeking. Under perfect competition when all agents are homogeneous, we prove that although individual agents have concave indifference curves (corresponding to risk seeking), the aggregate economy has a convex (linear) indifference curve that corresponds to risk aversion. When agents are heterogeneous in their initial endowment, aggregating risk seeking individual agents under perfect competition leads to an aggregate indifference curve that is strictly convex indicating that the economy demands a risk premium. We prove that the converse is also true. For an economy that in the aggregate exhibits risk aversion we can construct an economy of all risk seeking agents that in the aggregate produces the given indifference curve.

JEL Classification: G10, D11, D80.
Keywords: Risk aversion, risk seeking, investor sentiment, risk premium.

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Abstract

Several landmark papers (Friedman and Savage 1948, Markowitz 1952, Kahneman and Tversky 1979, Tversky and Kahneman 1992) have suggested that utility functions of individual investors have convex regions corresponding to risk seeking behavior. At the same time, the degree of individual risk aversion required to explain aggregate equity returns must be high (the “equity premium puzzle” of Mehra and Prescott 1985). Can both be true? Can individual investors exhibit risk seeking behavior and at the same time, in the aggregate, demand a high premium for holding risky assets? We study aggregation properties in an economy where all agents are risk seeking. Under perfect competition when all agents are homogeneous, we prove that although individual agents have concave indifference curves (corresponding to risk seeking), the aggregate economy has a convex (linear) indifference curve that corresponds to risk aversion. When agents are heterogeneous in their initial endowment, aggregating risk seeking individual agents under perfect competition leads to an aggregate indifference curve that is strictly convex indicating that the economy demands a risk premium. We prove that the converse is also true. For an economy that in the aggregate exhibits risk aversion we can construct an economy of all risk seeking agents that in the aggregate produces the given indifference curve.

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1 Introduction

Individual investors are averse to risk and when they trade in financial markets, they set the aggregate prices in such a way as to receive compensation for systematic risk. This main pillar of modern finance faces at least two challenges. One challenge comes from the research on investor behavior that argues that individuals are not always averse to risk. The other challenge is the equity premium puzzle (Mehra and Prescott 1985) which states that with commonly used assumptions on preferences, the degree of individual aversion to risk must be very high to explain aggregate equity returns.1 How can both challenges co-exist? Can individual investors exhibit risk seeking behavior and at the same time, in the aggregate, demand a high positive rate of return for holding risky assets? Is the reconciliation of these two phenomenon related to the aggregation of individual utility functions?

The premise that individuals are averse to risk dates back to at least the work of Bernoulli (1738).2 In canonical models, diminishing marginal utility plus maximization of expected utility imply that individuals always have to be paid to bear risk. This is represented by a concave utility function (Figure 1). At the same time, investigation of individual investor behavior reveals that people often prefer to hold risk if the risk comes with a probability of a large payoff. In their classic work, Friedman and Savage (1948) point out that individuals seem to exhibit aversion to some risks, and at the same time pay to participate in risky gambles. Friedman and Savage (1948) suggest that the fact that investors buy insurance, buy lottery tickets, and buy both insurance and lottery tickets simultaneously, plus the fact that most lotteries have more than one big prize, imply that

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the utility function must have two concave regions (corresponding to risk aversion) with a convex region (corresponding to risk seeking) in between, as represented in Figure 2.³

Markowitz (1952) argues that the Friedman and Savage utility function should be modified so that the inflection point where the concave region turns into the convex region is located exactly at the individual’s current wealth. Markowitz (1952) also suggests that the utility of wealth function has three inflection points. The utility function is monotonically increasing but bounded; it is first convex, then concave, then convex, and finally concave. The middle inflection point is defined to be at the current level of wealth. The first inflection point is below, the third inflection point is above, current wealth (Figure 3).

Perhaps the most well-known class of value function is the prospect theory S-shaped function suggested by Kahneman and Tversky. Based on their experimental results with bets that are either negative or positive, Kahneman and Tversky (1979) and Tversky and Kahneman (1992) suggest that the value function is concave in the domain of gains and convex in the domain of losses. This function has one inflection point located at the current level of wealth.

Gul (1991) develops a theory of disappointment aversion that includes expected utility theory as a special case and is consistent with the Allais Paradox. In this theory, an individual can be risk averse with respect to even chance gambles and gambles which yield a large loss with small probability. However, the agent will be risk loving with respect to gambles that involve winning a large prize with small probability provided his initial wealth is relatively low.

In addition to lottery purchases and experimental results, there is evidence obtained from asset prices that points in the direction of convex investor preferences, at least over some region of the utility function. The price behavior of Danish lottery bonds in Florentsen and Rydqvist (2002), and Swedish lottery bonds in Green and Rydqvist (1997, 1999) is consistent with convexity of investor utility function: In some instances, investors appear to pay to hold lottery risk instead of requiring a risk premium.

Although much research has been done on the individual utility functions with convex regions, less work has been done on rigorous aggregation of such utility functions. If an economy consists of individuals with utility functions as proposed by Friedman and Savage (1948), Markowitz (1952), or Kahneman and Tversky (1979), what are the aggregate properties? What do such utility functions

³Also, see Hakansson (1970) and Gregory (1980).
imply about the aggregate risk premium? In this paper we address the problem of aggregation of individual utility functions with convex regions.

This paper makes several contributions. First, it addresses in a rigorous fashion the question of aggregation of utility functions that are convex in consumption. All agents in our model have a convex utility function implying that they are risk takers. By making such an extreme assumption on preferences, the paper is able to focus on the roles that perfect competition and the budget constraint play in aggregation. We give precise mathematical meaning to the heuristic economic arguments used by Rothenberg (1960) who essentially suggests that concavities in individual indifference curves will disappear when aggregated over a large number of individual economic agents. We begin with the case of identical risk seeking agents. We prove that although individual agents have concave indifference curves that correspond to risk seeking, under perfect competition and identical initial endowments, the aggregate economy has a convex (linear) indifference curve indicating that the economy demands a risk premium. We formally model perfect competition by increasing the number of agents in a way that the influence of each individual participant remains negligible. To achieve this, we are careful to take limits in a fashion that increases the competitiveness of the economy without altering the endowment. Thus, the formal approach to perfect competition that we employ is in the spirit of Aumann (1964) who showed that a mathematical model of perfect competition should contain infinitely many economic agents.

Our second conclusion concerns the roles of perfect competition and initial wealth distribution in models of aggregate behavior toward risk. We study the case of risk seeking individual agents with different initial endowments. We show that if there exists a closed and bounded continuum of wealth classes then the aggregate indifference curve is strictly convex and differentiable. This result highlights the importance of the wealth distribution in any model that employs utility functions with convex regions. It shows that the initial wealth distribution may be of the same importance as the shape of the utility function, at least when it comes to the model’s implications for the aggregate data, and the equity risk premium in particular.

Third, we present a new theorem that shows that for any convex aggregate indifference curve (that corresponds to the case of aggregate aversion to risk) there exists a distribution of wealth among risk seeking individual agents (with concave individual indifference curves) that will replicate the convex curve. We start with an economy that in the aggregate exhibits risk aversion. We then
construct an economy of risk seekers with a specifically chosen wealth distribution such that they aggregate to the given curve. This result has implications for empirical studies based on aggregate data. It shows that such studies can potentially be consistent with a wide variety of individual investor behavior specifications. Recent studies of investor behavior have taken a new route by using consumer level data in addition to aggregate level securities market data (Brav, Constantinides, and Geczy 2002, and Vissing-Jorgensen 2002).

The remainder of the paper is organized as follows. Section 2 develops the aggregation results. Subsection 2.2 studies the case of identical agents. Subsection 2.3 analyzes the case when agents have different initial wealth. Section 3 concludes. All proofs are in the Appendix.

2 The Model of Utility Aggregation

2.1 Convex Utility

In a typical model, an agent solves the expected utility maximization problem,

$$\max \mathbb{E} \left[ u(C_0, \tilde{C}_1) \right],$$

where the utility of consumption is assumed concave and additively separable:

$$u(C_0, C_1) = U(C_0; 0) + U(C_1; 1).$$

In this paper we depart from the concavity (risk aversion) assumption made about the utility function of individual agents in the economy. Several landmark papers argue that individual utility functions have a convex region. We take this assumption to the extreme by studying aggregate properties of an economy where all individuals have identical convex additively separable utility of consumption:

$$u(x_i, y_i) = U(x_i; 0) + U(y_i; 1),$$

where $x_i$ represents consumption at time 0 and may be referred to as the “X consumption good,” and $y_i$ represents (uncertain) consumption at time 1 and may be referred to as the “Y consumption good.” We assume that $u(x_i, y_i)$ is twice continuously differentiable and increasing in all arguments. We work with indifference curves. Individual agents are assumed to have convex utility functions,
which correspond to concave indifference curves (Figure 4). We show, however, that the aggregate economy is characterized by convex indifference curves, which correspond to concave utility function (Figure 5).

2.2 Identical Agents

We begin with an economy with \( N \) identical agents all with the same utility function \( u(x_i, y_i) \) and all having the same initial level of wealth. Each agent has a strictly convex utility function that corresponds to strictly concave indifference curves. What are the properties of the aggregate economy? That is, what are the characteristics of the aggregate indifference curves under perfect competition? Will the efficient allocation be an interior solution?

The economy is initially endowed with \( Y_{\text{max}} > 0 \) units of \( Y \) and \( X = 0 \) of \( X \) to be allocated efficiently among the \( N \) identical agents. It is clear that all agents will be initially endowed with the same quantity of \( Y \), \( y_i = \frac{Y_{\text{max}}}{N} \) for all \( i \). What is the optimal allocation when \( X \in (0, X_{\text{max}}] \)?

Suppose there is a positive amount of \( X \) consumption good, \( X > 0 \). We want to determine the efficient allocation of \( X \) and \( Y \) by finding the maximum amount of \( Y \) the agents are willing to give up in order to hold all of \( X \) while ensuring all agents are as well off as before \( X \) was introduced. By doing this, we are essentially tracing the indifference curve. Each agent currently has utility level \( k \). Thus, to find the efficient allocation, we must solve the optimization problem:

\[
\begin{align*}
\text{Max} & \quad Y_{\text{max}} - \sum_{i=1}^{N} y_i \\
\text{subject to} & \quad u(x_i, y_i) = k \\
& \quad \sum_{i=1}^{N} x_i = X \\
& \quad x_i \geq 0, y_i \geq 0 \text{ for all } i.
\end{align*}
\]

By the Implicit Function Theorem, the equation \( u(x_i, y_i) = k \) defines implicit function, \( y_i = \Psi(x_i; k) \). Then, the optimization problem can be stated as:

\[
\begin{align*}
\text{Min} & \quad \sum_{i=1}^{N} \Psi(x_i; k) \\
\text{subject to} & \quad \sum_{i=1}^{N} x_i = X \\
& \quad x_i \geq 0, y_i \geq 0 \text{ for all } i.
\end{align*}
\]
Proposition 1 (Efficient Allocation) Let $\bar{X} = (n-1)x_{\text{max}} + r$, for $n = 1, 2, ..., N$ and $r \in (0, x_{\text{max}}]$, be the quantity of $X$ (time 0 consumption good) available and define $x_{\text{max}}$ as such value of $x$ that solves: $u(x,0) = k$. Then $x_{\text{max}}$ is the maximum amount of $X$ each individual agent can hold. The solution to (1) is

\[
(x_i, y_i) = \begin{cases} 
  (x_{\text{max}}, 0) & \text{for } i = 1 \text{ to } n - 1 \\
  (r, \Psi(r;k)) & \text{for } i = n \\
  (0, \Psi(0;k)) & \text{for } i = n + 1 \text{ to } N.
\end{cases}
\]

Proposition 1 shows that in an economy where both $Y$ and $X$ are held, some agents will only hold $X$, one agent will hold both $X$ and $Y$, and the remaining agents will only hold $Y$. It also shows that this allocation is efficient.

Now that the efficient allocation for each agent has been determined, we can turn our attention to aggregate demand under perfect competition. This is found by aggregating the demands from Proposition 1. Recall that $\bar{X} = (n-1)x_{\text{max}} + r$ where $n$ is an integer. This way of writing $\bar{X}$ essentially decomposes the value into the number of agents who can hold $x_{\text{max}}$ and a remainder that represents the amount of $X$ held by the $n^{th}$ agent. From this we can write $n - 1 = \text{int}\left[\frac{\bar{X}}{x_{\text{max}}}\right]$ for the total number of agents holding $x_{\text{max}}$ and $r = \bar{X} - x_{\text{max}}\text{int}\left[\frac{\bar{X}}{x_{\text{max}}}\right]$ as the amount of $X$ held by the $n^{th}$ agent where $\text{int}[\cdot]$ denotes the integer part of the value in brackets. Hence, the aggregate indifference curve (the trade-off relation between the goods $Y$ and $X$) for the $N$ agents is,

\[
Y_N(X) = \Psi(r;k) + \left(N - 1 - \text{int}\left[\frac{\bar{X}}{x_{\text{max}}}\right]\right) \Psi(0;k), \text{ for } X \in [0, x_{\text{max}}],
\]

where $y_i = \Psi(x_i;k)$, $r = X - x_{\text{max}}\text{int}\left[\frac{X}{x_{\text{max}}}\right]$ and $x_{\text{max}} = \Psi^{-1}(0;k)$.

To determine the demand under perfect competition, we consider the limit of the aggregate demand function as $N \to \infty$ while holding the supply of $Y$ and $X$ constant. By taking the limit in this fashion we keep the focus on the increasing competitiveness of the economy. Perfect competition is modeled by increasing the number of agents in such a way that the influence of each individual participant remains negligible. This notion of perfect competition is consistent with Aumann (1964) who formally argues that a mathematical model of perfect competition should contain infinitely many participants.

Our limiting results increase competitiveness of the economy. To pass to the limit, we introduce variable $z, z > 0$. Start with an integer $N^* > 0$ that represents the number of agents. Agents in
this initial economy achieve level of utility \( k_0 \). The goal is to pass to the limit by increasing the number of agents, but maintain \( Y_{\text{max}} = N^* \cdot \Psi (0; k_0) \) fixed. Holding the endowment \( Y_{\text{max}} \) fixed and increasing the total number of agents in the economy implies that each agent will be initially endowed with less \( Y \). Hence, the initial level of utility decreases. We will write \( zN^* \) instead of \( N \).

Then there is a function \( k : z \to k(z) \) defined as implicit function through the equation

\[
Y_{\text{max}} = zN^* \cdot \Psi (0; k(z))
\]

where \( Y_{\text{max}} = N^* \cdot \Psi (0; k_0) \).

This definition of \( k(z) \) assures that passing to the limit, \( z \to +\infty \) and increasing the number of agents (passing to the case of perfect competition) does not cause the endowment in the economy to increase without bound. It must be the case that \( \frac{dk(z)}{dz} < 0 \).

Similarly, in the initial economy \( x_{\text{max}} = \Psi^{-1} (0; k_0) \), and in general \( x_{\text{max}} (z) = \Psi^{-1} (0; k(z)) \), and \( X_{\text{max}} = N^*\Psi^{-1} (0; k_0) \). In order to fix the amount of \( X \) as we increase the number of agents, we introduce function \( f(z) \) such that as we write \( zN^* \), \( X_{\text{max}} = \frac{X_{\text{max}}(z)}{f(z)} \) stays fixed. Solving for \( f(z) \) gives

\[
f(z) = \frac{zN^*\Psi^{-1}(0; k(z))}{N^*\Psi^{-1}(0; k_0)} = \frac{z\Psi^{-1}(0; k(z))}{\Psi^{-1}(0; k_0)}.
\]

Then

\[
Y_z(X) = \Psi \left( X f(z) - \Psi^{-1} (0; k(z)) \int \left[ \frac{X f(z)}{\Psi^{-1} (0; k(z))} \right] k(z) \right)
+ \left( zN^* - 1 - \int \left[ \frac{X f(z)}{\Psi^{-1} (0; k(z))} \right] \Psi (0; k(z)) \right)
= \Psi \left( X \frac{z\Psi^{-1}(0; k(z))}{\Psi^{-1}(0; k_0)} - \Psi^{-1} (0; k(z)) \int \left[ \frac{zX}{\Psi^{-1}(0; k_0)} \right] k(z) \right)
+ \left( zN^* - 1 - \int \left[ \frac{zX}{\Psi^{-1}(0; k_0)} \right] \Psi (0; k(z)) \right)
\]

for \( X \in [0, X_{\text{max}}] \).

**Proposition 2** For fixed integer \( N^* \), as \( z \to \infty \), \( Y_z \) converges to \( Y_\infty (X) = -\frac{\Psi(0; k)}{\Psi^{-1}(0; k)} X + N^*\Psi (0; k) \) uniformly on \([0, X_{\text{max}}]\) where \( X_{\text{max}} = N^*\Psi^{-1}(0; k) \).

In Proposition 2, we start with an economy with a fixed number of agents, \( N^* \), and this economy achieves some initial level of individual utility. Then, it is shown that as the economy approaches perfect competition, the aggregate indifference curve becomes a straight line. This
proposition states that though individual agents have concave indifference curves, the aggregate economy, under perfect competition and identical initial wealth endowments, has a convex (linear) indifference curve.

### 2.3 Agents with Different Initial Wealth

We now consider the case where the agents all have the same convex utility function, but initial wealth is different across agents. Each agent will be initially endowed with a different quantity of $Y$.

We begin with a simple case. Suppose there are $N_1 + N_2$ agents divided into two wealth classes. There are $N_1$ type 1 agents (the poor) who have initial wealth less than the $N_2$ type 2 agents (the rich). Wealth is divided in such a way that the efficient allocation of $Y_{\text{max}}$ between the two types causes the type 1 agent to be on the indifference curve $k_1$ and the type 2 agent to be on the $k_2$ indifference curve where $k_1 < k_2$. $Y$ is initially allocated efficiently. Hence, all type 1 agents are allocated the same quantity and all the type 2 agents are allocated the same quantity but more than the type 1 agents. We now ask the same question posed previously. If $X \in [0, X_{\text{max}}]$ is introduced into the economy, what is the allocation of $X$ and $Y$ so that the maximum amount of $Y$ is exchanged for all available $X$ and so that all agents are as well off as they were prior to the introduction of $X$?

This question is exactly the same as the previous problem except that agents are no longer identical. As before, due to the concavity of the indifference curves, the efficient allocation will be a corner solution. In the previous problem, we distributed $X$ to one agent until that agent received the maximum amount, $x_{\text{max}}$. Then we gave $X$ to the second agent until he received $x_{\text{max}}$, and so on until either all agents held only $X$ or the supply of the $X$ was depleted. This procedure results in a correct allocation because the rate at which each agent is willing to give up $Y$ for $X$ increases with $X$. Hence, an agent is willing to give up an increasing amount of $Y$ for each increment of $X$. The same is true in this case except we must be careful in selecting who receives $X$ first. That is, which type, the rich or the poor, has the greatest substitution rate for each increment of $X$?

Without loss of generality, we order the agents such that $i = 1, \ldots, N_1$ are type 1 and $i = N_1 + 1, \ldots, N_1 + N_2$ are type 2. All agents have identical twice continuously differentiable, monotonically increasing convex utility function. Type 1 agents achieve utility level $k_1$ and type 2 agents achieve
Let the number of type 1 and type 2 agents be \( N_1 \) and \( N_2 \). By Implicit Function Theorem, the equation \( u(x_i, y_i) = k \) defines implicit function, \( y_i = \Psi(x_i; k) \). The problem to be solved is

\[
\text{Min } \quad \sum_{i=1}^{N_1} \Psi(x_i; k_1) + \sum_{i=N_1+1}^{N_1+N_2} \Psi(x_i; k_2)
\]

subject to

\[
\sum_{i=1}^{N_1+N_2} x_i = \underline{X}
\]

and \( x_i \geq 0, y_i \geq 0 \) for all \( i \).

**Proposition 3** Let the number of type 1 and type 2 agents be \( N_1 \) and \( N_2 \), respectively. The type 1 agents have an initial utility of \( k_1 \) and the type 2 agents have initial utility \( k_2 > k_1 \). If \( \underline{X} = (n - 1)x_{1,\text{max}} + r \leq N_1x_{1,\text{max}}, \) where \( n = 1, 2, \ldots, N_1, r_1 \in (0, x_{1,\text{max}}] \) and \( x_{1,\text{max}} = \Psi^{-1}(0; k_1) \), then the solution to problem (3) is

\[
(x_{1,i}, y_{1,i}) = \begin{cases} 
(x_{1,\text{max}}, 0) & \text{for } i = 1 \text{ to } n - 1 \\
(r_1, \Psi(r_1; k_1)) & \text{for } i = n \\
(0, \Psi(0; k_1)) & \text{for } i = n + 1 \text{ to } N_1
\end{cases}
\]

\[
(x_{2,i}, y_{2,i}) = (0, \Psi(0; k_2)) \quad \text{for } i = N_1 + 1 \text{ to } N_1 + N_2.
\]

If \( N_1x_{1,\text{max}} > \underline{X} \geq N_1x_{1,\text{max}} + N_2x_{2,\text{max}} \) and \( \underline{X} = N_1x_{1,\text{max}} + (n - 1)x_{2,\text{max}} + r_2 \) where \( r_2 \in (0, x_{2,\text{max}}] \) and \( x_{2,\text{max}} = \Psi^{-1}(0; k_2) \) then the efficient allocation is

\[
(x_{1,i}, y_{1,i}) = (x_{1,\text{max}}, 0) \quad \text{for } i = 1, \ldots, N_1
\]

\[
(x_{2,i}, y_{2,i}) = \begin{cases} 
(x_{2,\text{max}}, 0) & \text{for } i = N_1 + 1 \text{ to } N_1 + n - 1 \\
r_2, \Psi(r_2; k_2) & \text{for } i = N_1 + n \\
(0, \Psi(0; k_2)) & \text{for } i = N_1 + n + 1 \text{ to } N_1 + N_2
\end{cases}
\]

Notation \( x_{1,\text{max}} = \Psi^{-1}(0; k_1) \) means that \( x_{1,\text{max}} \) is the value of \( x \) that solves \( u(x, 0) = k_1 \), and notation \( x_{2,\text{max}} = \Psi^{-1}(0; k_2) \) means that \( x_{2,\text{max}} \) is the value of \( x \) that solves \( u(x, 0) = k_2 \).

The efficient allocation in Proposition 3 is very similar to the allocation described in Proposition 1. It should not be surprising that the allocation under perfect competition is very similar to the result described in Proposition 2. Where as Proposition 2 described a linear aggregate indifference
curve with slope $-\frac{\Psi(0;k_2)}{\Psi^{-1}(0;k_2)}$, this aggregate indifference curve will be piecewise linear where the first linear segment has slope $-\frac{\Psi(0;k_1)}{\Psi^{-1}(0;k_1)}$ for $X \in [0, N_1x_1, \text{max})$ and the second piece has slope $-\frac{\Psi(0;k_2)}{\Psi^{-1}(0;k_2)}$ for $X \in (N_1x_1, \text{max}, N_1x_1, \text{max} + N_2x_2, \text{max})$.

To assure that when joined the indifference curves form a convex aggregate indifference curve, we introduce a condition on the rate of substitution. For two wealth classes the condition takes the form $\frac{\Psi(0;k_1)}{\Psi^{-1}(0;k_1)} > \frac{\Psi(0;k_2)}{\Psi^{-1}(0;k_2)}$ for $k_1 < k_2$. An example of a simple convex utility function that satisfies this condition is $u(x_i, y_i) = \alpha x_i^2 + \beta y_i^4$, $\alpha > 0$, $\beta > 0$. This is, of course, not the only utility function that satisfies this condition.

Since $k_1 < k_2$, the slope is steepest over the first region, $[0, N_1x_1, \text{max}]$. Therefore, we should expect that the aggregate indifference curves are continuous and convex but not differentiable at the point where the two linear segments connect.

**Proposition 4** The aggregate demand of an economy with $N_1$ type 1 agents with utility $k_1$ and $N_2$ type 2 agents with utility $k_2$ such that $k_1 < k_2$ is

$$Y(X) = \begin{cases} 
-\frac{\Psi(0;k_1)}{\Psi^{-1}(0;k_1)} N_1 \Psi^{-1}(0;k_1) + N_2 \Psi(0;k_2), & X \in [0, N_1x_1, \text{max}] \\
-\frac{\Psi(0;k_2)}{\Psi^{-1}(0;k_2)} (X - N_1 \Psi^{-1}(0;k_1)) + N_2 \Psi(0;k_2), & X \in [N_1x_1, \text{max}, N_1x_1, \text{max} + N_2x_2, \text{max}] 
\end{cases}$$

where $N_1x_1, \text{max} = N_1 \Psi^{-1}(0;k_1)$ and $N_2x_2, \text{max} = N_2 \Psi^{-1}(0;k_2)$.

We now extend this result to $n$ wealth classes. For simplicity, we assume that $N_i = N$ for $i = 1$ to $n$. This is without loss of generality. Order the agents such that $k_i < k_{i+1}$. Then, for wealth classes $i = 1$ to $n$, the system of linear equations is:

$$Y_i(X) = -\frac{\Psi(0;k_i)}{\Psi^{-1}(0;k_i)} [X - x_{i-1}] + y_{i-1} \quad \text{for} \quad X \in [x_{i-1}, x_i],$$

\(i = 1, 2, \ldots, n\)

where $x_i = N \sum_{j=1}^{i} \Psi^{-1}(0;k_j)$ and $y_i = Y_{\text{max}} - N \sum_{j=1}^{i} \Psi(0;k_j)$ with $x_0 = 0$ and $y_0 = Y_{\text{max}}$. The maximum amount of $X$ and $Y$ that this economy can hold is $X_{\text{max}} = N \sum_{j=1}^{n} \Psi^{-1}(0;k_j)$ and $Y_{\text{max}} = N \sum_{j=1}^{n} \Psi(0;k_j)$. This system of equations creates a continuous, convex, but not differentiable indifference curves. The indifference curves are not differentiable at the points where the linear segments connect, $(x_i, y_i)$ for $i = 1$ to $n$. However, the indifference curves can be made sufficiently smooth by infinitely increasing the number of wealth classes so as to create a continuum of $k$ values over a range $k_{\text{min}}$ to $k_{\text{max}}$ while holding $X_{\text{max}}$ and $Y_{\text{max}}$ constant.
Theorem 5 Let there exist \([k_{\text{min}}, k_{\text{max}}]\), a continuum of utility levels with \(k_{\text{min}} > 0\) and \(k_{\text{max}} < \infty\), and for each utility level \(k_i \in [k_{\text{min}}, k_{\text{max}}]\) there exists an infinite number of agents so that the \(i\)-th agent type has a demand function described by (4). If, for all \(k_i < k_{i+1}\), \(\Psi(0;k_i) > \Psi(0;k_{i+1})\) and fixing the maximum amount of \(X\) and \(Y\) at \(X_{\text{max}}\) and \(Y_{\text{max}}\), respectively, the aggregate indifference curve for the economy is strictly convex and differentiable over \([0,X_{\text{max}}]\). If for all \(k_i < k_{i+1}\), \(\Psi(0;k_i) = \Psi(0;k_{i+1})\) then the aggregate indifference curve is convex (linear).

This theorem states that if there exists a closed and bounded continuum of wealth classes then the aggregate indifference curve is strictly convex and differentiable. This result allows us to create a variety of social indifference curves. Consider the following example.

Example 6 Consider an economy consisting of a poor, middle and rich classes all with the same convex additively separable utility function that satisfies the assumption on the rate of substitution, \(\Psi(0;k_i) > \Psi(0;k_{i+1})\) for all \(k_i < k_{i+1}\). The poor consist of a range of wealth associated with the range of utility levels \([k_{p1}, k_{p2}]\). Likewise, the middle and rich classes consist of a range of wealth associated with the range of utility levels \([k_{m1}, k_{m2}]\) for the middle class and \([k_{r1}, k_{r2}]\) for the rich class. The three types are distinct implying that \(k_{p2} \ll k_{m1}\) and \(k_{m2} \ll k_{r1}\). Suppose, further, that the maximum amount of \(X\) the poor, middle and rich can hold are \(X_p\), \(X_m\) and \(X_r\), respectively. Let \(X_{\text{max}} = X_p + X_m + X_r\) be the maximum amount of \(X\) that the entire economy can hold. What does the social indifference curve look like? From Proposition 3, we know that \(X\) is distributed to the agents from low \(k\) to high \(k\). Therefore, the poor receive the first portion of \(X\). From Theorem 5, since we have a continuum of wealth for the poor, the social indifference curve from \([0,X_p]\) is continuous, strictly convex and differentiable. \(X\) is then distributed to the middle class in order of low to high \(k\) over the interval \([X_p, X_p + X_m]\). Finally, the rich receive their portion of \(X\) over the range \([X_p + X_m, X_{\text{max}}]\). However, the social indifference curve is not differentiable at the points where each class’ demand curves connect. Consider the derivative at \(X = X_p\) from the right and from the left. By the same argument as in the proof of Proposition 3, we have

\[
\left| \frac{dY^-}{dX} (X_p) \right| > \left| \frac{dY^+}{dX} (X_p) \right|. 
\]

The same is true at \(X = X_p + X_m\). Hence, gaps in wealth create kinks in the social indifference curve.
So far we have shown that an economy where all individual agents are risk seeking is characterized by an aggregate indifference curve that is consistent with risk aversion if there are many different wealth classes of agents. We can still go one step further. We can start with an aggregate convex indifference curve (that corresponds to the case of risk aversion) and construct an economy of all risk seeking agents that in the aggregate produces the same curve. For any decreasing, convex function \( g : [0, X_{\text{max}}] \rightarrow [0, Y_{\text{max}}] \), there exists a distribution of wealth classes that will replicate the convex curve. Said differently, any convex indifference curve can be created by an economy, in perfect competition, of investors who individually have concave indifference curves. All that is needed is an appropriate wealth distribution.

**Theorem 7** Consider an economy of risk averse agents with aggregate demand defined by \( Y = g(X) \) with the properties \( g'(X) < 0, g''(X) > 0, 0 < g'(X) < \infty \) for all \( X \in [0, X_{\text{max}}] \), \( g(0) = Y_{\text{max}} \) and \( g(X_{\text{max}}) = 0 \). There exists a distribution of wealth such that an economy of risk seeking agents, with indifference curves \( Y = \Psi(X; k_i) \) satisfying the property \( \frac{\Psi(0; k_i)}{\Psi^{-1}(0; k_i)} > \frac{\Psi(0; k_{i+1})}{\Psi^{-1}(0; k_{i+1})} \) for \( k_i < k_{i+1} \), has the same aggregate demand as the economy of risk averse agents.

**Remark 8** Mathematically, Theorem 7 is equivalent to the following statement. Let the function \( g(X) \) be such that \( g'(X) < 0, g''(X) > 0, 0 < g'(X) < \infty \) for all \( X \in [0, X_{\text{max}}] \), \( g(0) = Y_{\text{max}} \) and \( g(X_{\text{max}}) = 0 \). Then there exists a sequence of \( k_i \)'s, \( i = 1 \) to \( n \), such that as \( n \rightarrow \infty \), the system of equations 4, along with the condition that \( \frac{\Psi(0; k_i)}{\Psi^{-1}(0; k_i)} > \frac{\Psi(0; k_{i+1})}{\Psi^{-1}(0; k_{i+1})} \) for \( k_i < k_{i+1} \), converges to \( g(X) \).

**Remark 9** This remark applies to both previous theorems. The results of the theorems do not require that there is perfect competition within each wealth class. It is sufficient to have an infinite number of wealth classes. The assumption of perfect competition within each wealth class is made merely to rely on previously established results.

**3 Conclusion**

This paper investigates aggregate properties of an economy where all investors have convex utility function. First, it is shown that when all investors have identical wealth, under perfect competition the aggregate indifference curve is convex (linear). This result shows that the assumption of perfect competition may be of the same degree of importance as the assumption on the shape of the individual utility function.
We then consider an economy where all agents have convex utility function, but different initial levels of wealth. If there is a finite number of different wealth classes then the aggregate indifference curve is continuous, convex, but not differentiable because it consists of finite number of linear segments. We prove that with a continuum of wealth classes, each being characterized by perfect competition, the aggregate indifference curve is convex and differentiable. The theorem shows that an economy that consists of small (atomistic) risk-seeking individual investors in the aggregate is characterized by an indifference curve consistent with risk aversion. The result shows how aggregation of individual agents each with a convex utility function can yield an indifference curve consistent with concave utility function. We then go a step further. We start with an aggregate convex indifference curve (which corresponds to the case of risk aversion). We show that there exists an economy composed of risk seeking individuals and a distribution of wealth such that in the aggregate the economy produces the same risk averse indifference curve as given. This result shows that empirical studies based on aggregate data can potentially be consistent with a wide variety of individual investor behavior specifications, even the ones based on utility functions with convex regions. Therefore, caution must be taken when drawing conclusions about individual behavior based on aggregate data.
A Appendix

A.1 Utility Aggregation: Identical Agents

Proof of Proposition 1. Efficient Allocation for Identical Agents. The maximum quantity of $X$ that any agent can hold is $x_{\text{max}} : u(x_{\text{max}}, 0) = k$. Notice that each $y_i = \Psi(x_i; k)$ is concave and continuous for all $x_i \in [0, x_{\text{max}}]$. This is so because, by Implicit Function Theorem,

$$
\frac{d^2 y_i}{dx_i^2} = -\frac{u_{xx}u_y - 2u_{xy}u_xu_y + u_{yy}u_x^2}{u_y^3} < 0.
$$

Note that $u_{xy} = 0$ because we study additively separable utility function. Since the sum of concave functions is concave, the interior optimum of (2) is a maximum and not a minimum. Due to the non-negativity constraints, the set of feasible solutions is compact and therefore, the solution exists and the solution must be located on the boundary of the feasible solution set.

Suppose $X = x_{\text{max}}$. Then the only corner solution available is associated with $x_{\text{max}}$ being given to only one agent. Hence, the first agent holds $(x_1, y_1) = (x_{\text{max}}, 0)$ and the remaining $N - 1$ agents hold $(x_i, y_i) = (0, \Psi(0; k))$ for $i = 2, \ldots, N$.

Now suppose the economy has $X = 2x_{\text{max}}$. We already know that it is better to allocate to the first agent $(x_1, y_1) = (x_{\text{max}}, 0)$ than to divide $x_{\text{max}}$ amongst several. This leaves us with an economy with $N - 1$ agents and only $x_{\text{max}}$ left to distribute. But this is the same as the original problem (except with one less agent). Hence, the second agent is allocated $(x_2, y_2) = (x_{\text{max}}, 0)$. All remaining agents receive $(x_i, y_i) = (0, \Psi(0; k))$ for $i = 3, \ldots, N$.

Finally, suppose $X = (n - 1)x_{\text{max}} + r$ is available. By the above argument, the first $n - 1$ agents will receive $(x_{\text{max}}, 0)$. The problem is now reduced to allocating $r$ amongst the remaining $N - n + 1$ agents. The corner solution is no longer associated with $(x_{\text{max}}, 0)$ since $r \leq x_{\text{max}}$. The corner is now at the point $(x, y) = (r, \Psi(r; k))$ and one agent will receive all of $r$ while the remaining agents receive none of the endowed $X$.

Therefore, for $X = (n - 1)x_{\text{max}} + r$, for $n = 1, \ldots, N$, $n - 1$ agents will hold $(x_i, y_i) = (x_{\text{max}}, 0)$, the $n^{th}$ agent will hold $(x_i, y_i) = (r, \Psi(r; k))$, and the remaining $N - n - 1$ agents will hold $(x_i, y_i) = (0, \Psi(0; k))$. This is the only corner solution such that all $X_{\text{max}}$ is allocated and hence, this must be the optimal allocation. This allocation is efficient by construction.

Proof of Proposition 2: Aggregate Indifference Curve. To prove that as $z \to \infty$,
$Y_z \rightarrow Y \infty$ uniformly, we must show that for any $\epsilon > 0$ there exists a $z^* > 0$ such that for all $z \geq z^*$, $|Y_z - Y_N| < \epsilon$. So, for any $\epsilon > 0$, we will show that such a $z^*$ exists. Consider the following:

$$|Y_z - Y_N| = \left| \Psi \left( X \frac{\Psi^{-1}(0; k(z))}{\Psi^{-1}(0; k_0)} - \Psi^{-1}(0; k(z)) \int \frac{zX}{\Psi^{-1}(0; k_0)} \right) ; k(z) \right|$$

$$+ \left| \Psi \left( 0; k(z) \right) \left( zN^* - 1 \right) - \int \frac{zX}{\Psi^{-1}(0; k_0)} \right| + \frac{\Psi(0; k_0)}{\Psi^{-1}(0; k_0)} X - N^* \Psi \left( 0; k_0 \right)$$

$$= \left| \Psi \left( X \frac{\Psi^{-1}(0; k(z))}{\Psi^{-1}(0; k_0)} - \Psi^{-1}(0; k(z)) \int \frac{zX}{\Psi^{-1}(0; k_0)} \right) ; k(z) \right|$$

$$- \Psi \left( 0; k(z) \right) - \Psi \left( 0; k_0 \right) \int \frac{zX}{\Psi^{-1}(0; k_0)} X \right|$$

The second equality above comes from choosing $k(z)$ so that $zN^* \Psi \left( 0; k(z) \right) = N^* \Psi \left( 0; k_0 \right)$. Now, we can decompose $z$ as $z = I \frac{\Psi^{-1}(0; k_0)}{X} + r_e$ where $I$ is any non-negative integer and $r_e \in [0, \frac{\Psi^{-1}(0; k_0)}{X}]$. Decomposing $z$ in this way, we can now write

$$\int \frac{zX}{\Psi^{-1}(0; k_0)} = \int \left( I \frac{\Psi^{-1}(0; k_0)}{X} + r_e \right) \int \frac{zX}{\Psi^{-1}(0; k_0)}$$

$$= \int I + \int \frac{X}{\Psi^{-1}(0; k_0)} r_e$$

$$= I \left( z - r_e \right) \frac{X}{\Psi^{-1}(0; k_0)}.$$

Therefore,

$$|Y_z - Y_N| = \left| \Psi \left( X \frac{\Psi^{-1}(0; k(z))}{\Psi^{-1}(0; k_0)} - \Psi^{-1}(0; k(z)) \left( z - r_e \right) \frac{X}{\Psi^{-1}(0; k_0)} \right) ; k(z) \right|$$

$$- \Psi \left( 0; k(z) \right) - \Psi \left( 0; k_0 \right) \int \frac{zX}{\Psi^{-1}(0; k_0)} X \right|$$

$$= \left| \Psi \left( X \frac{\Psi^{-1}(0; k(z))}{\Psi^{-1}(0; k_0)} - X \frac{\Psi^{-1}(0; k(z))}{\Psi^{-1}(0; k_0)} - X \frac{r_e}{\Psi^{-1}(0; k_0)} ; k(z) \right) \right|$$

$$- \Psi \left( 0; k(z) \right) - \Psi \left( 0; k_0 \right) \int \frac{zX}{\Psi^{-1}(0; k_0)} X \right|$$

$$= \left| \Psi \left( X \frac{\Psi^{-1}(0; k(z))}{\Psi^{-1}(0; k_0)} - \Psi(0; k(z)) \frac{X}{\Psi^{-1}(0; k_0)} + \frac{\Psi(0; k_0)}{\Psi^{-1}(0; k_0)} X \right) \right|$$

$$= \left| \Psi \left( X \frac{r_e}{\Psi^{-1}(0; k_0)} ; k(z) \right) - \Psi \left( 0; k(z) \right) \frac{X}{\Psi^{-1}(0; k_0)} + \frac{\Psi(0; k_0)}{\Psi^{-1}(0; k_0)} X \right|$$

The third equality again uses the fact that $zN^* \Psi \left( 0; k(z) \right) = N^* \Psi \left( 0; k_0 \right)$. Since $r_e$ is bounded and $\Psi \left( x_i; k \right)$ is closed and bounded, there exists an $r^*_e$ that maximizes $\Psi \left( X \frac{r_e}{\Psi^{-1}(0; k_0)} ; k(z) \right) - \Psi \left( 0; k(z) \right) + \frac{\Psi(0; k_0)}{\Psi^{-1}(0; k_0)} X$. The same argument holds for the existence of $X^*$ that maximizes the expression. Hence, the above inequality holds for $r_e$. Finally, to finish the proof, we note that as $z \rightarrow \infty$, $\Psi \left( x_i; k(z) \right) \rightarrow 0$ monotonically. Thus, by the intermediate value theorem there exists a
\[ z^* \text{ such that for } \epsilon > 0 \]
\[
\left| \Psi \left( X^* \frac{r^*_e}{\Psi^{-1}(0; k_o)}; k(z^*) \right) - \Psi \left( 0; k(z^*) \right) + \frac{r^*_e \Psi \left( 0; k(z^*) \right)}{\Psi^{-1}(0; k_o)} X^* \right| = \epsilon.
\]

Hence, for all \( z > z^* \) and for all \( X \) we have
\[
\left| \Psi \left( X^* \frac{r^*_e}{\Psi^{-1}(0; k_o)}; k(z) \right) - \Psi \left( 0; k(z) \right) + \frac{r^*_e \Psi \left( 0; k(z) \right)}{\Psi^{-1}(0; k_o)} X^* \right| < \epsilon.
\]

\[\Box\]

A.2 Utility Aggregation: Agents with Different Initial Wealth

Proof of Proposition 3: Two Classes of Agents. The proof for the first part is exactly the same as the proof of Proposition 1 with the exception that we must show that the type 1 agents are allocated \( X \) before the type 2 agents. As before, each type 1 agent can hold a maximum of \( x_{1,\text{max}} \) of \( X \) and each type 2 agent can hold a maximum \( x = x_{2,\text{max}} \) of \( X \). Since \( k_1 < k_2 \), then \( x_{1,\text{max}} < x_{2,\text{max}} \). Suppose \( \overline{X} = \epsilon \) where \( \epsilon > 0 \) is an arbitrarily small real number. We already know that the efficient allocation is to give \( \epsilon \) to one agent. According to the objective function, we choose the agent by determining which is willing to give up more of his allocation of \( Y \) for \( \epsilon \). Said differently, we give \( \epsilon \) to the agent who has the greatest rate of substitution. Thus, since we have \( |y_{1i}(0)| > |y_{2i}(0)| \), the first \( \epsilon \) goes to a type 1 agent.

The inequality \( |y'_{1i}(0)| > |y'_{2i}(0)| \) holds for an arbitrary \( u(x_i, y_i) \) that satisfies our conditions of differentiability, monotonicity, and convexity. Since utility function is additively separable, \( u_x (\cdot) \) is a function of \( x \) only, and \( u_y (\cdot) \) is a function of \( y \) only. Both partial derivatives are positive because \( u(x_i, y_i) \) is an increasing function. By the implicit function theorem,
\[
\left. \frac{dy_{1i}}{dx} \right|_{x=0, k=k_1} = -\frac{u_x (x)}{u_y (y)} \text{ and } \left. \frac{dy_{2i}}{dx} \right|_{x=0, k=k_2} = -\frac{u_x (x)}{u_y (y)}.
\]

Since the derivative is evaluated at \( x = 0 \), and the agents in the two classes have different utility with \( k_1 < k_2 \), it must be that the only difference is in the allocation of \( y \). Then, the nominators are equal because they are a function of \( x \) only. By convexity of \( u(x_i, y_i) \), the partial derivative \( u_y (y) \) is higher for higher values of \( y \). Then, the denominator is larger in the second ratio and therefore
\[
|y'_{1i}(0)| = \left| \frac{dy_{1i}}{dx} \right|_{x=0, k=k_1} > |y'_{2i}(0)| = \left| \frac{dy_{2i}}{dx} \right|_{x=0, k=k_2}.
\]
Now, suppose an additional $\epsilon$ is added to the economy. Then, we can give this additional amount to the type 1 agent who just received $\epsilon$ or to a type 2 agent. Since each agents’ demand function is concave and decreasing, the rate at which each agent is willing to give $Y$ for each additional increment of $X$ increases,

$$|y'_{1i}(\epsilon)| = \left| \frac{dy_{1i}}{dx} \right|_{x=\epsilon, k=k_1} > |y'_1(0)| = \left| \frac{dy_{1i}}{dx} \right|_{x=0, k=k_1} > |y'_{2i}(0)|.$$

The first inequality holds true because $u_x(x)$ is increasing by convexity of $u(x_i, y_i)$. Therefore, the $X$ consumption good will be allocated to the type 1 agents until all agents receives $x_{1,\text{max}}$ or the supply of $X$ has been depleted. If all type 1 agents have $x_{1,\text{max}}$, then the remaining supply of $X$ will be allocated to the type 2 agents one agent at a time with each type 2 agent holding no more than $x_{2,\text{max}}$. ■

**Proof of Proposition 4: Aggregate Indifference Curve for Two Classes of Agents.**

The proof is identical to the proof of Proposition 2. Let $X$ be the amount of the $X$ good available. If $X \leq N_1 x_{1,\text{max}}$, then the problem is the same as having only the type 1 agents since the type 2 agents will not receive any $X$. When $X > N_1 x_{1,\text{max}}$, all the type 1 agents have $x_{1,\text{max}}$ and so the problem reduces to having only the type 2 agents. ■

**Proof of Theorem 5: Convexity of Aggregate Indifference Curve.** We start the proof by choosing some partition of $k_i$’s, for $i = 1$ to $n$, over the interval $[k_{\text{min}}, k_{\text{max}}]$ such that $k_i < k_{i+1}$. As $n$ increases, we choose a new partition such that $\lim_{n \to \infty} |k_{i-1} - k_i| = dk_i$, $\lim_{n \to \infty} k_i = k_{\text{min}}$, $\lim_{n \to \infty} k_n = k_{\text{max}}$.

Define a continuous and differentiable function $f : [0, X_{\text{max}}] \to [k_{\text{min}}, k_{\text{max}}]$ such that $f(0) = k_{\text{min}}$, $f(X_{\text{max}}) = k_{\text{max}}$, and $f'(X) > 0$ with $0 < f'(X) < \infty$ for all $X \in [0, X_{\text{max}}]$.

Define $k_i = f(x_i)$. Hence, the partition of $k_i$’s is associated with some partition of $x_i$’s on $[0, X_{\text{max}}]$ defined by $f^{-1}(k_i)$ (note that the inverse does exist since $f$ is strictly increasing).

By the continuity of $f^{-1}(k)$, since $\lim_{n \to \infty} |k_{i-1} - k_i| = dk_i$, then $\lim_{n \to \infty} \left| f^{-1}(k_{i-1}) - f^{-1}(k_{i-1}) \right| = \lim_{n \to \infty} |x_{i-1} - x_i| = dx_i$.

For each $i$, we have

$$Y(X) = -\frac{\psi(0; f(x_i))}{\psi^{-1}(0; f(x_i))} [X - x_{i-1}] + y_{i-1} \quad \text{for} \quad X \in [x_{i-1}, x_i].$$

Evaluating this expression at $X = x_i$ and letting $n \to \infty$ we get

$$dY(x_i) = -\frac{\psi(0; f(x_i))}{\psi^{-1}(0; f(x_i))} dx_i \quad \text{for} \quad x_i \in [0, X_{\text{max}}].$$
That is, the partition on $X$ becomes infinitely fine and each $Y_i$ is defined over an infinitely small interval. Thus $Y$ becomes differentiable. Further, convexity comes directly from the assumption

$$\frac{\Psi(0; k_i)}{\Psi^{-1}(0; k_i)} > \frac{\Psi(0; k_{i+1})}{\Psi^{-1}(0; k_{i+1})}$$

for $k_i < k_{i+1}$. As $x \in [0, X_{\text{max}}]$ increases, $f(x)$ increase and $\frac{dy}{dx}$ approaches zero. Therefore, $\frac{dY}{dx} = -\frac{\Psi(0; f(x))}{\Psi^{-1}(0; f(x))}$ decreases at a decreasing rate. Hence, the function, $Y(x)$, is convex and differentiable on the interval $[0, X_{\text{max}}]$. 

**Proof of Theorem 7: Construction of Economy with Risk Seeking Agents.** In the proof we will show how to start with an aggregate convex indifference curve and construct wealth distribution in an economy where each individual has a concave indifference curve. Define a partition over $[0, X_{\text{max}}]$ as $x_i = i\frac{X_{\text{max}}}{n}$, for $i = 0$ to $n$. For each $x_i$, let $y_i = g(x_i)$ where $g$ is as defined in the theorem. We will show that the function $g(x)$ can be reproduced from a system of concave indifference by carefully choosing a distribution of wealth, $k_i$s. From 4, we can write

$$Y_i(X) = -\frac{\Psi(0; k_i)}{\Psi^{-1}(0; k_i)} [X - x_{i-1}] + y_{i-1} \quad \text{for} \quad X \in [x_{i-1}, x_i].$$

We choose $k_i$ so that each function $Y_i(X)$ is a chord connecting the points $(x_{i-1}, g(x_{i-1}))$ and $(x_i, g(x_i))$. Hence,

$$-\frac{\Psi(0; k_i)}{\Psi^{-1}(0; k_i)} = \frac{g(x_i) - g(x_{i-1})}{x_i - x_{i-1}}.$$

The above statement simply states that choosing $k_i$ carefully, the line segment $Y(x_i)$ can be made to have the same slope as the chord connecting $(x_{i-1}, g(x_{i-1}))$ and $(x_i, g(x_i))$. Such a $k_i$ exists since from the conditions $\frac{\Psi(0; k_i)}{\Psi^{-1}(0; k_i)} > \frac{\Psi(0; k_{i+1})}{\Psi^{-1}(0; k_{i+1})}$ and $0 < g'(X) < \infty$, we can choose a range of $k_i$s sufficiently large so as to match any slope $\frac{g(x_i) - g(x_{i-1})}{x_i - x_{i-1}}$ for all $x_i \in [0, X_{\text{max}}]$. That is, we can find a $k_i$ sufficiently small (large) so as to make the slope $\frac{\Psi(0; k_i)}{\Psi^{-1}(0; k_i)}$ as steep (shallow) as we need.

Substituting into $Y_i(X)$ gives

$$Y_i(X) = \left(\frac{g(x_i) - g(x_{i-1})}{x_i - x_{i-1}}\right) [X - x_{i-1}] + g(x_{i-1}) \quad \text{for} \quad X \in [x_{i-1}, x_i].$$

$Y_i$, for $i = 1$ to $n$, represents a point in a sequence of systems of linear equations. Each system of equations consists of $n$ chords connecting points on $g(x)$. 

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We now need to show that this choice of $k_i$ causes the sequence of systems to converge to $g(X)$ as $n \to \infty$. For any $i$, choose any $x \in (x_{i-1}, x_i)$. We can rewrite this point as $x = x_i - 1 + r \left( \frac{X_{\max}}{n} \right)$ for $r \in (0, 1)$. For all $\delta > 0$, let $n > n_0 = \frac{X_{\max}}{\delta}$

$$
| x_{i-1} - x | = | x_{i-1} - x_{i-1} - r \left( \frac{X_{\max}}{n} \right) |
\leq \left( \frac{X_{\max}}{n} \right)
\leq \left( \frac{X_{\max}}{n_0} \right)
\leq \delta.
$$

Hence, for all $\delta > 0$, there exists an $n_0$ such that for all $n > n_0$, $| x_{i-1} - x | < \delta$. By the continuity of $g(X)$, for every $\epsilon > 0$, there exists a $\delta(n_0) > 0$ such that if $| x_{i-1} - x | < \delta$, then $| g(x_{i-1}) - g(x) | < \epsilon$. Said differently, for every $\epsilon > 0$, there exists an $n_0$ such that for all $n > n_0$, $| x_{i-1} - x | < \delta(n_0)$, and hence, $| g(x_{i-1}) - g(x) | < \epsilon$. So, let $\epsilon > 0$ and choose $n_0$ to satisfies the continuity definition. For all $n > n_0$,

$$
| Y_i(x) - g(x) | = \left( \frac{g(x) - g(x_{i-1})}{x - x_{i-1}} \right) [x - x_{i-1}] + g(x_{i-1}) - g(x)
\leq \left( \frac{g(x) - g(x_{i-1})}{x - x_{i-1}} \right) [x - x_{i-1}] + (g(x_{i-1}) - g(x))
\leq \left( \frac{g(x) - g(x_{i-1})}{x - x_{i-1}} \right) \delta + \epsilon
\leq \epsilon.
$$

The last inequality is true since $g'(X) < 0$. But, this is true for any $i$ and for any $x \in (x_{i-1}, x_i)$. Therefore, for $n$ large enough, each chord, $Y_i$, can be made arbitrarily close to $g(X)$ and hence, the sequence of systems of linear equations approaches $g(X)$ on $[0, X_{\max}]$ uniformly as the sequence approaches infinity.
References


Figure 1: Concave Utility of Wealth Function

Figure 2: Friedman and Savage Utility of Wealth Function
Figure 3: Markowitz Utility of Wealth Function

Figure 4: Convex Utility: Indifference Curves
Figure 5: Concave Utility: Indifference Curves