Pricing Discrete Barrier Options with an Adaptive Mesh Model

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Abstract

Many exotic derivatives do not have closed-form valuation equations, and must be priced using approximation methods. Where they can be applied, standard lattice techniques based on binomial and trinomial trees will achieve correct valuations asymptotically. They can also generally handle American exercise. But for many problems, including pricing barrier options, convergence may be slow and erratic, producing large errors even with thousands of time steps and millions of node calculations. Options with price barriers that are only monitored at discrete points in time present additional difficulty for lattice models. Standard tree methods increase accuracy by shrinking the time and price step size throughout the lattice, but this increases the number of calculations sharply and much of the additional computation is in regions of the tree where it makes little difference to accuracy. A previous paper, Figlewski and Gao [1999], introduced the Adaptive Mesh Model (AMM), a very flexible approach that greatly increases efficiency in trinomial lattices. Coarse time and price steps are used in most of the tree, but small sections of finer mesh are constructed to improve resolution in specific critical areas. This paper presents an especially effective AMM structure for pricing options with discrete barriers. In a basic example, an AMM with 60 time steps is ten times more accurate than a 5000-step trinomial, but runs more than 1000 times faster.
Introduction

As derivatives theory has developed in recent years, a great variety of so-called "exotic" contracts have been proposed and analyzed. Most such instruments remain exotic and little seen in the real world, but barrier options are a significant exception. A barrier option's payoff depends both on where the price of the underlying asset ends up relative to the strike price at expiration, but also on whether the asset price has hit a specified price barrier at any point during the option's lifetime.

A "down and out" call option is a typical example. Let $S$ be the asset price, $K$ be the strike price and $H$, with $H < S$, be the barrier price, or "out strike." A (European) down and out call will pay $\max[S_T - K, 0]$ at its maturity date $T$ just like an ordinary European call, as long as $S$ stays above $H$ throughout the option's entire life. But if $S \leq H$ for any $t < T$, the option is "knocked out" and expires worthless, regardless of where the asset price is at expiration.

Barrier options have become quite popular in the foreign exchange markets. One appeal is that a barrier option is cheaper than a regular option, but a trader with a speculative opinion on the market may regard the conditions under which it will not pay off while a straight call will as being quite unlikely. Or a hedger may buy a barrier contract to hedge a position that itself has a natural barrier, e.g., the foreign currency exposure on a deal that will only take place if the exchange rate remains above a certain level.

In many cases, the barrier option does not become worthless when the asset price hits the barrier, but simply expires and pays a fixed "rebate" at that point. There are also other kinds of derivative contracts with barriers, such as capped options, ladder options, interest rate corridors, and many others.\(^1\)

Another set of variations on the barrier theme involves the nature of the barrier itself, both its form and how frequently the barrier condition is checked. A closed-form valuation equation for the basic European down and out call with a fixed out strike that is continuously monitored was first derived by Merton [1973]. As usual, however, American exercise eliminates the possibility of a useful closed-form valuation equation.

The standard binomial and trinomial lattice models can easily be adapted to the barrier option calculation, but problems arise with convergence because of interaction between the discrete placement of the nodes in the lattice and the location of the barrier. Boyle and Lau [1994] describe the difficulty in the context of a binomial model. If the lattice with a given price step size has a layer of nodes at a price just beyond the barrier (e.g., the knock out barrier is at 90 and the lattice has a layer of nodes at 89.95), then the probability of being knocked out is well approximated by the probability of hitting an 89.95 node. But when the analyst tries to make the

\(^{1}\) Rubinstein [1991] describes a number of exotic option contracts, many of which have barriers. Brief explanations of a vast array of traded derivative products, including many with various types of barrier features are given in Gastineau and Kritzman [1996].
lattice approximation more accurate by increasing the number of time steps and reducing the price step just a little, the same layer of nodes may now be at 90.05 and it will require one more full down move in the price before the option is knocked out. Thus a very small change in the time step within the lattice can produce a significant drop in the probability of being knocked out and a sharp jump in the estimated option value. A tree with 343 time steps might give a very good estimate of the option's true value, while one with 344 steps could be way off.

Figlewski and Gao [1999] describe two different types of approximation error that arise in a lattice model. "Distribution error" stems from the use of a discrete binomial or trinomial probability distribution to approximate the continuous lognormal distribution produced by a diffusion process. Distribution error disappears rapidly as the number of steps in the tree increases. The second error is "nonlinearity error," which arises when the option value function is highly nonlinear or discontinuous in some region, such as at the strike price at the expiration date, or at the barrier of a barrier option. The error described in the previous paragraph is an example of nonlinearity error.

Jumps in the model price due to nonlinearity error can be quite large, and take a long time to die out as the number of time steps in the tree is increased. Boyle and Lau's solution is to calculate the number of time steps N needed to produce a tree of (approximately) the desired fineness with a layer of nodes in the right place, and only build lattices with those values of N. This is a clever solution, but in fact, the binomial model is not well-suited to deal with complex payoff structures like this. It has no extra degrees of freedom that would allow the tree to accommodate a second barrier, or discontinuity in the barrier. For barrier option problems the trinomial model is much more useful.

Ritchken [1996] offers an approach in the context of a trinomial model. In order to force a layer of nodes to lie on the barrier, he introduces a "stretch" parameter into the lattice, which changes the price step just enough to place nodes in the desired location. Ritchken's trinomial has enough flexibility to match a second barrier, as well, but time-varying barriers (e.g., a knock out price that rises by 1 point each month) present a problem. Cheuk and Vorst [1996] also introduce a deformation of the trinomial tree, but of a different type. The extra degrees of freedom in a trinomial lattice allow price nodes to be placed more or less where the analyst chooses, with the branch probabilities adjusting to match the desired drift and volatility. This permits one to adjust the location of the nodes differently in each time period, and allows great flexibility in matching a time-varying barrier.

One major difficulty all of these techniques face is pricing an option correctly and efficiently when the initial asset price is close to the barrier. In order for the lattice to represent the probability of

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2 Gao, Huang, and Subrahmanyam [1999] obtain an analytical representation for the value and hedge parameters of an American barrier option using the decomposition technique of separating the European option value from the early exercise premium, which depends critically on the early exercise boundary. Since the boundary does not depend on the initial asset price, the method is free of the difficulty that lattice methods have in dealing with spot prices near the barrier.
hitting the barrier properly, there must be at least one price step between the initial asset price and the barrier. But if the asset price starts too close, the tree will require a very short time step to produce a small enough price step, and the total number of nodes in the tree will become extremely large. Figlewski and Gao [1999] present the Adaptive Mesh Model (AMM) as an approach to deal efficiently with this problem. The AMM is a trinomial model in which most of the tree is set up with relatively coarse time and price steps, but sections of higher resolution lattice are grafted onto it in the areas where greater accuracy is important. This permits pricing a barrier option accurately very close to the barrier without greatly increasing the amount of calculation required to solve the tree. In typical cases, efficiency may be improved by several orders of magnitude relative to alternative methods.

The barriers for the contracts described so far run for the whole life of the option and are continuously monitored. Some barrier options have only "partial" barriers, that are continuous but may only apply during the first part or the last part of the option's life. Heynen and Kat [1994] examine such contracts and derive a closed-form valuation equation in the same spirit as Merton's formula.

A more serious valuation problem occurs when the barrier is only monitored at discrete intervals, for example, at the close on the last trading day of each month. Such a structure allows the asset price to penetrate the barrier level in between monitoring dates without triggering the barrier event. Figure 1 illustrates the difference between a continuous and a discrete barrier for a down and out call. The barrier level is set at H = 35, but for the discrete barrier contract, it is monitored only once, at date $T_k = 0.5$. Thus the continuous barrier is the horizontal line at H = 35, while the discrete barrier consists of only the vertical bar at the monitoring date. If the asset price follows the path shown in the figure, the continuous barrier option would be knocked out when the barrier is hit, and would expire worthless. But the discrete barrier option would not be knocked out, because the asset was above the barrier at the monitoring date, and it would have a positive payoff when it finished in the money at $T = 1$.³

Discreteness reduces the probability of hitting the barrier, which makes an "out" option worth more and an "in" option worth less than in the continuous case. The impact of barrier discreteness on option value is surprisingly large. Cheuk and Vorst [1996] show that even hourly versus continuous monitoring can make a significant difference in option value. Building a lattice that deals effectively with discrete barriers is problematical, Cheuk and Vorst propose a variant of their continuous barrier technique discussed above. However, with a discrete barrier, it is no longer desirable for a layer of nodes to lie on the barrier. Rather, (based on experimentation) they suggest that the barrier should lie about halfway between two node layers.

³ Strictly speaking, there can be no truly continuous barrier observation, because prices are generated by transactions that occur at discrete points in time during trading hours. More problematical is the fact that if a market is closed at night, there will be periods during which the (unobserved) asset price might breach the barrier without triggering the barrier event. A continuous barrier pricing equation does not allow for that possibility.
The relatively limited work that has been done on discrete barrier options has focused more on analytic solutions than on lattice methods. Consider, for example, a European down and out call with only one monitoring date $T_k$. One can write the expected payoff as

$$E[\max(0, S_T - K) \mid S_{T_k} > H],$$

which is an expression that can easily be evaluated analytically. The solution involves bivariate normal distributions, and is similar to Geske's [1979] compound option model. Heynen and Kat [1996] adopt this approach and derive closed-form valuation equations. However, implementation of this kind of model becomes infeasible once the number of monitoring points grows beyond just a few, because of the difficulty in evaluating a multivariate cumulative normal distribution function with more than a small number of dimensions.

Wei [1998] offers an approximation approach based on interpolating between the formula for a barrier option with the highest number of monitoring points that can be handled with the analytic formula, and the continuous case (infinite monitoring dates). Broadie, Glasserman and Kou [1997] devise a very interesting and surprisingly accurate method of approximating the discrete barrier option value by simply altering the value for the barrier in the continuous formula. Both of these techniques, however, can be used only for European options, and are limited in other ways in the kinds of problems that can be handled.

In the next section, we present a new kind of Adaptive Mesh Model to deal with discrete barrier options, that has several advantages over the current approaches. First, it is based on a trinomial lattice, which allows pricing of American options and great flexibility in the kinds of contracts and barriers that can be handled. Second, it is an approximation technique that can be made arbitrarily accurate by using more time periods and shorter price steps. Third, like the other AMM models examined in Figlewski and Gao [1999], it is highly efficient because it economizes on the amount of calculation required by using very fine time and price steps only in the regions where they are needed.

II. The Model

The essence of the AMM is to use a relatively coarse lattice throughout most of the time and price state space, but to create a small amount of finer mesh in critical regions where greater resolution is needed. This tends to be in areas of the time and price space where the derivative value is highly nonlinear or discontinuous, in this case, near the discrete price barriers.

It is important for the fine mesh structure to be isomorphic so that additional, still finer, sections of mesh can be added using the same procedure. This permits increasing the resolution in a given section of the lattice as much as one wishes without requiring the step size to change elsewhere. For an option with a continuous barrier, constructing high resolution mesh in the region directly next to the barrier produces very significant decrease in the execution time required reach a given level of accuracy. Here we present an isomorphic AMM structure that will be applied around each segment of a discrete barrier.
The base lattice is a trinomial, set up to approximate the risk neutralized price process for the underlying asset. The asset price $S$ follows the standard diffusion process

**Equation 1**

$$d \ln S = \left( r - q - \frac{\sigma^2}{2} \right) dt + \sigma dW$$

where $r$ is the continuously compounded riskless interest rate, $q$ is the rate of dividend payout, $\sigma$ denotes the volatility and $dW$ is standard Brownian motion.

It is convenient to define $X = \ln S$ and the drift $\mu = (r - q - \frac{\sigma^2}{2})$. Thus

**Equation 2**

$$d X = \mu dt + \sigma dW$$

This process is discretized and approximated by a trinomial process

**Equation 3**

$$X_{t+k} - X_t = \begin{cases} 
\alpha k + \sigma h, & \text{with probability } p_u = \frac{k}{2h^2}, \\
\alpha k, & \text{with probability } p_m = 1 - \frac{k}{h^2}, \\
\alpha k - \sigma h, & \text{with probability } p_d = \frac{k}{2h^2},
\end{cases}$$

where $k$ is the discrete time step and $h$ is the price step. The probabilities for the up, middle, and down branches are $p_u$, $p_m$, and $p_d$, respectively.

The option value at a given asset price and time, $V(X, t)$ is computed from the values at the three successor nodes, as

**Equation 4**

$$V(X, t) = \exp(-rk)[p_u V(X + \alpha k + \sigma h, t + k) + p_m V(X + \alpha k, t + k) + p_d V(X + \alpha k - \sigma h, t + k)]$$

This model works for any positive $h$ and $k$ with $h = O(\sqrt{k})$. If we set $h = \sqrt{3k}$,
then \( p_u = p_d = 1/6 \), \( p_m = 2/3 \) and the model becomes the High Order Trinomial (HOT) model proposed by Gao (1997), for which the discrete trinomial process matches the first five moments of the continuous lognormal diffusion that it is designed to approximate.

Significant improvement in performance for a lattice approximation to an option valuation problem can often be obtained very cheaply by recognizing that even an option with complex contingencies during its lifetime can generally be priced as if it were European in the final time step prior to expiration (“early” exercise of an American option is no longer possible, for example). Using the analytic European formula eliminates the nonlinearity error produced by the discreteness of the lattice around the option’s strike price at maturity. We therefore construct our trees using Gao’s [1997] Analytic - High Order Trinomial (A-HOT) model: The pricing lattice is built to match as many moments of the underlying lognormal diffusion as possible at each point, and sets the option price in the last period before expiration equal to the analytic formula value for the equivalent European option at that date. Note that this means the “standard” trinomial model that the AMM will be compared to below is significantly more efficient than the most basic trinomial structure in common use. An alternative would be to eliminate the nonlinearity error at expiration with an AMM procedure, as described in Figlewski and Gao [1999].

The base trinomial tree has \( N \) periods, making the time step \( k = T / N \), where \( T \) is the maturity of the option. The finer layers of lattice are set up with the price step at each level being one half the size of the step at the previous level. To cut the price step in half, while maintaining the same relationship between the time and price step sizes, the time step must be set to one quarter the size of the next higher level step. Thus, if \( M \) is the level of the mesh, with the base lattice as level 0, then \( h_M = h/2^M \) and \( k_M = k/4^M \).

Figure 2 shows a section of the base trinomial tree around a discrete barrier. The coarse mesh nodes are labeled \( A_{i,j} \), where \( i \) denotes the time period and \( j \) the price step. Note that the indices here are specified relative to the diagram. For example, node \( A_{1,0} \) denotes the node on the barrier (\( i = 1 \)) for the lowest price shown here (\( j = 0 \)). Within the whole tree, of course, there may be many periods prior to these time steps and many prices within the tree above and below the segment shown here. It is important that the tree be set up so that it hits the barriers at integer values for the number of time steps. In our first example, we will examine a single discrete barrier placed halfway to option maturity. This requires the base trinomial tree to be built with an even number of time steps.

The standard trinomial approximation would increase accuracy by shrinking the lattice time and price steps everywhere. To halve the price step, the number of time steps would be quadrupled, and the total number of nodes in the tree would go up by a factor of 16. The AMM approach adds a section of fine mesh lattice just in the vicinity of the barrier, with price and time steps of \( h/2 \) and \( k/4 \), respectively, and branch probabilities given by Equation 3 with these values substituted in for \( h \) and \( k \). It will cover the two coarse time steps before and after the barrier, and within a small number of price steps around it.

Figure 3 shows the fine mesh that is constructed in the time period before the barrier. Nodes in
the (first layer of) fine mesh are denoted as $B_{i,j}$, again relative to the diagram. Thus, nodes $A_{1,4}$ and $B_{4,8}$ correspond to the same date and price. Only selected B-level nodes are labeled.

To increase the lattice's resolution at the barrier, the fine mesh must cover all coarse mesh $A_{0,j}$ nodes from which there will be both fine mesh paths that hit the barrier and paths that avoid it. For example, the lowest time 0 fine mesh path begins at $A_{0,1}$, since all such paths starting from node $A_{0,0}$ or below would hit the barrier. Likewise, the highest price time 0 fine mesh node will coincide with $A_{0,4}$ ($= B_{0,6}$), because all fine paths starting at $A_{0,5}$ or above would miss the barrier. There are therefore only four time 0 B-level nodes.

The next step is to connect the fine mesh lattice back to the coarse lattice at date 2. Figure 4 shows how this is done in the complete discrete AMM structure with one level of fine mesh around the barrier. The B-level nodes such as $B_{4,10}$ that overlap A-level nodes at date 1 present no problem, but the nodes like $B_{1,9}$ that fall in between two coarse lattice nodes do. One possibility would be to use quadrinomial branching, as described in Figlewski and Gao [1999], and connect $B_{4,9}$ to $A_{2,4}$, $A_{2,5}$, $A_{2,6}$ and $A_{2,7}$. However, that would make it somewhat more complicated to add finer levels of mesh. Instead, we break the second (coarse mesh) time period into two subperiods, the first of length $k/4$ and the second of length $3k/4$. That is, the first subperiod is one fine mesh time step, and the second is $3/4$ of a coarse time step.

Branching for the first subperiod is the same as at the other B-level nodes. This leads to two kinds of $B_{5,*}$ nodes. For those lying on a price step that coincides with a price in the coarse mesh, such as $B_{3,4}$, trinomial branching to the neighboring date 2 A-level nodes is straightforward. The node values are obtained from Equation 3, with a price step of $h$ and a time step of $3k/4$. This leads to branch probabilities of $p_u = p_d = 1/8$ and $p_m = 3/4$.

For the B-level nodes that lie in between A-level prices, we can use quadrinomial branching. For example, we allow branching from $B_{5,3}$ to $A_{2,4}$, $A_{2,5}$, $A_{2,6}$, and $A_{2,7}$. Let $k' = 3k/4$. Adopting the same approach as before, we set the moments of the quadrinomial equal to those of the lognormal it is trying to approximate. Equating the mean, volatility and kurtosis, and constraining the probabilities to sum to 1 gives four equations in four unknowns. The solution, interestingly, collapses to binomial branching, just to the two middle nodes, as follows:

**Equation 5**

$$X_{t+1} - X_{t+1/4} = \begin{cases} 
\alpha k' + 3\sigma h / 2 & \text{with probability } p_{uu} = 0, \\
\alpha k' + \sigma h / 2 & \text{with probability } p_u = 1/2, \\
\alpha k' - \sigma h / 2 & \text{with probability } p_d = 1/2, \\
\alpha k' - 3\sigma h / 2 & \text{with probability } p_{dd} = 0.
\end{cases}$$

The isomorphic structure of the fine mesh allows us to add the next layer, with price and time steps $h_C = h/4$ and $k_C = k/16$, using exactly the same procedure as just described. The C-level mesh is constructed within the two B-level time steps around the barrier, starting from four B-
level nodes in the vicinity of the barrier, one period before it. Implementation is facilitated by the fact that the same computational subroutines can be simply called recursively. Figure 5 illustrates the resulting lattice structure.

III. Model Performance

An Analytical Solution for the Discrete Barrier Option

Under the usual "Black-Scholes" assumptions of continuous trading, frictionless and complete markets, and so on, the price of a discrete barrier option can be represented analytically by a formula involving a combination of cumulative multivariate normal distribution functions. For example, under risk neutrality, the value of a down and out call option with strike price K, maturity T, and a single barrier at date T_k with out-strike H and rebate R_b if the barrier is hit, is given by

Equation 6

\[ C(S, K, H, R_b, T_k, T) = R_b e^{-rT_k} \Pr(S_{T_k} \leq H) + e^{-rT}E_0[(S_T - K)|S_{T_k} > H] \]

\[ = R_b e^{-rT_k} N[-d_2(S, H, T_k)] + e^{-rT}N_2[d_1(S, H, T_k), d_1(S, K, T), \rho] - Ke^{-rT}N_2[d_2(S, H, T_k), d_2(S, K, T), \rho] \]

Here \( E_0[.] \) denotes the expectation as of time 0, \( N[.] \) and \( N_2[.,..] \) represent the cumulative normal and bivariate normal distribution functions respectively, \( \rho = \sqrt{T_k / T} \), \( d_1(.) \) and \( d_2(.) \) are defined as follows

\[ d_1(x, y, t) = \frac{1}{\sigma \sqrt{t}}[\ln(x / y) + (r - q + \frac{1}{2} \sigma^2) t] \]

\[ d_2(x, y, t) = \frac{1}{\sigma \sqrt{t}}[\ln(x / y) + (r - q - \frac{1}{2} \sigma^2) t] \]

and other variables are as previously defined.

The problem with this type of "analytic" solution is that it is difficult to compute reliable estimates for the multivariate normal distribution once the dimension is greater than about 4. This is because the cumulative normal distribution function itself can only be evaluated by a numerical approximation, which becomes much more computationally intensive and produces less accurate results than alternative approximation techniques, as the dimension increases. However, for an
option with a single discrete barrier, we will be able to use this exact solution as a benchmark to examine the performance of our AMM approximation.

**Example 1:**
We first consider pricing a single European down and out call option. The relevant parameters are:

- **Asset price:** $S = 40$
- **Strike price:** $K = 40$
- **Maturity:** $T = 1$ year
- **Barrier level:** $H = 35$
- **Rebate at barrier:** $R_b = 0$
- **Barrier date:** $T_b = T/2 = 1/2$ year
- **Riskless interest rate:** $r = 0.0488$ (equivalent to 5.00% simple interest)
- **Volatility:** $F = 0.40$
- **Dividend yield:** $q = 0$

Figure 6 shows the convergence for three lattice techniques to the true value of $C = 7.7766$ obtained from Equation 6, as the number of time steps $N$ increases. Convergence for the standard (A-HOT) trinomial is both slow and choppy, as is typical for lattice approximations for this type of problem.

The AMM-2 model is constructed as described above, with two levels of fine mesh at the barrier. As shown in Figure 4, each mesh level adds 62 new nodes to the tree, or 124 in total for the AMM-2 model. This is a negligible increase, given that the total number of nodes in a trinomial with $N$ time steps is $(N+1)^2$, e.g., 40,401 for a 200 step tree. The AMM-8 model, with eight levels of fine mesh that bring the price step at the barrier down to 1/256 of its value elsewhere in the tree can be seen to reach convergence for a very small value of $N$.

These results show that for a given number of time steps in the lattice, the AMM is considerably more accurate than the standard trinomial in valuing this option. To examine more closely the average efficiency gains obtained with the AMM structure, we now consider pricing a portfolio of options. This permits us to compare average accuracy and computation time across models, which is the true measure of efficiency for such an approximation method.

**Example 2:**
Here we examine a portfolio of single discrete barrier down and out call options. We consider a total of 108 options, consisting of all combinations of three initial asset prices, four maturities, three values for the knockout date, and three volatilities.

**Variable parameters:**

- **Asset price:** $S = 35, 40, 45$
- **Maturity:** $T = 0.25, 0.5, 0.75, 1$ year
Barrier date: \( T_k = T/4, \ T/2, \ 3T/4 \)

Volatility: \( F = 0.20, 0.30, 0.40 \)

Fixed parameters:
- Strike price: \( K = 40 \)
- Barrier level: \( H = 35 \)
- Rebate at barrier: \( R_b = 0 \)
- Riskless interest rate: \( r = 0.10 \)
- Dividend yield: \( q = 0 \)

Absolute and relative pricing errors are defined as \( |C_{\text{model}} - C_{\text{benchmark}}| \) and \( |C_{\text{model}} / C_{\text{benchmark}} - 1| \), respectively. The models are compared on the basis of absolute and relative root mean squared error, denoted ARMSE and RRMSE. We also compute the maximum absolute error (MAE) and the maximum absolute relative error (MRE) among the 108 options, as well as total CPU time to price the whole portfolio using a personal computer with a Pentium Pro 200 MHz processor.

Table 1 shows the results for standard trinomial models with 1000 and 5000 time steps (about 1 million and 25 million nodes, respectively), and AMM models with 60 time steps in the base lattice and 2, 4, or 8 levels of fine mesh at the barrier. The "Equivalent steps" value \( N_{\text{eq}} \) indicates how many time steps a standard trinomial model would require to have the same step size as the finest AMM level throughout the lattice.

The AMM-2 model with 60 time steps in the base lattice is only slightly less accurate than a standard trinomial with 1000 steps, but it runs about 60 times faster. Adding further levels of fine mesh improves performance substantially, with only a very small increase in execution time. Thus, the AMM-8 model, which is "equivalent" to a standard trinomial with nearly 4 million time steps, produces results that are about ten times more accurate than a standard trinomial with 5000 time steps, yet takes less than 1/1000th of the execution time.

The examples so far have considered options with a single discrete barrier. Although the AMM structure produces very significant increases in performance in valuing them, these option are not typical of those encountered in practice. The next example is a two-year down and out call with a knockout barrier that is tested at the end of each quarter. That is, there are seven discrete barriers.

**Example 3:**
The parameters are:

- Asset price: \( S = 40 \)
- Strike price: \( K = 40 \)
- Maturity: \( T = 2 \) years
- Barrier level: \( H = 35 \)
- Rebate at barrier: \( R_b = 0 \)
- Barrier dates: \( T_k = 0.25, 0.5, 0.75, 1.0, 1.25, 1.50, 1.75 \) years
Riskless interest rate: \( r = 0.10 \)
Volatility: \( \sigma = 0.40 \)
Dividend yield: \( q = 0 \)

With seven barrier dates before expiration, the analytic solution is no longer practically feasible, so we have no exact benchmark for comparison. However, all of the models examined here are consistent, in that they will reach the true option value in the asymptotic limit as the number of time steps is increased. We may, therefore, gauge model's accuracy for a given option by how quickly the value it produces settles down to its limiting value. Figure 7 plots the values produced by the standard trinomial, the AMM-2 and the AMM-8 models for this quarterly knockout call as the number of time steps grows.

The standard trinomial exhibits a more complex pattern of convergence for this option than for the single barrier contract in the first example, but the general properties of choppiness and relatively slow convergence are similar, and the model does not settle down to a clear value even with 1000 time steps. The AMM-2 model shows much tighter range of variability and produces option prices within a penny of the asymptotic value within a couple hundred time steps. But, as we saw before, the AMM-8 model converges almost immediately, even with less than 100 time steps.

As a final example, we consider a nonstandard option with a discretely observed barrier that varies over time. Other approximation techniques for barrier options are typically limited in the type of barrier that can be handled. One important advantage of the AMM approach is that it can be tailored relatively easily to fit a great variety of option structures. For discrete barriers, for example, each barrier is dealt with individually, so that it is not necessary for all of them to be at the same price, or at constant intervals in time.

Example 4:
We consider a two-year down and out call with a quarterly barrier that begins at a price of 38 and is reduced by one point each quarter.

| Asset price: | \( S = 40 \) |
| Strike price: | \( K = 40 \) |
| Maturity: | \( T = 2 \) years |
| Barrier level: | \( H = 38, 37, 36, 35, 34, 33, 32 \) in sequence |
| Rebate at barrier: | \( R_b = 0 \) |
| Barrier dates: | \( T_k = 0.25, 0.5, 0.75, 1.0, 1.25, 1.50, 1.75 \) years |
| Riskless interest rate: | \( r = 0.10 \) |
| Volatility: | \( \sigma = 0.40 \) |
| Dividend yield: | \( q = 0 \) |

Like the previous example, we have no analytic solution but we may examine convergence behavior as the number of time steps increases. Figure 8 shows the results. The standard trinomial appears to have a great deal of difficulty with this problem. It exhibits large swings in
the model valuation that do not seem to die out even with 1000 time steps. By contrast, the AMM-8 converges extremely rapidly.

The AMM-8 model with between 800 and 1000 time steps produces an average value for this option of 0.23573, and the largest difference from this value for any N in this range was only 0.00013. Taking this price as our benchmark allows us to compare the accuracy of the different models with smaller N.

Even with the smallest number of coarse time steps considered (32), the AMM-8 model obtains an option value well within .001 of the benchmark. For N over 100, the maximum discrepancy is less than 0.0002. The standard trinomial, on the other hand, can be off by more than 0.02, i.e., 2 cents or nearly 10 percent, with over 900 time steps.

IV. Conclusion

Barrier options are among the most common "exotic" option contracts in the marketplace. Although a closed-form valuation equation has long been known for European options with a continuous barrier, both discreteness of the barrier and American exercise--common features of real-world barrier options--present serious difficulties for such analytic formulas. Lattice models and other approximation techniques in current use are also strictly limited in their ability to deal with these features.

Figlewski and Gao [1999], develops the Adaptive Mesh technology and shows how it can be applied with considerable success to several basic option valuation problems, including pricing continuous barrier options. This paper has adapted the new approach to pricing barrier options when the barrier is not continuous, but is only monitored at discrete intervals. Because the model is based on a trinomial lattice, it can be readily adapted to incorporate American exercise, as well as other more complex contingencies. We therefore anticipate that AMM models will prove to be very useful for dealing with barrier options, as well as with a wide variety of other real-world derivative structures.
References


Gao, B., J. Huang, and M. G. Subrahmanyam, 1999, "The Valuation of American Barrier Options Using the Decomposition Technique.", working paper, New York University}


# Table 1
## Performance Comparison for Lattice Models Relative to an Exact Benchmark

<table>
<thead>
<tr>
<th>Model</th>
<th>Time steps N</th>
<th>Equivalent steps N&lt;sub&gt;eq&lt;/sub&gt;</th>
<th>ARMSE</th>
<th>RRMSE</th>
<th>MAE</th>
<th>MRE</th>
<th>CPU time (seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard Trinomial</td>
<td>1000</td>
<td>1000</td>
<td>0.0315</td>
<td>0.0040</td>
<td>0.1622</td>
<td>0.0144</td>
<td>39.99</td>
</tr>
<tr>
<td>Standard Trinomial</td>
<td>5000</td>
<td>5000</td>
<td>0.0148</td>
<td>0.0019</td>
<td>0.0838</td>
<td>0.0065</td>
<td>908.3</td>
</tr>
<tr>
<td>AMM-2</td>
<td>60</td>
<td>960</td>
<td>0.0350</td>
<td>0.0042</td>
<td>0.1769</td>
<td>0.0170</td>
<td>0.671</td>
</tr>
<tr>
<td>AMM-4</td>
<td>60</td>
<td>15,360</td>
<td>0.0113</td>
<td>0.0011</td>
<td>0.0618</td>
<td>0.0047</td>
<td>0.691</td>
</tr>
<tr>
<td>AMM-8</td>
<td>60</td>
<td>3.9 million</td>
<td>0.0014</td>
<td>0.0002</td>
<td>0.0079</td>
<td>0.0007</td>
<td>0.721</td>
</tr>
</tbody>
</table>

Notes: The table reports model performance in pricing a portfolio of 108 down and out call options with a single discrete barrier, as described in the text. Absolute and relative pricing errors are defined as \( |C_{\text{model}} - C_{\text{benchmark}}| \) and \( |C_{\text{model}} / C_{\text{benchmark}} - 1| \), respectively. The models are compared on the basis of absolute and relative root mean squared error, denoted ARMSE and RRMSE. The maximum absolute error (MAE) and maximum absolute relative error (MRE) across the 108 options are also shown, as well as total CPU time to price the whole portfolio using a personal computer with a Pentium Pro 200 MHz processor. The AMM models use 60 time steps for the coarsest mesh, and 2, 4 or 8 levels of fine mesh at the barrier. N<sub>eq</sub> is the number of time steps that would be required for an "equivalent" standard trinomial model to have the same size time step as the finest AMM level throughout the lattice.
Figure 1
Stock Path with Continuous and Discrete Barriers

![Graph showing stock price over time with continuous and discrete barriers.]
Figure 2

The Coarse Mesh in the Two Periods around a Discrete Barrier
Figure 3

Building the Fine Mesh in the Period Before a Discrete Barrier
Figure 4

Building the Fine Mesh in the Period After a Discrete Barrier
Figure 5

Adding the Second Level of Finer Mesh at the Barrier
Figure 6
Model Convergence for a Single Discrete Barrier Down and Out Call

Model Option Price

Trinomial
AMM-2
AMM-8

Exact value
C = 7.7766
Figure 7
Model Convergence for a Two-Year Down and Out Call with Quarterly Barriers

![Graph showing model convergence for a two-year down and out call with quarterly barriers. The x-axis represents the number of time steps, ranging from 0 to 1000, and the y-axis represents model option price, ranging from 0.1 to 0.3. The graph compares three models: Trinomial, AMM-2, and AMM-8. The Trinomial model is represented by a dashed line, AMM-2 by a dotted line, and AMM-8 by a solid line. The convergence is observed at around 500 time steps.](image-url)
Figure 8
Convergence for Two-Year Down and Out Call with Quarterly Declining Barrier

Model Option Price

Number of Time Steps

Benchmark Value
C = 0.23573