Distribution Free Bounds for Service Constrained (Q, r) Inventory Systems

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Abstract: A classical and important problem in stochastic inventory theory is to determine the order quantity (Q) and the reorder level (r) to minimize inventory holding and backorder costs subject to a service constraint that the fill rate, i.e., the fraction of demand satisfied by inventory in stock, is at least equal to a desired value. This problem is often hard to solve because the fill rate constraint is not convex in (Q, r) unless additional assumptions are made about the distribution of demand during the lead-time. As a consequence, there are no known algorithms, other than exhaustive search, that are available for solving this problem in its full generality. Our paper derives the first known bounds to the fill-rate constrained (Q, r) inventory problem. We derive upper and lower bounds for the optimal values of the order quantity and the reorder level for this problem that are independent of the distribution of demand during the lead time and its variance. We show that the classical economic order quantity is a lower bound on the optimal ordering quantity. We present an efficient solution procedure that exploits these bounds and has a guaranteed bound on the error. When the Lagrangian of the fill rate constraint is convex or when the fill rate constraint does not exist, our bounds can be used to enhance the efficiency of existing algorithms. © 2000 John Wiley & Sons, Inc. Naval Research Logistics 47: 635–656, 2000

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1. INTRODUCTION

A classical and important problem in stochastic inventory theory is to determine the order quantity (Q) and the reorder level (r) to minimize inventory holding and backorder costs subject to a service constraint that the fraction of demand satisfied by inventory in stock (i.e., the fill rate), is at least equal to a desired value. This optimization problem is usually difficult to solve without making additional assumptions regarding the distribution of demand during the lead-time (see Zipkin [19]). As a consequence, no algorithm other than exhaustive search is available for solving this problem in its full generality. In this paper, we derive upper and lower bounds for the optimal values of Q and r for this problem and in particular show that the classical Economic Order Quantity (EOQ) is a lower bound for the optimal Q. We also present an efficient solution procedure that exploits these bounds and produces a solution that has a guaranteed error bound.

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Several heuristics have been proposed for solving this service constrained (Q, r) inventory problem. Platt, Robinson, and Fruend in [10] provide a comprehensive review of these procedures. Heuristics adopt one or both of two approaches for solving this problem. They simplify either the expression for the fill rate, or the expression for the average inventory. Such simplifications are introduced by assuming that there is at most one order outstanding at any time (Section 4-2 of Hadley and Whitin [7], Yano [16], Silver and Wilson [14], and Rosling [12]), or by assuming that the duration of backorders is negligible (Section 4-2 of [7], [16], and [14]), or by neglecting a part of the integral in the expression for the fill rate. The advantage of heuristic procedures lies in the ease with which a user can determine the values of Q and r. For example, Yano [16] provides an iterative heuristic for normally distributed lead-time demand that is guaranteed to converge. Platt, Robinson, and Freund [10] propose and compare two elegantly motivated closed form heuristics for solving this problem. Our bounds can be used to enhance the efficiency of these algorithms.

The problem has been found to be intractable when the above simplifications are not made. This is because the optimization domain shifts from convex optimization to general nonlinear optimization. The difficulty is created by the fact that the fill rate constraint is not in general a convex function of the problem parameters, Q and r. Hadley and Whitin [7] realized the difficulty of this problem and stated "the program (algorithm) determines a relative minimum but does not provide any guarantee that the minimum so obtained is the absolute minimum (p. 188)."

Obtaining optimal parameters might appear to be inconsequential, Zipkin [19] showed that the penalty of making simplifications in order to convexify the fill rate expression can be quite substantial in certain instances. Zipkin [19] proved that the average level of backorders was convex in Q and r. He also derived sufficient conditions that can be imposed on the lead-time demand distribution to guarantee that the expression for the fill rate is convex in (Q, r). The condition (which is also provided by Zhang in [17]) states that fill rate expression is convex in (Q, r) if the probability density function of lead-time demand $f(x)$ is decreasing for $x \geq r$. This condition restricts r to large positive values. In practical problems r can be negative or small when either the setup cost is high, backorder cost is low, demand variance is small, or the lead-time is short.

Rosling in a series of recent papers ([11] and [12]) has analyzed several inventory cost rate functions. His approach unifies the analysis of different inventory models (continuous review, period review, etc.) with the notion of quasi-convexity (see below) of the cost rate function. Our model without the fill rate constraint is the closest to model 4 in [11]. In this paper, Rosling shows that the cost (rate) function for model 4 is quasiconvex if the distribution function of lead-time demand is logconcave. (An accessible summary of logconcavity property and applications can be found in [4].) He uses this result in [12], and shows that if the cost rate function is quasiconvex for all nonnegative cost coefficients, then the Lagrangian that incorporates the fill rate constraint is quasiconvex (p. 14). When there is a service constraint, Rosling also comments ([12], p. 20): "It is also taken for granted that an optimal finite policy exists, but the exact conditions for this are not well known for the problem with a service constraint." He too does not establish the existence of a finite optimal solution. In [12], Rosling proposes an algorithm that assumes the quasiconvexity of the cost rate function for the service level constrained case and the existence of a finite optimal solution. He also observes that algorithms proposed in the literature are not always guaranteed to converge, similar to observations made by Platt, Robinson, and Freund [10].

Our bounds are useful even when the quasiconvexity property of the Lagrangian of the fill rate is established by the lead time demand distribution being logconcave. Notice that a function $g(x)$ from $R^n \to R$ is said to be strictly quasiconvex if for all x, y, in R^n and $\theta \in (0, 1), g((1 - \theta)x + \theta y) < \max\{g(x), g(y)\}$. This property implies that a local optimum is a global optimum (see, e.g., [2]). Therefore, in order to use an algorithm that exploits this property, it is essential to know a good upper bound for the decision variables (in our case the order quantity and the reorder level). This explains the need for the assumption by Rosling that a finite optimal policy exists. As Rosling comments, "An algorithm is initiated at $Q_w < Q^*$, it follows . . . that Q increases for some time until it possibly overshoots Q^*, after which it might oscillate so that convergence is not monotonic" ([12], p. 22)). Therefore, establishing the existence of an upper bound that is not too large when compared to the lower bound increases the efficiency of algorithms for solving the fill-rate constrained problem even when the Lagrangian is quasiconvex.

Our bounds are new for the service constrained version of the problem. Bounds are available for the *unconstrained* version of the problem (see Zheng [18], Axsater [1], and Gallego [6]). Zheng showed that the deterministic EOQ (modified for the case when backorders are allowed) is a lower bound for the optimal value of Q. He also established that the use of the EOQ as the order quantity does not lead to substantial deviation from the optimal cost. Axsater sharpened the bound on cost provided by Zheng. Federgruen and Zheng provided an algorithm in [5] for determining the optimal values of Q and r for the unconstrained problem that is also applicable to the constrained problem in special cases. Discussion of this aspect is given in Section 2. Gallego provided an upper bound for Q for the *unconstrained* problem. The upper bound given by Gallego grows with the standard deviation of the lead-time demand, whereas our bounds are distribution-free (and variance free) as explained below. As Nahmias and Smith [9] stress, in many retail settings the variance to mean ratio can be very high (ranging from 3 to 500 in their retail study). Thus our bounds can be useful in these scenarios even for the unconstrained problem.

Our analysis is based on the exact formulation of the objective function, as well as the exact expression for the fill rate. Interest in (Q, r) inventory systems using the exact formulation of the objective function, as given by Hadley and Whitin (in Section 4:7 of [7]), was rekindled by the work of Zipkin and Zheng. The bounds derived by us are independent of the distribution of the lead-time demand in the following sense. We show that in the worst case the search for the optimal values of Q and r can be restricted to *four* intervals, the first of which is given in terms of the reorder level and the others in terms of the order quantity. In each interval, regardless of the distribution of the lead-time demand, the ratio of the largest to the smallest value within the interval is less than 4. The solution algorithm uses this feature by searching along the values of r in the first interval and along the values of Q in the others. We provide an epsilon-optimal solution algorithm which is guaranteed to converge in polynomial time. Moreover, these bounds can be easily incorporated into existing algorithms in order to improve the efficiency of the search for Q, even when the fill rate constraint is well behaved (i.e., convex or quasiconvex) or not present.

We introduce a novel way of dealing with the fill rate constraint. We express the cost function as a sum of a convex function of (Q, r) and a nonconvex function. It has the following special property that, for a given Q, if the reorder level $r(Q)$ minimizes the convex portion of the cost, then the set of all such $(Q, r(Q))$ pairs is exactly equal to the set of solutions that satisfy the fill rate constraint. This property plays an important role in guaranteeing an epsilon-optimal solution within finite time. Another novelty in our analysis is that it does not use the two-point distribution approach taken by earlier researchers (such as [1] and [6]) to establish bounds for the unconstrained problem. The main feature of our approach is to bypass the tail integral with regard to the demand distribution. This property allows us to produce distribution free (and variance independent) bounds. Our methods are attractive when viewed as alternate methods of establishing bounds for the stochastic inventory problem.

In Section 2, we specify the problem and derive structural properties that are used to transform the problem. We then derive bounds in Section 3 and provide an algorithm that can be used to determine (Q, r) in Section 4.

2. PROBLEM FORMULATION

We consider the classical formulation of the continuous review stochastic inventory control problem. The decision variables are the parameters r and Q. The objective is to minimize the sum of setup, holding and backorder costs subject to the constraint that the fill rate should be at least equal to F (as given by Hadley and Whitin in [7]). The setup cost per order is given by K. The holding cost per unit per unit time is denote as h and the back order penalty per unit per unit time as p. In order for the optimal value of Q to be nonzero and the optimal values of r and Q to be finite, we assume that F, h, and K are strictly greater than zero and that the penalty p is greater than or equal to zero. The lead-time is assumed to be independent of the inventory position. The distribution function of the lead-time demand is denoted as $\mathcal{F}$, the complement of $\mathcal{F}$ as $\mathcal{F}^c(\cdot)$, and the density function of the lead-time demand is denoted as $f(\cdot)$. We assume that the density function $f(\cdot)$ is strictly positive on $[0, \infty)$. The average arrival rate of demand is denoted as λ and the average demand during the lead-time is given by μ. We assume that the inventory position is uniformly distributed in the interval $[r, r+Q]$ and independent of the lead time demand (see Browne and Zipkin [3], Serfozo and Stidham [13], and Zipkin [20]) for discussion of the conditions under which this assumption holds). In this model all unfilled demand is backordered. The case when there are either lost sales or partially lost sales poses greater difficulty in the analysis as described in [9] and [8]. We plan to analyze the lost sales model in future work.

Denote the average inventory and the average backorder level by $\bar{I}(Q,r)$ and $\bar{S}(Q,r)$. It follows from the assumption that inventory position is uniformly distributed,

$$\bar{I}(Q,r) = \frac{\int_r^{r+Q} \left[\int_0^y (y-x) f(x)\, dx\right] dy}{Q}$$

and

$$\bar{S}(Q,r) = \frac{\int_r^{r+Q} \left[\int_y^\infty (x-y) f(x)\, dx\right] dy}{Q}.$$

Define

$$G(y) = h\left[\int_0^y (y-x) f(x)\, dx\right] + p\left[\int_y^\infty (x-y) f(x)\, dx\right].$$

The sum of the average holding cost and the average cost of backorder can be expressed in terms of $G(y)$ as

$$h\bar{I}(Q,r) + p\bar{S}(Q,r) = \frac{\int_r^{r+Q} G(y)\, dy}{Q}.$$

The optimization problem is given by

Problem P:

$$\min_{Q,r} C(Q,r) = \min_{Q,r}\left(\frac{\lambda K}{Q} + \frac{\int_r^{r+Q} G(y)\, dy}{Q}\right) \tag{1}$$

subject to the fill rate constraint,

$$1 - \frac{\int_r^{r+Q} \mathcal{F}^c(x)\, dx}{Q} \geq F. \tag{2}$$

It is known that the objective function of problem $\mathbf{P}$ is jointly convex in Q and r (see Zipkin [19] or Zheng [18]), that $G(y)$ is convex, $G(\infty) = \infty$, and that $\lim_{y\to\infty} G'(y) = h$, where $G'(y)$ is the derivative of $G(y)$. In addition, if $f(\cdot) > 0$ on $[0, \infty)$, then $C(Q, r)$ is strictly and jointly convex in (Q, r).

REMARK: One approach to solving $\mathbf{P}$ would be to form the Lagrangian and search for the optimal multiplier employing the algorithm proposed by Federgruen and Zheng in [5] (denoted as the FZ algorithm). It can be shown that the Lagrangian will be of the same form as the objective function except that $G(y)$ will be replaced by $G(y) - \pi F(y)$, where π is the multiplier. Unfortunately, $G(y) - \pi F(y)$ is not always unimodal—a condition that is necessary to ensure the optimality of (Q, r) found using the FZ algorithm. The condition holds when the lead-time demand is Poisson. It is also worth mentioning that the complexity of an algorithm that uses the Lagrangian approach to incorporate the fill rate constraint is as yet unknown. Moreover, by using the structure of the fill rate constraint, it might be possible to use weak duality theory to develop bounds on the solution produced by the Lagrangian approach. As mentioned in the Introduction, Rosling [12] shows that if the distribution of lead-time demand is logconcave, then the Lagrangian (that incorporates the fill rate constraint) is quasiconvex, but a finite optimal policy is still not guaranteed. Therefore, bounds are required on the values of Q and r to facilitate the search for these parameters.

Let (Q^*, r^*) be an optimal solution to Problem $\mathbf{P}$. Let (Q_u^*, r_u^*) minimize $C(Q, r)$ (i.e., (Q_u^*, r_u^*) achieve the unconstrained minimum).

Define the fill rate achieved with a given set of (Q, r) to be $\Phi(Q, r)$. Thus,

$$\Phi(Q, r) = 1 - \frac{\int_r^{r+Q} \mathcal{F}^c(x)\, dx}{Q}. \tag{3}$$

Let $r_u(Q)$ be the value of r that minimizes $C(Q, r)$ for a given value of Q.

LEMMA 1: $\Phi(Q_u^*, r_u^*) \geq F$ iff $p/(p + h) \geq F$.

PROOF: $C(Q, r)$ can be expressed as (see Zheng [18]):

$$C(Q, r) = \frac{\lambda K}{Q} + \frac{\int_r^{r+Q} [(h + p) \int_0^y \mathcal{F}(x)\, dx + p(\lambda L - y)]\, dy}{Q}.$$

From the convexity of $C(Q, r)$, $r_u(Q)$ is obtained by solving for r in

$$\frac{\partial C(Q, r)}{\partial r} = 0.$$

Therefore,

$$(h + p) \int_0^{r_u(Q)+Q} \mathcal{F}(x)\, dx + p(\lambda L - r_u(Q) - Q) - (h + p) \int_0^{r_u(Q)} \mathcal{F}(x)\, dx - p(\lambda L - r_u(Q)) = 0.$$

This implies that

$$\frac{h+p}{p}\left[\int_0^{r_u(Q)+Q}\mathcal{F}(x)\,dx-\int_0^{r_u(Q)}\mathcal{F}(x)\,dx\right]=Q.$$

Consolidating the terms in the integrals, we obtain

$$\frac{h+p}{p}\left[\int_{r_u(Q)}^{r_u(Q)+Q}(1-\mathcal{F}^c(x))\,dx\right]=Q.$$

This implies that

$$\frac{h+p}{p}\left[Q-\int_{r_u(Q)}^{r_u(Q)+Q}\mathcal{F}^c(x)\,dx\right]=Q.$$

Therefore,

$$\left[1-\frac{\int_{r_u(Q)}^{r_u(Q)+Q}\mathcal{F}^c(x)\,dx}{Q}\right]=\Phi(Q,r_u(Q))=\frac{p}{p+h}. \tag{4}$$

This in turn yields

$$\Phi(Q_u^*,r_u(Q_u^*))=\Phi(Q_u^*,r_u^*)=\frac{p}{p+h}. \tag{5}$$

If $\frac{p}{p+h}\geq F$, then from Eq. (5) we obtain that $\Phi(Q_u^*,r_u^*)\geq F$. On the other hand, if $\Phi(Q_u^*,r_u^*)\geq F$, then from Eq. (5) it follows that $\frac{p}{p+h}\geq F$. □

REMARK: A similar result to Lemma 1 is derived in [6] in the context of the unconstrained problem, but we need the ‘‘if and only if’’ argument, which is new in this paper, to characterize the solution set for the constrained problem.

For a given order quantity Q, let $r(Q)$ be the reorder level that satisfies the fill rate constraint as an equality, i.e.,

$$1-\frac{\int_{r(Q)}^{r(Q)+Q}\mathcal{F}^c(x)\,dx}{Q}=\Phi(Q,r(Q))=F. \tag{6}$$

Given Q, let $r^*(Q)$ to be the reorder level that minimizes $C(Q,r)$ and satisfies the fill rate constraint. The function $r^*(Q)$ is characterized in the lemma given below.

LEMMA 2: If $p/(p+h)<F$, then $r^*(Q)=r(Q)$, i.e., $r^*(Q)$ satisfies the fill rate constraint as an equality. If $p/(p+h)\geq F$, then $r^*(Q)=r_u(Q)$.

PROOF: From equation (3) we know that $\Phi(Q,r)$ is non-decreasing in r. Therefore, the set of reorder points, S_F, that satisfy equation (2) for a given value of Q is given by

$$\mathcal{S}_F=\{r:r\geq r(Q)\}.$$

Thus, $r^*(Q)$ can be represented as:

$$r^*(Q) = \arg\min_{r \in \mathcal{S}_F} C(Q, r).$$

The minimum value of $C(Q, r)$ for a given Q is achieved at $r_u(Q)$. It follows from equation (4) that for a given value of Q, if $p/(p+h) < F$ then $\Phi(Q, r_u(Q)) < F$. From the fact that $\Phi(Q, r)$ is non-decreasing in r, it follows that $r(Q) > r_u(Q)$ when $p/(p+h) < F$. Therefore, from the convexity of $C(Q, r)$ in r, it follows that $C(Q, r) \geq C(Q, r(Q))$ for $r \in \mathcal{S}_F$. This finally implies that $r^*(Q) = r(Q)$.

The second part of the lemma follows from Lemma 1. □

REMARK: It should be noted that the set $\{(Q, r) : \Phi(Q, r) \geq F, \text{ and } Q \geq 0\}$ need not be convex (see Zipkin [19]). Lemmas 1 and 2 narrow down the set in which an optimal solution of $\mathbf{P}$ can be found as follows.

1. If $p/(p+h) \geq F$ then the unconstrained solution, (Q_u^*, r_u^*), is optimal for $\mathbf{P}$.
2. If $p/(p+h) < F$ then the search for the optimal solution to problem $\mathbf{P}$ can be restricted to the set $\{(Q, r(Q)) : Q \geq 0\}$.

We will now use these properties to modify the objective function in problem $\mathbf{P}$. As stated in Section 1 we express the cost as a sum of two functions. The first function has the property that for a given Q, the reorder level $r(Q)$ that satisfies the fill rate constraint in equality (as defined earlier), also minimizes this portion of the cost. This set of $(Q, r(Q))$ is identical to the set of solutions that satisfy the fill rate constraint (as shown in Lemma 4 below). Moreover we will prove in Lemma 5 that this portion is a convex function of Q when r is replaced by $r(Q)$. Let $K_1 = \frac{K}{p+h}$ and $F_u = \frac{p}{p+h}$.

LEMMA 3: Problems **P1** and **P** are equivalent, where

Problem P1:

$$\min_{Q,r} C_1(Q, r) = \min_{Q,r} \left(\frac{\lambda K_1}{Q} + F\bar{S}(Q, r) + (1-F)\bar{I}(Q, r) + (F - F_u)(r + Q/2 - \mu) \right) \tag{7}$$

subject to

$$1 - \frac{\int_r^{r+Q} \mathcal{F}^c(x)\, dx}{Q} \geq F. \tag{8}$$

PROOF: The objective function of problem **P** (see equation (1)) can be written as

$$\begin{aligned}
&(p+h)\left[\frac{\lambda K}{(p+h)Q} + \frac{p}{p+h}\bar{S}(Q, r) + \frac{h}{p+h}\bar{I}(Q, r)\right] \\
&= (p+h)\left[\frac{\lambda K_1}{Q} + F_u\bar{S}(Q, r) + (1-F_u)\bar{I}(Q, r)\right] \\
&= (p+h)\left[\frac{\lambda K_1}{Q} + F\bar{S}(Q, r) + (1-F)\bar{I}(Q, r) + (F - F_u)(\bar{I}(Q, r) - \bar{S}(Q, r))\right].
\end{aligned}$$

Further,

$$\begin{aligned}(\bar{I}(Q,r) - \bar{S}(Q,r)) &= \frac{\int_r^{r+Q} [\int_0^y (y-x)f(x)\,dx - \int_y^\infty (x-y)f(x)\,dx]\,dy}{Q} \\ &= \frac{\int_r^{r+Q} (y-\mu)\,dy}{Q} = \frac{\frac{(r+Q-\mu)^2}{2} - \frac{(r-\mu)^2}{2}}{Q} \\ &= r + Q/2 - \mu. \qquad \square \end{aligned} \tag{9}$$

LEMMA 4:

(1) The mapping from Q to $r(Q)$ is one-to-one.
(2) $-1 \le r'(Q) \le 0$.
(3) $r(Q) + Q$ is increasing in Q.
(4) $\lim_{Q\to\infty}(r(Q) + Q) = \infty$.

PROOF: By assumption $f(\cdot) > 0$. Therefore the fill rate, which is given by $1 - \frac{\int_r^{r+Q} \mathcal{F}^c(x)\,dx}{Q}$, is strictly increasing in r. Therefore given Q, the fill rate can be satisfied only for one value of r. This proves part (1) of the lemma. For proving parts (2), (3) and (4), we know from Lemma 2 and the first part of this lemma that $r(Q)$ is the solution to

$$\frac{\partial}{\partial r}\left(\frac{\lambda K_1}{Q} + F\bar{S}(Q,r) + (1-F)\bar{I}(Q,r)\right) = 0. \tag{10}$$

Therefore, the results obtained in Lemma 3.3 and 3.4 in Zheng [18] correspond to claims (2), (3), and (4). $\square$

3. BOUNDS ON Q^*

We will now restrict our attention to the case $p/(p+h) < F$, that is when the fill rate constraint is binding at the optimal solution to problem $\mathbf{P}$ (or $\mathbf{P1}$). As a result of Lemmas 2 and 3, problem $\mathbf{P1}$ can be solved by finding the value of Q that minimizes:

$$C_1(Q) = \frac{\lambda K_1}{Q} + \frac{\int_{r(Q)}^{r(Q)+Q} G_1(y)\,dy}{Q} + (F - F_u)(r(Q) + Q/2 - \mu) \tag{11}$$

where,

$$G_1(y) = (1-F)\left[\int_0^y (y-x)f(x)\,dx\right] + F\left[\int_y^\infty (x-y)f(x)\,dx\right]. \tag{12}$$

Notice that,

$$\frac{\int_{r(Q)}^{r(Q)+Q} G_1(y)\,dy}{Q} = F\bar{S}(Q,r(Q)) + (1-F)\bar{I}(Q,r(Q)).$$

Also, note that by specifying r to be equal to $r(Q)$ in the cost function (11) we ensure that the fill rate constraint is satisfied. Define,

$$H(Q,r) = \frac{\int_r^{r+Q} G_1(y)\,dy}{Q}.$$

As a matter of notation, we denote derivatives using a prime wherever there is no scope for confusion. We shall refer to $H(Q, r(Q))$ by $H(Q)$. The properties of $H(Q)$ derived in the next lemma will be used to obtain bounds on Q.

LEMMA 5:

(1) $H'(Q) \geq 0$.
(2) $H''(Q) \geq 0$.
(3) $H'(Q) \leq (1-F)/2$, for all $Q \geq 0$.

PROOF: Please see Appendix (also see Zheng [18]).

THEOREM 1: A lower bound for Q^* is given by the Economic Order Quantity (EOQ), i.e., $Q^* \geq \sqrt{\frac{2\lambda K}{h}}$.

PROOF: Note that as $K > 0, Q^* > 0$, and because $h > 0, Q^* < \infty$. Therefore, an optimal solution to **P1** has to satisfy the first-order condition for optimality, i.e., $\frac{\partial}{\partial Q} C_1(Q) = 0$. Therefore, we get

$$\frac{\lambda K_1}{(Q^*)^2} = H'(Q^*) + (F - F_u)(r'(Q^*) + 1/2).$$

From Lemma 4 we know that $r'(Q) \leq 0$; therefore,

$$\frac{\lambda K_1}{(Q^*)^2} \leq H'(Q^*) + \frac{F - F_u}{2}. \tag{13}$$

From part 3 of Lemma 5, $H'(Q^*) \leq (1-F)/2$; therefore,

$$\frac{\lambda K_1}{(Q^*)^2} \leq \frac{1-F}{2} + \frac{F-F_u}{2} = \frac{1-F_u}{2}.$$

Substituting for the values of K_1 and F_u, we get

$$\frac{\lambda K}{(p+h)(Q^*)^2} \leq \frac{h}{(p+h)2} \quad \text{or} \quad Q^* \geq \sqrt{\frac{2\lambda K}{h}} = \text{EOQ}. \qquad \square$$

REMARK: This is an important and useful lower bound on Q as it is not a function of the backorder cost rate p, because in many practical situation p is difficult to find and many papers in the literature (such as [10], [16]) have analyzed the fill rate constrained (Q, r) problem without a backorder cost. It can be shown that when the variance of demand approaches zero, $p \to 0$, and $F \to 1$, then $Q^* \to \sqrt{2\lambda K/h}$. Therefore, the lower bound is also *tight*. (At first sight, this example looks contradictory because p goes to zero while the fill rate goes to 1. We wish to emphasize that this is not unusual if the decision maker wishes to only specify a fill rate and does not wish to specify a value for p.) In contrast the lower bound on Q for the *unconstrained* problem is $\sqrt{2\lambda K(p+h)/hp}$ (see [6] and [17]) and is not applicable to the constrained problem.

We now present Lemmas 6–8 that will be used to derive an upper bound for the value of Q^*. Define

$$J(Q) = \frac{\lambda K_1}{Q} + H(Q) - B,$$

where

$$B = H(0).$$

It can be verified by the use of L'Hospital's rule that $H(0) = G_1(r(0))$.

LEMMA 6:

(1) $J(0) = J(\infty) = \infty$.
(2) $J(Q)$ is strictly convex.

PROOF: (1) As $K > 0$, it follows that $J(0) = \infty$, and as $H(\infty) = \infty$ (see Zheng [18]) it follows that $J(\infty) = \infty$. (2) The proof follows from Lemma 5 given in Zheng [18]. □

Let Q_m be a value of Q that satisfies the first order condition $\frac{\partial}{\partial Q}(\frac{J(Q)}{Q}) = 0$.

LEMMA 7:

(1) Q_m is unique.
(2) $J(Q)/Q$ is decreasing and convex for $0 \leq Q \leq Q_m$ and increasing for $Q > Q_m$. Thus, Q_m minimizes $J(Q)/Q$.

PROOF: Please see the Appendix.

LEMMA 8: $[H(Q) - B]/Q$ is increasing in Q.

PROOF: Please see the Appendix. □

As B is constant, it follows from Eq. (11) that problem **P1** can be solved by finding the value of Q that minimizes the redefined objective function

$$C_n(Q) = \frac{\lambda K_1}{Q} + F\bar{S}(Q, r(Q)) + (1-F)\bar{I}(Q, r(Q)) - B + (F - F_u)(r(Q) + Q/2 - \mu). \quad (14)$$

Define Q_r to be the value of Q such that $r(Q_r) = Q_r$. Denote the lower bound on Q^* derived in Eq. (13) by Q_l. Also, define $\tilde{Q}$ to be such that $r(\tilde{Q}) = 0$. The bounds will be derived in two theorems given below. The first theorem determines the bounds for values of the fill rate greater than 62.5%. The second theorem determines bounds for fill rates less than 62.5%.

THEOREM 2: If the required fill rate F exceeds 62.5%, then:

(I) If $Q_l < Q_r$, then the optimal solution will be found in one of the three intervals (1) $Q_r \leq Q^* \leq \min(4Q_r, \tilde{Q})$, (2) $\tilde{Q} \leq Q^* \leq 4\tilde{Q}$, or (3) $r(Q_r) \leq r^* \leq 2r(Q_r)$.

(II) If $Q_l > Q_r$, then the optimal solution will fall in one of the two intervals, (1) $Q_l \le Q^* \le \min(4Q_l, \tilde{Q})$, or (2) $\tilde{Q} \le Q^* \le 4\tilde{Q}$.

PROOF:

CASE I, $Q_l \le Q_r$: This case will be proved in three parts. Case I.1 provides the upper bound on Q^* in terms of Q_r, while Case I.2 provides the upper bound on r^* in terms of $r(Q_r)$. Case I.3 analyzes the region when r is negative and provides the bound for Q^* as a multiple of $\tilde{Q}$.

CASE I.1: Consider any $Q_2 \ge Q_r$. Define Q_u to be the smallest value of Q_2 such that $C_n(Q_2) \ge C_n(Q_r)$ for $Q_2 \ge Q_u$. Let Q_2 be such that $Q_r \le Q_2 \le Q_u$. Then by assumption

$$\frac{\lambda K_1}{Q_r} + H(Q_r) - B + (F - F_u)\left(r(Q_r) + \frac{Q_r}{2} - \mu\right) \ge \frac{\lambda K_1}{Q_2} + H(Q_2) - B + (F - F_u)\left(r(Q_2) + \frac{Q_2}{2} - \mu\right). \quad (15)$$

Therefore,

$$1 \le \frac{\frac{\lambda K_1}{Q_r} + H(Q_r) - B + (F - F_u)(r(Q_r) + \frac{Q_r}{2})}{\frac{\lambda K_1}{Q_2} + H(Q_2) - B + (F - F_u)(r(Q_2) + \frac{Q_2}{2})} = \frac{Q_r[\frac{\lambda K_1}{Q_r^2} + \frac{(H(Q_r) - B)}{Q_r} + (F - F_u)(1 + 1/2)]}{Q_2[\frac{\lambda K_1}{Q_2^2} + \frac{(H(Q_2) - B)}{Q_2} + (F - F_u)(\frac{r(Q_2)}{Q_2} + 1/2)]}. \quad (16)$$

Define

$$\mathbf{A} = \frac{\frac{\lambda K_1}{Q_r^2} + \frac{(H(Q_r) - B)}{Q_r} + (F - F_u)(1 + 1/2)}{\frac{\lambda K_1}{Q_2^2} + \frac{(H(Q_2) - B)}{Q_2} + (F - F_u)(\frac{r(Q_2)}{Q_2} + 1/2)}. \quad (17)$$

As $\frac{Q_r}{Q_2}\mathbf{A} \ge 1$ by assumption, an upper bound on $\mathbf{A}$ can be used to bound Q_u in terms of Q_r. Assume that $r(Q_2) \ge 0$. The case where the reorder level is negative is analyzed in I.2. To get the upper bound on $\mathbf{A}$, we substitute $r(Q_2)/Q_2 = 0$ in Eq. (17). Therefore,

$$\mathbf{A} \le \frac{\frac{\lambda K_1}{Q_r^2} + \frac{(H(Q_r) - B)}{Q_r} + \frac{3}{2}(F - F_u)}{\frac{\lambda K_1}{Q_2^2} + \frac{(H(Q_2) - B)}{Q_2} + \frac{1}{2}(F - F_u)} = \frac{\frac{J(Q_r)}{Q_r} + \frac{3}{2}(F - F_u)}{\frac{J(Q_2)}{Q_2} + \frac{1}{2}(F - F_u)}. \quad (18)$$

Cases I.1.1, I.1.2, and I.1.3 will now provide upper bounds on $\mathbf{A}$ depending on the relative values of Q_l, Q_r, and Q_m.

CASE I.1.1, $Q_l \le Q_r \le Q_m$: From Lemma 7, $Q_r \le Q_m$ implies $J(Q_m)/Q_m \le J(Q_2)/Q_2$. Therefore, we can substitute $J(Q_m)/Q_m$ for $J(Q_2)/Q_2$ in (18) without affecting the direction of the inequality. From the first-order condition $J(Q_m)/Q_m = J'(Q_m)$, we obtain

$$-\frac{\lambda K_1}{Q_m^2} + H'(Q_m) = \frac{\lambda K_1}{Q_m^2} + \frac{(H(Q_m) - B)}{Q_m}.$$

This implies

$$\frac{\lambda K_1}{Q_m^2} = \frac{1}{2}\left(H'(Q_m) - \frac{(H(Q_m) - B)}{Q_m}\right).$$

Therefore,

$$\frac{J(Q_m)}{Q_m} = \frac{\lambda K_1}{Q_m^2} + \frac{(H(Q_m) - B)}{Q_m} = \frac{1}{2}\left(H'(Q_m) + \frac{(H(Q_m) - B)}{Q_m}\right). \tag{19}$$

By the definition of Q_l, $\lambda K_1/Q_r^2 \le \lambda K_1/Q_l^2$. Thus, we can substitute $\lambda K_1/Q_l^2$ for $\lambda K_1/Q_r^2$ in (18) without affecting the direction of the inequality. Substituting the value of $J(Q_m)/Q_m$ from Eq. (19) in the denominator of (18) and the value of $\lambda K_1/Q_l^2$ from Eq. (13) in the numerator of (18), we get

$$\mathbf{A} \le \frac{H'(Q_l) + \frac{F-F_u}{2} + \frac{(H(Q_r)-B)}{Q_r} + \frac{3}{2}(F - F_u)}{\frac{1}{2}(H'(Q_m) + \frac{(H(Q_m)-B)}{Q_m}) + \frac{1}{2}(F - F_u)}.$$

As $Q_r \le Q_m$, it follows from Lemma 8 that $(H(Q_r) - B)/Q_r \le (H(Q_m) - B)/Q_m$. As $Q_l \le Q_m$, it follows from Lemma 5 that $H'(Q_l) \le H'(Q_m)$. Therefore,

$$\mathbf{A} \le \frac{\frac{4}{2}(F - F_u)}{\frac{1}{2}(F - F_u)} = 4.$$

CASE I.1.2, $Q_l \le Q_m \le Q_r$ and CASE I.1.3, $Q_m \le Q_l \le Q_r$: As $Q_2 \ge Q_r \ge Q_m$, it follows from Lemma 7 that $J(Q_r)/Q_r \le J(Q_2)/Q_2$. Therefore, from (18)

$$\mathbf{A} \le \frac{\frac{J(Q_r)}{Q_r} + \frac{3}{2}(F - F_u)}{\frac{J(Q_2)}{Q_2} + \frac{1}{2}(F - F_u)} \le \frac{\frac{3}{2}(F - F_u)}{\frac{1}{2}(F - F_u)} = 3.$$

CASE I.2: We shall now consider the values of Q for which the reorder point is negative, i.e., $Q \ge \tilde{Q}$. Similar to inequality (16), we can define $Q_2 \ge \tilde{Q}$ such that

$$\begin{aligned} 1 \ &\le \ \frac{\frac{\lambda K_1}{\tilde{Q}} + H(\tilde{Q}) - B + (F - F_u)(r(\tilde{Q}) + \frac{\tilde{Q}}{2})}{\frac{\lambda K_1}{Q_2} + H(Q_2) - B + (F - F_u)(r(Q_2) + \frac{Q_2}{2})} \\ &= \ \frac{\frac{\lambda K_1}{\tilde{Q}} + H(\tilde{Q}) - B + (F - F_u)(\frac{\tilde{Q}}{2})}{\frac{\lambda K_1}{Q_2} + H(Q_2) - B + (F - F_u)(r(Q_2) + \frac{Q_2}{2})}, \end{aligned} \tag{20}$$

as $r(\tilde{Q}) = 0$.

From Lemma 10 given in the Appendix it follows that

$$\frac{\partial}{\partial Q}\left(r(Q) + \frac{Q}{2}\right) = -\frac{1}{2} + \frac{F}{\mathcal{F}(r + Q)} \qquad \text{for} \qquad r \le 0. \tag{21}$$

Therefore, for $F \ge .625$, $(r(Q) + Q/2)$ is increasing in Q. Thus, when

$$1 \le \frac{\frac{\lambda K_1}{\tilde{Q}} + H(\tilde{Q}) - B}{\frac{\lambda K_1}{Q_2} + H(Q_2) - B} \tag{22}$$

is true, then inequality (20) is also true. This implies

$$1 \leq \frac{\tilde{Q}[\frac{\lambda K_1}{\tilde{Q}^2} + \frac{(H(\tilde{Q})-B)}{\tilde{Q}}]}{Q_2[\frac{\lambda K_1}{Q_2^2} + \frac{(H(Q_2)-B)}{Q_2}]}. \tag{23}$$

Therefore, similar to Eq. (17), define **A3** as

$$\mathbf{A3} = \frac{\frac{\lambda K_1}{\tilde{Q}^2} + \frac{(H(\tilde{Q})-B)}{\tilde{Q}}}{\frac{\lambda K_1}{Q_2^2} + \frac{(H(Q_2)-B)}{Q_2}}. \tag{24}$$

The first-order condition for the optimality must hold at Q^*, so

$$\begin{aligned}\frac{\lambda K_1}{(Q^*)^2} &= H'(Q^*) + \frac{\partial}{\partial Q}(F - F_u)\left(r(Q^*) + \frac{Q^*}{2}\right) \\ &\geq H'(Q^*) + (F - F_u)\left(F - \frac{1}{2}\right).\end{aligned} \tag{25}$$

where inequality (25) follows from (21). As $Q_l \leq \tilde{Q}$ and $Q_2 \leq Q^*$, it follows from equations (13), (24), and (25) that

$$\begin{aligned}\mathbf{A3} &\leq \frac{\frac{\lambda K_1}{Q_l^2} + \frac{(H(\tilde{Q})-B)}{\tilde{Q}}}{\frac{\lambda K_1}{(Q^*)^2} + \frac{(H(Q_2)-B)}{Q_2}} \\ &\leq \frac{H'(Q_l) + (F - F_u)/2 + \frac{(H(\tilde{Q})-B)}{\tilde{Q}}}{H'(Q^*) + (F - F_u)(F - 1/2) + \frac{(H(Q_2)-B)}{Q_2}}.\end{aligned} \tag{26}$$

By assumption $Q^* \geq \tilde{Q}$ and as $Q_2 \geq \tilde{Q}$, it follows from Lemmas 5 and 8 that

$$\mathbf{A3} \leq \frac{(F - F_u)/2}{(F - F_u)(F - 1/2)} \tag{27}$$

$$\leq \frac{1/2}{(.625 - 1/2)} = 4. \tag{28}$$

CASE I.3: Consider any $r(Q) \geq r(Q_r)$. From Lemma 4 we know that if $r(Q) \geq r(Q_r)$, then $r(Q) + Q \leq r(Q_r) + Q_r$. This implies that $r(Q) \leq 2r(Q_r)$.

CASE II, $Q_l > Q_r$: The optimal Q lies in the range $[Q_l, \infty)$. This part of the theorem will be proved in Case II.1 which provide bounds for Q^* in terms of Q_l for the case when the optimal r is positive and Case II.2 which provide bounds for Q^* in terms of $\tilde{Q}$ for the case when the optimal r is negative.

CASE II. 1: Define **A1** to be such that $\frac{C_n(Q_l)}{C_n(Q_2)} = \frac{Q_l}{Q_2}\mathbf{A1}$. The upper bound on **A1** gives the upper bound on Q^* in terms of Q_l. We obtain, similar to Eq. (18),

$$\mathbf{A1} \leq \frac{\frac{\lambda K_1}{Q_l^2} + \frac{(H(Q_l)-B)}{Q_l} + \frac{3}{2}(F - F_u)}{\frac{\lambda K_1}{Q_2^2} + \frac{(H(Q_2)-B)}{Q_2} + \frac{1}{2}(F - F_u)} = \frac{\frac{J(Q_l)}{Q_l} + \frac{3}{2}(F - F_u)}{\frac{J(Q_2)}{Q_2} + \frac{1}{2}(F - F_u)}. \tag{29}$$

This case will now be proved in three parts II.1.1, II.1.2, and II.1.3.

CASE II.1.1, $Q_r \le Q_m \le Q_l$ and Case II.1.2, $Q_m \le Q_r \le Q_l$: As $Q_2 \ge Q_l \ge Q_m$ it follows from Lemma 7 that $\frac{J(Q_l)}{Q_l} \le \frac{J(Q_2)}{Q_2}$. Therefore, from (29) we obtain

$$\mathbf{A1} \le \frac{\frac{J(Q_l)}{Q_l} + \frac{3}{2}(F - F_u)}{\frac{J(Q_2)}{Q_2} + \frac{1}{2}(F - F_u)} \le \frac{\frac{3}{2}(F - F_u)}{\frac{1}{2}(F - F_u)} = 3.$$

CASE II.1.3, $Q_r \le Q_l \le Q_m$: The proof of this part is similar to Case I.1.1. Substituting the value of $J(Q_m)/Q_m$ from Eq. (19) in the denominator and the value of $\lambda K_1/Q_l^2$ from Eq. (13) in the numerator of (29), we get

$$\mathbf{A1} \le \frac{H'(Q_l) + \frac{F-F_u}{2} + \frac{(H(Q_l)-B)}{Q_l} + \frac{3}{2}(F - F_u)}{\frac{1}{2}(H'(Q_m) + \frac{(H(Q_m)-B)}{Q_m}) + \frac{1}{2}(F - F_u)}.$$

As $Q_l \le Q_m$, it follows from Lemma 8 that $(H(Q_l) - B)/Q_l \le (H(Q_m) - B)/Q_m$ and from Lemma 5 that $H'(Q_l) \le H'(Q_m)$. Therefore,

$$\mathbf{A1} \le \frac{\frac{4}{2}(F - F_u)}{\frac{1}{2}(F - F_u)} = 4.$$

CASE II.2: If $\tilde{Q} \ge Q_l$, the proof of this case is identical to Case I.2. If $\tilde{Q} < Q_l$, then the proof is similar to Case I.2, and we bound the value of Q_2 such that $C_n(Q_l)/C_n(Q_2) \le 1$. □

The reader will note that the bounds on Q^* for positive values of $r(Q)$ given in Theorem 2 are valid for *all* values of $F > 0$. Therefore, only bounds on Q^* when $r(Q)$ is negative are presented in Theorem 3.

THEOREM 3: When the optimal r is negative and the required fill rate is less than 62.5%, then the optimal Q falls in one of the following intervals:

(1) If $\tilde{Q} \le \mu/4$, then $\tilde{Q} \le Q^* \le 4\tilde{Q}$
(2) If $\tilde{Q} > 4\mu$, then $\tilde{Q} \le Q^* \le 4\tilde{Q}$
(3) If $\mu/4 < \tilde{Q} \le 4\mu$, then $\tilde{Q} \le Q^* \le 16\mu$.

PROOF: (1) We consider the values of Q for which the reorder point is negative, i.e., $Q \ge \tilde{Q}$. Similar to inequality (16) we define $Q_2 \ge \tilde{Q}$ such that

$$\begin{aligned} 1 \;&\le\; \frac{\frac{\lambda K_1}{\tilde{Q}} + H(\tilde{Q}) - B + (F - F_u)(r(\tilde{Q}) + \frac{\tilde{Q}}{2} - \mu)}{\frac{\lambda K_1}{Q_2} + H(Q_2) - B + (F - F_u)(r(Q_2) + \frac{Q_2}{2} - \mu)} \\ &=\; \frac{\tilde{Q}}{Q_2}\left[\frac{\frac{\lambda K_1}{\tilde{Q}^2} + \frac{(H(\tilde{Q})-B)}{\tilde{Q}} + \frac{1}{2}(F - F_u) - (F - F_u)(\frac{\mu}{\tilde{Q}})}{\frac{\lambda K_1}{Q_2^2} + \frac{(H(Q_2)-B)}{Q_2} + \frac{1}{2}(F - F_u) - (F - F_u)(\frac{-r(Q_2)}{Q_2} + \frac{\mu}{Q_2})}\right]. \end{aligned} \quad (30)$$

From the proof for Case I.1.1 of Theorem 2 we know that

$$\frac{\frac{\lambda K_1}{\tilde{Q}^2} + \frac{(H(\tilde{Q})-B)}{\tilde{Q}} + \frac{1}{2}(F-F_u)}{\frac{\lambda K_1}{Q_2^2} + \frac{(H(Q_2)-B)}{Q_2} + \frac{1}{2}(F-F_u)} \leq 2. \tag{31}$$

Using inequality (31), it can be verified[1] that inequality (30) cannot be true for $Q_2 \geq 2\tilde{Q}$ if

$$\frac{\frac{\mu}{\tilde{Q}}}{(\frac{-r(Q_2)}{Q_2} + \frac{\mu}{Q_2})} \geq 2. \tag{32}$$

From Eq. (60) given in the Appendix, we get

$$-\frac{r(Q)}{Q} = (1-F) - \frac{\int_0^{r(Q)+Q} \mathcal{F}^c(x)\,dx}{Q} \leq 1-F. \tag{33}$$

Using inequality (33) it follows that inequality (32) holds true if

$$\frac{\frac{Q_2\mu}{\tilde{Q}}}{(Q_2(1-F)+\mu)} \geq 2. \tag{34}$$

Rearranging terms in inequality (34), we obtain

$$\frac{Q_2}{\tilde{Q}} \geq \frac{2}{1-\frac{2\tilde{Q}}{\mu}(1-F)}. \tag{35}$$

As $\frac{\tilde{Q}}{\mu} \leq 1/4$ it follows that $1 \geq 1 - \frac{2\tilde{Q}}{\mu}(1-F) > 0$. The maximum value of $\frac{2}{1-(2\tilde{Q}/\mu)(1-F)}$ is 4 when $\tilde{Q}/\mu = 1/4$ and $F = 0$. Therefore, if $Q_2/\tilde{Q} \geq 4$, inequality (30) cannot hold true for any value of $Q_2 \geq 2\tilde{Q}$. Thus, when $\tilde{Q}/\mu \leq 1/4$, inequality (30) cannot hold for values of $Q_2 \geq 4\tilde{Q}$.[1]

(2) $\tilde{Q} \geq 4\mu$: Similar to inequality (20) in the proof of Theorem 2, we get

$$1 \leq \frac{\frac{\lambda K_1}{\tilde{Q}} + H(\tilde{Q}) - B + (F-F_u)(r(\tilde{Q}) + \frac{\tilde{Q}}{2})}{\frac{\lambda K_1}{Q_2} + H(Q_2) - B + (F-F_u)(r(Q_2) + \frac{Q_2}{2})}. \tag{36}$$

From Eq. (60) in the Appendix, we get

$$-\frac{r(Q_2)}{Q_2} = (1-F) - \frac{\int_0^{r(Q_2)+Q_2} \mathcal{F}^c(x)\,dx}{Q_2} \tag{37}$$

[1]For positive $A, B, X, Y, A-X$ and $B-Y$, if $A/B \leq 2$ and $X/Y \geq 2$, then $(A-X)/(B-Y) \leq 2$.

and

$$0 = (1 - F) - \frac{\int_0^{r(\tilde{Q})+\tilde{Q}} \mathcal{F}^c(x)\,dx}{\tilde{Q}}. \tag{38}$$

Subtracting Eq. (37) from (38), we obtain

$$\frac{r(Q_2)}{Q_2} = -\frac{\int_0^{r(\tilde{Q})+\tilde{Q}} \mathcal{F}^c(x)\,dx}{\tilde{Q}} + \frac{\int_0^{r(Q_2)+Q_2} \mathcal{F}^c(x)\,dx}{Q_2} \tag{39}$$

$$\geq -\frac{\int_0^{r(\tilde{Q})+\tilde{Q}} \mathcal{F}^c(x)\,dx}{\tilde{Q}} \geq -\frac{\mu}{\tilde{Q}} \tag{40}$$

Since $\tilde{Q} \geq 4\mu$ in this case it follows from inequality (40) that $r(Q_2)/Q_2 \geq -\frac{1}{4}$. Using this in the denominator of inequality (36), the proof now follows in a manner similar to Case I.1.1 in Theorem 2.

(3) $\mu/4 \leq \tilde{Q} \leq 4\mu$: The proof follows from part 1 and 2. □

4. ALGORITHM

We utilize the bounds derived in the previous section in the algorithm given below for determining the optimal values of Q and r. The algorithm is for the case when the required fill rate is greater than 62.5%. A similar algorithm can be constructed when the fill rate required is less than 65.5%; and the details are omitted due to considerations of space. Define

$$C_{con}(Q) = \frac{\lambda K_1}{Q} + H(Q) + (F - F_u)\left(r(Q) + \frac{Q}{2}\right).$$

This function differs from the original cost function in only a constant that is

$$C_{con}(Q) = C_1(Q) + (F - F_u)\mu.$$

Let ϵ be the maximum desired error in $C_{con}(Q)$. Let

$$N = \left\lceil \frac{2\log(2)}{\log(1+\epsilon)} \right\rceil \tag{41}$$

and

$$\Delta = \epsilon\tilde{Q}/2. \tag{42}$$

Algorithm for Determining (Q, r) to Satisfy a Given Fill Rate F

1. If $p/(p+h) \geq F$, utilize the FZ algorithm to compute the optimal values of (r, Q). Otherwise, proceed to step 2.

2. Determine the $EOQ = \sqrt{2\lambda K/h}$ and the value of Q_r. A simple line search can be used to determine the value of Q_r, i.e., to find the value of Q such that $r(Q) = Q$. Determine the exact value of the lower bound for Q by solving the convex program

$$Q_l = \arg\min_Q \left\{ \frac{\lambda K_1}{Q} + H(Q) + (F - F_u)\frac{Q}{2} \right\}.$$

3. If $EOQ < Q_r$, then perform a search (see step 3.1 and 3.2) in the two intervals, $I_1 = [r(Q_r), 2r(Q_r)]$ and $I_2 = [Q_r, 4Q_r]$. Otherwise, let $I_3 = [EOQ, 4EOQ]$ and execute step 3.3.

   3.1. *Search in* I_1: Let $r_i = r(Q_r) * 2^{(i/N)}, i = 1, 2, \ldots, N$. Determine Q_i's corresponding to each of the r_i's such that the fill rate constraint is met [see Eq. (2)]. There are three cases to consider. If $Q_l > Q_N$, go to step 2. If $Q_l < Q_1$, set $i_0 = 1$. Else, if $Q_1 \leq Q_l \leq Q_N$, determine i_0 such that $Q_{i_0} \leq Q_l < Q_{i_0+1}$. Evaluate $C_{con}(Q)$ at each of the (Q_i, r_i) pairs, for $i = i_0, i_0 + 1, \ldots, N$.

   3.2. *Search in* I_2: Let $Q_i = Q_r * 2^{(i/N)}, i = 1, 2, \ldots, 2N$. Determine r_i's corresponding to each of the Q_i's such that the fill rate constraint is met [see Eq. (2)]. Determine the value of i_0 as done in step 3.1. If $r(Q_N) \geq 0$, then let $M = N$. Else, let

$$N_0 = \max i : r(Q_i) \geq 0,$$

$$M = N_0 + \left\lceil \frac{6}{\epsilon} \right\rceil,$$

   and

$$Q_i = \tilde{Q} + [i - 1 - N_0]\Delta, \qquad i \geq N_0 + 1.$$

   Evaluate $C_{con}(Q)$ at each of the (Q_i, r_i) pairs, for $i = i_0, i_0 + 1, \ldots, M$. Proceed to step 4.

   3.3. *Search in* I_3: Let $Q_i = EOQ * 2^{(i/N)}, i = 1, 2, \ldots, 2N$. Determine r_i's corresponding to each of the Q_i's such that the fill rate constraint is met [see Eq. (2)]. Determine the value of i_0 as done in step 3.1 and the value of M and Q_i as in step 3.2. Evaluate $C_{con}(Q)$ each of the (Q_i, r_i) pairs, for $i = i_0, i_0 + 1, \ldots, M$.

4. Choose the (Q, r) pair that gives the lowest value of $C_{con}(Q)$ in steps 3.1 and 3.2 or step 3.3.

end

Let the algorithm return the order quantity Q_k. Define the error from using Q_k as δ, i.e.,

$$\delta = \frac{C_{con}(Q_k)}{C_{con}(Q^*)}.$$

LEMMA 9: The (Q, r) pair produced by the above algorithm gives an error with respect to $C_{con}(Q^*)$ that is at most ϵ percent for required fill rates greater than 62.5%.

PROOF: Theorem 2 guarantees that the optimal values of (Q, r) will lie in the ranges I_1 and I_2 or in I_3 as the case might be. Suppose the optimal order quantity lies between two adjacent values of Q's as determined by the algorithm. Let these values of Q be Q_j and Q_{j+1}, with $Q_j < Q^* < Q_{j+1}$. Notice that by convexity and the choice of Q_l the function $(\lambda K_1/Q + H(Q) + (F - F_u)(r(Q) + Q/2))$ is increasing in Q for $Q \geq Q_l$. Thus,

$$\begin{aligned}\delta \;&\leq\; \frac{\lambda k/Q_j + H(Q_j) + (F - F_u)(r_j + Q_j/2)}{\lambda k/Q^* + H(Q^*) + (F - F_u)(r^* + Q^*/2)} - 1 \\ &\leq\; \frac{(F - F_u)(r_j - r^*)}{\lambda k/Q^* + H(Q^*) + (F - F_u)(r^* + Q^*/2)}. \qquad (43)\end{aligned}$$

If the cost minimizing solution lies in I_1 then the error bound follows from the choice of the values of r_i's, i.e., from (41) and (43)

$$\begin{aligned}\delta \;&\leq\; \frac{r_j - r_{j+1}}{r_{j+1}} \\ &\leq\; \epsilon.\end{aligned}$$

If the cost minimizing solution lies in I_2 or I_3, we note that from Lemma 4 that

$$\begin{aligned}Q_j + r_j \;&\leq\; Q^* + r^* \\ &\Rightarrow\; r_j - r^* \leq Q^* - Q_j \leq Q_{j+1} - Q_j. \qquad (44)\end{aligned}$$

If $r(Q_N) \geq 0$, then from (41), (43), and (44),

$$\begin{aligned}\delta \;&\leq\; \frac{2(Q_{j+1} - Q_j)}{Q_j} \\ &\leq\; 2(\sqrt{1+\epsilon} - 1) \leq \epsilon.\end{aligned}$$

If $r(Q_N) \leq 0$, then, from Lemma 10 in the Appendix and from the fact that $F \geq 62.5\%$, it follows that $(r(Q_i) + Q_i/2)$ is increasing in i for $i \geq N_0 + 1$, where N_0 is as defined in steps 3.2 and 3.3 of the algorithm. Also, by definition (step 3.2 of the algorithm) $Q_{N_0+1} = \tilde{Q}$ and $r(\tilde{Q}) = 0$. Thus,

$$r(Q_i) + \frac{Q_i}{2} \geq r(\tilde{Q}) + \frac{\tilde{Q}}{2} = \frac{\tilde{Q}}{2} \qquad \text{for } i \geq N_0 + 1. \qquad (45)$$

Therefore, from (42), (43), (44), and (45),

$$\begin{aligned}\delta \;&\leq\; \frac{(F - F_u)(Q_{j+1} - Q_j)}{(F - F_u)(\tilde{Q}/2)} \\ &=\; \frac{\epsilon\tilde{Q}/2}{\tilde{Q}/2} \leq \epsilon. \qquad \square\end{aligned}$$

5. CONCLUSION

In the preceding sections, we considered a general form of the problem of determining the optimal values of (Q, r) subject to a fill rate constraint. The bounds are also applicable to two special cases. First, the majority of the literature deals with the special case in which there is no backorder penalty, i.e., $p = 0$. This implies that $F_u = 0$. All our bounds are applicable for this case as well since the bounds are independent of p. Second, if the fill rate constraint were not imposed, then F could be allowed to approach F_u from above in Theorem 2. As a consequence and from Lemma 1, the bounds will continue to hold.

APPENDIX

PROOF OF LEMMA 5: (1) By differentiating $H(Q)$ with respect to Q we get

$$H'(Q) = \left(\frac{G_1(r(Q)+Q)}{Q} - \frac{\int_{r(Q)}^{r(Q)+Q} G_1(y)\,dy}{Q^2} \right) + \frac{(G_1(r(Q)+Q) - G_1(r(Q)))}{Q} r'(Q). \tag{46}$$

As $r(Q)$ is the solution to

$$\frac{\partial}{\partial r} \left(\frac{\lambda K_1}{Q} + \frac{\int_r^{r+Q} G_1(y)\,dy}{Q} \right) = 0,$$

it follows that

$$G_1(r(Q)+Q) - G_1(r(Q)) = 0. \tag{47}$$

Therefore,

$$H'(Q) = \left(\frac{QG_1(r(Q)+Q)}{Q^2} - \frac{\int_{r(Q)}^{r(Q)+Q} G_1(y)\,dy}{Q^2} \right). \tag{48}$$

From Eq. (47) and because $G_1(\cdot)$ is convex, it follows that $G_1(r(Q)+Q) \geq G_1(y)$ for $y \in [r(Q), r(Q)+Q]$. This proves that $H'(Q)$ is nonnegative.

(2) By definition,

$$H(Q, r) = F\bar{S}(Q, r) + (1-F)\bar{I}(Q, r).$$

From Eq. (9), we get $\bar{I}(Q, r) = \bar{S}(Q, r) + Q/2 + r - \mu$. Therefore,

$$H(Q, r) = \bar{S}(Q, r) + (1-F)(Q + r - \mu).$$

Zipkin [19] shows that if $f(\cdot) > 0$, then $\bar{S}(Q, r)$ is strictly convex in (Q, r). Therefore, $H(Q, r)$ is strictly convex for in (Q, r), i.e., for a given $\alpha \in (0, 1)$, $(Q_1, r(Q_1))$, and $(Q_2, r(Q_2))$,

$$\alpha H(Q_1, r(Q_1)) + (1-\alpha)H(Q_2, r(Q_2)) > H(\alpha Q_1 + (1-\alpha)Q_2, \alpha r(Q_1) + (1-\alpha)r(Q_2)). \tag{49}$$

From Eq. (10) we know that $r(Q)$ minimizes $H(Q, r)$, for a given Q; hence

$$H(\alpha Q_1 + (1-\alpha)Q_2, \alpha r(Q_1) + (1-\alpha)r(Q_2)) \geq H(\alpha Q_1 + (1-\alpha)Q_2, r(\alpha Q_1 + (1-\alpha)Q_2)). \tag{50}$$

It follows from (49) and (50) that

$$\alpha H(Q_1, r(Q_1)) + (1-\alpha)H(Q_2, r(Q_2)) > H(\alpha Q_1 + (1-\alpha)Q_2, r(\alpha Q_1 + (1-\alpha)Q_2)).$$

This proves that $H(Q)$ is a strictly convex function of Q.

(3) It follows from the convexity of $H(Q)$ that $H'(Q) \leq \lim_{Q\to\infty} H'(Q)$, for all $Q > 0$. From Eq. (48)

$$\lim_{Q\to\infty} H'(Q) = \lim_{Q\to\infty} \frac{G_1(r(Q)+Q)}{Q} - \lim_{Q\to\infty} \frac{\int_{r(Q)}^{r(Q)+Q} G_1(y)\,dy}{Q^2}. \tag{51}$$

We know from Lemma 4 that $\lim_{Q\to\infty} G_1(r(Q)+Q) = G(\infty) = \infty$. Therefore, we use L'Hospital's rule for computing one sided limits (see Theorem 4, page 264 in [15]) to compute the limit for the first quantity on the right-hand side of Eq. (51). Differentiating the numerator and denominator separately, we obtain

$$\lim_{G\to\infty} \frac{G_1(r(Q)+Q)}{Q} = \frac{\lim_{Q\to\infty} G_1'(r(Q)+Q)(1+r'(Q))}{1}. \tag{52}$$

Similarly,

$$\lim_{Q\to\infty} \frac{\int_{r(Q)}^{r(Q)+Q} G_1(y)\,dy}{Q^2} = \lim_{Q\to\infty} \frac{G_1(r(Q)+Q)}{2Q} = \frac{\lim_{Q\to\infty} G_1'(r(Q)+Q)(1+r'(Q))}{2}. \tag{53}$$

From Eqs. (52) and (53),

$$\lim_{Q\to\infty} H'(Q) = \frac{\lim_{Q\to\infty} G_1'(r(Q)+Q)(1+r'(Q))}{2}. \tag{54}$$

It follows from Zheng [18] that $\lim_{Q\to\infty} G_1'(Q) = 1-F$. From Lemma 4 we get $1+r'(Q) \le 1$. Therefore, $\lim_{Q\to\infty} H'(Q) \le (1-F)/2$. □

PROOF OF LEMMA 7: (1) By the definition of Q_m,

$$\frac{\partial}{\partial Q}\left(\frac{J(Q)}{Q}\right)_{Q=Q_m} = \frac{1}{Q_m}\left[J'(Q_m) - \frac{J(Q_m)}{Q_m}\right] = 0.$$

This implies that

$$J'(Q_m) = \frac{J(Q_m)}{Q_m}. \tag{55}$$

Let us suppose that Q_m is not unique and that there are two values Q_1 and Q_2 that satisfy Eq. (55). Assume without loss of generality that $Q_2 > Q_1$. It follows from Eq. (55) that

$$Q_2 J'(Q_2) = J(Q_2)$$

and

$$Q_1 J'(Q_1) = J(Q_1).$$

These equations imply that

$$J(Q_2) - J(Q_1) = Q_2 J'(Q_2) - Q_1 J'(Q_1). \tag{56}$$

From Lemma 6, $J(Q)$ is strictly convex. Therefore,

$$J(Q_2) - J(Q_1) < J'(Q_2)(Q_2 - Q_1). \tag{57}$$

Subtracting Eq. (57) from Eq. (56), we get

$$0 > (J'(Q_2) - J'(Q_1))Q_1.$$

This is a contradiction as $J'(Q_2) \ge J'(Q_1)$ due to the convexity of $J(Q)$. This shows that Q_m is unique.

(2) To prove the second part of this lemma

$$\frac{\partial^2}{\partial Q^2}\left(\frac{J(Q)}{Q}\right) = \frac{J''(Q)}{Q} + \frac{2}{Q^2}\left[\frac{J(Q)}{Q} - J'(Q)\right]. \tag{58}$$

From Lemma 6, $J''(Q)/Q \ge 0$. From the first part of this lemma Q_m is unique; therefore $(\partial/\partial Q)(J(Q)/Q)$ can change sign only once. Therefore, $(\partial/\partial Q)(J(Q)/Q) \le 0$ for $Q \le Q_m$, which implies that $J(Q)/Q \ge J'(Q)$ for $Q \le Q_m$. Hence using (58) we get that $(\partial^2/\partial Q^2)(J(Q)/Q) > 0$, for $0 \le Q \le Q_m$. Finally, from the first part of this lemma, $(\partial/\partial Q)(J(Q)/Q) \ge 0$, for $Q \ge Q_m$. This proves the second part of the lemma. □

PROOF OF LEMMA 8:

$$\frac{\partial}{\partial Q}\left(\frac{H(Q)-B}{Q}\right) = \frac{1}{Q}\left(H'(Q) - \frac{H(Q)-B}{Q}\right).$$

As $H(Q)$ is an increasing convex function (from Lemma 5); therefore, for any $Q > 0$, we get

$$QH'(Q) \ge (H(Q) - H(0)). \tag{59}$$

As $B = H(0)$, it follows from (59) that

$$\frac{\partial}{\partial Q}\left(\frac{H(Q)-B}{Q}\right) \ge 0. \quad \square$$

LEMMA 10: When $r \leq 0$,

$$r'(Q) = -1 + \frac{F}{\mathcal{F}(r+Q)} \geq -1 + F.$$

PROOF: From the definition of $r(Q)$ in Eq. (6) we obtain

$$\begin{aligned}(1-F)Q &= \int_{r(Q)}^{r(Q)+Q} \mathcal{F}^c(x)\,dx \\ &= \int_{r(Q)}^{0} \mathcal{F}^c(x)\,dx + \int_{0}^{r(Q)+Q} \mathcal{F}^c(x)\,dx \\ &= -r(Q) + \int_{0}^{r(Q)+Q} \mathcal{F}^c(x)\,dx. \end{aligned} \tag{60}$$

Taking the derivative of both sides of Eq. (60) with respect to Q, we obtain

$$\begin{aligned}(1-F) &= -r'(Q) + \mathcal{F}^c(r(Q)+Q)(1+r'(Q)) \\ &= -r'(Q)\mathcal{F}(r(Q)+Q) + \mathcal{F}^c(r(Q)+Q). \end{aligned} \tag{61}$$

Rearranging the terms in (61), we get

$$r'(Q) = -1 + \frac{F}{\mathcal{F}(r+Q)} \geq -1 + F. \qquad \square \tag{62}$$

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