IIE Transactions (2000) **32**, 245–262

Policy mechanisms for supply chain coordination

MICHAEL MOSES and SRIDHAR SESHADRI

Operations Management Department, Leonard N. Stern School of Business, New York University, NY 10012, USA
E-mail: mmoses@stern.nyu.edu or sseshadr@stern.nyu.edu

Received April 1998 and accepted March 1999

The problem is to determine a review period and stocking policy that are mutually beneficial to a producer and a retailer. In our model, the retailer uses a periodic review, base stock policy for ordering the item from the producer's Distribution Center (DC). Excess customer demand is assumed to be lost. A make-to-order production system supplies to the DC. We show that given a review period, unless the manufacturer agrees to share the cost of carrying a fraction of the safety stocks at the retailer, the two will not agree upon the level of stocks to be carried in the store. We prove that there is an equilibrium value for this fraction, such that the retailer and the manufacturer are always in agreement with regard to the stocking level. We then show that complete coordination on the stocking level as well as the review period can be achieved solely through carrying out negotiations on credit terms. These theoretical results are used to construct an algorithm for calculating the optimal policy parameters for a supply chain. As part of the analysis we suggest a modification of the base stock policy for the positive lag lost sales case of periodic review inventory models that consistently outperforms the base stock policy in our numerical studies.

1. Introduction

In this paper, we study the problem of determining both a review period and a stocking policy that are mutually beneficial to a producer and a retailer. In our model, the retailer faces stationary stochastic demand with independent increments for an item. Excess customer demand is considered to be lost. The retailer uses a periodic review inventory system and a base stock policy for ordering the item from the producer's Distribution Center (DC). The selling price to customers is fixed. The producer uses a make to order system for manufacturing the item. The item is scheduled for production such that it arrives at the DC in advance of the anticipated shipment date. The lead-time from the producer's DC to the retailer can be random but is surely less than the review period. The price charged by the producer is fixed and the producer extends credit for a fixed duration on each reorder to the retailer.

We show that in this model, given a review period, unless the producer agrees to share the cost of carrying stocks with the retailer, the two will not agree as to the base stock level. On the other hand, if the producer offers to share a fraction β of the cost of carrying safety stock, then there is an equilibrium value for β, denoted as β_e, such that both the retailer and the producer will agree to the same base stock level. This equilibrium value of the base stock level is denoted as S_e. The properties of our solution are that: (i) the sharing fraction β_e is independent of the distribution of demand; and (ii) the base stock level S_e is jointly optimal, it represents a Nash equilibrium, and it is independent of both the credit terms as well as the value of β_e. Using these results, we prove that *complete agreement* on the base stock level as well as the length of the review period can be achieved by the producer offering credit to the retailer as a function of the length of the review period. The calculation of the credit terms is based on a "robust design" principle proposed by us in Section 5. As part of the analysis we suggest a modification of the base stock policy in our numerical studies.

Our study was motivated by an example from the cosmetic industry. The problem as stated by the producer, who is a leader in the cosmetic industry, was to develop stocking guidelines at the retail level for over 600 Stock Keeping Units (SKU's) stocked at nearly 2000 locations in the US. The problem of determining stocking guidelines surfaced when a major account started to use an EDI system for placing orders electronically; and insisted upon using automatic replenishment formulae for placing store orders. Using field data, we demonstrated to the producer that the use of automatic replenishment formulae within the proposed EDI system without regard to the variability of demand can lead to excessive lost sales. We suggested that the use of better forecasting techniques in combination with making more frequent deliveries will mitigate the extent of lost sale[1], especially when these suggestions are implemented with the modified base stock formula described in Section 4. The solutions suggested by us would have reduced the inventories carried by retailers without impacting lost sales. However, the two parties were initially unable to reach an agreement either on stocking levels or the review period. Gupta and Neel [1] and Wang and Seidmann [2] point out that final customers and the retailer may be the only ones to gain from EDI, and that it may be in the interest of the retailer to subsidize the producer. This study not only provides the theoretical framework for understanding the reasons for their disagreement but also gives guidelines for writing contracts in such situations.

In Section 2, we review the literature on supply chain coordination. In section 3, we describe the model, and present the results on stocking levels in Section 4. In Section 5, we describe review period coordination and describe an algorithm that can be used to calculate the policy parameters. In Section 6, we briefly describe how the extension to multiple SKU's can be made.

2. Review of literature

In this section, we review and where appropriate contrast our work with related literature on quantity discounts, game theoretic models, and multi-echelon inventory theory.

2.1. *Quantity discounts*

Quantity discounts offer a method for coordinating the order quantities between a retailer and a producer. (We use the terms producer and retailer to be consistent within our framework. However in the literature on quantity discounts, the terms seller and buyer are used instead). The motivation for giving quantity discounts could be either price discrimination or coordinating order quantities, see Goyal and Gupta [3]. A survey of the price discrimination literature can be found in Dolan [4]. Goyal and Gupta survey the literature that focuses on the use of quantity discounts for increasing the retailer's order quantity. The modeling assumptions used in the study of quantity discounts are those underlying the Economic Order Quantity (EOQ) model, namely that demand is deterministic, stationary and independent of price changes, no shortages or backlogs are allowed, lead-times are deterministic and that the producer has full knowledge of the retailers's holding and ordering costs. The important insights provided by this literature and relevant to our work are:

- there is an incentive for the producer to make the retailer purchase more than the optimal quantity [5];
- the producer may choose to use a production cycle that is different from the retailer's ordering cycle [6];
- a necessary and sufficient condition for quantity discount pricing to be Pareto efficient is that the holding cost of the producer should be less than that of the retailer [7].

There have been several extensions to the basic model described above. For example, Lee and Rosenblatt [8] incorporate the economics of production, price sensitive demand is studied in Weng [9], and Kohli and Park [10] extend the model to multiple products. In contrast to the work on quantity discounts: (i) we assume stochastic demand; (ii) model the base stock method of replenishing stocks; (iii) incorporate credit terms; and (iv) analytically investigate the problem of determining the "optimal" base stock level and review frequency. We prove that when demand is stochastic, it is impossible to achieve a Nash equilibrium with respect to the base stock level by using quantity discounts alone. Thus our results indicate that there is need to develop contract mechanisms that are different from those found in practice and studied in the past.

2.2. *Game theoretic models of coordination*

An early application of game theory to manufacturer-retailer coordination is the problem of Double Marginalization due to Spengler [11]. Some recent applications of game theoretic models to study production and inventory control problems include the work of Anupindi and Bassok [12], Parlar [13], Wang and Parlar [14], Ernst and Cohen [15], Ernst and Powell [16], and Pasternak [17]. The last three papers are the ones that are the most relevant to our study. Ernst and Cohen [15] propose a model in which customer demand is a function of the service level. They conclude that a manufacturer and a dealer can gain substantially through coordinating their decisions and that such an arrangement is also more stable in the long run. They however do not design a contract mechanism to coordinate the decisions. Ernst and Powell [16] also model the effect of service level on demand. The retailer in their model uses a periodic review, base stock policy and holds safety stock that is computed using a newsboy approach. The manufacturer has full knowledge of the retailer's ordering rule and costs. The demand over the order period and the manufacturer's lead-time are assumed to be normally distributed. The manufacturer makes a side payment to the retailer to induce the retailer to maintain higher service

[1] The description of the empirical study and some of the results in this paper were presented in the conference, "Managing Buyer-Seller Relationships: EDI & Long Term Partnerships," at Stern School of Business, New York University, NY 10012, November 10, 1995. Details of the study are available on request from the authors.

levels. It is assumed that the retailer is a "follower", i.e., he or she takes the side payment as given and solves for the optimal order up to level. Ernst and Powell solve this problem numerically. Their numerical results indicate that most of the gain from coordination goes to the manufacturer (the leader); however they also state that there are still higher gains to be had from joint decision making. In Section 4, we show that the cost sharing mechanism performs better than joint decision without cost sharing.

Pasternak [17] describes a single period model in which a producer of a commodity with a short shelf life has to decide the optimal pricing and return policy, also see Kandel [18] and Padmanabhan and Png [19]. Pasternak proves that channel coordination can be achieved when the producer extends a partial credit for unsold goods. This result is similar to our result on cost sharing.

2.3. *Multi-echelon inventory theory*

Amongst the early studies of multi-echelon systems were those of Allen [20] and Clark and Scarf [21,22]. Surveys of the work on two-echelon systems can be found in Federgruen and Zipkin [23] and Nahmias [24]. In two echelon models it is assumed that the DC and retail outlets (or retail warehouses) are under single ownership. In contrast to these models we assume that the DC and retail outlets are not under single ownership. The other major differences are that we use a lost-sales model as is appropriate in our setting and we assume a make to order situation because retailer orders are not combined for production in our problem setting. Despite these differences, as explained in Section 4, our results on sharing the cost of carrying safety stocks are general and extend to most two echelon models found in the literature in which the echelons are not under single ownership. McGavin *et al.* [25] and Nahmias and Smith [26] model lost sales in multi-echelon systems. McGavin *et al.* model an N retailer system, in which they assume that the demand is generated by a stationary process with independent increments. The demand process is assumed to be i.i.d. across retailers. Their analysis is restricted to one review period of the DC (though it can be extended to multiple periods). The DC receives stock at the beginning of its review period and uses a base stock policy. The delivery to the DC is from an external supplier and is assumed to be instantaneous. The review period is divided into two intervals. Part of the stock is allocated to the retailers at the beginning of the first interval and the rest of the stock at the beginning of the second interval. Excess demand is considered to be lost. The lead-time from the DC to the retailers is assumed to be zero. They determine an allocation method to minimize the system-wide expected lost sales.

Nahmias and Smith [26] model partial lost sales and focus on the optimization of base stock level at the retailer and the DC. The objective is to minimize the sum of the holding cost, the cost of lost sales and the cost of expedited shipments. In their model, part of the sales is lost and the rest satisfied, if there is enough DC stocks, through the use of special orders. The retailers order every period, whereas the DC orders every m periods. The demand distribution is assumed to be negative binomial, independent across retailers and also independent across time periods. Nahmias and Smith use a normal approximation for the demand seen by the DC. They assume that there is no stock out at the retailers during periods 1 through $(m-1)$. This assumption is similar in spirit to the assumption that sales are lost only during the procurement lead-time, see Hadley and Whitin [27] and Section 4. Nahmias and Smith (like McGavin *et al.*) assume a zero lead-time to supply the retailer from the DC. They conclude that significant savings may accrue from using their approach depending on the nature of the demand for the item.

3. The model

In this section we first describe the sequence of ordering, production, and shipping events, after which we describe the decision variables and the parameters of the model. Our model is a multi-period model for a single item. The extension to multiple items is discussed in Section 6. Several variations of the production policy can be accommodated by changing the parameters of the model. These suggestions are summarized at the end of this section. The base stock level is denoted as S, and the length of the review period is T years. There is a set up for production once every m review periods. The sequence of ordering, production, and shipping events is as follows.

- A seasonal plan, typically for 6 months, is jointly developed by the producer and the retailer.
- Orders are transmitted by the retailer to the producer's DC at the beginning of each review period and might deviate from the seasonal plan. Typically, while there might be period to period deviations from the seasonal plan, we have observed that the total quantity actually shipped during a season is quite close to the planned quantity (within 10%).
- The plant produces according to the seasonal plan and sets up for production once every m periods. The goods are sent from the plant to the producer's DC every m periods and timed to reach the DC, αT (with $0 \leq \alpha < 1$) years before the first installment of the order is shipped from the DC. For example, if the seasonal plan is for 10 cases, with five cases to be shipped on Jan 1 and five on Feb 1, then the plant produces the 10 cases and ships them to the DC so that they reach αT years before Jan 1. Any cost associated with carrying work in progress can be cap-

tured by changing the value of αT. For example, if the average work-in-progress is 10 days of demand then the 10 days are added to αT. This in turn is reflected in the holding cost to the producer, please see Equation (2) below.

- The goods are produced as per the sales forecast, reach the DC, and await shipment. Orders are received at the beginning of each review period from the retailer. Goods are shipped from stock immediately upon receipt of the retailer's order. Small deviations in the actual orders from the sales forecast are accommodated using system level safety stocks. We neglect the cost of carrying such safety stocks in our model (however also see the discussion following Equation (2) below). The fill rate is defined to be the fraction of the retailer's orders that is shipped from the shelf at the DC. Due to the availability of system level safety stocks, the fill rate from the DC is assumed to be close to 100%.

We now describe the decision variables and the parameters of the model.

Decision variables: the decision variables are the review period, T, the base stock level at the store for the item, S, the frequency of setting up for production, m, and the credit terms extended by the producer to the retailer.

Demand: the demand for the item is stationary with independent increments. The demand distribution over the review period T is given by the function $F_T(x)$. The mean and standard deviation of demand when $T = 1$ are μ and σ respectively. The demand during a period of length γ (γ could be either random or deterministic) is denoted as X_γ. X_γ has the distribution $F_\gamma(x)$. Given a random variable Y, $E[Y]$ will denote the expected value of Y. The notation Y^+ and Y^- will be used to denote the positive and negative parts of Y.

Production: production is make to order. The plant sets up to produce the item for the retailer once every m review periods. Production is scheduled such that the consignment for the retailer reaches the producer's DC αT units of time before the first instalment is shipped to the retailer.

DC/Producer: we assume that there is a fixed setup cost, B, of producing the item, the production cost per unit is c_p, the holding cost per dollar per unit time is i_p, and the opportunity cost of capital per dollar per unit time is f_p. In addition to these costs, the producer incurs a fixed setup cost, A_p, in shipping each order from the DC to the retailer. The producer also incurs a linear penalty of $(c_r - c_p)$ per unit of lost sale by the retailer as well as the cost of extending credit for a duration of τ_c to the retailer on each shipment. The price charged by the producer is c_r per unit.

Lead-time: the lead-time between the shipment of an order from the DC and its receipt by the retailer could be either random or fixed, but surely lies in the interval $[0,T]$. Let L_i be the lead-time of the ith shipment. Then the L_i's are assumed to be i.i.d. and distributed as the random variable L.

Retailer: the retailer incurs a fixed setup cost of A_r per order. The retailer has a holding cost per dollar per unit time of i_r, and an opportunity cost of capital per dollar per unit time of f_r. The retailer experiences a linear penalty of $(p - c_r)$ per unit of lost sales at the store. The selling price per unit is P. The retailer uses a base stock policy to reorder the item when the lead-time is zero. We assume that the retailer follows a modified version of the base stock policy in the non-zero lead-time case for reasons given in Section 4.

4. Analysis of the retailer's stocking level

In this section, the review period and the production frequency, i.e., T and m, are assumed to be given. In Section 4.1 we analyze the case when the lead-time (also called lag) is equal to zero. The positive lag case is analyzed in Section 4.2. The cost sharing arrangement is described in Section 4.3.

4.1. *Lead-time equals zero*

The retailer's annual expected cost is given by

$$\begin{aligned} C_r(T,S) = {} & \frac{A_r}{T} \quad \text{(fixed cost of reorder)} \\ & + \tfrac{1}{2}(S + E[S - X_T]^+)c_r i_r \\ & \quad \text{(holding cost assessed on the average of opening and ending inventories)} \\ & + \frac{(p - c_r)}{T} E[X_T - S]^+ \quad \text{(lost sales penalty)} \\ & - \frac{\tau_c}{T}(S - E[S - X_T]^+)c_r f_r. \\ & \quad \text{(benefit of credit extended by producer)} \end{aligned} \tag{1}$$

The annual expected cost to the producer is given by

$$\begin{aligned} C_p(T,m,S) = {} & \frac{A_p + B/m}{T} \\ & \quad \text{(fixed cost of shipping plus setup cost of production)} \\ & + (S - E[S - X_T]^+)\left(\frac{m-1}{2} + \alpha\right) c_p i_p \\ & \quad \text{(holding cost – see below)} \end{aligned}$$

$$+\frac{(c_r - c_p)}{T}E[X_T - S]^+$$
(lost sales penalty)
$$+\frac{\tau_c}{T}(S - E[S - X_T]^+)c_r f_p.$$
(cost of credit on reorder) (2)

The functional form of the cost to the producer differs from that of the cost to the retailer only in the second term on the right hand side of Equation (2). We explain this difference below. The retailer uses a base stock policy and therefore orders a quantity equal to S minus the inventory on hand at the time of the review. Therefore, the average size of the retailer's order is $(S - E[S - X_T]^+)$. The plant schedules for production a quantity equal to m times the retailer's order quantity. This quantity reaches the producer's DC, αT years before shipping commences. Therefore, the average stock held at the DC is $(\alpha m + (m-1) + (m-2) + \cdots + 1)/m$ times the average order quantity, thereby resulting in an average inventory level at the DC of $((m-1)/2 + \alpha)(S - E[S - X_T]^+)$. By using the standard transformation

$$(S - E[S - X_T]^+) = \mu T - E[X_T - S]^+,$$

We can rewrite Equation (2) as

$$C_p(T, m, S) = \frac{A_p + B/m}{T} + \mu T\left(\frac{m-1}{2} + \alpha\right)c_p i_p + \mu\tau_c c_r f_p + E[X_T - S]^+\left((c_r - c_p) - \tau_c c_r f_p - \left(\frac{m-1}{2} + \alpha\right)c_p i_p\right)\Big/ T. \quad (3)$$

Let $\zeta = ((m-1)/2 + \alpha)$.

The reader should note that the calculation of the incentive required to coordinate the review period will depend on the production (or re-supply) method. However, many different production methods can be accommodated within our model. In particular, our model can cater to *make to stock situations* by additionally assuming that the value of α is independent of the fill rate at the DC. These situations are modeled by selecting appropriate values for m and α. For example:

(i) Monahan [25] sets $\alpha = -1/2$ to model a situation in which instead of the manufacturer there is a wholesaler who buys the item from an outside supplier. In this model, the wholesaler buys a large quantity and ships smaller lots to retailers. Thus the investment in the initial "large" quantity of inventory is treated by Monahan as a sunk cost. The value of α is negative because the wholesaler benefits due to the reduction inventory upon shipment to the retailer.

(ii) Joglekar [6] sets $\alpha = (1 - d/p)$, where d is the demand rate and p the production rate, to reflect the fact that consumption and production are simultaneous. In other words, the continuous replenishment version of the Economic Order Quantity model is used to compute the average inventory (see Hadley and Whitin [27]).

(iii) By setting $\alpha = (k-1)/2$, Lee and Rosenblatt [8] model a situation in which a multiple k of the retailer's order quantity is produced every k periods.

(iv) The value of α can be determined by using a queueing model to compute the inventory necessary for ensuring a given fill rate from the DC (see Buzacott and Shanthikumar [28]).

Proposition 1.

(i) *If* $p > c_r$ *the optimal base stock level for the retailer,* $S_r^*(T)$, *is obtained by solving*

$$F_T(S_r^*(T)) = \frac{((p - c_r) + \tau_c c_r f_r - c_r i_r T/2)}{((p - c_r) + \tau_c c_r f_r + c_r i_r T/2)}. \quad (4)$$

(ii) *If*

$$(c_r - c_p) - \varsigma c_p i_p T - c_r f_p \tau_c > 0, \quad (5)$$

then (a) *the average cost per unit time to the producer* $C_p(T, m, S)$ *is decreasing and convex in* S; *and* (b) *therefore the producer and the retailer will not agree upon a base stock level.*

(iii) *Centralized Decision Making*: *If the producer and the retailer were to jointly optimize their combined costs, the optimal base stock level,* $S_J^*(T, m)$, *is obtained by solving*:

$$F_T(S_J^*(T,m)) = \frac{((p-c_r) + \tau_c c_r f_r - c_r i_r T/2 + ((c_r - c_p) - \tau_c c_r f_p - \varsigma c_p i_p T)))}{((p-c_r) + \tau_c c_r f_r + c_r i_r T/2 + ((c_r - c_p) - \tau_c c_r f_p - \varsigma c_p i_p T)))}. \quad (6)$$

(iv) *If the condition in Equation* (5) *is met, then the optimal base stock level under centralized decision making will not be smaller than the retailer's choice of* $S_r^*(T)$.

Proof. The proof of parts (i), (iia) and (iii) follows the analysis of the standard newsboy problem, as is discussed for example by Silver and Peterson [29]. Part (iib) follows from parts (i) and (iia).

Comparing Equations (4) and (6), because $(c_r - c_p) - \varsigma c_p i_p T - \varsigma c_r f_p \tau_c > 0$, the right hand side of Equation (4) will be strictly smaller than the right hand side of Equation (6). Part (4) then follows from the properties of a distribution function. ■

The main result is part (iib) of the Proposition, which states that if the condition given in Equation (5) holds then the two parties will not agree on the base stock level. This result is not surprising if we consider the re-arranged form of the producer's cost function given in Equation (3). The last term in that equation has the form, (expected lost sales) $\times$ $\left(\left((c_r - c_p) - \varsigma c_p i_p T - c_r f_p \tau_c\right)/T\right)$, implying that the manufacturer will seek to minimize the lost sales at the store if $((c_r - c_p) - \varsigma c_p i_p T - c_r f_p \tau_c) \geq 0$, which is condition (5).

The condition in (5) will hold if the producer makes a profit because the equation states that the net margin should be non-negative, namely the gross margin $(c_r - c_p)$, *minus* the cost of holding an item at the DC *minus* the cost of extending credit per unit should be non-negative. Joint optimization will not result in agreement because the producer will still want to hold larger stocks at the outlets than what the retailer desires to hold. Before we address this issue we shall first model the effect of a positive lead-time, L.

4.2. *Lead-time is greater than zero*

The positive lag (lead-time) lost sales problem is difficult to solve, see for example Karlin and Scarf [30], Kaplan [31], Morton [32], Nahmias [33], Ehrhardt [34], and Zipkin [35]. The text book approach for incorporating a non-zero lead-time, see for example Hadley and Whitin [27] and Hax and Candea [36], is to extend the newsboy solution to cover such situations. We shall first demonstrate that this method of computing the base stock level results in inconsistencies. We shall modify the text book approach to obtain a uniform formula for both the zero lag as well as the positive lag cases.

In order to extend the newsboy model to the positive lag case, one of the many assumptions made is that orders are received in the sequence in which they are placed, i.e., there is no cross over of orders. This "no cross over" assumption is valid in our context due to the retailer's policy of insisting on deliveries within the same review period. Given that this assumption is valid, the text book approach is to simply *add* the safety stock to the "cycle stock" to get the average inventory. Thus following the development in Hadley and Whitin [27], the retailer's

$$\begin{aligned} \textit{Average inventory} &= \textit{cycle stock} + \textit{safety stock} \\ &= \mu T/2 + E[S - X_{T+L}]^+ \\ &= \mu T/2 + [(S - \mu E[L] \\ &\quad - \mu T + E[X_{T+L} - S]^+)]. \end{aligned} \tag{7}$$

Notice that the second term on the right hand side of (7) is the safety stock (namely the base stock (S), minus the average consumption during the lead-time and the review period $(\mu E[L] + \mu T)$, plus lost sales $(E[X_{T+L} - S]^+)$). If we were to use this expression to compute the average inventory then the reader can verify that the value of S that minimizes the cost for the retailer should satisfy the equation

$$F_{T+L}(S_r^*(T)) = \frac{((p - c_r) + (\tau_c - E[L])c_r f_r - c_r i_r T)}{((p - c_r) + (\tau_c - E[L])c_r f_r + c_r i_r T)}. \tag{8}$$

There are two problems with adopting this approach. First, the approximation for the average in inventory given in (7) can be quite poor for highly volatile demand as is discussed by Nahmias and Smith [26]. Second, comparing the expressions on the right hand sides of (4) and (8), shows that in the limit, as the lead-time L goes to zero, the holding cost term $(c_r i_r T)$ in Equation (8) does not equal to $(c_r i_r T/2)$ as found in (4). Thus the base stock levels given in Equation (4) and (8) *do not agree* in the limit when the lead-time goes to zero. To remedy both these problems we suggest a modification of the base stock policy. Let I be the inventory on hand at the time of review and X_L be the demand during the lead-time. Let $Q(I)$ be the quantity ordered when the inventory is I at the time of review. We shall first trace the cause of the two problems identified above.

If the retailer were to follow a base stock policy, then a quantity given by $Q(I) = (S - I)$, is ordered at the time of review. The quantity on hand just after the receipt of the order will be $(S - I + [I - X_L]^+)$. This is a random quantity. It is assumed in textbooks that the expected value of this quantity is $(S - \mu E[L] + E[X_{(T+L)} - S]^+)$ (see Equation (7)). However, the reader will notice that

$$E[S - I + (I - X_L)^+] \neq (S - \mu E[L] + E[X_{(T+L)} - S]^+).$$

This creates the error in the estimate of the average inventory.

To remedy this shortcoming, we propose that the retailer should order a quantity, $Q(I)$, so that the on hand quantity just after a delivery is as "close" as possible to $(S - \mu E[L])$. If the metric of closeness is chosen to be the squared deviation from $(S - \mu E[L])$, then $Q(I)$ must be chosen to minimize

$$E[((Q(I) + [I - X_L]^+) - (S - \mu E[L]))^2].$$

From elementary statistics, we should choose $Q(I)$ to minimize the variance of the inventory just after a delivery around the desired mean value of $(S - \mu E[L])$. However, such a choice of $Q(I)$ requires the knowledge of the distribution of I, whereas the distribution of I is not known *a priori*. As a compromise we suggest it is meaningful to minimize the conditional expectation of

$$E[(Q(I) + [I - X_L]^+) - (S - \mu E[L]))^2 | I].$$

Therefore set

$$Q(I) = S - \mu E[L] - E[[I - X_L]^+ | I].$$

Notice that with this modification the average opening inventory just after the receipt of an order will be given by

$$\begin{aligned} E[Q(I) + [I - X_L]^+] &= E[S - \mu E[L] - E[[I - X_L]^+ | I] \\ &\quad + [I - X_l]^+] \\ &= S - \mu E[L], \end{aligned}$$

as desired. The functional form for $Q(I)$ also satisfies the conditions for $Q(I)$ given in Theorem 4 of Karlin and

Scarf [30] for the case when the lead time is exactly equal to one review period. The expression for Q is complex and as a simplification we propose taking the expectation into the $[\cdot]^+$ operation, giving,

$$Q(I) = S - \mu E[L] - [I - E[X_L]]^+.$$

This is a modified version of the base stock policy. In this modification, $Q(I)$ is adjusted *downwards* for the estimated amount of lost sales when compared to the order quantity given by the base stock policy. Moreover, due to the logic used in deriving this formula for $Q[I]$, we expect the inventory to be close to $(S - \mu E[L])$ after each delivery. This should intuitively reduce the extent of lost sales by cutting down on an unnecessary and avoidable source of variability, namely the variance of the inventory just after a delivery. If this modification were to result in a consistent level of inventory *after* each delivery (i.e., $(S - \mu E[I])$) then the expected value of the inventory just *before* the receipt of an order should be equal to $E[S - X_{T+L}]^+$.The average order quantity will be given by $(\mu T - E[X_{T+L} - S]^+)$, i.e., the demand during the review period *minus* the expected lost sales. Therefore the retailer's cost can be approximated using (compare with Equation (1)),

$$\begin{aligned} C_r(T,S) = &+\frac{A_r}{T} \quad \text{(fixed cost of reorder)} \\ &+\frac{1}{2}(S - \mu E[L] + E[S - X_{T+L}]^+)c_r i_r \\ &\text{(holding cost as weighted average of opening and ending inventory)} \\ &+\frac{(p-c)}{T}E[X_{T+L} - S]^+ \quad \text{(lost sale penalty)} \\ &-\frac{(\tau_c - E[L])}{T}(\mu T - E[X_{T+L} - S]^+)c_r f_r. \\ &\text{(credit on reorder less carrying cost during the lead-time)} \end{aligned} \tag{9}$$

The cost to the producer can be similarly approximated using (compare with Equation (3)),

$$C_p(T,m,S) = \frac{A_p + B/m}{T} + \mu T \varsigma c_p i_p + \mu \tau_c c_r f_p + E[X_{T+L} - S]^+ \times \left(\frac{(c_r - c_p)}{T} - \frac{\tau_c}{T}c_r f_p - \varsigma c_p i_p\right). \tag{10}$$

Based on Equation (9), the optimal base stock level for the retailer will be given by:

$$F_{T+L}(S_r^*(T)) = \frac{((p - c_r) + (\tau_c - E[L])c_r f_r - c_r i_r T/2)}{((p - c_r) + (\tau_c - E[L])c_r f_r + c_r i_r T/2}.$$

This formula is exactly like the formula for the zero lag case in the sense that as the lead-time goes to zero the formulae become identical. Simulation was used to compare the performance of the base stock policy to that of the modified base stock policy. We have studied the performance of the two policies for several scenarios chosen from our case study and found that: (i) the modified policy gives as good or better results compared to the base stock policy; (ii) it unifies the formulae for the zero and positive lag cases, and more important to our purpose; (iii) the cost expression in (9) provides a reasonable approximation to the true cost. (Details of the numerical study can be obtained by writing to the authors).

It immediately follows via arguments that were used to establish Proposition 1 and from the approximate cost functions given in (9) and (10), that the disagreement on the value of S remains. The source of the disagreement is the difference in value attached by the two parties to holding stock at the store. For the producer, the retailer's stock that remains unsold at the end of a period costs nothing, but this is not so for the retailer. A useful way of visualizing the conflict is shown in Fig. 1(a). If the demand during the period $(T + L)$ is less than S, the producer loses nothing, whereas the retailer has borne the cost of unsold stock. On the other hand, when demand exceeds S, lost sales adversely impact both parties. Quantity discounts can not be used to achieve coordination, when both parties wish to choose the optimal S. The only method of arriving at a mutually acceptable value of S is for the producer to share in the cost when there is unsold stock at the store. We remark here that the expected value of the unsold stock is the *safety* stock, and that sharing the cost of carrying this part of the inventory can be considered to be equivalent to ''risk sharing'', see Wilson [37].

4.3. *Cost sharing arrangement*

We now suggest a method for obtaining agreement on the base stock level. Assume that the producer agrees to bear a (fixed) fraction β of the cost of safety stock carried by the retailer. Assume that the arrangement is initiated by the producer *giving* the fraction β of safety

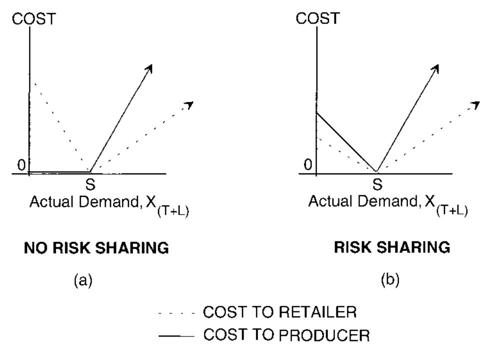

Fig. 1. Cost to retailer and producer as a function of the realized demand.

stock at no cost to the retailer on the condition that the producer receives payment once this is sold. There after, each period the manufacturer "tops up" this part of the stock based on actual sales. Therefore, this arrangement has the effect of increasing or decreasing credit in a dynamic fashion. As a consequence of this cost sharing arrangement, whenever there is unsold stock the producer has to bear a fraction of the cost of carrying this stock, see Fig. 1(b). The cost functions are then modified as follows (compare with Equations (9) and (10))

$$\begin{aligned} C_r(T,S,\beta) = {} & \frac{A_r}{T} + \frac{1}{2}(S - \mu E[L] + E[S - X_{T+L}]^+)c_r i_r \\ & + \frac{(p-c_r)}{T}E[X_{T+L} - S]^+ - \frac{(\tau_c - E[L])}{T} \\ & \times ((\mu T - E[X_{T+L} - S]^+))c_r f_r \\ & - \beta E[S - X_{T+L}]^+ c_r f_r, \end{aligned} \tag{11}$$

and

$$\begin{aligned} C_p(T,m,S,\beta) = {} & \frac{A_p + B/m}{T} + \mu T \varsigma c_p i_p + \mu \tau_c c_r f_p \\ & + E[X_{T+L} - S]^+ \\ & \times \left(\frac{(c_r - c_p)}{T} - \frac{\tau_c}{T}c_r f_p - \varsigma c_p i_p\right) \\ & + \beta E[S - X_{T+L}]^+ c_r f_p. \end{aligned} \tag{12}$$

In Equations (11) and (12), the terms involving $\beta E[S - X_{T+L}]^+$ reflect the cost sharing arrangement. This arrangement costs the producer $\beta E[S - X_{T+L}]^+ c_r f_p$ whereas it creates a benefit $\beta E[S - X_{T+L}]^+ c_r f_r$ for the retailer. Given values for T, m, and β, denote the optimal base stock level for the retailer and the producer to be $S_r^*(T,\beta)$ and $S_p^*(T,m,\beta)$. In our analysis, we shall assume that

$$\beta \le i_r/f_r \text{ and } (p - c_r) - c_r i_r T/2 + (\tau_c - E[L])c_r f_r \ge 0. \tag{13}$$

The second condition given in (13) implies that the gross margin per unit *minus* the average cost of holding a unit of the product (in inventory and the pipeline) *plus* the credit offered is non-negative. The reader will observe that this condition is also necessary for the retailer to make a profit from selling the item. We shall show that the first condition in (13) holds good for the equilibrium value of β, (denoted as β_e). (In general, it is true that i_r will be smaller than f_r but *a priori* we do not know whether the equilibrium value of β is less than one. Therefore, we cannot avoid stating this condition initially, but after our analysis find that it is not necessary). As described below, the cost sharing arrangement resolves the tension between the producer and the retailer.

Proposition 2. *If the conditions given in* (5) *and* (13) *hold, then*

(i) *The optimizing retailer will choose a base stock level given by,*

$$F_{T+L}(S_r^*(T,\beta)) = \frac{((p-c_r) + (\tau_c - E[L])c_r f_r - c_r i_r T/2)}{((p-c_r) + (\tau_c - E[L])c_r f_r + c_r i_r T/2 - \beta c_r f_r T)}.$$

(ii) *The optimizing producer will prefer a base stock level given by,*

$$F_{T+L}(S_p^*(T,m,\beta)) = \frac{(c_r - c_p) - \tau_c c_r f_p - \varsigma c_p i_p T}{(c_r - c_p) - \tau_c c_r f_p - \varsigma c_p i_p T + \beta c_r f_p T}.$$

(iii) *The retailer and the producer will agree upon a stocking level if the sharing fraction, β, is chosen to satisfy,*

$$\beta_e(T,\tau_c) = \frac{c_r i_r((c_r - c_p) - \tau_c c_r f_p - \varsigma c_p i_p T)}{c_r f_r((c_r - c_p) - \varsigma c_p i_p T) + c_r f_p((p - c_r) - E[L]c_r f_r - c_r i_r T/2)},$$

and the first part of condition (13) *holds for this equilibrium value of β.*

(iv) *Given the cost sharing agreement, the optimal base stock level is independent of the credit terms.*

(v) *At the equilibrium value of β_e, the ratios of the stock out cost to the holding cost are equal for the producer and retailer* (*i.e., ratio of the slopes of the lines on either side of S in Fig.* 1(b) *are equal*).

Proof. Please see Appendix.

Part (iii) of Proposition 2 shows that given T there exists a Nash equilibrium for the base stock level if the correct value of β were chosen for implementation. Part (iv) then shows that this stocking level is independent of the credit terms. This is the key result used in the next section to analyze review period coordination and to develop an algorithm for calculating the optimal review period. Here after we shall denote the optimal base stock level as $S_e^*(T)$. The value of β_e does not depend on the demand parameters, and will be nearly the same for a group or family of SKU's that have similar margins, set up costs and production frequency. Thus the extension of the sharing formula to a family of similar SKU's is quite straightforward. We have dropped the dependence on the production frequency m for simplicity in exposition. The implications of cost sharing are described in the next proposition.

Proposition 3.

(i) *If the conditions given in* (5) *and* (13) *hold then the base stock level $S_e^*(T)$ minimizes the joint cost of the retailer and the producer.*

(ii) *If $\tau_c \ge 0$ and $c_r f_r \ge c_r f_p$, then*: (a) *cost sharing leads to a lower combined cost compared to centralized decision* *making* (*but without cost sharing*); (b) *the combined cost of the producer and retailer is a linear and non-increasing function of* τ_c; (c) *the average cost to the producer* (*retailer*) *is a linear and non-decreasing* (*non-increasing*) *function of* τ_c; (d) *this result is general for a given value of T so long as the average order quantity and ending inventory are unaffected by the credit terms offered.*

Proof. Please see Appendix.

We observe that the reduction in the joint cost is due to the difference in the cost per unit to the retailer and producer. This is a benefit that cannot be obtained by using quantity discounts. We remark that the condition in part (ii) of Proposition 3 is similar to the condition given in Kohli and Park [38] for the Pareto optimality of quantity discounts, see Section 2.

From Fig. 1(a), it follows that unless a cost sharing scheme is put into place, regardless of the modeling assumptions, the producer and retailer will not agree with regard to base stock level. The solution proposed by us to resolve the conflict is widely applicable. For example, partial or complete backordering can be accommodated in this framework. As Pasternak has shown, returns of unsold goods could also be modeled. Moreover if myopic ordering policies based on newsboy type solutions are used then non-stationary demand or lead-times can be incorporated, (see Morton and Pentico [39] for examples of myopic ordering rules). The robustness of the cost sharing rule is a consequence of our applying the robust design principle explained in Section 5. The solution proposed by us results in a *linear sharing rule*. Linear sharing rules are known to be robust, see Hart and Holmstrom [40]. Wilson [37] shows that restricting the preferences of the producer and retailer to exponential utility functions, and performing joint optimization will lead to linear sharing rules. Thus our "form" of the cost sharing contract can be extended to a cooperative game theoretic framework in which the two players jointly maximize their expected utility.

Propositions that demonstrate that selfishly optimal individual decisions will result in efficient aggregate outcomes are called *decentralization results*. Farrell [41] finds such results to be interesting because they are surprising, provide a taxonomy of inefficiency, and also serve as arguments against government intervention. There is yet another benefit of decentralization, namely when the underlying parameters change, decentralization saves the trouble and expense of renegotiating contracts. The cost sharing scheme results in decentralization in two ways. At the firm level, the producer and retailer even though optimizing in their own best interests arrive at the same base stock level. Therefore, the stocking decision can be made independent of the decisions on credit terms and cost sharing.

5. Analysis of the review period

We assume that there exist discrete increasing numbers say $[t_1, t_2, \dots, t_n]$, over which the choice of the review period has to be made (see Silver and Peterson [29]). Given the falling cost of set up and order placement for retailers, it is likely that t_1 is the preferred value for the retailer, and that t_n is the preferred value for the producer. From Proposition 2, part (iv) we know that the optimal base stock level is independent of the credit terms. We can therefore assume without loss of generality that the retailer maintains a safety stock of $K(T)\sigma\sqrt{T+E[L]}$, where $K(T)$ is obtained by solving for the base stock level for *any arbitrary positive value of* τ_c. The important point for our analysis is that the value of $K(T)$ is independent of everything else except the length of the review period. It follows that for a given value of T, the average inventory, average lost sales, average safety stock, and the average number of orders processed are all functions of the length of the review period. Therefore given a value of m, we assume below (see Equations (14) and (15)) that the credit terms and the cost sharing fraction, β_e are functions of T alone. It is important to keep in mind that we are attempting to express the credit terms as a function of the length of the review period, i.e., to determine the credit terms as $\tau_c(T)$. Our objective is to construct a function $\tau_c(T)$ so that both the producer as well as the retailer agree with regard to the length of the review period. Assume that the value of m is fixed. Then we can write,

$$C_r(T) = A_r/T + \tfrac{1}{2}\mu T c_r i_r + g_r(T), \tag{14}$$

$$C_p(T) = (A_p + B/m)/T + \mu T \varsigma c_p i_p + g_p(T), \tag{15}$$

where, the functions $g_r(T)$ and $g_p(T)$ represent the benefits and costs associated with carrying safety stock, extending credit, and sharing the cost of holding safety stock. The expressions for $g_r(T)$ and $g_p(T)$ are given in the Appendix. The first order necessary condition for coordination at a common value of the review period, say T_o, can be derived as follows. (We shall discuss below the conditions under which our approach is sufficient to guarantee coordination). The retailer differentiates the relevant cost with respect to the length of the review period and sets $\partial C_r(T)/\partial T = 0$, yielding (see(14))

$$-\frac{A}{T^2} + \frac{1}{2}\mu c_r i_r + \frac{\partial g_r(T)}{\partial T} = 0 \quad \text{or} \quad T^2 = A_r \bigg/ \left(\frac{1}{2}\mu c_r i_r + \frac{\partial g_r}{\partial T}\right).$$

Similarly, the producer optimizes with respect to T and sets $\partial C_p(T)/\partial T = 0$, yielding (see (15))

$$-(A_p + B/m)/T^2 + \mu\varsigma c_p i_p + \frac{\partial g_p(T)}{\partial T} = 0 \quad \text{or}$$

$$T^2 = (A_p + B/m)\bigg/\left(\mu\varsigma c_p i_p + \frac{\partial g_p(T)}{\partial T}\right).$$

Therefore, for the retailer and the producer to agree to a review period equal to T_0 we require

$$A_r \Big/ \left(\frac{1}{2}\mu c_r i_r + \frac{\partial g_r}{\partial T_0}\right) = (A_p + B/m) \Big/ \left(\mu \varsigma c_p i_p + \frac{\partial g_p}{\partial T_0}\right). \tag{16}$$

5.1. *The robust design principle*

Equation (16) gives the condition for coordination as an *equality* that has to be satisfied at a particular value of $T = T_0$. Therefore any contract that we write should satisfy this condition at some value T_0. There are two problems with determining such a contract. First, if the length of the review period were to change then the incentive scheme might have to be rewritten to satisfy Equation (16). Second observe that (16) provides only a partial specification for the contract, it does not tell us how to obtain $\tau_c(T)$. There are potentially an uncountable number of functions $\tau_c(T)$ that will satisfy Equation (16). To see this, observe that the credit terms are assumed to be a function of T. Therefore, given a particular value of T, we can always determine a suitable differentiable function $\tau_c(T)$ such that (16) holds uniquely at that value of T. We overcome these drawbacks by appealing to a "robust" design principle. (This principle is based on a discussion in Hart and Holmstrom [40]). Simply stated the contract should be so designed that a small change to the length of the review period does not result in a large change in the credit terms offered. We apply this principle and treat (16) to be a *differential equation* for all values of T. In other words, instead of treating (16) to be an equation that has to be satisfied at the equilibrium value, T_0, we require it to be satisfied for every T in the open interval $[0, \infty)$.

This approach satisfies the design principle because once we are able to solve the differential equation, and obtain a sufficiently smooth solution $\tau_c(T)$, then small changes in the length of the review period will not affect the value of the optimal credit terms substantially. Our way out of the dilemma posed by (16) is closely related to the concept of refinement of a Nash equilibrium discussed by van Damme [42]. Viewed in the game theoretic framework, any two cost functions g_r and g_p that satisfy (16) constitute an equilibrium. However, by requiring that small deviations from the optimal contract parameters do not change the equilibrium value of T we have refined the equilibrium to obtain, a pair of cost functions that are unique up to a constant (see below). We state without proof that the robust design principle of treating equality of derivatives as a differential equation will yield the linear cost sharing rule of Section 4. Using this principle we have the following proposition.

Proposition 4. *Based on the robust design principle,*

(i) *The optimal credit terms can be determined from the equation,*

$$\begin{aligned}&\left(A_r \mu \varsigma c_p i_p - (A_p + B/m)(\tfrac{1}{2}\mu c_r i_r)\right)T \\ &\quad = (A_p + B/m)g_r(T) - A_r g_p(T) + \Sigma,\end{aligned} \tag{17}$$

where Σ is a constant of integration.

(ii) *Credit has to be extended over a longer duration when demand is more volatile or when lead-times are longer or when higher service levels are required.*

(iii) *If the differential equation has a solution then our solution is sufficient to guarantee coordination at some value of T.*

Proof. Please see Appendix. The analysis of the cost functions given there also illustrates the calculations for a simple case.

The constant of integration, Σ, is subject to negotiations between the retailer and producer. In practice a non-negative value for this constant might be required to keep the credit terms positive. An algorithm to determine the optimal review period, the base stock level, the credit terms and the sharing fraction is described next. For ease implementation, we employ the formula for $\tau_c(T)$ given in the Appendix in Equation (A21), but if the lost sales are not negligible, then an exact formula should be used in its place. The value of m is assumed to be given. If needed, this assumption can be replaced with a search for the optimal value of m.

Algorithm for determining policy parameters for supply chain coordination

Step 0. Input problem parameters, and the set of review periods, $[t_1, t_2, \ldots, t_n]$. Set $\Sigma = 0$.

Step 1. do for each value of t_i

- 1.1 Assume *any* positive value for τ_c. Find the corresponding value of β_e using the formula given in Proposition 2.
- 1.2 Find the value of $S(t_i)$ that minimizes the joint cost of producer and retailer, using the formula in Proposition 2.
- 1.3 Find the value of $K(t_i)$ from $S = \mu(t_i + E[L]) + K(t_i)\sigma\sqrt{t_i + E[L]}$.
- 1.4 Determine τ_c using (A21). If $\tau_c < 0$, adjust Σ accordingly to maintain non-negativity.
- 1.5 Determine β_e using τ_c and the equation given Proposition 2.
- 1.6 Determine the expected cost for the producer and the retailer using Equations (11) and (12).

Step 2. Choose the value of t_i that yields the lowest expected cost for the producer (if the values of the t_i's are sufficiently close to one another then this will also give the lowest expected cost for the retailer because the solution will be a coordinated one).

end

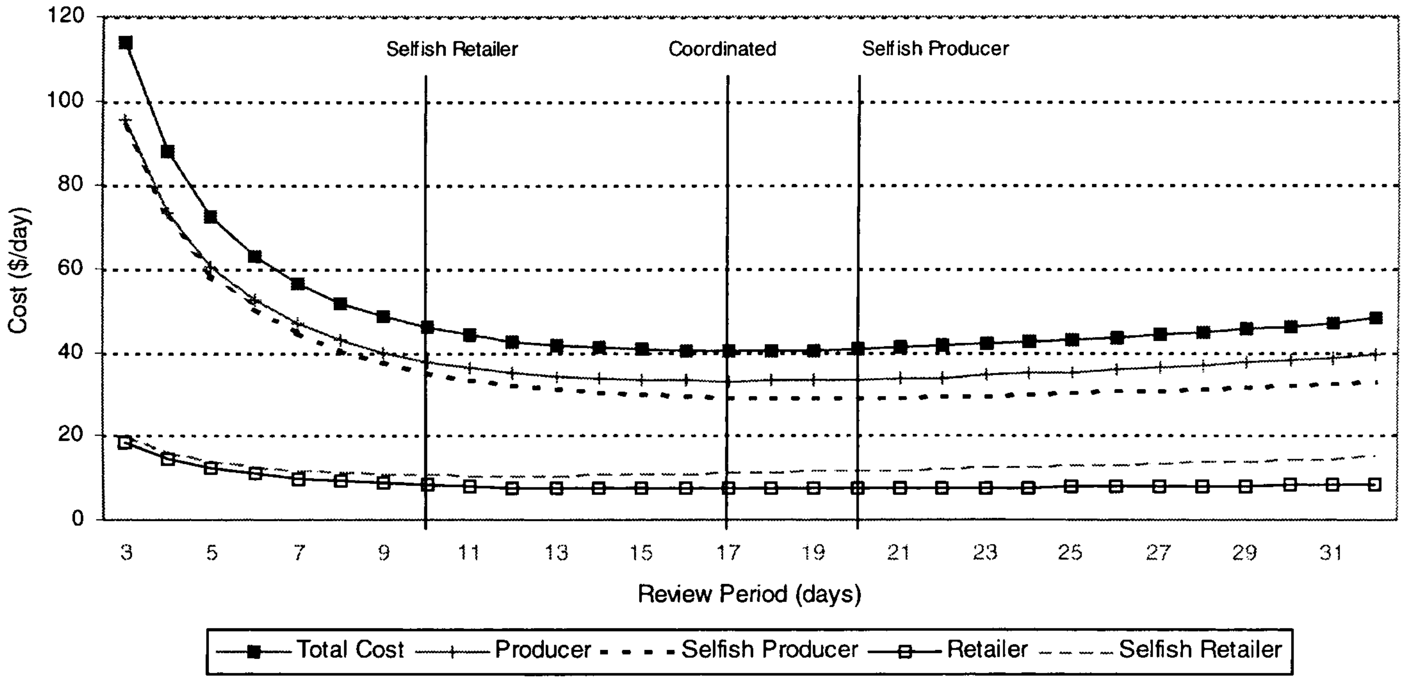

Fig. 2. Average cost versus review period when $p = 70$, $c_r = 49$, $c_p = 35$, $m = 2$, $\alpha = 0.8$, $A_r = 50$, $A_p = 150$, $B = 250$, $i_r = i_p = 0.3/\$/\text{year}$, $f_r = f_p = 0.24/\$/\text{year}$.

The results of applying the algorithm to a problem instance are shown in Figs. 2 and 3. The cost data for this example is based on the parameters for a family of SKU's in our field study. The average inventory, lost sales and safety stock were obtained by simulating customer arrivals according to a Poisson process of rate 20. The average inventory is calculated by time averaging the simulated inventory values. If the producer were to offer no incentives (neither credit nor sharing the cost of carrying safety stocks) then the retailer prefers the length of the review period to be 10 days. The producer prefers that the product is ordered every 20 days.

These points are labeled as "Selfish Retailer" and "Selfish Producer" in Fig. 2. Offering both the incentives to the retailer shifts the average cost curve for the retailer down and to the right, and the producer's total cost curve

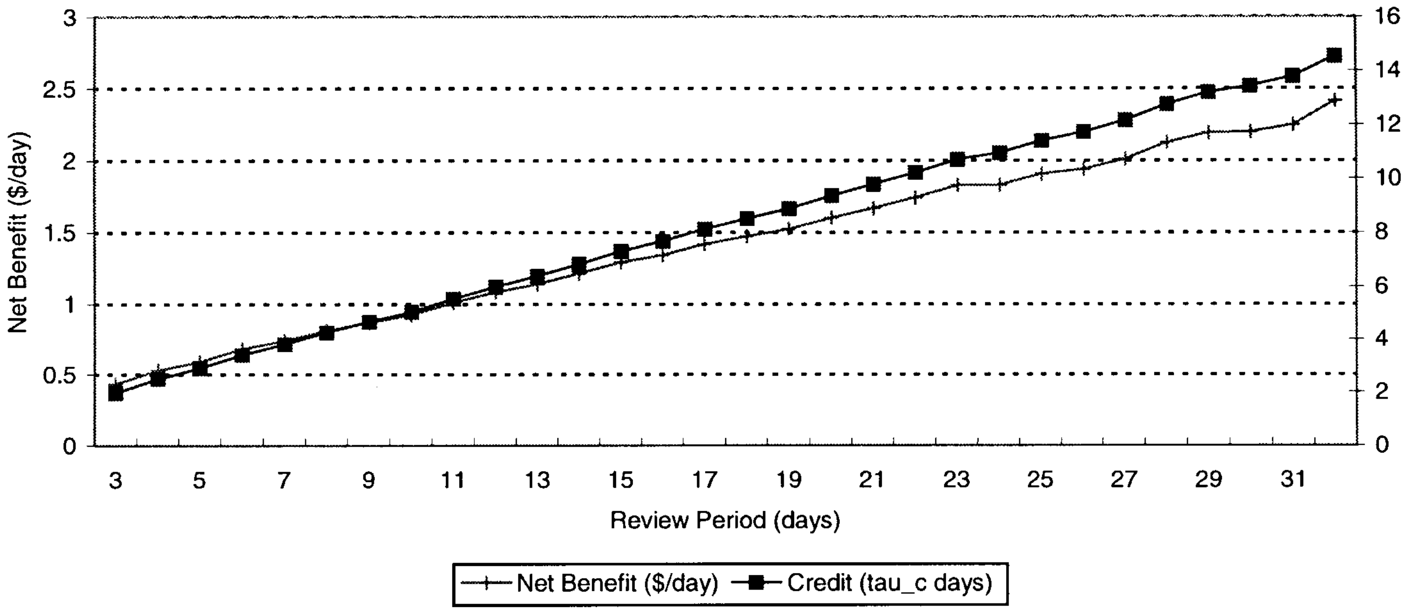

Fig. 3. Credit and net benefit versus review period when $p = 70$, $c_r = 49$, $c_p = 35$, $m = 2$, $\alpha = 0.8$, $A_r = 50$, $A_p = 150$, $B = 250$, $i_r = i_p = 0.3/\$/\text{year}$, $f_r = f_p = 0.24/\$/\text{year}$.

up. The combined effect is to achieve a jointly optimal and Nash solution at $T=17$ days, see Fig. 2. In this solution, the benefit to the retailer is more than the cost to the producer, and thus negotiations over the value of Σ might be necessary. These negotiations do not affect the optimal length of the review period.

The ''selfish''solutions are defined to be the best that the producer or the retailer can do without regard to the consequences to the other. In Fig. 2, the sum of the selfish costs of the producer and the retailer (see dashed lines at 10 and 20 days) is *larger* than the combined cost at the coordinated solution of 17 days. This coordinated solution is indeed ''strongly'' optimal because the selfish solutions cannot be achieved simultaneously.

The *net benefit* is defined to be the difference in the benefit accruing to the retailer from the credit and the cost sharing incentives *minus* the cost to the producer of extending these incentives to the retailer. In Fig. 3, the values of τ_c is plotted along with the net benefit in the supply chain. The credit terms and the net benefit are seen to be increasing almost linearly with the length of the review period. The sharing fraction β_e (not shown in the figure) is the least affected due to the change in the length of the review period. It declines from 0.727 to 0.696 as the length of the review period increases from 3 to 20 days.

6. Conclusion

We have developed a method for achieving coordination in a supply chain using just one policy variable, namely, the credit terms. An optimization method has been developed to solve for all the policy parameters of the supply chain. A natural extension to multiple SKU's would be to aggregate products into product families before calculating the cost sharing fraction or credit terms. Including multiple retailers, demand surges due to promotions, and make to stock production systems are other interesting extensions of our model.

Acknowledgements

We thank Professor Eric Denardo of Yale University, and Professors Vipul Agrawal and Mike Pinedo of New York University for several suggestions on an earlier version of this manuscript. We are grateful to Professor Steve Nahmias, Santa Clara University, for his comments on an earlier version of the modified ordering policy given in Section 3, that led to the present formula. We also thank two anonymous referees and the Departmental Editor for several helpful comments. A reviewer's comment led to the analysis of the two cost functions and the proof that our approach is sufficient to ensure coordination.

References

[1] Gupta, Y.P. and Neel, G.A. (1992) The origin of EDI and changes associated with its implementation. *IIE Solutions*, **24**(8), 25–29.

[2] Wang, E.T.G. and Seidmann, A. (1995) Electronic data interchange: competitive externalities and strategic implementation policies. *Management Science*, **41**, 401–418.

[3] Goyal, S.K. and Gupta, Y.P. (1989) Integrated inventory models: the buyer-vendor coordination, *European Journal of Operational Research*, **41**, 261–169.

[4] Dolan, R.J. (1987) Quantity discounts: managerial issues and research opportunities, *Marketing Science*, **30**, 1–22.

[5] Monahan, J.P. (1984) A quantity discount pricing model to increase vendor profits. *Management Science*, **30**, 720–726.

[6] Joglekar, P.N. (1988) Comments on ''A quantity discount pricing model to increase vendor profits. *Management Science*, **34**, 1393–1398.

[7] Dada, M. and Srikanth, K.N. (1987) Pricing policies for quantity discounts, *Management Science*, **33**, 1247–1252.

[8] Lee, H.L. and Rosenblatt, M.J. (1986) A generalized quantity discount pricing model to increase supplier's profits. *Management Science*, **32**, 1177–1185.

[9] Weng, Z.K. (1995) Channel coordination and quantity discounts. Management Science, **41**, 1509–1522.

[10] Kohli, R. and Park, H. (1994) Coordinating buyer-seller transactions across multiple products. *Management Science*, **40**, 1145–1150.

[11] Spengler, J. (1950) Vertical integration and anti-trust policy. *Journal of Political Economy*, **58**, 347–352.

[12] Anupindi, R. and Bassok, Y. (1994) Centralization of stocks: retailer vs. manufacturer. Technical Report, J. L. Kellog School of Management, Northwestern University, Evanston, IL 60208.

[13] Parlar, M. (1988) Game theoretic analysis of the substitutable product inventory problem with random demands. *Naval Research Logistics*, **35**, 397–409.

[14] Wang, Q. and Parlar, M. (1994) A three-person game theory model arising in stochastic inventory control theory. *European Journal of Operational Research*, **76**(1), 83–97.

[15] Ernst, R. and Cohen, M.A. (1992) Coordination alternatives in a manufacturer/dealer/inventory system under stochastic demand. *Production and Operations Management*, **1**, 254–268.

[16] Ernst, R. and Powell, S.G. (1992) Manufacturer's incentives to improve retail service levels. Working Paper number 282, (1987) *The Amos Tuck School of Business Administration*, *Dartmouth College*, *NH 03755*.

[17] Pasternak, B.A. (1985) Optimal pricing and return policies for perishable commodities. *Marketing Science*, **4**, 166–176.

[18] Kandel, E. (1996) The right to return, *Journal of Law and Economics*, **39**, 329–56.

[19] Padmanabhan, V. and Png, I.P.L. (1997) Manufacturer's return policies and retail competition. *Marketing Science*, **16**, 81–94.

[20] Allen, S.G. (1958) Redistribution of total stock over several user locations, *Naval Research Logistics Quarterly*, **5**, 51–59.

[21] Clark, A.J. and Scarf, H.E. (1960) Optimal policies for a multiehelon inventory problem. *Management Science*, **6**, 475–490.

[22] Clark, A.J. and Scarf, H.E. (1962) Approximate solutions to a simple multi-echelon inventory problem in *Studies in Applied Probability and Management Science*, Arrow, K.J., Karlin S. and Scarf, H. (eds.), Stanford University Press, Stanford, CA, pp. 88–100.

[23] Federgruen, A. and Zipkin, P. (1984) Approximations of dynamic multilocation production and inventory problems. *Management Science*, **30**(1), 69–84.

[24] Nahmias, S. (1993) Mathematical models of retailer inventory systems: an overview, in *Perspectives in Operations Management:*

Essays in Honor of Elwood S. Buffa, Sarin, R.K. (ed.), Kluwer Academic Press, Boston, MA. pp. 249–278.
[25] McGavin, E.J., Schwarz, L.B. and Ward, J.E. (1993) Two-interval inventory allocation policies in a one-warehouse N-identical retailer distribution system. *Management Science*, **39**, 1092–1107.
[26] Nahmias, S. and Smith, S.A. (1994) Optimizing inventory levels in a two- echelon retailer system with partial lost sales. *Management Science*, **40**, 582–596.
[27] Hadley, G. and Whitin, T.M. (1963) *Analysis of Inventory Systems*, Prentice Hall, Englewood Cliffs, NJ, pp. 237–242.
[28] Buzacott, J.A. and Shanthikumar, J.G. (1993) *Stochastic Models of Manufacturing Systems*, John Wiley, NY.
[29] Silver, E.A. and Peterson, R. (1985) *Decision Systems for Inventory Management and Production Planning*, 2nd. ed., John Wiley, NY, p. 290.
[30] Karlin, S. and Scarf, H.E. (1958) Optimal inventory policy for the Arrow-Harris-Marschak dynamic model with a time lag in *Studies in the Mathematical Theory of Inventory and Production*, Arrow, K.J., Karlin, S. and Scarf, H. (eds.), Stanford University Press, Stanford, CA, Ch. 10, pp. 155–178.
[31] Kaplan, R. (1970) A dynamic inventory model with stochastic lead times, *Management Science*, **16**, 491–507.
[32] Morton, T.E. (1971) The near-myopic nature of the lagged- proportional-cost inventory problem with lost sales. *Operations Research*, **19**, 1708–1716.
[33] Nahmias, S. (1979) Simple approximations for a variety of dynamic lead-time lost-sales inventory models. *Operations Research*, **27**, 904–924.
[34] Ehrhardt, R. (1984) (s, S) policies for a dynamic inventory model with stochastic lead-times. *Operations Research*, **32**, 121– 132.
[35] Zipkin, P. (1986) Stochastic lead-times in continuous-time inventory models. *Naval Research Logistics Quarterly*, **33**, 763–774.
[36] Hax, A.C. and Candea, D. (1984) *Production and Inventory Management*, Prentice-Hall, Englewood Cliffs, NJ.
[37] Wilson, R. (1968) The theory of syndicates. *Econometrica*, **36**, 119–132.
[38] Kohli, R. and Park, H. (1989) A cooperative game theory model of quantity discounts. *Management Science*, **35**, 693–707.
[39] Morton, T.E. and Pentico, D. (1995) The finite horizon nonstationary stochastic inventory problem: near myopic bounds, heuristics and testing. *Management Science*, **41**, 334–343.
[40] Hart, O. and Holmstrom, B. (1987) The theory of contracts, in *Advances in Economic Theory*, Bewley T. (ed), Cambridge University Press, Cambridge, UK, Ch. 3, pp. 71–159.
[41] Farrell, J. (1987) Information and the Coase theorem, *Economic Perspectives*, **1**, 113–129.
[42] van Damme, E. (1996) *Stability and Perfection of Nash Equilibrium*, 2nd edn., Springer-Verlag, Berlin.
[43] Elsgolts, L. (1970) *Differential Equations and the Calculus of Variations*, Mir Publishers, Moscow.

Appendix

Proof of Proposition 2

$$\frac{\partial}{\partial S}C_r(T,S,\beta) = \frac{1}{2}c_r i_r(1 + F_{T+L}(S)) - \frac{1}{T}(p - c_r)F^c_{T+L}(S) + \frac{1}{T}(-1 + F_{T+L}(S))(\tau_c - E[L])c_r f_r - \beta c_r f_r F_{T+L}(S). \quad \text{(A1)}$$

$$\frac{\partial^2}{\partial S^2}C_r(T,S,\beta) = \frac{1}{2}c_r i_r f_{T+L}(S) + \frac{1}{T}(p - c_r)f_{T+L}(S) + f_{T+L}(S)(\tau_c - E[L])c_r f_r/T - \beta c_r f_r f_{T+L}(S). \quad \text{(A2)}$$

From (A2) and the conditions in (13) that $\beta \le i_r/f_r$ and $(p - c_r) - c_r i_r T/2 + (\tau_c - E[L])c_r f_r \ge 0$, it follows that $C_r(T,S,\beta)$ is a convex function of S. Similarly,

$$\frac{\partial}{\partial S}C_p(T,S,m,\beta) = -\left(\frac{1}{T}(c_r - c_p) - \varsigma c_p i_p - \frac{\tau_c}{T}c_r f_p\right)F^c_{T+L}(S) + \beta c_r f_p F_{T+L}(S). \quad \text{(A3)}$$

$$\frac{\partial^2}{\partial S^2}C_p(T,S,\beta) = +\left(\frac{1}{T}(c_r - c_p) - \varsigma c_p i_p - \frac{\tau_c}{T}c_r f_p\right)f_{T+L}(S) + \beta c_r f_p f_{T+L}(S). \quad \text{(A4)}$$

It follows from (A4) that if the condition given in (5) holds then $C_p(T,S,m,\beta)$ is a convex function of S. Setting the derivatives given in (A1) and (A3) equal to zero gives the first two parts of the proposition (the calculations are omitted). For part (iii), equating $F_T(S^*_p(T,m,\beta))$ and $F_T(S^*_r(T,\beta))$, yields:

$$\frac{(c_r - c_p) - \tau_c c_r f_p - \varsigma c_p i_p T}{(c_r - c_p) - \tau_c c_r f_p - \varsigma c_p i_p T + \beta_e(T,\tau_c)c_r f_p T} = \frac{((p - c_r) + (\tau_c - E[L])c_r f_r - c_r i_r T/2)}{(p - c_r) + (\tau_c - E[L])c_r f_r + c_r i_r T/2 - \beta_e(T,\tau_c)c_r f_r T}.$$

$$\Leftrightarrow (c_r i_r - \beta_e(T,\tau_c)c_r f_r)((c_r - c_p) - \tau_c c_r f_p - \varsigma c_p i_p T) = \beta_e(T,\tau_c)c_r f_p((p - c_r) + (\tau_c - E[L])c_r f_r - c_r i_r T/2).$$

$$\Leftrightarrow c_r i_r((c_r - c_p) - \tau_c c_r f_p - \varsigma c_p i_p T) = \beta_e(T,\tau_c)(c_r f_p((p - c_r) + (\tau_c - E[L])c_r f_r - c_r i_r T/2) + c_r f_r((c_r - c_p) - \tau_c c_r f_p - \varsigma c_p i_p T)).$$

$$\Leftrightarrow \beta_e(T,\tau_c) = \frac{c_r i_r((c_r - c_p) - \tau_c c_r f_p - \varsigma c_p i_p T)}{c_r f_r((c_r - c_p) - \varsigma c_p i_p T) + c_r f_p((p - c_r) - E[L]c_r f_r - c_r i_r T/2)} \quad \text{(A5)}$$

This completes part (iii) of the proposition. As long as the credit terms are positive, the equilibrium value of β is less than i_r/f_r. And in any case β_e does not depend on the demand parameters. For part (iv),

$$\frac{\partial}{\partial \tau_c}F_T(S^*_p(T,\beta_e(T,\tau_c))) = \frac{((c_r - c_p) - \tau_c c_r f_p - \varsigma c_p i_p T)(c_r f_p - (\partial\beta_e(T,\tau_c)/\partial\tau_c)c_r f_p T)}{((c_r - c_p) - \tau_c c_r f_p - \varsigma c_p i_p T + \beta_e(T,\tau_c)c_r f_p T)^2} - \frac{c_r f_p}{((c_r - c_p) - \tau_c c_r f_p - \varsigma c_p i_p T + \beta_e(T,\tau_c)c_r f_p T)}. \quad \text{(A6)}$$

Using (A5), we obtain

$$\frac{\partial \beta_e(T,\tau_c)}{\partial \tau_c} = \frac{-c_r i_r c_r f_p}{c_r f_r((c_r-c_p)-\varsigma c_p i_p T)+c_r f_p((p-c_r)-E[L]c_r f_r - c_r i_r T/2)} \quad \text{(A7)}$$

Using Equations (A6) and (A7), we get

$$\frac{\partial}{\partial \tau_c} F_T(S_p^*(T,\beta_e(T,\tau_c))) = \frac{((c_r-c_p)-\tau_c c_r f_p-\varsigma c_p i_p T)\big(c_r f_p+((c_r f_p)c_r i_r c_r f_p T)/(c_r f_r((c_r-c_p)-\varsigma c_p i_p T)+c_r f_p((p-c_r)-E[L]c_r f_r-c_r i_r T/2))\big)}{((c_r-c_p)-\tau_c c_r f_p-\varsigma c_p i_p T+\beta_e(T,\tau_c)c_r f_p T)^2} - \frac{(c_r f_p)(((c_r-c_p)-\tau_c c_r f_p-\varsigma c_p i_p T+\beta_e(T,\tau_c)c_r f_p T))}{(((c_r-c_p)-\tau_c c_r f_p-\varsigma c_r i_p T+\beta_e(T,\tau_c)c_r f_p T))^2}. \quad \text{(A8)}$$

The numerator on the right hand side of (A8) evaluates to

$$(c_r f_p)^2 T\left(\frac{c_r i_r((c_r-c_p)-\tau_c c_r f_p-\varsigma c_p i_p T)}{c_r f_r((c_r-c_p)-\varsigma c_p i_p T)+c_r f_p((p-c_r)-E[L]c_r f_r-c_r i_r T/2)} - \beta_e(T,\tau_c)\right)=0,$$

where the final equality follows from (A5). A simpler proof of this can be obtained by adding the numerators and denominators (separately of course) of $(f_r/f_p)F_T(S_p^*(T,m,\beta))$ and $F_T(S_r^*(T,\beta))$.

For part (v), holding costs per unit (at the store) are $(c_r i_r - \beta_e c_r f_r)$ for the retailer and $\beta_e c_r f_p$ for the producer. Using (A5), the ratio of these two unit holding costs is

$$\begin{aligned}(c_r i_r - \beta_e c_r f_r)/\beta_e c_r f_p &= \frac{c_r i_r - (c_r f_r c_r i_r((c_r-c_p)-\tau_c c_r f_p-\varsigma c_p i_p T))/(c_r f_r((c_r-c_p)-\varsigma c_p i_p T)+c_r f_p((p-c_r)-E[L]c_r f_r-c_r i_r T/2))}{(c_r f_p c_r i_r((c_r-c_p)-\tau_c c_r f_p-\varsigma c_p i_p T))/(c_r f_r((c_r-c_p)-\varsigma c_p i_p T)+c_r f_p((p-c_r)-E[L]c_r f_r-c_r i_r T/2))} \\ &= \frac{((p-c_r)+(\tau_c-E[L])c_r f_r-c_r i_r T/2)}{((c_r-c_p)-\tau_c c_r f_p-\varsigma c_p i_p T)} \\ &= \text{(profit per unit to retailer)/(profit per unit to producer).}\end{aligned}$$

■

Part (ii): we are given τ_c. From (6) and (7), and the condition $c_r f_r \geq c_r f_p$,

$$C_r(T,S,\beta)+C_p(T,S,m,\beta) = C_r(T,S,0)+C_p(T,S,m,0) - \beta(c_r f_r - c_r f_p)E[S-X_{T+L}]^+$$

$$C_r(T,S,\beta)+C_p(T,S,m,\beta) \leq C_r(T,S,0)+C_p(T,S,0). \quad \text{(A9)}$$

The proof of part (ii) (a) follows from A(9) because,

$$\begin{aligned}&C_r(T,S_e^*(T),\beta_e(T,\tau_c))+C_p(T,S_e^*(T),m,\beta_e(T,\tau_c))\\ &\leq C_r(T,S,\beta)+C_p(T,S,m,\beta)\\ &\leq C_r(T,S,0)+C_p(T,S,0).\end{aligned}$$

For the proof of the remaining parts, let $OQ(S) = (\mu T - E[X_{T+L}] - S]^+)$, denote the expected order quantity. Let $I_{\text{END}}(S) = E[S-X_{T+L}]^+$ be the expected ending inventory. Using part (iv) of Proposition 2,

$$\begin{aligned}&\frac{\partial}{\partial \tau_c}(C_r(T,S_e^*(T)+C_p(T,S_e^*(T),m,\beta_e(T,\tau_c))\\ &= (c_r f_r - c_r f_p)(-OQ(S_e^*(T))/T)+(c_r f_r - c_r f_p)\\ &\times\left(-\frac{\partial \beta_e(T,\tau_c)}{\partial \tau_c} I_{\text{END}}(S_e^*(T))\right).\end{aligned} \quad \text{(A10)}$$

Using (A7) and (A10), we get,

Proof of Proposition 3

Part (i): we know that

$$\begin{aligned}&\min_S(C_r(T,S,\beta)+C_p(T,S,m,\beta))\\ &\geq \min_S C_r(T,S,\beta)+\min_S C_p(T,S,m,\beta).\end{aligned}$$

However, given T, τ_c and $\beta_e(T,\tau_c)$, the optimum value of S for both the retailer as well as the producer are the same. Therefore $S_e^*(T)$ is jointly optimal.

$$\frac{\partial}{\partial \tau_c}(C_r(T,S_e^*(T)+C_p(T,S_e^*(T),m,\beta_e(T,\tau_c)) = \frac{(c_r f_r - c_r f_p)}{T}\left(-OQ(S_e^*(T))+\frac{(c_r i_r c_r f_p T)I_{\text{END}}(S_e^*(T))}{c_r f_r((c_r-c_p)-\varsigma c_p i_p T)+c_p f_p((p-c_r)-E[L]c_r f_r - c_r i_r/2)}\right). \quad \text{(A11)}$$

The expression in parentheses in (A11) can be written as,

$$-\left[\frac{OQ(S_e^*(T))(c_rf_r((c_r-c_p)-\varsigma c_p i_p T)+c_rf_p((p-c_r)-E[L]c_rf_r-c_ri_r/2))-(c_ri_rc_rf_pT)I_{\mathrm{END}}(S_e^*(T))}{c_rf_r((c_r-c_p)-\varsigma c_p i_p T)+c_rf_p((p-c_r)-E[L]c_rf_r-c_ri_r/2)}\right]. \quad \text{(A12)}$$

The numerator of the expression in (A12) can be re-expressed as,

$$\begin{aligned}&(c_rf_pT[((p-c_r)+(\tau_c-E[L])c_rf_r-c_ri_r/2)OQ(S_e^*(T))/T\\&\quad-(c_ri_r-\beta_e(T,\tau_c)c_rf_r)I_{\mathrm{END}}(S_e^*(T))]\\&\quad+c_rf_rT[((c_r-c_p)-\tau_cc_rf_p-\varsigma c_pi_pT)OQ(S_e^*(T))/T\\&\quad-(\beta_e(T,\tau_c)c_rf_p)I_{\mathrm{END}}(S_e^*(T))].\end{aligned} \quad \text{(A13)}$$

We know that, $((p-c_r)+(\tau_c-E[L])c_rf_r-c_ri_r/2)$ is the per unit margin to the retailer *plus* the per unit benefit from credit *less* the cost of holding an item in the pipeline *less* the cost of holding an item in the pipeline *less* the cost per unit of carrying an item as cycle stock. Thus we observe that the first term in (A13) has the form,

$c_rf_pT\times$ [retailer's gross profit per year −retailer's annual cost of holding safety stocks]. (A14)

Similarly, the second term in (A13), has the form,

$c_rf_rT\times$ [producer's gross profit per year− producer's share of the annual cost of holding safety stocks at the store]. (A15)

From (A10), both $C_r(T,S_e^*(T)$ and $C_p(T,S_e^*(T),m,\beta_e(T,\tau_c)$ are linear functions of τ_c. From (A11)–(A15), if both the retailer as well as the producer make profits, and, if $(c_rf_r-c_rf_p)\geq 0$, then $(C_r(T,S_e^*(T)+C_p(T,S_e^*(T),\ m,\beta_e(T,\tau_c))$ is a linear and non-increasing functions of τ_c. It also follows that $C_p(T,S_e^*(T)$ is a linear and non-decreasing function of τ_c. This result is quite general because given the value of T, it holds whenever the values of the average order quantity and ending inventory are unaffected by the credit terms offered. ■

Proof of Proposition 4

We assume with out loss of generality (as explained in the text) that the retailer carries a safety stock of $K(T)\sigma\sqrt{T+E[L]}$. In this expression $K(T)$ can be any continuously differentiable function of T.

$$\begin{aligned}C_r(T,S,\beta_e)=&\frac{A_r}{T}+\frac{1}{2}(S-\mu E[L]+E[S-X_{T+L}]^+)c_ri_r\\&+\frac{(p-c_r)}{T}E[X_{T+L}-S]^+-\frac{(\tau_c(T)-E[L])}{T}\\&\times((\mu T-E[X_{T+L}-S]^+))c_rf_r-\beta_eE[S-X_{T+L}]^+c_rf_r\end{aligned}$$

$$\begin{aligned}C_r(T,S,\beta_e)\approx&A_r/T+\frac{1}{2}\mu Tc_ri_r+K(T)\sigma\sqrt{T+E[L]}(c_ri_r-\beta_ec_rf_r)\\&-(\tau_c(T)-E[L])\mu c_rf_r,\end{aligned} \quad \text{(A16)}$$

where the approximation is based on the assumption that lost sales are small. Let

$$g_r(T)=C_r(T,S,\beta_e)-\frac{A_r}{T}-\frac{1}{2}\mu Tc_ri_r.$$

Based on the approximation

$$g_r(T)\approx K(T)\sigma\sqrt{T+E[L]}\,(c_ri_r-\beta_ec_rf_r)-\tau_c(T)\mu c_rf_r.$$

Similarly,

$$\begin{aligned}&C_p(T,m,S,\beta_e)\\&=\frac{A_p+B/m}{T}+\mu T\varsigma c_pi_p+\mu\tau_c(T)c_rf_p\\&\quad+E[X_{T+L}-S]^+\left(\frac{(c_r-c_p)}{T}-\frac{\tau_c(T)}{T}c_rf_p-\varsigma c_pi_p\right)\\&\quad+\beta_eE[S-X_{T+L}]^+c_rf_p\\&\approx\frac{A_p+B/m}{T}+\mu T\varsigma c_pi_p+K\sigma\sqrt{T}\beta_ec_rf_p+\mu\tau_c(T)c_rf_p,\end{aligned} \quad \text{(A17)}$$

and define

$$\begin{aligned}g_p(T)&=C_p(T,m,S,\beta_e)-\frac{A_p+B/m}{T}-\mu T\varsigma c_ri_p\\&\approx K(T)\sigma\sqrt{T+E[L]}\beta_ec_rf_p+\mu\tau_c(T)c_rf_p.\end{aligned}$$

Let $\beta_e(T,\tau_c)$

$$\begin{aligned}&=\frac{c_ri_r((c_r-c_p)-\tau_c(T)c_rf_p-\varsigma c_pi_pT)}{c_rf_r((c_r-c_p)-\varsigma c_pi_pT)+c_rf_p((p-c_r)-E[L]c_rf_r-c_ri_rT/2)}\\&=\frac{A(T)-\tau_c(T)c_ri_rc_rf_p}{B(T)}.\end{aligned} \quad \text{(A18)}$$

Exact equation: the condition for equilibrium is obtained by equating the first order necessary conditions for an optima at a value T_0,

$$A_r\Bigg/\left(\frac{1}{2}\mu c_ri_r+\frac{\partial g_r}{\partial T_0}\right)=\left(A_p+B/m\right)\Bigg/\left(\mu\varsigma c_pi_p+\frac{\partial g_p}{\partial T_0}\right). \quad \text{(A19)}$$

We are seeking a robust solution, therefore we treat this equation as a differential equation. Rearranging equation (A19) we get,

$$A_r\mu\varsigma c_p i_p - (A_p + B/m)\left(\frac{1}{2}\mu c_r i_r\right) = (A_p + B/m)\frac{\partial g_r(T)}{\partial T} - A_r\frac{\partial g_p(T)}{\partial T}. \quad \text{(A20)}$$

Integrating (A20) gives (17).

Approximation: for part (ii), using (A16), (A17) and (A19) and integrating gives,

$$A_r\left(K(T)\sigma\sqrt{T+E[L]}\beta_e c_r f_p + \mu\tau_c(T)c_r f_p\right) - (A_p+B/m)(K(T)\sigma\sqrt{T+E[L]}(c_r i_r - \beta_e c_r f_r) - \tau_c(T)\mu c_r f_r) = ((A_p+B/m)\tfrac{1}{2}c_r i_r - A_r\varsigma c_p i_p)\mu T + \Sigma,$$

where Σ is a constant of integration. Solving for the credit terms, gives

$$\tau_c = \frac{((A_p+B/m)(c_r i_r/2) - A_r\varsigma c_p i_p)\mu T + \Sigma + ((A_p+B/m)(c_r i_r - A(T)/B(T)c_r f_r) - A_r(A(T)/B(T)c_r f_p))K(T)\sigma\sqrt{T+E[L]}}{(A_r c_r f_p + (A_p+B/m)c_r f_r)(\mu - (K(T)\sigma\sqrt{T+E[L]}\ c_r i_r c_r f_p/BT))} \quad \text{(A21)}$$

where

$$A(T) = c_r i_r((c_r - c_p) - \varsigma c_p i_p T),$$

and

$$B(T) = c_r f_r((c_r - c_p) - \varsigma c_p i_p T) + c_r f_p((p - c_r) - E[L]c_r f_r - c_r i_r T/2).$$

$$\text{safety stock} = \sigma\sqrt{T}\ \Phi^{-1}\left(\frac{1}{1 + ((c_r i_r c_r f_p T)/(crf_r((c_r - c_p) - \varsigma c_p i_p T) + c_r f_p((p - c_r) - c_r i_r T/2)))}\right). \quad \text{(A24)}$$

From the form of the approximate cost functions, we can show that there is a unique cost minimizing T (the proof is omitted). Therefore (A21) is a sufficient condition for coordination, when the lost sales are small, also see the next section. The conclusions given in part (ii) follow from (A21), and the assumptions that $K(T)$ is nearly constant for small changes in the volatility of demand or the lead-time, and that it is larger for higher service levels. ∎

Analysis of the cost functions

We establish if the differential equation has a solution then solving it to deduce the credit terms leads to a *sufficient* condition for optimality We also solve a simple example to illustrate the calculations.

Simple example: in the simple example we assume that the demand in an interval $[0, T]$ is normally distributed with mean equal to μT and standard deviation equal to $\sigma\sqrt{T}$. Let $\phi(\cdot)$ and $\Phi(\cdot)$ stand for the density and distribution function of a random variable that has the normal distribution with mean equal to zero and standard deviation equal to one. let $\Phi(\cdot)^{-1}$ stand for the inverse of the function $\Phi(\cdot)$. Assume that B and the lead-time are equal to zero, and that ς is strictly positive.

Derivation of safety stock for simple example: from Proposition 2, the retailer carries a safety stock of given by

$$\sigma\sqrt{T}\ \Phi^{-1}\left(\frac{((p - c_r) + \tau_c c_r f_r - c_r i_r T/2)}{((p - c_r) + \tau_c c_r f_r + c_r i_r T/2 - \beta_e c_r f_r T)}\right). \quad \text{(A22)}$$

We know that the safety stock is independent of the credit terms. Thus setting the credit terms equal to zero, solving for the sharing fraction and substituting in (A22) we get:

$$\beta_e(T,0) = \frac{c_r i_r((c_r - c_p) - \varsigma c_p i_p T)}{c_r f_r((c_r - c_p) - \varsigma c_p i_p T) + c_r f_p((p - c_r) - c_r iT/2)}, \quad \text{(A23)}$$

and

Using (A16) the expected cost to the retailer is

$$C_r(T,S,\beta_e) = \frac{A_r}{T} + \frac{1}{2}(S + E[S - X_T]^+)c_r i_r + \frac{(p - c_r)}{T}E[X_T - S]^+ - \frac{\tau_c(T)}{T}((\mu T - E[X_T - S]^+))c_r f_r - \beta_e E[S - X_T]^+ c_r f_r. \quad \text{(A25)}$$

Similarly,

$$C_p(T,m,S,\beta_e) = \frac{A_p}{T} + \mu T\varsigma c_p i_p + \mu\tau_c(T)c_r f_p + E[X_T - S]^+\left(\frac{(c_r - c_p)}{T} - \frac{\tau_c(T)}{T}c_r f_p - \varsigma c_p i_p\right). + \beta_e E[S - X_T]^+ c_r f_p. \quad \text{(A26)}$$

Properties of the cost functions (simple and general case)

(i) In the simple case, both cost functions have continuous derivatives of every order and satisfy a suitable Lipschitz

condition when T is bounded away from zero. Therefore the solution to the differential equation (16) is unique up to a constant of integration for T greater than some $\varepsilon > 0$, as is discussed by Elsgolts [43]. (In any case, we are not interested in values of T that are extremely small.)
(ii) Moreover, if the differential equation has a solution then in *general* the first derivatives of the producer's and retailer's cost functions are multiples of one another. It is not straightforward to deduce this. To see this, from (A20):

$$\begin{aligned}(A_p + B/m)\frac{\partial C_r(T,S,\beta_e)}{\partial T} &= (A_p + B/m)\left(\frac{\partial g_r(T)}{\partial T} - \frac{A_r}{T^2} + \frac{1}{2}\mu c_r i_r\right),\\ &= A_r\frac{\partial g_p(T)}{\partial T} + A_r\mu\varsigma c_p i_p - (A_p + B/m)\left(\frac{1}{2}\mu c_r i_r\right)\\ &\quad + (A_p + B/m)\left(-\frac{A_r}{T^2} + \frac{1}{2}\mu c_r i_r\right),\\ &= A_r\frac{\partial g_p(T)}{\partial T} + A_r\mu\varsigma c_p i_p - (A_p + B/m)\frac{A_r}{T^2},\\ &= A_r\left(\frac{\partial g_p(T)}{\partial T} + \mu\varsigma c_p i_p - \frac{(A_p + B/m)}{T^2}\right)\\ &= A_r\frac{\partial C_p(T,m,S,\beta_e)}{\partial T}.\end{aligned}$$

(iii) In the simple case, because the differential equation (16) is satisfied at all T and due to (ii) above, the first derivative of the function $\Gamma(T) = C_r(T,S,\beta_e) + (f_r/f_p)C_p(T,0,S,\beta_e)$ should be zero at the values of T at which the first derivative of the two costs equal zero.

The new function is,

$$\begin{aligned}\Gamma(T) = \frac{A_r}{T} + \frac{f_r}{f_p}\left[\frac{A_p}{T} + \mu T\varsigma c_p i_p\right] + \frac{1}{2}(S + E[S - X_T]^+)c_r i_r\\ + \left[\frac{(p - c_r)}{T} + \frac{f_r}{f_p}\left(\frac{(c_r - c_p)}{T} - \varsigma c_p i_p\right)\right]E[X_T - S]^+.\end{aligned} \tag{A27}$$

We can simplify

$$\begin{aligned}E[S - X_T]^+ &= \int_{-\infty}^{S}(S - x)\frac{1}{\sigma\sqrt{2\pi T}}\exp\left(-\frac{1}{2}\left(\frac{x - \mu T}{\sigma\sqrt{T}}\right)^2\right)\mathrm{d}x\\ &= (S - \mu T)\Phi\left(\frac{S - \mu T}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}\phi\left(\frac{S - \mu T}{\sigma\sqrt{T}}\right).\end{aligned} \tag{A28}$$

From the expressions for $F_T(S_p^*(T,m,\beta))$ and $F_T(S_r^*(T,\beta))$, (A24), (A27), and (A28), we obtain:

$$\begin{aligned}\Gamma(T) = \frac{A_r}{T} + \frac{f_r}{f_p}\left[\frac{A_p}{T} + \mu T\varsigma c_p i_p\right] + \frac{1}{2}\mu T c_r i_r + \sigma\phi\left(\frac{S - \mu T}{\sigma\sqrt{T}}\right)\\ \left[(p - c_r) + \frac{f_r}{f_p}((c_r - c_p) - \varsigma c_p i_p T) + \frac{1}{2}c_r i_r T\right]\Big/\sqrt{T}.\end{aligned} \tag{A29}$$

$$\begin{aligned}C_r(T,S,\beta_e) &= \frac{A_r}{T} + \sigma\sqrt{T}\phi\left(\frac{S - \mu T}{\sigma\sqrt{T}}\right)\\ &\quad\times\left(\frac{(p - c_r)}{T} + \frac{1}{2}c_r i_r - \beta_e c_r f_r + \frac{\tau_c(T)}{T}c_r f_r\right)\\ &\quad + \frac{1}{2}\mu T c_r f_r - \tau_c(T)\mu c_r f_r,\\ C_p(T,m,S,\beta_e) &= \frac{A_p}{T} + \sigma\sqrt{T}\phi\left(\frac{S - \mu T}{\sigma\sqrt{T}}\right)\\ &\quad\times\left[\frac{(c_r - c_p)}{T} - \varsigma c_p i_p + \beta_e c_r f_p - \frac{\tau_c(T)}{T}c_r f_p\right]\\ &\quad + \mu T\varsigma c_p i_p + \tau_c(T)\mu c_r f_p.\end{aligned}$$

Coordination in the simple and in the general case

From (ii) above, if the differential equation (16) has a solution then it is immediately seen that the retailer's cost function can be written as an increasing affine transformation of the producer's cost function. Therefore, *in general* offering credit terms based on the solution to the differential equation is sufficient to guarantee coordination.

In the simple case, it can be shown using (A24) and (A29) and the function $\Gamma(T)$ increases beyond some finite value of T, thus the values of T for which the first derivative of this function is zero are all finite. We can also conclude that *if* the value of $\phi((S - \mu T)/\sigma\sqrt{T})$ is relatively small or relatively unchanged in the range of T's we are interested in when compared to the other terms in this expression (this will be true for example, if the lost sales are small), *then* there is a unique value of T at which the first derivative of the function $\Gamma(T)$ equals zero.

Biographies

Michael Moses received a B.S. in Mathematics from Worcester Polytechnic Institute in 1962, and a Ph.D. in Industrial Engineering and Management Science from Northwestern University in 1968. He is interested in problems where the firms operating and strategic visions need alignment. He is also interested in the financial returns associated with art investment over the last 50 years. He has published articles on the design and implementation of computer based planning systems as well as the use of optimization models in strategic planning. He is also part of the authoring team for HOM, a software system for helping firms and students analyze problems related to gaining strategic advantage from the operating system of the firm.

Sridhar Seshadri received a Bachelor of Technology degree from the Indian Institute of Technology, Madras, India, in 1978, a Post Graduate Diploma in Management from the Indian Institute of Management, Ahmedabad, India, in 1980 and a Ph.D. in Management Science from the University of California at Berkeley in 1993. His research interests are in the area of stochastic modeling and optimization, with applications to manufacturing, distribution, telecommunications, database design, insurance and finance. He is an Associate Editor for *Naval Research Logistics* and *Telecommunication Systems*, and on the Editorial Review Board of *Production and Operations Management*. He has worked as an engineer in a medium and heavy engineering firm in India, as a contractor in the Middle East as well as in India, and as a Professor for 5 years in the Operations Management Area, at the Administrative Staff College of India, Hyderabad, India. He is currently an Associate Professor with the Operations Management Department, Leonard N. Stern School of Business, New York University.

Contributed by the Engineering Statistics and Applied Probability Department