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Web Based Capacity Allocation Strategies for Customers with Heterogeneous Preferences

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Abstract

We study different mechanisms for the pricing and allocation of capacity to customers with heterogeneous and unknown preferences. The mechanisms we study include posting of prices, auctions, with and without the possibility of resale. We compare the performance of each mechanism with the optimal selling procedure in terms of expected seller's profit and efficiency. We also discuss the feasibility of web based implementation.

Keywords: auction theory, combinatorial auctions, revenue management

1. Introduction

We consider an allocation problem in which customers with heterogeneous preferences are to be assigned to capacity slots. In the general version of the problem, a seller has K slots, and each customer (buyer) has a demand for at most one slot. Slot i is worth V_{ij} to customer j. We assume that the seller and the buyers are risk neutral. Each buyer's willingness to pay is his/her private information. The number of customers that require capacity slots is uncertain and the arrival sequence of customers is also random. Our goal is to design pricing and allocation mechanisms for assigning slots to customers. The objectives in the mechanism design problem are: (i) to maximize the seller's expected profits and (ii) to evaluate the ensuing allocation in terms of the total expected welfare. The total expected welfare is given by the total of the seller's expected profit and the customers' expected surplus. Without loss of generality we assume that the number of slots is greater than or equal to the number of customers. In this paper we present an optimal pricing mechanism, and numerical examples for a two slot problem that provide insight into more general problems of this nature. We also present a practical algorithm for allocating slots labeled Bundle Value Based Pricing (BVBP). We show that it performs better than posting a price for each slot. We illustrate how BVBP can be implemented using an Internet based trading mechanism.

The capacity allocation problem described above is directly applicable to situations in which a sub-contractor (seller) has to determine how to price and allocate his capacity to different customers. Examples of this version of the problem are found in industries with multi-tiered supplier structure such as the automotive and hi-tech industries. The model

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is also applicable in industries in which the selling season is short, for example those that produce seasonal products such as apparel or promotional goods. A direct example can be found in the semiconductor industry where in which a foundries are required to allocate capacity for manufacturing custom designed integrated circuits. The circuits are typically designed by specialist firms that do not own a foundry (the so-called fabless firms), see [Chatterjee et al., 6]. Another potential application is in capital intensive industries, such as paper and steel, that produce goods to customer orders. In these industries the set up times and costs to change from one customer order to another are high. As a result manufacturers prefer to allocate distinct blocks of time on their production lines to customers. We have observed that some manufacturers now allow customers to use web links to connect to their (manufacturers') ERP systems and directly reserve time slots in the Manufacturing Production Schedule (MPS), see [Vollmann, Berry, and Whybark, 23]. Other applications are found in the allocation of specific time slots to customers in time share resorts, allocation of advertising slots on television, allocation of capacities in telecommunication networks, allocation of airport landing slots [Rassenti, Smith, and Bulfin, 19] and peak load pricing and capacity allocation for utilities [Elmaghraby and Oren, 9].

In all these examples potential buyers have different values for the same time slot (of capacity). They can choose from multiple slots but incur a penalty if their preferred slot is either unavailable or is too expensive. Moreover, the slot preference information is usually private and depends on the business realities of each buyer. Hence it is not easy to identify the high value buyers for a given time slot from the low value buyers. At present, posting of prices for individual time slots is the dominant mechanism to assign the slots to customers in many of the examples cited above. There are exceptions to this practice. In certain industries the time preference of customers is highly correlated with the value. For example many "high value" customers for an airline do not stay over weekends and most high value users of phone services call during the day. Therefore posting high prices during predictable high value time periods can yield the highest profit to the seller. On the other hand, if the seller has no idea who the high value customer is or what the high value customer's preference will be or is unsure when such a customer will arrive then posting prices for individual slots can be inefficient. When the high value customers arrives concurrently with low value customers posting fixed slot prices results in either loss of revenue or lost demand. Even in the airline case, the setting of high prices just before the departure of a flight results in unsold seats—seats that could have been sold to low value customers.

Using a two slot case we show that posting prices is not the best mechanism to use in these environments because it results in lost demand (when the value differential between customers is high) or in lost revenue (when the value differential between customers is small). The other popular mechanism for allocation of capacity is the use of auctions. For example auctions are used in the electric utility markets to buy and sell capacity. The environments we consider are not conducive to the use of a general auction mechanism for several reasons including:

(1) *Liquidity*. The capacity allocation examples that were described earlier involve business to business trades between a seller and few customers. In most of these industries the total capacity is either greater than or equal to the total demand.

(2) *Preference structure*. The time preference is a function of individual customer requirements. This presents an opportunity to the seller to mix and match the assignment so that complementarities can be realized to the extent possible amongst customers.
(3) *Random arrival pattern*. The sequential nature of arrival of customers makes it difficult to implement the auction mechanism.

Most of the examples mentioned above allow for capacities to be reserved, purchased or traded over the web. The Internet provides a unique medium to facilitate the trading and allocation of capacities. The Internet is different from the traditional channels of selling capacity due to the following factors:

1. Real time information exchange from many-to-many: information on arrival of customers, information pertaining to capacity availability, current prices/bids, and reservation of capacity are available in real time.
2. The Internet allows for greater ease in facilitating trade between buyers. The resale market for many goods (on web sites such as e-Bay) not only exists but also is seen to be thriving. Therefore sellers have to take into account the resale possibilities by buyers (even) in the primary market. Moreover, buyers and sellers can remain anonymous in trades conducted over the web. This is a safeguard against the strategic use of information. This is an important feature because at times trades could occur between competitors.
3. Payment systems enable reasonably secure and authenticated trading.

As a result the Internet has given rise to a wide variety of market mechanisms that already co-exist or can potentially co-exist for the same industry. For example, multiple and sequential auctions of many varieties and dynamic pricing based selling mechanisms co-exist in the electric utility market. The new marketplaces on the Internet can be seller based (a steel or paper mill or the US government) or comprised of third party intermediaries (e-Bay, e-Steel, Freemarkets, and Priceline) or buyer based (Covisint.com, myaircraft.com) that match buyers and sellers. The emergence of web based auction mechanisms (e-Bay, Yahoo auctions, Priceline, etc.) have also accentuated the need for research into mechanisms that can be employed in real time and also guarantee integrity and fairness to bidders. Optimal mechanism designs provide insights into the maximum expected value that a seller can expect to achieve under each of a broad spectrum of assumptions about the information possessed by the seller and the buyers. These solutions serve as a benchmark for comparing the performance of any given design. This paper therefore focuses on the optimal design and provides some comparisons across other mechanisms.

In Section 2, we characterize the revenue maximizing (i.e., optimal) selling procedures of slots to two potential buyers with unknown willingness to pay. In Section 3, we consider several selling procedures:

1. First, we study the case in which the seller simply posts prices for slots. Customers arrive randomly. The slots are offered and sold on a First Come First Served basis. We study two methods of determining the slot price. In the simple case, the seller determines the distribution of the price for each slot on a stand alone basis, and uses

the newsboy formula to set the price. In the more complicated case, the seller jointly optimizes the slot prices thus has to determine the vector of slot prices.
2. Then we analyze the case in which customers create an after market. In this case, after all customers have arrived they negotiate amongst themselves and reallocate the slots in order to maximize their joint surplus. Such a cooperative outcome is possible—namely, that the core of the equivalent cooperative game is not empty under mild assumptions. The seller's problem is to determine the optimal vector of slot prices so that his expected revenue is maximized when the after market mechanism operates.
3. Finally we characterize the outcomes of various selling procedures that use the auction format.

In Section 4 we present a practical selling procedure and compare its efficiency to other selling procedures.

1.1. Relevant literature

There is substantial and fast increasing literature on dynamic pricing and yield management models. These models consider a single product but incorporate demand uncertainty and heterogeneity in the value for the good or service, for example see [Braden and Oren, 5; Feng and Gallego, 10; Gallego and van Ryzin, 11; Stokey, 21]. In this body of research the decision is to determine the pricing strategy as a function of the elapsed time (in a finite horizon) and the realized demand. Most of this research is concerned with selling multiple goods of the same type. Moreover, the seller has information on whether the high value customers arrive earlier (for products such as fashion apparel or consumer electronics) or arrive later (air travel, hotels, car rentals) when compared to the low value customers.

News-vendor type models with pricing (Agrawal and Seshadri [1]), and dynamic pricing and inventory (yield management) models are examples wherein an optimal price or a set of prices is chosen to maximize either the expected utility or the expected profit of the seller. When there is a single object and several buyers the objective can be achieved by the use of a simple auction mechanism. When there are several objects, and buyers have use for only a subset of these objects, then little is known about the theoretical properties of the optimal mechanism.

Wilson [24] discusses the problem of setting prices (multi-dimensional tariffs) in contexts similar to the ones described in this paper. Relevant models for auctions and auction theory are presented in [Armstrong, 3; Cremer and McLean, 8; McMillan, 14; Milgrom and Weber, 15; Myerson, 17; Rochet and Chone, 20]. Much is known on the optimality and the efficiency of many standard auction formats for the case in which one indivisible object is up for sale. However, the properties of auctions in situations in which many objects are to be sold are not yet fully understood. For example, the revenue maximizing selling procedure is not known in general. Existing results are tentative and only for special cases (e.g., 2 objects and 2 buyers, with only two possible values for each buyer's willingness to pay for each object). It is also known that with risk neutral buyers, the slightest degree of correlation among the buyers' values enables the seller to fully extract all gains from trade, see [Cremer and McLean, 8] and [McAfee and McMillan, 13]. A mathematical programming formulation of the seller's revenue maximization problem can always be constructed and solved by imposing standard incentive compatibility and individual rationality constraints into the maximization problem. However, these formulations become extremely large very soon as the number of objects or types of buyers increases. Thus, closed form solutions describing the optimal selling procedure are highly desirable.

In this paper we focus on the case in which each buyer has no use for more than one object, and the buyer's willingness to pay is not known to the seller. This problem is a special case of the general multi-object mechanism design problem.

Market mechanisms can be used to develop a schedule for producing a given set of jobs using a given set of resources. For example, by announcing the price for the nth job to be produced on a single machine, we can schedule a set of jobs that have time value. See [Pinedo, 18] for an extensive discussion of such mechanisms. Also see [Banks, Ledyard, and Porter, 4] and [Kim et al., 12]. As Kim et al. [12] say, "An important class of economic problems that arise naturally in several applications is the allocation of multiple resources when there are uncertainties in demand or supply, unresponsive supplies (no inventories and fixed capacities), and significant demand indivisibilities (rigidities). The two most common mechanisms used to deal with this problem—markets and administrative procedures—can perform at low efficiencies. Thus, new mechanisms need to be designed that more efficiently allocate resources in these environments." They examine the efficiency of single good auction mechanisms when they are adapted to work in a multi-object environment.

2. The model

In this section we restrict attention to a two slot problem with two buyers. We determine the structure of the optimal selling mechanism for buyers with symmetric and uncorrelated preferences. We also recast the problem in the tradition of the literature that deals with mechanism design.

The owner of two indivisible objects V and W faces two potential buyers. For simplicity, we assume that each object has the same value $c > 0$ for the seller. Each buyer $i = 1, 2$ has use for at most one object, and quasilinear utility function:

$$U^i = v_i q^i_{\{V\}} + w_i q^i_{\{W\}} + \max\{v_i, w_i\} q^i_{\{V,W\}} - p^i,$$

where v_i and w_i denote i's willingness to pay for object V and W, respectively, and q^i_B denotes the probability that buyer i is awarded the objects in the set $B \subset \{V, W\}$.

Buyer i knows his two values (v_i, w_i), while both the other buyer $3-i$ and the seller perceive the pair (v_i, w_i) as the realization of a bidimensional random vector, distributed independently of (v_{3-i}, w_{3-i}), according to a symmetric density f with support in $[\tau_m, \tau_M]^2$. We assume that $c \le \tau_m$.

It will be convenient to think of the pair (a_i, w_i), where $a_i := v_i - w_i$, as buyer i's 'type.' The joint distribution of (a_i, w_i) is $j(a_i, w_i) \equiv f(a_i + w_i, w_i)$, with support $\Theta_1 := \{(a, w) \in \mathbb{R}^2 \mid w, w + a \in [\tau_m, \tau_M]\}$. To eliminate subscripts from types, we also define $(a_1, w_1) := (a, w)$ and $(a_2, w_2) := (b, y)$.

2.1. Ex-post efficient outcome functions

In this environment, an *allocation function* specifies a probability distribution

$$\chi(\cdot \mid a, w, b, y)$$

over all possible ways of distributing the objects among the three traders, for each profile of buyers' types $(a, w, b, y) \in \Theta_1^2$. An *outcome function* consists of an allocation function and payment functions M_i, specifying buyer i's expected payment $M_i(a_i, w_i)$ for each type $(a_i, w_i) \in \Theta_1$, for each $i = 1, 2$.

An outcome function is said to be *ex post efficient* if no other outcome function exist such, for any type profile $(a_1, w_1, a_2, w_2) \in \Theta_1^2$, all traders are at least as well off, and some traders are strictly better off.

Since we have assumed that all traders have quasi-linear utility functions, in any ex post efficient outcome function the objects must be allocated in order to maximize the difference between the buyers' total willingness to pay and the seller's total cost. Moreover, because no buyer has use for more than one object, and the seller's value for each object c is positive but below the lowest possible buyers' value τ_m, exactly one object must be assigned to each buyer. This leaves only two ways of assigning the objects, which we denote as (V, W) and (W, V), where the ith entry denotes the object which is assigned to the ith buyer. The total surplus generated by (V, W) is $a + w + y - 2c$, while (W, V) generates $w + b + y - 2c$. Since $a + w + y - 2c > (=, <) w + b + y - 2c$ if and only if $a > (=, <) b$, it follows that an allocation function χ^* is part of an ex post efficient outcome function only if each object is assigned to a buyer with the highest difference between his values, i.e.

$$\chi^*\big((V, W) \mid (a, w, b, y)\big) = \mathbf{1}_{[a>b]}, \quad \text{and} \quad \chi^*\big((W, V) \mid (a, w, b, y)\big) = \mathbf{1}_{[a<b]}.$$

Denoting as $Q^*_{ij}(a_i, w_i)$ the corresponding probability that buyer $i = 1, 2$ is awarded object $j = V, W$, conditional on her type (a_i, w_i), we have

$$Q^*_{1V}(a, w) = 1 - Q^*_{1W}(a, w) = \int_{\tau_m}^{\tau_M} \int_{\tau_m}^{\tau_M} \mathbf{1}_{[b<a]} f(b + y, y)\, db\, dy = G(a)$$

and similarly

$$Q^*_{2V}(b, y) = 1 - Q^*_{2W}(b, y) = G(b).$$

In general, given any allocation function $\chi(\cdot \mid \cdot)$, the implied probabilities that each buyer i is awarded each object j are called *interim assignment* functions. Since the buyers' types are private information, an outcome function can be feasible only if each buyer, given her type, has no incentive to mimic the behavior of any other type in Θ_1. Formally, the corresponding interim assignment functions $Q_{ij}(a_i, w_i)$; $i = 1, 2$; $j = V, W$; and the payment functions $M_i(a_i, w_i)$, for $i = 1, 2$, must satisfy the following "incentive compatibility" (IC) constraints:

$$\begin{aligned}&(a_i + w_i)Q_{iV}(a_i, w_i) + w_i Q_{iW}(a_i, w_i) - M_i(a_i, w_i)\\ &\quad\geq (a_i + w_i)Q_{iV}(a'_i, w'_i) + w_i Q_{iW}(a'_i, w'_i) - M_i(a'_i, w'_i),\\ &\qquad\text{for all } (a_i, w_i), (a'_i, w'_i) \in \Theta_1. \end{aligned}\tag{IC}$$

Thus, in particular, the objects can be allocated efficiently, only if one can find payment functions M_i, $i = 1, 2$ such that

$$\begin{aligned}&(a_i + w_i)G(a_i) + w_i\big[1 - G(a_i)\big] - M_i(a_i, w_i)\\ &\quad\geq (a_i + w_i)(a'_i) + w_i\big[1 - G(a'_i)\big] - M_i(a'_i, w'_i),\end{aligned}\tag{1}$$

or equivalently

$$a_i G(a_i) - M_i(a_i, w_i) \geq (a_i)G(a'_i) - M_i(a'_i, w'_i),$$

for all $(a_i, w_i), (a'_i, w'_i) \in \Theta_1$.

The following well-known result in mechanism design theory, which we record here as Lemma 0, provides a useful characterization of the (IC) constraints, see Armstrong [3] and Rochet and Chone [20].

Lemma 0. The functions $Q_{ij} : \Theta_1 \to [0, 1]$; and $M_i : \Theta_1 \to \mathbb{R}$, $j = V, W$, $i = 1, 2$, satisfy (IC) if and only if the surplus functions $S_i(a, w)$ defined as

$$S_i(a_i, w_i) \equiv (a_i + w_i)Q_{iV}(a_i, w_i) + w_i Q_{iW}(a_i, w_i) - M_i(a_i, w_i)$$

is convex, hence differentiable almost everywhere, and such that

$$\frac{\partial S_i(a_i, w_i)}{\partial a} = Q_{iV}(a_i, w_i) \quad\text{and}\quad \frac{\partial S_i(a_i, w_i)}{\partial w} = Q_{iV}(a_i, w_i) + Q_{iW}(a_i, w_i)$$

almost everywhere.

Lemma 0 immediately implies the extension of Myerson's *Revenue Equivalence Theorem* to the present model: the expected payment functions M_i of any outcome function with interim assignment functions $Q_{iV}(a_i, w_i)$ and $Q_{iW}(a_i, w_i)$ must satisfy the following 'envelope condition':

$$\begin{aligned}M_i(a_i, w_i) &= a_i Q_{iV}(a_i, w_i) + w_i\big[Q_{iV}(a_i, w_i) + Q_{iW}(a_i, w_i)\big] - S_i(a_i, w_i)\\ &= a_i Q_{iV}(a_i, w_i) + w_i\big[Q_{iV}(a_i, w_i) + Q_{iW}(a_i, w_i)\big]\\ &\qquad - \int_{(0,\tau_m)}^{(a_i,w_i)} \big[Q_{iV}(z', z)\, dz' + Q_{iW}(z', z)\, dz\big] - S_i(0, \tau_m)\end{aligned}\tag{EC}$$

where the integral can be taken along any connected path from $(0, \tau_m)$ to (a_i, w_i). The first equality in (EC) expresses buyer i's expected payment as the difference between her willingness to pay and her net surplus. The second equality uses Lemma 0 to express her net surplus in terms of her interim assignment functions.

The envelope condition (EC) can be used to find the payment of each buyer as a function of her interim assignment functions and the net surplus of type $(0, \tau_m)$. If the object are allocated efficiently, i.e., if $Q^*_{iV}(a_i, w_i) = 1 - Q^*_{iW}(a_i, w_i) = G(a_i)$, then

$$\begin{aligned} M^*_i(a_i, w_i) &\equiv (a_i, w_i)G(a_i) + w_i\big[1 - G(a_i)\big] - S^i(a_i, w_i) \\ &= a_i G(a_i) + w_i - S^i(a_i, w_i). \end{aligned}$$

For $a_i > 0$, we have $S^i(a_i, w_i) = S^i(a_i, \tau_m) + w_i$ (since the derivative along on the 45° line is $Q_{iV}(0, w_i) + Q_{iW}(0, w_i) = 1$), hence

$$\begin{aligned} M_1(a_i, w_i) &= a_i G(a_i) + w_i - S^i(a_i, \tau_m) - w_i \\ &= a_i G(a_i) - S^i(a_i, \tau_m) \\ &= a_i G(a_i) - S^i(0, \tau_m) - \int_0^{a_i} G(b)\, db \\ &= \int_0^{a_i} b\, dG(b) - S^i(0, \tau_m). \end{aligned}$$

The last expression indicates that the efficient allocation can be implemented with a sequence of one-object English auctions: in the auction for the first object, say V, each bidder bids until the price reaches the difference between his two values. In the second auction, only the loser of the first auction participates and wins the object for the minimum price.

In the next subsection we characterize the revenue maximizing selling procedure, under a regularity condition on the values distribution F, and under the restriction that both object are always sold.

2.2. *Optimal selling procedures*

The main result of this subsection is that under a regularity condition on the values' distribution, the ex post efficient outcome function also maximizes the seller's expected revenue among all feasible selling procedures in which both object are always sold. By the Revelation Principle, (see Myerson [17]) the search for an optimal mechanism can be restricted without loss of generality to the class of all incentive compatible and individually rational direct revelation mechanisms, i.e., measurable functions

$$q_{ij} : \Theta_1 \times \Theta_1 \to [0, 1],\ \ M_i : \Theta_1 \to \mathbb{R}; \quad i = 1, 2 \text{ and } j = V, W;$$

such that, for each $i = 1, 2$,

$$\begin{aligned} &(a + w)Q_{iV}(a, w) + wQ_{iW}(a, w) - M_i(a, w) \\ &\quad \geq (a + w)Q_{iV}(a', w') + wQ_{iW}(a', w') - M_i(a', w') \end{aligned} \tag{IC}$$

for all $(a, w), (a', w') \in \Theta_1$, and

$$(a + w)Q_{iV}(a, w) + wQ_{iW}(a, w) - M_i(a, w) \geq 0, \tag{IR}$$

all $(a, w) \in \Theta_1$, where $Q_{ij}(a_i, w_i) \equiv \int_{\Theta_1} q_{ij}(a_1, w_1, a_2, w_2) f(a_i + w_i, w_i)\, dw_i\, da_i$, each $j = V, W, i = 1, 2$.

We are now ready to state the main result.

Proposition 1. If the function

$$z(a) \equiv a - \frac{1 - G(a)}{g(a)}$$

is nondecreasing, the seller's expected revenue is maximized, among all feasible selling mechanisms in which each buyer always receives one object, by the ex-post efficient outcome function, i.e. by the allocation function χ^* and the payment functions

$$M_i^*(a, w) = \int_0^{|a|} z\, dG(z), \quad i = 1, 2.$$

Proof. See Appendix. □

The key idea in the proof of Proposition 1 consists in using of the restriction that each buyer always receives one object to reformulate the seller's program in a one-dimensional format (Lemma 1, in the Appendix). This enables us to adopt standard mechanism design techniques to find the optimal mechanism.

Extensions to more general cases, i.e. with more that two objects and more buyers, require the analysis of mechanisms design programs with multidimensional types, which are know to be substantially more difficult to solve in closed form, and to yield optimal mechanisms that are quite sensitive to the distribution the buyers' values. (See Armstrong [3] and Rochet and Chone [20].)

Example. If f is the *uniform density, with support* $[0, 1]^2$, (and $c = 0$) the ex post efficient assignment functions imply

$$1 - Q_{1V}(a, w) = Q_{1W}(a, w) = \begin{cases} \frac{1}{2}(1 - a)^2, & a \geq 0, \\ \frac{1}{2}(1 + a)^2, & a < 0. \end{cases}$$

The corresponding surplus function satisfies

$$S^1(a, 0) = S^1(0, 0) + \int_0^a \left(1 - \frac{1}{2}(1 - t)^2\right) dt = S_1(0, 0) + \frac{1}{2}a + \frac{1}{2}a^2 - \frac{1}{6}a^3,$$

for any $a \in [0, \tau_M - \tau_m]$, and

$$S_1(a, w) = S_1(a, 0) + w = w + \frac{1}{2}a + \frac{1}{2}a^2 - \frac{1}{6}a^3,$$

for any $(a, w) \in \Theta_1$, such that $a > 0$. Hence the resulting interim-expected *payment* function is:

$$M_1(a, w) = \frac{1}{2}a^2 - \frac{1}{3}a^3.$$

Since the density of a conditional on $a > 0$ is

$$\frac{1-|a|}{\int_0^1 (1-|a|)\,da} = 2\big(1-|a|\big),$$

the *ex-ante expected revenue from buyer 1*, *conditional on* $a > 0$, is

$$E\big[M_1(a) \mid 0 < a\big] = 2\int_0^1 \left(\left(\frac{1}{2}a^2 - \frac{1}{3}a^3\right)(1-a)\right) da = \frac{1}{20}.$$

By symmetry, the seller's *expected revenue* generated *by buyer 1* with the efficient allocation is also $\frac{1}{20}$.

To compute the ex-ante expected social surplus, we need the buyer's expected surplus. Since

$$E[w \mid a] = \int_0^{1-a} \frac{w}{1-a} dw = \frac{1}{2}(1-a),$$

the expected net surplus of one buyer, given his difference a, is

$$E\big[S(a,w) \mid a\big] = \frac{1}{2}(1-a) + \frac{1}{2}a + \frac{1}{2}a^2 - \frac{1}{6}a^3 = \frac{1}{2} + \frac{1}{2}a^2 - \frac{1}{6}a^3.$$

Thus the expected consumer surplus, conditional on $a > 0$, is

$$E\big[S_1(a,w) \mid 0 < a < 1\big] = 2\int_0^1 \left(\left(\frac{1}{2} + \frac{1}{2}a^2 - \frac{1}{6}a^3\right)(1-a)\right) da = \frac{17}{30} = \frac{34}{60}.$$

Thus, again by symmetry, the ex ante *expected surplus* of buyer 1 is $\frac{34}{60}$.

The social surplus generated by one buyer is $\frac{34}{60} + \frac{1}{20} = \frac{37}{60}$. Based on the above numerical results we may conclude that: that when the uncertainty regarding the true value to the buyer is high the expected surplus to the seller can be quite small and thus any inefficiency will result in even smaller profits to the seller.

In concluding this section we observe that the key simplification used in the proofs is that each buyer obtains an object. The theoretical results also suggest that the objects can be sold through an auction. We also observe that if we remove the restriction that each buyer always receives one object, it may be optimal for the seller to bundle the two slots and sell them as a single object. The revenue of this procedure depends on how the resale market works. In particular, if the buyer who buys the two slots from the buyer can make take it or leave if offers, then the seller can extract his monopoly rent ex-ante. This observation motivates a heuristic selling procedure for the general case and is discussed in Section 4.

Before proceeding to more general cases, we note that the results of Proposition 1 can be extended to selling multiple objects as follows to create a welfare optimal solution. Assume that the number of buyers equals the number of slots. Also assume that it is optimal that each buyer gets a slot. Given the type of the first buyer and a slot taken by him, assume that an oracle can provide the expected value of the remaining slots. Thus, if there are originally K slots, the oracle generates K *bundle* valuations, where each bundle valuation equals the expected value of the $K - 1$ slots plus the personal value of the slot kept by the first buyer. The buyer then offers to sell any one of K groupings of $K - 1$ slots. Each group is offered at a price equal to the expected value of the corresponding bundle minus the minimum expected value of all bundles. The second buyer will obviously purchase a bundle of $K - 1$ slots. Now, assume that the first buyer and the second buyer decide to split any additional benefits through cooperation (which is defined as the difference in the profit if they sold cooperatively versus buyer 2 selling alone) and ask the oracle to compute for every two slots that they kept the expected value of the remaining $K - 2$ slots. They determine bundle valuations as before as the sum of the $K - 2$ slot value plus the personal value of the two slots they keep. And they offer each group of $K - 2$ slots at the value of the corresponding bundle minus the minimum value of all such bundles. This process is repeated until all slots are sold.

It can be seen that the eventual outcome will be the optimal assignment of slots to buyers, see also [Clarke, 7; Milgrom, 16; Vickery, 22]. Furthermore, the seller can sell the whole bundle at some minimum price to the first buyer. Moreover, for computing the bundle values (by acting as the oracle) the seller can obtain a share in the profit from cooperative sales at each step. This procedure does not necessarily maximize the seller's profit but maximizes the total welfare and is therefore welfare optimal. In addition it has three desirable features: the early buyer obtains more of the profit due to his or her participation in all successive auctions—this prevents holdouts, at each stage the buyer at that stage is forced to (partially) reveal his or her type by selecting a bundle, and finally each buyer is made to signal his or her commitment because he or she has to purchase a bundle to get into the remaining auctions. These features make the procedure quite attractive but several simplifications as well as testing for optimality from the seller's viewpoint are necessary before it can be adopted in practice. A restricted version of this mechanism is tested in Section 4.

3. Pricing strategy examples with 2 slots

In this section we analyze a simple example with 2 slots and 2 customers. We compare the net profit to the seller under different "intuitive" pricing and allocation mechanisms. The treatment in Section 2 assumed that the values of the slots took a continuous distribution. Here for ease of exposition, we restrict the values to be distributed according to the discrete distribution shown in Table 1.

Table 1. TOY Example: Distribution of values with 2 slots and 2 customers.

Type	Probability	Value slot 1	Value slot 2
A	0.5	10	6
B	0.5	6	10

Table 2. TOY Problem: Optimal assignment under FBS.

Arrival	Probability	Allocation	Value
AB	0.25	A1 B2	20
AA	0.25	A1 A2	16
BA	0.25	B2 A1	20
BB	0.25	B2 B1	16

Table 3. TOY Problem: FCFSopt when the price vector is (10, 10).

Arrival	Probability	Allocation	Profit to seller
AB	0.25	A1 B2	20
AA	0.25	A1 A0	10
BA	0.25	B2 A1	20
BB	0.25	B2 B0	10

The optimal assignment: First Best Solution (FBS). In this case we assume that the seller has full information about the valuation of slots by the customers. This mechanism is used as a benchmark case to compare the performance of other mechanisms. The optimal allocation is shown in Table 2. The optimal expected profit to the seller is 18. In the notation used in Tables 2–5, the slot allocations to customers under different scenarios is denoted as customer-slot combination. Therefore A1 denotes that buyer A is assigned slot 1. Slot 0 denotes no allocation to the customers.

3.1. First-Come-First-Served with announced prices: FCFSopt

The seller announces a price for each slot. Customers arrive according to a specified distribution. The customers pick the slot that is the most valuable to them from the set of available slots.

The problem for the seller is to determine the price vector that yields the highest expected profit. The solution methodology is simulation and greedy assignment to evaluate the expected profit for the seller. Complete enumeration is necessary to find the price vector that maximizes seller's expected profit. In the next section we also consider a simpler method of pricing slots called "single slot" (or FCFSss) pricing in which the value distribution of a slot is used to price that slot separately (it is not optimal for the seller because the seller does not take into account the effect of the posted price on the buyer behavior). The pricing strategies to be considered are (10, 10), (10, 6), (6, 10), and (6, 6).

1. *Price* (10, 10). The expected profit to the seller is 15 and the expected surplus to customers is zero, see Table 3.
2. *Price* (10, 6). The expected profit to the seller is 13.5 and the expected surplus to the customers is 2, see Table 4.
3. *Price* (6, 6). The expected profit to the seller is 12 and the expected surplus to the customers is 6.

Table 4. TOY Problem: FCFSopt when the price vector is (10, 6).

Arrival	Probability	Allocation	Profit to seller
AB	0.25	A1 B2	16
AA	0.25	A1 A2	16
BA	0.25	B2 A1	16
BB	0.25	B2 B0	6

Thus, we conclude that the optimal price vector is (10, 10) and the maximum expected profit is 15. The expected surplus to customers is zero.

3.2. *Announced prices and After Market (AM)*

In this mechanism the seller announces the prices. Customers arrive, and after every customer has arrived, negotiate amongst themselves to assign the slots in order to maximize their joint surplus. We can prove that such a cooperative outcome is possible—the core is not empty under mild assumptions. The problem is to determine the slot prices, P_k, such that

$$\max E\Big[\sum P_k x_{ik}\Big],$$

where x_{ik} solves

$$W = \max\Big[\sum\sum (V_{ik} - P_k)x_{ik}\Big]$$

$$\text{subject to:}\quad x_{ik} \le 1 \quad \forall i, k,$$

$$\sum_i x_{ik} \le 1,$$

$$\sum_i (V_{ik})x_{ik} \ge P_k.$$

The seller has to determine the profit maximizing price vector. Simulation and an optimal assignment algorithm are used to evaluate the expected profit for the seller. Complete enumeration is necessary to find the optimal (that maximizes seller's expected profit) price vector (denoted by AMopt in the next section). In the TOY Problem the possible price vector combinations are (10, 10), (10, 6), (6, 10), (6, 6). The optimal price is (10, 10) and the expected profit is equal to 15.

The pricing policy and the optimal profit do not change with the addition of the after market in this toy problem. But it can be different if the number of customers or slots is larger or if the slots have a different value distribution. One can conjecture that the secondary market is more useful when the value distribution amongst customers is not symmetric (i.e., for example, in the two slot example the two types of customers have values (10, 5) and (10, 9) respectively for slots 1 and 2).

It is easy to construct a counter example to show that the secondary market can actually reduce the manufacturer's profits. One such example is described here:

Table 5. Results of 5 slot example.

Mechanism	Exp. No. of slots sold	Seller's profit	Buyers' profit	Total profit	% welfare	Slot prices
Optimal allocation	4.00			37.607	100	10, 10, 10, 8, 8
FCFSSopt	3.46	31.613	1.937	33.55	89.2	10, 10, 10, 8, 8
AMopt	3.15	31.57	0	31.575	84.0	10, 10, 10, 10, 10

Assume there are 5 capacity slots to be sold. Customers arrive in a random order order and the total number of customers that arrive is uniformly distributed over integers between 1 and 10. The maximum value a customer has for his preferred slot is 10. Furthermore, there is penalty of 2 times the distance from the optimal slot. So if a customer's preferred slot is slot 4, then her value distribution for the 5 slots is (4, 6, 8, 10, 8). The preferred slot of a customer (type of customer) has a probability distribution of (0.15, 0.35, 0.65, 0.85,1). So a customer's preferred slot is slot 1 with probability (w.p.) 0.15, slot 2 w.p. 0.20, slot 3 with probability 0.3, etc. The results of the simulation are summarized in Table 5. The optimal allocation values are derived by solving an assignment problem for each scenario (i.e., for each realization of customer arrivals and slot preferences).

Based on numerical study of the after market mechanism for several such five slot problems (that exhibit similar outcomes) we summarize that:

1. For the same prices, when compared to FCFSSopt, both greater expected profit accrues to customers and the seller's expected profit is smaller under AMopt.
2. The optimal prices under AMopt are higher compared to the optimal prices under FCFSopt because the seller wants to prevent customers with very low values for slots from occupying them.
3. Expected number of slots sold is lower with AMopt because the seller's prices are higher.
4. Therefore, the after market mechanism is not necessarily beneficial for the seller or from a welfare perspective.

3.3. FCFS with dynamic pricing

Under dynamic pricing the posted prices change as time passes or as slots get sold. By definition dynamic pricing should achieve higher profit than FCFSopt because the pricing policies under FCFSopt are a subset of the dynamic pricing strategies.

In our TOY Problem the dynamic pricing strategy consists of determining two sets of prices: one before the arrival of the first customer and one after the arrival of the first customer. The optimal pricing strategy is to announce a starting price of (10, 10). Once the first customer picks her preferred slot, the price for the other slot should be changed to 6. This strategy yields 16 as the expected profit to the seller.

3.4. Bundle pricing

The seller announces prices for the K slot bundle or individual slots. If a buyer purchases the bundle of K slots, then the buyer in turn announces prices for the K slot bundle (buyer

Table 6. TOY Problem: bundle pricing outcomes.

Arrival	Probability	Allocation	Total value to the first customer
AB	0.25	A1 B2	20
AA	0.25	A1 A2	16
BA	0.25	B2 A1	20
BB	0.25	B2 B1	16

sells his own slot also), prices for $K - 1$ slot bundles (buyer keeps one and sells the rest) and prices for individual slots (buyer stays in the market). The cycle repeats after each subsequent buyer makes his/her decision. The bundle prices are calculated by backward induction beginning with the optimal price for the last slot.

In the TOY Problem the bundle pricing strategy works as follows. The seller sells both slots to the first customer at 18. The first customer purchases the bundle and in turn prices each of the two slots at 10. If the second customer is of the same type as the first customer then the first customer takes the less valuable slot (valued at 6). Otherwise both customers enjoy their preferred slot. The outcomes are shown in Table 5.

Therefore the expected value to the first customer from the purchase of the two slot bundle is 18. Thus, the seller prices the bundle of two slots at 18 and extracts the maximum possible surplus (FBS).

3.5. *Solution to the seller's optimization problem*

A mathematical program can be constructed to determine the seller's ex-post announced prices and allocation rules for type (value) revelation by the customers as shown below. Let α be the set of possible allocations, namely, the set $\{00, 0a, 0b, a0, ab, b0, ba\}$, where the first (second) position stands for the slot allocated to buyer 1 (2) and takes value 0 if no slot is allocated, value a if slot 1 is allocated and b if slot 2 is allocated. Notice that compared to the problem solved in Section 2, we have to expand the set of outcomes to account for no sale(s). Let the type of buyer 1 (2) be denoted as t_1 (t_2). Let $\chi(t_1, t_2, \alpha)$ stand for the probability of the seller choosing the allocation α given that the buyers declare their types as t_1 and t_2. The payment made by buyer i is independent of the other buyer's type, and is given by $M(i, t_i)$. The decision variables are $M()$ and $\chi()$. The data items are the probability of different types, $f(t_1)$ and $f(t_2)$ and the value of the allocation, $v(i, t_i, \alpha)$, $i = 1, 2$, to the buyers. The optimization problem can be written as:

$$\max \sum_{t_1} f(t_1) M(1, t_1) + \sum_{t_2} f(t_2) M(2, t_2) \tag{3}$$

subject to

$$\sum_{t_2} f(t_2) \sum_{\alpha} \big(\chi(t_1, t_2, \alpha) v(1, t_1, \alpha)\big) \geq M(1, t_1) \quad \forall t_1, \tag{4}$$

$$\sum_{t_1} f(t_1) \sum_{\alpha} \big(\chi(t_1, t_2, \alpha) v(2, t_2, \alpha)\big) \geq M(2, t_2) \quad \forall t_2, \tag{5}$$

$$\sum_{t_2} f(t_2) \sum_{\alpha} \big(\chi(t_1, t_2, \alpha) v(1, t_1, \alpha) - M(1, t_1)\big)$$
$$\geq \sum_{t_2} f(t_2) \sum_{\alpha} \big(\chi(t, t_2, \alpha) v(1, t_1, \alpha) - M(1, t)\big) \quad \forall t_1,\ \forall t, \tag{6}$$
$$\sum_{t_1} f(t_1) \sum_{\alpha} \big(\chi(t_1, t_2, \alpha) v(2, t_2, \alpha) - M(2, t_2)\big)$$
$$\geq \sum_{t_1} f(t_1) \sum_{\alpha} \big(\chi(t_1, t, \alpha) v(2, t_2, \alpha) - M(2, t)\big) \quad \forall t_2,\ \forall t, \tag{7}$$
$$0 \leq \chi(t_1, t_2, \alpha) \leq 1, \tag{8}$$
$$\sum_{\alpha} \chi(t_1, t_2, \alpha) = 1, \quad \forall t_1,\ \forall t_2. \tag{9}$$

The formulation maximizes the seller's expected revenue which is computed by adding up the payments over the player types as shown in (3). This is subject to constraints that players should participate, should have the incentive to tell the truth, and that the allocation should be a probability distribution. The participation constraint for buyer 1 is given by (4) and for buyer 2 in (5). These constraints ensure that the expected profit to the buyer of the slot (net of the purchase price) should be positive. The truth revealing constraints for the two buyers are (6) and (7), respectively. These ensure that each buyer makes the maximum profit by revealing her true type. The constraints (8) and (9) ensure that given the player types the allocation is a probability distribution. The allocation can be made deterministic by constraining the variables $\chi(t_1, t_2, \alpha)$ to take either the value zero or the value one.

In the example if the allocation is restricted to be deterministic, i.e., a particular slot if purchased at a given price is allocated to a customer when that customer identifies herself to be of a certain type, then the maximum expected value to the seller is 16. This allocation is welfare optimal but the seller does not extract the full surplus. On the other hand if allocation is allowed to be probabilistic, i.e., the customer is assigned a probability of allocation of particular slots when the customer pays the given price upon announcing type, then the seller can extract the maximum expected profit of 18. This profit corresponds to the Bundle Pricing mechanism and FBS profit. In general, the optimal expected profit can not be derived from the Bundle Pricing mechanism, but the mechanism is quite promising and therefore deserves further study. Moreover, the revelation mechanism's rules are rather complex. Therefore solving large problems can be computationally expensive. In

Table 7. TOY Problem: summary comparison of pricing mechanisms.

Pricing mechanism	Max. expected value to seller	Surplus to buyers
FBS	18	0
FCFSopt	15	0
AMopt	15	0
FCFS dynamic pricing	16	2
Bundle pricing	18	0
Optimal: deterministic alloc.	16	2
Optimal: stochastic alloc.	18	0

Section 4 we present a simple mechanism based on Bundle Pricing and show how it can be implemented.

4. A practical bundle pricing mechanism

The following criteria should be met by a practical allocation and pricing mechanism.

(1) The market mechanism should be implementable:

  (a) Can various types of trades be executed particularly resale by the buyers?
  (b) Do the information requirements of the buyers and sellers match our assumptions?

(2) The computational intensity for the buyers (or sellers) should be manageable:

  (a) Is the number of prices to be evaluated by the buyer small?
  (b) Can the buyers and sellers easily understand the pricing structure? Can they determine the expected value of each alternative when there is demand and value uncertainty?

Both the general bundle pricing and the mathematical programming based optimal revelation mechanism do not satisfy these criteria. The most general dynamic bundle pricing strategies with resale could encompass many of the pricing strategies that were discussed in the previous section. For example, the seller may offer a menu of price for all types of bundles comprising of $K, K-1, \ldots, k, \ldots, 1$ slots. A customer may purchase one of the bundles of k slots and in turn offer a menu of prices for bundles consisting of all combinations of $k, k-1, \ldots, 1$ slots. Therefore the number of bundles in each iteration is $k!$. This will place high computational as well as informational burden on the buyers and thus is not practical.

The optimal selling policy can be determined in principle by solving a mathematical programming problem as described in the previous section. The solution of such a program is not easy to implement because the buyers are expected to signal their type by viewing the price schedule corresponding to each type of buyer. For example, if there are 1024 different types of potential buyers, then there could be 1024 different prices. It is moreover, difficult to express the price that the buyer pays in a closed form expressions such as the one shown in Section 2.2. Thus, we need a simple mechanism that develops on the findings in Section 2.2.

In this section we propose a simplified version of the bundle pricing mechanism. As mentioned previously, the bundle pricing approach seems to have the greatest potential for ease of implementation and for maximizing the seller's profit. In designing a practical mechanism for bundle pricing, we need to keep in mind the following requirements:

1. An intermediary to manage the allocation process might be desirable to ensure fairness and anonymity, and to prevent the strategic use of revealed preferences. We label the intermediary as the "exchange."
2. The buyers know their own preferences and can increase the value to each other (and thus to the seller) by agreeing voluntarily to swap slots if necessary.

3. The seller and the exchange probably have better information regarding the demand for slots and the value distribution over the slots. This information should be effectively communicated to the potential buyers.
4. The exchange is better equipped to compute on behalf of the buyer the implication of different resale policies.

In our scheme, called Bundle Value Based Pricing (BVBP), a buyer buys a bundle of k slots through the exchange. Thereafter she puts all combinations of $k - 1$ slots for sale on the exchange. Therefore, under BVBP each buyer retains exactly one slot and at any given time there is only one owner of the slots on sale. This is the key feature that simplifies bundle pricing.

The function of the exchange is critical. In addition to executing buying and selling transactions, exchange provides information and makes pricing suggestions to buyers and sellers. The exchange charges a percentage of the total dollar sales for its services. It can also provide insurance to buyers of a bundle of slots as a protection against unsold slots in case the number of customers is too small or if buyers are risk averse (see [Agrawal and Seshadri, 2]). The exchange also performs a qualification step to register buyers. This step ensures that buyers have genuine requirements and an ability to pay. Moreover, it also ensures that a buyer does not participate in the purchase process for a given set of slots a multiple times with different identities.

The working of a generic capacity-exchange can be visualized as shown in Figures 1–4. Assume that there are 4 capacity slots (for example, 4 weeks of a given month). The owner (seller) of the time slots puts the slots up for sale on the independent capacity-exchange. The exchange uses historical demand data (or seller supplied distributions) and suggests a pricing strategy to the owner (seller) regarding the optimal price for a bundle consisting of all slots and optimal individual prices for each slot. In the scenario shown in Figure 1 the owner has already sold the four slot bundle to customer 1. Customer 1 has in turn sold slots 1, 3 and 4 to customer 2.

In Figure 1, customer 3 upon logging into the exchange is shown the available individual slots and the bundles of slots that are on sale and their respective prices. Note that slot 2 is unavailable as it is occupied by customer 1. In order for customer 3 to purchase a two slot bundle, this customer needs the following information: the value distribution for the slots and the number of customers that are expected to arrive after customer 3. This is shown in Figure 2. In Figure 3, customer 3 inputs her valuation of individual slots to the exchange and the exchange returns with the value for each bundle and individual slot to the customer. This computation is done by the exchange using the valuation of slots entered by customer 3, the demand data and the bundle pricing algorithm. Customer 3 selects a two slot bundle from the list and in Figure 4 the exchange suggests the best prices to set for the resale of single slots.

We give several examples below that illustrate the efficiency of BVBP. In these examples, a customer has a preferred slot and is willing to pay a premium, say p, for that slot. The customer's valuation of slots other than the preferred slot is the same, denoted as v. All customers are alike except in their personal information regarding which is their preferred slot. To keep the exposition simple, let the distribution of which slot is preferred

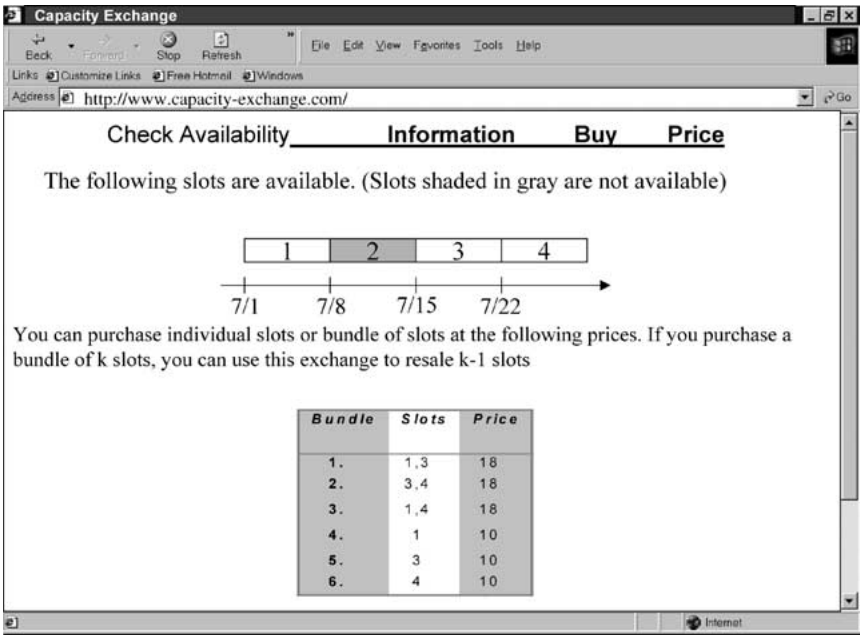

Figure 1. Capacity exchange: check availability and prices page.

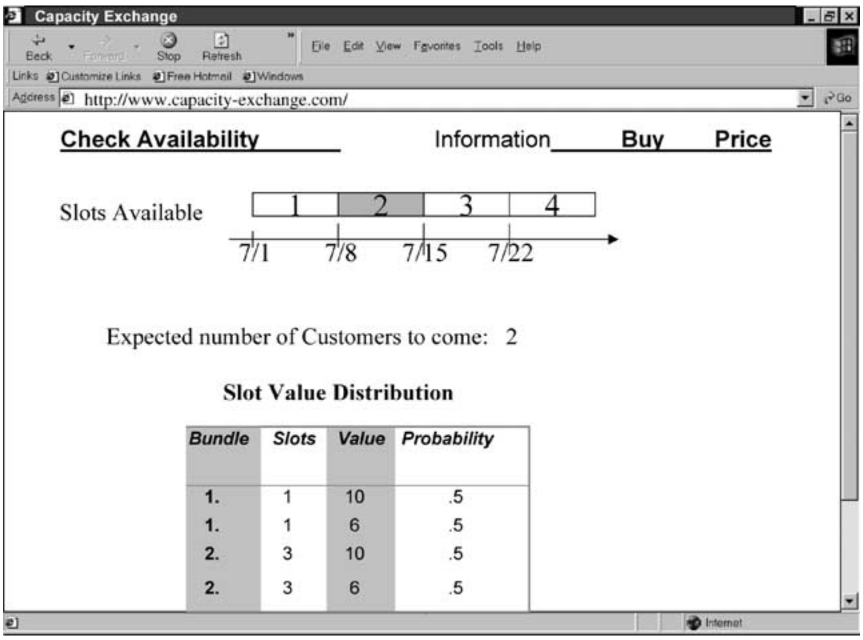

Figure 2. Capacity exchange: market information for a buyer.

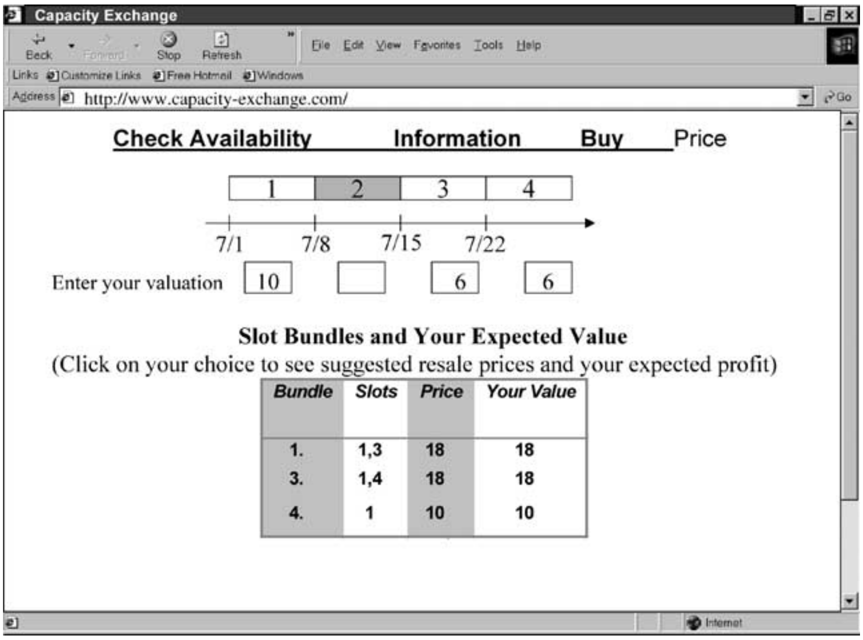

Figure 3. Capacity exchange: determine optimal bundle.

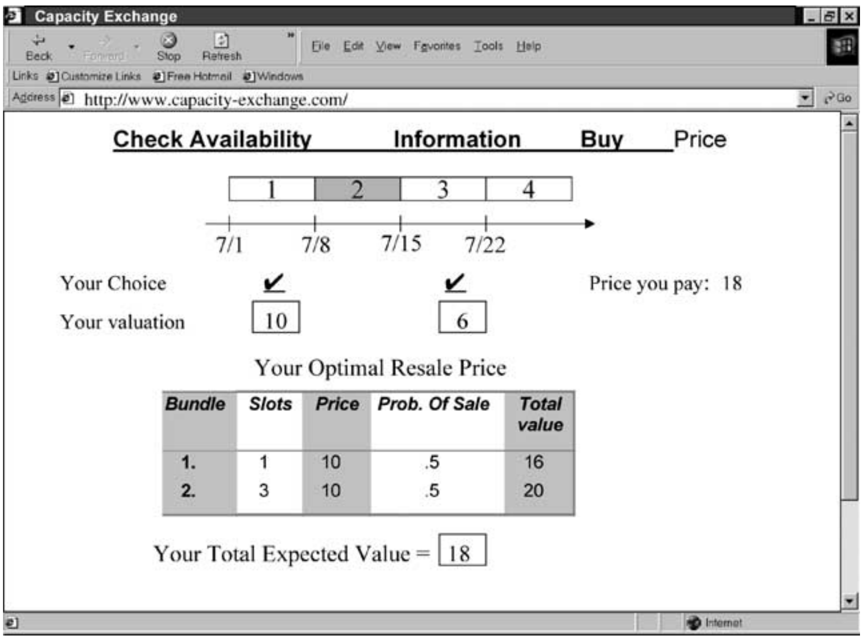

Figure 4. Capacity exchange: pricing of bundles for resale.

be uniform over all the slots that are originally available. Also assume that the number of customers is equal to the number of available slots. Thus, consider an example in which there are two slots, $p = 2$, and $v = 8$. The number of customers is equal to two. They each are equally likely to prefer the first or the second slot. If they prefer slot 1 (2) then they are willing to pay 10 (8) for the first slot and 8 (10) for the second one.

We compare the BVBP policy with five other policies. The first one is the policy in which the seller knows the buyer's preference, allocates slots and charges accordingly, i.e., this corresponds to the first best solution (FBS). The second is one in which the seller announces prices for each slot and buyers purchase them on a first come first served basis. The prices could either be computed optimally for this policy or decided for each slot separately. We denote these two policies as FCFSopt and FCFSss (where opt and ss stands for the optimal and for single slot price, respectively). The last two policies correspond to the after market mechanism. In these the seller announces prices as before but the buyers confer with each other and decide which if any slot to buy. We shall denote the two policies corresponding to the two different methods of determining prices as: AMopt and AMss, where AM denotes after market.

In principle, the optimal BVBP prices can be computed by explicit enumeration of all pricing possibilities. Examples of the logic used in the computation were given in the previous section. For example, in the two slot example the first buyer buys both slots for 19, offers each for 10. Thus, he obtains a profit of 20 w.p. and 18 w.p. 0.5 (this profit includes the value of his own slot), and obtains a value equal to the price of 19. An exhaustive search reveals that the optimal as well as the single slot prices are 10 and 10. When there are more than two slots we use a simpler algorithm to determine the profit under BVBP. This algorithm is based on worst case analysis and is described below. It is interesting to observe that even with more detailed analysis we were unable to increase the profit under BVBP. It is an open question to establish bounds for BVBP under worst case analysis with respect to the optimal pricing policy. The numerical results are shown in Table 8. The worst case analysis of the BVBP policy follows.

The bundle pricing policy for the two slot example was explained in Section 3.4 with $p = 4$ and $v = 6$. We enumerated all possible outcomes and showed why the first customer will buy both slots at 18.

Here is the logic for the BVBP with $p = 2$ and $v = 8$ for 3, 4, and 5 slot problems:

Table 8. Performance of BVBP.

No. of slots	v	$v + p$	FBS	BVBP	FCFSopt	FCFSss	AMopt	AMss
2	8	10	19.00	19.00	16.00	16.00	16.00	16.00
2	6	10	18.00	18.00	15.42	15.42	15.42	15.42
3	8	10	28.23	26.67	24.00	24.00	24.00	24.00
3	6	10	26.48	24.00	21.44	21.44	21.44	21.44
4	8	10	37.43	36.00	32.00	32.00	32.00	32.00
4	6	10	34.89	32.00	27.57	27.57	27.57	27.57
5	8	10	46.72	44.00	40.00	40.00	40.00	40.00
5	6	10	43.47	38.00	33.24	33.24	33.24	33.24

3 slot example. Let $v = 8$. With two remaining slots, say slots 1 and 2, the third customer's valuation of these two slots could be (10, 8) or (8, 10) or (8, 8) each w.p. $1/3$. If the second customer prefers the second slot, then he offers the individual slots at prices of 10 and 8. Thus, he gets an expected return given by 20 w.p. $1/3$ and 16 w.p. $2/3$, or a return of 17 $1/3$. Say the first customer likes the first slot. Then she offers the slot bundle (2, 3) at 17 $1/3$ and the bundles (1, 3) and (1, 2) also at 17 $1/3$. She gets a return of $(52/3 + 8)$ w.p. $1/3$ and $(52/3 + 10)$ w.p. $2/3$. Thus gets $52/3 + 28/3 = 26.67$. Notice that the first customer makes sure that the second one gets his preferred slot. (If $p = 4$ then the last two slots yield 20 w.p. $1/3$ and 12 w.p. $2/3$, or $44/3$. The first customer gets $(44/3 + 10)$ w.p. $2/3$ and $(44/3 + 8)$ w.p. $1/3$ or $44/3 + 28/3 = 72/3 = 24$.)

4 slot example. Let us start with the last two slots. It is now difficult to always make sure that the holder of these slots gets her preferred slot. So assume the worst case that she does not get it. The expected return is lower: $0.75 \cdot 16$ (or $0.75 \cdot 12$) and $0.25 \cdot 18$ (or $0.25 \cdot 16$), thus yielding her a return of 16.5 (or 13). The second buyer gets her preferred slot, and thus obtains 26.5, i.e., $16.5 + 10$ (or 23, i.e., $13 + 10$) because she is unconcerned about the valuation of the third buyer. The first buyer might have to yield to the second buyer ($1/4$ of the time) in order to give the second customer her preferred slot. Thus, the first buyer gets $26.5 + 0.25 \cdot 8 + 0.75 \cdot 10 = 36.00$ (or $23 + 0.25 \cdot 6 + 0.75 \cdot 10 = 32.00$).

5 slot example. The last two slots will now sell for $0.8 \cdot 16$ (or $0.8 \cdot 12$) and $0.2 \cdot 18$ (or $0.2 \cdot 16$) which equals 16.4 (or 12.8). We can not ensure that the third buyer gets his preferred slot, thus we can expect 24.4 (or 18.8) from him. Because we have employed worst case analysis, the second buyer pays 10 more, i.e., 34.4 (or 28.8). The first buyer gets a return of $34.4 + 0.2 \cdot 8 + 0.8 \cdot 10$ (or $28.8 + 0.2 \cdot 6 + 0.8 \cdot 10$) $= 44$ (or 38).

It is clear that BVBP even under worst case analysis is superior to the other pricing mechanisms. Moreover, it is practical and easily understood by buyers and sellers alike.

5. Future research

We have presented several methods for pricing and allocating capacity to customers with time preferences. Barring very special cases, it is difficult to obtain closed form expressions for the optimal prices to charge. We provided the analysis for a special case to illustrate the difficulties involved in determining the optimal mechanism. Many heuristic mechanisms can be used for pricing and allocation of capacity, and we provided a sampling of some of the intuitively appealing ones. We motivated bundle pricing based on the results for the special case. We also argued that bundle pricing offers a practical and effective mechanism. However, bundle pricing in its full generality is difficult to implement. Therefore, we made a key simplification step, and constructed the Bundle Value Based Pricing (BVBP) mechanism that is not only practical but proves to be quite effective in the examples considered by us.

There are several directions that can be pursued in future work. The first involves the analysis of BVBP particularly with regard to determining the optimal prices under different scenarios. Second, it is also important to bound the difference in the expected profit to the seller if BVBP is used instead of more general bundle value based selling procedures or the optimal procedure. Prior to doing this it is necessary to formalize the arguments given at the end of Section 2 for the multiple-object selling procedure. Third, as pointed out by a reviewer, BVBP places excessive burden on the early purchasers. The first buyer for example assumes the risk of not being able to sell the rest of the bundle. Unfortunately, unless the slots are sold as a bundle their value can not be extracted. Moreover, by selling the slots as a bundle the seller gets insured against not being able to sell the slots. The major portion of the risk of no-sale is thus transferred to the early buyers. One way to mitigate this risk is to offer insurance against large losses. The seller or a third party might be able to accomplish this. The fourth involves extending BVBP to cater to different situations. For instance, customers themselves may not know their true values but only have a distribution for it. For example, an auto manufacture who purchases steel on the exchange only has forecasts of steel consumption in different time periods which get updated with increasing accuracy as time progresses. The customer might wish to deviate from the estimates of the exchange with regard to the probability distributions. The customer might wish to purchase an option for purchasing slots that expires if it is not exercised prior to a certain time. In analyzing these generalizations the criteria outlined in this paper will be useful to balance the complexity involved versus the effectiveness of mechanisms.

Appendix: Proof of Proposition 1

We begin by establishing a key simplifying step, which follows from the restriction that each buyer always gets exactly one object.

Lemma 1. In any mechanism in which each buyer always gets exactly one object, the interim assignment functions Q_{iV}, $i = 1, 2$, must satisfy

$$Q_{iV}(a_i, w_i) = Q_{iV}(a_i, 0),$$

for almost all $(a_i, w_i) \in \Theta_1$ such that $a_i > 0$.

Proof of Lemma 1. Since each buyer is always getting exactly one object, we have $Q_{iV}(a_i, w_i) + Q_{iW}(a_i, w_i) = 1$. By Lemma 0, the difference between the surplus any type (a_i, w_i) with $a_i > 0$ and the surplus of type $(0, \tau_m)$ can be written as the integral along the following two paths, both consisting of two line segments. In the (v_i, w_i) space, the first path goes through $(0, \tau_m)$, (a_i, τ_m), and (a_i, w_i), i.e., first horizontally and then along a 45° line:

$$\begin{aligned} S^i(a_i, w_i) - S^i(0, \tau_m) &= \int_{\tau_m}^{a_i+\tau_m} Q_{iV}(\alpha, 0)\, d\alpha \\ &\quad + \int_{\tau_m}^{\tau_m+w_i} \big[Q_{iV}(a_i, y) + Q_{iW}(a_i, y)\big]\, dy \\ &= \int_{\tau_m}^{a_i+\tau_m} Q_{iV}(\alpha, 0)\, d\alpha + w_i; \end{aligned}$$

while the second path goes through $(0, \tau_m)$, $(0, w_i)$, and (a_i, w_i), i.e., first along the 45° diagonal and then horizontally:

$$\begin{aligned} S^i(a_i, w_i) - S^i(0, \tau_m) &= \int_{\tau_m}^{\tau_m + w_i} \big[Q_{iV}(0, y) + Q_{iW}(0, y)\big]\, dy \\ &\quad + \int_{\tau_m}^{a_i + \tau_m} Q_{iV}(\alpha, w_i)\, d\alpha \\ &= w_i + \int_{\tau_m}^{a_i + \tau_m} Q_{iV}(\alpha, w_i)\, d\alpha. \end{aligned}$$

The two equalities above imply

$$\int_{\tau_m}^{a_i + \tau_m} Q_{iV}(\alpha, 0)\, d\alpha = \int_{\tau_m}^{a_i + \tau_m} Q_{iV}(\alpha, w_i)\, d\alpha,$$

which in turn immediately implies the result, since the point (a_i, w_i) was arbitrary.

Writing buyer i's interim expected payment as

$$M^i(a_i, w_i) = a_i Q_{iV}(a_i, w_i) + w_i - S^i(a_i, w_i),$$

by Lemmas 0 and 1 we have

$$M^i(a_i, w_i) = a_i Q_{iV}(a_i, 0) - \int_0^{a_i} Q_{iV}(\alpha, 0)\, d\alpha - S_i(0, \tau_m)$$

for almost all (a_i, w_i) such that $a_i > 0$. Taking the expected value, conditional on $a_i > 0$, yields

$$\begin{aligned} &2 \int_{\Theta_1} M^i(a_i, w_i)\, dK(w_i|a_i)\, dG(a_i) \\ &= 2 \int_0^{\tau_M - \tau_m} a_i Q_{iV}(a_i, 0)\, dG(a_i) \\ &\quad - \int_0^{\tau_M - \tau_m} \int_0^{a_i} Q_{iV}(\alpha, 0)\, d\alpha\, dG(a_i) - S_1(0, \tau_m) \\ &= 2 \int_0^{\tau_M - \tau_m} \left(a_i - \frac{1 - G(a_i)}{g(a_i)}\right) Q_{iV}(a_i, 0)\, dG(a_i) - S^i(0, \tau_m) \\ &= 2 \int_{\Theta_1} \left(a_i - \frac{1 - G(a_i)}{g(a_i)}\right) Q_{iV}(a_i, w_i)\, dK(w_i|a_i)\, dG(a_i) - S^i(0, \tau_m) \\ &= 2 \int_{\Theta_1} z(a_i) Q_{iV}(a_i, w_i)\, dK(w_i|a_i)\, dG(a_i) - S^i(0, \tau_m), \end{aligned}$$

where K denotes the conditional c.d.f. of w_i, given a_i, $z(a_i) \equiv a_i - (1 - G(a_i))/g(a_i)$, and the second equality is obtained with an integration by parts.

Expanding the assignment function yields

$$
\begin{aligned}
&2\int_{\Theta_1} M^i(a_i, w_i)\, dK(w_i|a_i)\, dG(a_i)\\
&\quad = 2\int_{\Theta_1}\int_{\Theta_1} z(a_i)q_{iV}(a_i, w_i, a_{3-i}, w_{i3-i}) f(a_i + w_i, a_i)\\
&\qquad\qquad \times f(a_{3-i} + w_{3-i}, a_{3-i})\, dw_i\, da_i\, dw_{3-i}\, da_{3-i},
\end{aligned}
$$

which, given the symmetry of the density f, is buyer i's expected payment to seller. The total expected revenue is

$$
\begin{aligned}
&\int_{\Theta_1} M^1(a, w)\, d\Gamma(w|a)\, dG(a) + \int_{\Theta_1} M_2(b, y)\, dK(y|b)\, dG(b)\\
&\quad = -S^1(0, \tau_m) - S^2(0, \tau_m) + 2\int_{\Theta_1\times\Theta_1} \big[z(a)q_{1V}(a, w, b, y) + z(b)q_{2V}(a, w, b, y)\big]\\
&\qquad\qquad \times f(a + w, a) f(b + y, b)\, dw\, dy\, da\ db.
\end{aligned}
$$

Clearly, it is optimal to set both $S^1(0, \tau_m)$ and $S^2(0, \tau_m)$ to zero, and maximizing (pointwise) the last expression with respect to q_V^1 and q_V^2, subject to $q_V^1 + q_V^2 = 1$, yields

$$
q_{1V}^{**}(a, w, b, y) = 1 - q_{2V}^{**}(a, w, b, y) = 1 \quad \text{if } z(a) > z(b).
$$

By Lemma 0, these assignment functions, with the corresponding payment functions as determined by the envelope condition (EC), are also an optimal solution for the seller if the implied surplus functions S^i are convex. Since the function $z(\cdot)$ is assumed to be nondecreasing, the assignment functions $q_{ij}^{**}(a, w, b, y)$ coincide with the ex post efficient ones. It is optimal to assign the objects according the buyers' differences in values: i.e., object V is assigned to buyer i if $a_i = \max\{a_1, a_2\}$.

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