Strategic Allocation:  
The Role of Corporate Bond Indices?

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Appendix A Portfolio Allocations

A.1 Proof of Theorem 1

The generalized Bellman equation for this problem is

\[
0 = \sup_{\alpha_t} \left\{ J_t + J_W \left[ \alpha_t^\top (\mu - r_t) + r \right] W + \frac{1}{2} J_W W^2 \alpha_t \sigma \sigma^\top \alpha_t + J_W \Pi \pi_t + \frac{1}{2} J_W \Pi^2 \sigma \sigma^\top + J_X \mu_X + \frac{1}{2} \text{tr} \left( J_{XX} \sigma_X \sigma_X^\top \right) + J_W X W \sigma_X \sigma^\top \alpha_t + J_{W\Pi} W \sigma_{\Pi} \sigma_{\Pi} + J_{X\Pi} X \sigma_{\Pi} \sigma_{\Pi}^\top \right\} \tag{A.1}
\]

with the boundary condition \( J(W(T), \Pi(T), X(T), T, T) = U(W(T)/\Pi(T)) \). The first order conditions of optimality from the Bellman equation, that describe the optimal allocation as function of the value function, are

\[
\alpha = \left( -\frac{J_W}{W J_{WW}} \right) (\sigma \sigma^\top)^{-1} (\mu - r_t) + (\sigma \sigma^\top)^{-1} (\sigma \sigma_X^\top) \left( -\frac{J_{W\Pi}}{W J_{WW}} \right) + \Pi (\sigma \sigma^\top)^{-1} (\sigma \sigma_{\Pi}^\top) \left( -\frac{J_{W\Pi}}{W J_{WW}} \right) \tag{A.2}
\]

Substituting the first-order conditions into the Bellman equation (A.1), we get the following partial differential equation in \( J \):

\[
0 = J_t + J_W \Pi \pi_t + \frac{1}{2} J_W \Pi^2 \sigma \sigma^\top + J_X \mu_X + \frac{1}{2} \text{tr} \left( J_{XX} \sigma_X \sigma_X^\top \right) + J_{X\Pi} X \sigma_{\Pi} \sigma_{\Pi}^\top

+ J_W X W \sigma_X \sigma^\top \left[ \left( -\frac{J_W}{W J_{WW}} \right) (\sigma \sigma^\top)^{-1} (\mu - r_t) + (\sigma \sigma^\top)^{-1} (\sigma \sigma_X^\top) \left( -\frac{J_{W\Pi}}{W J_{WW}} \right) + \Pi (\sigma \sigma^\top)^{-1} (\sigma \sigma_{\Pi}^\top) \left( -\frac{J_{W\Pi}}{W J_{WW}} \right) \right]

+ J_{W\Pi} W \Pi \left[ \left( -\frac{J_W}{W J_{WW}} \right) (\mu - r_t)^\top (\sigma \sigma^\top)^{-1} + \left( -\frac{J_{W\Pi}}{W J_{WW}} \right) (\sigma \sigma_X^\top)^{-1} \left( \sigma \sigma^\top \right) \left( -\frac{J_{W\Pi}}{W J_{WW}} \right) \right]

+ J_W W \left[ \left\{ \left( -\frac{J_W}{W J_{WW}} \right) (\mu - r_t)^\top (\sigma \sigma^\top)^{-1} + \left( -\frac{J_{W\Pi}}{W J_{WW}} \right) (\sigma \sigma_X^\top)^{-1} \left( \sigma \sigma^\top \right) \right\} + \Pi \left( -\frac{J_{W\Pi}}{W J_{WW}} \right) (\sigma \sigma_{\Pi}^\top)^{-1} (\mu - r_t) + r \right]

+ \frac{1}{2} J_W W^2 \left[ \left( -\frac{J_W}{W J_{WW}} \right) (\mu - r_t)^\top + \left( -\frac{J_{W\Pi}}{W J_{WW}} \right) (\sigma \sigma_X^\top) + \Pi \left( -\frac{J_{W\Pi}}{W J_{WW}} \right) (\sigma \sigma_{\Pi}^\top) \right] (\sigma \sigma^\top)^{-1}

\left[ \left( -\frac{J_W}{W J_{WW}} \right) (\mu - r_t) + (\sigma \sigma_X^\top) \left( -\frac{J_{W\Pi}}{W J_{WW}} \right) + \Pi (\sigma \sigma_{\Pi}^\top) \left( -\frac{J_{W\Pi}}{W J_{WW}} \right) \right] \tag{A.3}
\]
Trying a functional form for $J$ of the form (14), the above partial differential equation collapses to the partial differential equation for $F$:

$$
0 = \frac{1}{1-\gamma} F_t - F\pi_t + \frac{2-\gamma}{2} F\sigma_{\Pi T} + \frac{1}{1-\gamma} F_{XX} + \frac{1}{2} \frac{1}{1-\gamma} \text{tr} \left( F_{XX}\sigma_X\sigma_T^\top \right) - F_{XX}\sigma_X\sigma_T^\top \\
+ F_X\left( \sigma_X\sigma_T^\top \right) \left[ \frac{1}{\gamma} \left( \sigma_T^\top \right)^{-1} (\mu - \nu) + (\sigma_T^\top)^{-1} (\sigma_X) \right] \left( \frac{F_X}{\gamma F} \right) + \left( \sigma_T^\top \right)^{-1} (\sigma_{\Pi T}) \left( \frac{\gamma - 1}{\gamma} \right) \\
+ (\gamma - 1) F \left[ \frac{1}{\gamma} (\mu - \nu)^T (\sigma_T^\top)^{-1} + \left( \frac{F_X}{\gamma F} \right) (\sigma_X^\top)^T (\sigma_T^\top)^{-1} + \left( \frac{\gamma - 1}{\gamma} \right) (\sigma_{\Pi T})^T (\sigma_T^\top)^{-1} \right] \sigma_{\Pi T} \\
+ F \left[ \left( \frac{1}{\gamma} (\mu - \nu)^T (\sigma_T^\top)^{-1} + \left( \frac{F_X}{\gamma F} \right) (\sigma_X^\top)^T (\sigma_T^\top)^{-1} + \left( \frac{\gamma - 1}{\gamma} \right) (\sigma_{\Pi T})^T (\sigma_T^\top)^{-1} \right) (\mu - \nu) + r \right] \\
- \frac{\gamma}{2} F \left[ \frac{1}{\gamma} (\mu - \nu)^T + \left( \frac{F_X}{\gamma F} \right) (\sigma_X^\top)^T + \left( \frac{\gamma - 1}{\gamma} \right) (\sigma_{\Pi T})^T \right] (\sigma_T^\top)^{-1} \left[ \frac{1}{\gamma} (\mu - \nu) + (\sigma_X) \left( \frac{F_X}{\gamma F} \right) \right] \\
+ \left( \sigma_{\Pi T} \right) \left( \frac{\gamma - 1}{\gamma} \right). \tag{A.4}
$$

Substituting in the first-order conditions the functional form for $J$ (14), we get the optimal allocation, and optimal consumption expressions shown in Theorem 1. \(\Box\)

### A.2 Proof of Corollary 1

By substituting in (A.4), $F$ of the form

$$
F(X(t), t, T) = \exp \left( \frac{1}{2} X_t^\top B_3(\tau) X_t + B_2(\tau) X_t + B_1(\tau) \right), \tag{A.5}
$$

where $\tau = T - t$, we can collapse the partial differential equation (A.4) into a set of ordinary differential equations for the matrices $B_3$, $B_2$, and $B_1$. This way we can verify that our guess about $H$ is correct, and also define it through matrices $B_3$, $B_2$, and $B_1$. In what follows we use the notation $\mu(t) - r(t) = \phi_0 + \phi_1 X_t$. Substitute the above equation for $F$, and (16), in (A.4), and collect terms for $X_t^\top [:\cdot] X_t$, $[\cdot \cdot] X_t$, and 1. Then the ordinary differential equations that define the $B_3$, $B_2$, and $B_1$ matrices in Corollary (1) are the following:

$$
B_3'(\tau) = \frac{B_3(\tau) + B_3(\tau)^T}{2} \left[ \sigma_X\sigma_X^\top + \frac{1 - \gamma}{\gamma} (\sigma_X^\top)^T (\sigma_T^\top)^{-1} (\sigma_X) \right] \frac{B_3(\tau) + B_3(\tau)^T}{2} \\
\quad + (B_3(\tau) + B_3(\tau)^T) \left[ A_1 + \frac{3(1 - \gamma)}{2\gamma} (\sigma_X^\top)^T (\sigma_T^\top)^{-1} \phi_1 \right] - \frac{1 - \gamma}{2\gamma} \phi_1^T (\sigma_T^\top)^{-1} (\sigma_X) (B_3(\tau) + B_3(\tau)^T) \\
\quad + \frac{1 - \gamma}{\gamma} \phi_1^T (\sigma_T^\top)^{-1} \phi_1 \tag{A.6}
$$
the state variables, and the inflation.

The estimation, Section 3.

The economy is affine, i.e. we have not yet imposed any restrictions on the matrices. This is done in
the expected excess returns on the assets ($\pi(t)$), the matrices that define the instantaneous expected return on inflation ($\pi_0$, $\pi_1$), the matrices that define the short term interest rate ($\nu_0$, $\nu_1$), and the volatility matrices of the assets, the state variables, and the inflation.

Another important point from the above equations is that all that matters to the investor are
the expected excess returns on the assets ($\mu(t) - r(t)\epsilon$), and the “real” short term interest rate
$r(t) - \pi(t)$.

In the above equations we have not imposed any structure in the economy other than that the
economy is affine, i.e. we have not yet imposed any restrictions on the matrices. This is done in
the estimation, Section 3. $\square$

\[ B'_2(\tau) = B_2(\tau) \left[ \sigma_X \sigma_X^\top + \frac{1-\gamma}{\gamma} (\sigma_X \sigma_X^\top \sigma_X \sigma_X^\top)^{-1} \right] \frac{B_3(\tau) + B_3(\tau)^\top}{2} \]

\[ + \left[ A_0^\top + (1-\gamma) \left[ \frac{1}{\gamma} \phi_0^\top + \frac{(\gamma-1)}{\gamma} (\sigma_{\Pi})^\top \right] (\sigma_X \sigma_X^\top)^{-1} \left( \phi_0^\top \right) \left[ (\nu_1 - \pi_1) \right] \]

\[ \frac{B_3(\tau) + B_3(\tau)^\top}{2} \]

\[ B'_1(\tau) = B_2(\tau) \left[ \frac{1}{2} \sigma_X \sigma_X^\top + \frac{1-\gamma}{2\gamma} (\sigma_X \sigma_X^\top \sigma_X \sigma_X^\top)^{-1} \right] \frac{B_3(\tau) + B_3(\tau)^\top}{2} \]

\[ + \frac{1}{4} \text{tr} \left( (B_3(\tau) + B_3(\tau)^\top) \sigma_X \sigma_X^\top \right) \]

\[ + \left[ A_0 + (1-\gamma) (\sigma_{\Pi})^\top \sigma_X \sigma_X^\top \sigma_X \sigma_X^\top)^{-1} \left[ \frac{1}{\gamma} \phi_0^\top + \frac{(\gamma-1)}{\gamma} (\sigma_{\Pi})^\top \right] \left( - (1-\gamma) \sigma_X \sigma_X^\top \right] \]

\[ + \left[ (\nu_0 - \pi_0) \frac{2-\gamma}{2} \sigma_{\Pi} \sigma_{\Pi}^\top + \frac{1}{2\gamma} \phi_0^\top (\sigma_X \sigma_X^\top)^{-1} \phi_0 + \frac{(\gamma-1)}{\gamma} (\sigma_{\Pi})^\top (\sigma_X \sigma_X^\top)^{-1} \phi_0 \]

\[ + \frac{(\gamma-1)^2}{2\gamma} \left( \sigma_{\Pi} \sigma_{\Pi}^\top \sigma_X \sigma_X^\top \right) \right] \]

\[ (A.7) \]

with the boundary conditions

\[ B_3(0) = 0_{4\times 4} \]

\[ B_2(0) = 0_{1\times 4} \]

\[ B_1(0) = 0 \]

\[ (A.9) \]

\[ (A.10) \]

\[ (A.11) \]
Appendix B  Indirect utility for sub-optimal strategies.

It is easy to see that any strategy that can arise as solution of our portfolio optimization problem for any parametrization \( \Theta \) can be written as a linear function of the state vector, with deterministic coefficients depending on the horizon length \( \tau := T - t \),

\[
\hat{\alpha}(t, \tau) := \hat{\alpha}(X(t), t, T) = \hat{\alpha}_0(T - t) + \hat{\alpha}_1(T - t) X(t). \tag{B.1}
\]

The optimal allocation for any parametrization \( \hat{\Theta} \) takes the form (17), where matrices \( \hat{B}_3, \hat{B}_2, \hat{B}_1 \) are given by the equations in Appendix 1 using parameters \( \hat{\Theta} \). In all generality, \( \hat{\alpha}_0(\tau) \) and \( \hat{\alpha}_1(\tau) \) are given by,

\[
\hat{\alpha}_0(\tau) = \frac{1}{\gamma} (\hat{\sigma} \hat{\sigma}^\top)^{-1} \hat{\phi}_0 + \left(1 - \frac{1}{\gamma}\right) (\hat{\sigma} \hat{\sigma}^\top)^{-1} (\hat{\sigma} \hat{\sigma}_\Pi^\top) + \frac{1}{\gamma} (\hat{\sigma} \hat{\sigma}^\top)^{-1} (\hat{\sigma} \hat{\sigma}_X^\top) \left(\hat{B}_2(\tau)^\top\right) \tag{B.2}
\]

\[
\hat{\alpha}_1(\tau) = \frac{1}{\gamma} (\hat{\sigma} \hat{\sigma}^\top)^{-1} \hat{\phi}_1 + \frac{1}{\gamma} (\hat{\sigma} \hat{\sigma}^\top)^{-1} (\hat{\sigma} \hat{\sigma}_X^\top) \left(\frac{\hat{B}_3(\tau) + \hat{B}_3(\tau)^\top}{2}\right). \tag{B.3}
\]

The indirect utility for any strategy, defined as \( \hat{J}_t = \mathbb{E}_t^{\hat{\Theta}}[U(W(T)/\Pi(T))] \) where the expectation is taken with respect to the correct beliefs/model, is a martingale and therefore has a zero drift. From the Markov property, it is a function of \( t, W(t), \Pi(t), X(t) \) and \( T \). Thus, the indirect utility corresponding to strategy \( \hat{\alpha} \) must satisfy the partial differential equation:

\[
\hat{J}_t + \mathcal{L} \hat{J} = 0, \tag{B.4}
\]

where \( \mathcal{L} \) is the infinitesimal generator based on the correct beliefs/model \( \Theta \) of \( \hat{J} \) given by:

\[
\mathcal{L} \hat{J} = \hat{J}_W W (\hat{\alpha}^\top (\mu - \rho \tau) + r) + \hat{J}_X \mu X + \hat{J}_\Pi \Pi \pi + \hat{J}_W W \sigma X \sigma^\top \hat{\alpha}(t, \tau) + \hat{J}_W W \Pi \sigma \sigma^\top \Pi + \hat{J}_X \Pi X \sigma \sigma^\top \Pi + \hat{J}_X \Pi X \sigma \sigma^\top X + \hat{J}_X \Pi \Pi X \sigma \sigma^\top X + \frac{1}{2} \hat{J}_W W W^2 \hat{\sigma} \sigma^\top \hat{\sigma} + \frac{1}{2} \hat{J}_W W \Pi^2 \Pi \sigma \sigma^\top \Pi + \frac{1}{2} \text{tr}(\hat{J}_X X \sigma \sigma^\top X) \tag{B.5}
\]

It follows from the partial differential equation (B.4) that indirect utility takes the form:

\[
\hat{J}(W(t), \Pi(t), X(t), t, T) = \frac{1}{1 - \gamma} \left(\frac{W(t)}{\Pi(t)}\right)^{1-\gamma} \hat{F}(X(t), t, T),
\]

where \( \hat{F}(X(t), t, T) \) satisfies the partial differential equation

\[
\hat{F}_t + (1 - \gamma) \hat{F} \left( \hat{\alpha}(t, \tau) (\mu - \rho \tau) + r - \pi - \frac{\gamma}{2} \hat{\alpha}(t, \tau)^\top \sigma \sigma^\top \hat{\alpha}(t, \tau) + \frac{2 - \gamma}{2} \sigma \sigma^\top \Pi - (1 - \gamma) \hat{\alpha}(t, \tau)^\top \sigma \sigma^\top \Pi \right) + \hat{F}_X \left( A_0 + A_1 X(t) + (1 - \gamma) \sigma X \sigma^\top \hat{\alpha}(t, \tau) - (1 - \gamma) \sigma X \sigma^\top \Pi + \frac{1}{2} \text{tr}(\hat{F}_X X \sigma \sigma^\top X) = 0. \tag{B.6}
\]

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When the trading strategy can be expressed as (B.1), it follows from (B.6) that \( \hat{F}(X(t), t, T) \) is exponential quadratic:

\[
\hat{F}(X(t), t, T) = \exp \left\{ \frac{1}{2} X(t)^\top \Gamma_3(t) X(t) + \Gamma_2(t) X(t) + \Gamma_1(t) \right\}.
\]

where \( \Gamma_3(t), \Gamma_2(t), \) and \( \Gamma_1(t) \) satisfy the following system of ordinary differential equations:

\[
\begin{align*}
\Gamma_3'(t) &= \frac{\Gamma_3(t) + \Gamma_3(t)^\top}{2} \sigma_X \sigma_X^\top \Gamma_3(t) + \frac{\Gamma_3(t)^\top}{2} + (\Gamma_3(t) + \Gamma_3(t)^\top) \left[ A_1 + (1 - \gamma) \sigma_X \sigma^\top \hat{\alpha}_1(t) \right] \\
&\quad + (1 - \gamma) \left[ 2\hat{\alpha}_1(t)^\top \phi_1 - \gamma \hat{\alpha}_1(t)^\top \sigma \sigma^\top \hat{\alpha}_1(t) \right] \quad \text{(B.7)}
\end{align*}
\]

\[
\begin{align*}
\Gamma_2'(t) &= \Gamma_2(t) \sigma_X \sigma_X^\top \left( \frac{\Gamma_3(t) + \Gamma_3(t)^\top}{2} \right) + \Gamma_2(t) \left[ A_1 + (1 - \gamma) \sigma_X \sigma^\top \hat{\alpha}_1(t) \right] \\
&\quad + \left[ A_0^\top + (1 - \gamma) \hat{\alpha}_0(t)^\top \sigma \sigma^\top - (1 - \gamma) \sigma \sigma^\top \hat{\alpha}_1(t) \right] \frac{\Gamma_3(t) + \Gamma_3(t)^\top}{2} \\
&\quad + (1 - \gamma) \left[ \hat{\alpha}_0(t)^\top \phi_1 + \nu_1 - \pi_1 + \phi_0^\top \hat{\alpha}_1(t) - (1 - \gamma) \sigma \sigma^\top \hat{\alpha}_1(t) - \gamma \hat{\alpha}_0(t)^\top \sigma \sigma^\top \hat{\alpha}_1(t) \right] \quad \text{(B.8)}
\end{align*}
\]

\[
\begin{align*}
\Gamma_1'(t) &= \frac{1}{2} \Gamma_2 \sigma_X \sigma_X^\top \Gamma_2^\top + \Gamma_2(t) \left[ A_0 + (1 - \gamma) \sigma_X \sigma^\top \hat{\alpha}_0(t) - (1 - \gamma) \sigma_X \sigma^\top \hat{\alpha}_1(t) \right] \\
&\quad + \frac{1}{2} \text{tr} \left( \frac{\Gamma_3(t) + \Gamma_3(t)^\top}{2} \sigma_X \sigma_X^\top \right) + (1 - \gamma) \left[ \hat{\alpha}_0(t)^\top \phi_0 + \nu_0 - \pi_0 - (1 - \gamma) \sigma^\top \hat{\alpha}_0(t) \right] \\
&\quad - \frac{\gamma}{2} \hat{\alpha}_0(t)^\top \sigma \sigma^\top \hat{\alpha}_0(t) + \frac{2 - \gamma}{2} \sigma \sigma^\top \hat{\alpha}_0(t) \right] \quad \text{(B.9)}
\end{align*}
\]

with the boundary conditions

\[
\begin{align*}
\Gamma_3(0) &= 0_{4 \times 4} \quad \text{(B.10)} \\
\Gamma_2(0) &= 0_{1 \times 4} \quad \text{(B.11)} \\
\Gamma_1(0) &= 0 \quad \text{(B.12)}
\end{align*}
\]
Appendix C  Estimation Methodology

C.1 Discretization of Continuous Time Systems

We demonstrate the method to estimate the system with only three assets for the sake of space. This method can be extended straightforwardly to estimate the whole system with all the corporate bonds at once. This is the method followed for our results shown in the tables in the paper.

Consider the augmented state vector

\[ \hat{X}(t) = \begin{bmatrix} X(t) \\ \log S(t) \\ \log B_G(t) \\ \log B_C(t) \\ \log \Pi(t) \end{bmatrix} \]

The continuous time dynamics of this vector are defined by

\[ d\hat{X}(t) = (\kappa_0 + \kappa_1 \hat{X})dt + \sigma \hat{X} dz, \]

where

\[
\kappa_1 = \begin{bmatrix} A_1 & 0_{4 \times 4} \\ \mu_{1,S} & 0_{1 \times 4} \\ \mu_{1,G} & 0_{1 \times 4} \\ \mu_{1,C} & 0_{1 \times 4} \end{bmatrix}, \quad \kappa_0 = \begin{bmatrix} A_0 \\ \mu_{0,S} - \frac{1}{2} \sigma_S \sigma_S^T \\ \mu_{0,G} - \frac{1}{2} \sigma_G \sigma_G^T \\ \mu_{0,C} - \frac{1}{2} \sigma_C \sigma_C^T \end{bmatrix}, \quad \sigma = \begin{bmatrix} \sigma_X \\ \sigma_S \\ \sigma_G \\ \sigma_C \end{bmatrix}
\]

Applying Ito’s lemma to the process \( e^{-\kappa_1 t} \hat{X}_t \), it follows that:

\[ \hat{X}(T) = e^{\kappa_1 (T-t)} \hat{X}_t + \int_t^T e^{\kappa_1 (T-s)} \kappa_0 ds + \int_t^T e^{\kappa_1 (T-s)} \sigma \hat{X} dz(s). \]

Which shows that \( \hat{X}_T \) is normally distributed conditional on \( \hat{X}_t \). The last equation can be written in general as

\[ \hat{X}(T) = C_0 + C_1 \hat{X}_t + \epsilon(T), \]

where \( \epsilon \) are conditionally normal correlated disturbances (\( \epsilon \sim \mathcal{N}(0, V) \)). Equation (C.3) can be estimated as a VAR system, therefore we can obtain estimates of the matrices \( C_0, C_1 \), and the conditional variance-covariance of the disturbances \( V \). The structure of the VAR system will depend on the structure implied by the matrices \( \kappa_1, \kappa_0 \), and \( \sigma \) onto \( C_1 \), and \( C_0 \) and the variance-covariance matrix of the disturbances \( \epsilon \). Next section shows how to convert the discrete time
VAR estimates $C_0$, $C_1$, and the variance-covariance matrix $V$, into estimates of the continuous time VAR system as described in (C.1).

### C.2 Conversion from Discrete Time Variables to Continuous Time Counterparts

Comparing (C.2) and (C.3), we get the following equations:

\[
  C_0 = \int_t^T e^{\kappa_1(T-s)} \kappa_0 \, ds, \quad C_1 = e^{\kappa_1(T-t)}, \quad \text{and} \quad \text{(C.4)}
\]

\[
  Var_t(\epsilon(T)) \equiv V = Var_t \left( \int_t^T e^{\kappa_1(T-s)} \sigma_X \, dz(s) \right). \quad \text{(C.5)}
\]

Going from estimates of $C_0$, $C_1$, and $V$ to estimates of $\kappa_0$, $\kappa_1$, and $\sigma_X$, is in general tedious. A general technique is to solve the equations defined by (C.4) and (C.5) using numerical methods.

The estimation of the full fledged continuous time VAR system was executed using these numerical techniques. However, for the structural VAR, the conversion can be done in analytical closed form. To see that let us first convert the $C_1$ matrix into estimates for the $\kappa_1$ matrix. By exponentiating the $\kappa_1$ matrix we establish that the matrix $H = e^{\kappa_1(T-t)}$ takes the form: Where,

\[
  H(i,i) = e^{-k_i(T-t)}, \quad \text{for all } i=1,...,6. \\
  H(7,i) = \frac{\mu_{1,S}(i)}{k_i} (1 - e^{-k_i(T-t)}), \quad H(7,7) = 1 \\
  H(8,i) = \frac{\mu_{1,C}(i)}{k_i} (1 - e^{-k_i(T-t)}), \quad H(8,8) = 1 \quad \text{(C.6)} \\
  H(9,i) = \frac{\mu_{1,C}(i)}{k_i} (1 - e^{-k_i(T-t)}), \quad H(9,9) = 1 \\
  H(10,i) = \frac{\pi_{1}(i)}{k_i} (1 - e^{-k_i(T-t)}), \quad H(10,10) = 1
\]

From this equation we can completely define the $\kappa_1$ matrix.\(^1\) Next, the constant vector $G = \begin{bmatrix} X, r_S, r_G, r_C, I \end{bmatrix}^\top$.

\(^1\)We estimated the VAR system of the six state variables and the returns on the assets and the inflation. This system is equivalent to (C.3), since the ones in matrix (C.6) allow us to take the logarithms of the asset prices and the logarithm of the price level on the left hand side, and essentially create the equivalent system where the variables are log-differences, or returns $\hat{X} = [X, r_S, r_G, r_C, I]^\top$. 


\[ \int_t^T e^{\kappa_1(T-s)} \kappa_0 \, ds \text{ takes the form:} \]

\[ G(i, i) = \frac{1}{k_i} (1 - e^{-k_i(T-t)}), \quad \text{for all } i=1,\ldots,6. \]

\[ G(7, i) = \frac{\mu_1.S(i)}{k_i} (T - t - \frac{1}{k_i} (1 - e^{-k_i(T-t)})), \quad G(7, 7) = T - t \]  

\[ G(8, i) = \frac{\mu_1.G(i)}{k_i} (T - t - \frac{1}{k_i} (1 - e^{-k_i(T-t)})), \quad G(8, 8) = T - t \]  

\[ G(9, i) = \frac{\mu_1.C(i)}{k_i} (T - t - \frac{1}{k_i} (1 - e^{-k_i(T-t)})), \quad G(9, 9) = T - t \]

\[ G(10, i) = \frac{\pi_1(i)}{k_i} (T - t - \frac{1}{k_i} (1 - e^{-k_i(T-t)})), \quad G(10, 10) = T - t \]

Finally the conditional variance-covariance matrix \( V \) is given as a function of the \( \hat{\sigma}_X \) matrix through the following equality:

\[
\text{Var}_t(\hat{X}(T)) = E_t \left[ \left( \int_t^T e^{\kappa_1(T-u)} \sigma_X d\tau_u \right) \left( \int_t^T e^{\kappa_1(T-u)} \sigma_X d\tau_u \right)^\top \right] = E_t \left[ \int_t^T e^{\kappa_1(T-u)} \sigma_X \sigma_X^\top e^{\kappa_1(T-u)}^\top \, du \right] = \int_t^T e^{\kappa_1(T-u)} \sigma_X \sigma_X^\top e^{\kappa_1(T-u)}^\top \, du. \quad \text{(C.8)}
\]

Doing more algebra like we did above, we are able to derive 36 different equations that define the elements of \( V \) as functions of the 36 elements of \( \sigma_X \). Solving sequentially this system of equations we were able to get estimates of the 36 elements of the volatility matrix \( \sigma_X \). Then using these estimates and the estimates for \( \kappa_1 \) derived by \( C_1 \) and matrix (C.6), we use matrix (C.7) and \( C_0 \) to obtain estimates for the vector \( \kappa_0 \). Even though algebraically intense, this method is able to give us analytical closed form estimates for the continuous time VAR system, given estimates of the discrete time VAR system, without incurring any computational errors due to numerically solving non-linear equations. For the full system estimated in the paper the matrix is consisted of 120 parameters resulting to 120 equations that all of them are solved with zero error.

Alternatively, and especially useful when we cannot obtain simple closed forms of the above matrices, we could employ numerical methods in solving the equations. In that the following manipulations, in forming the conditional expectation and conditional variance, are particularly handy decreasing the computational error in the equations solving algorithms.

Assume that \( \kappa_1 \) is diagonalizable. Let \( U \) be such that

\[ \kappa_1 = U D U^{-1}, \quad D \text{ diagonal.} \]
From the definition of the matrix exponential and (C.2), it follows that

$$E_t\left(\hat{X}(T)\right) = e^{\kappa_1(T-t)}\hat{X}(t) + \left(\int_t^T U e^{D(T-s)}U^{-1} ds\right)\kappa_0.$$ 

Note that $e^{D(T-t)} = \left(e^{d_i(T-t)}\right)_i$. Performing the integration element-by-element produces:

$$E_t\left(\hat{X}(T)\right) = e^{\kappa_1(T-t)}\hat{X}(t) + U\left(f(d_i, T-t)\right)_i U^{-1}\kappa_0.$$ 

where

$$f(d_i, T-t) = \begin{cases} -\frac{1}{d_i}(1 - e^{d_i(T-t)}) & d_i \neq 0 \\ T - t & d_i = 0 \end{cases}$$

This completes the derivation of the conditional mean.

Let $\Omega = U^{-1}\hat{X}^\top\hat{X}(U^{-1})^\top$. Then, equation C.8 can be re-written as:

$$\text{Var}_t(\hat{X}(T)) = \int_t^T U e^{D(T-u)}\Omega e^{D(T-u)}U^\top du = U \left[g(d_i, d_j, T-t)\Omega\right]_{i,j} U^\top,$$

where

$$g(d_i, d_j, T-t) = \begin{cases} -\frac{1}{d_i+d_j}(1 - e^{(d_i+d_j)(T-t)}) & d_i \neq 0 \text{ or } d_j \neq 0 \\ T - t & d_i = d_j = 0 \end{cases}$$

This completes the derivation of the conditional variance-covariance matrix.