Rollover Risk and Market Freezes

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Abstract

We present a model that can explain a sudden drop in the amount of money that can be borrowed against an asset, even in the absence of asymmetric information or fears about the value of the collateral. Three features of the model are essential: (i) the debt has a much shorter tenor than the assets and needs to rolled over frequently; (ii) in the event of default by the borrower, the collateral is sold by the creditors and there is a (small) liquidation cost; (iii) a significant fraction of the potential buyers of the collateral also relies on short-term debt finance. Under these conditions, the debt capacity of the assets (the maximum amount that can be borrowed using the securities as collateral) can be much less than the fundamental value, and in fact, equal the minimum possible value of the asset. This is true even if the fundamental value of the assets is currently high. In particular, a small change in the fundamental value of the assets can be associated with a sudden collapse in the debt capacity. The crisis of 2007-09 was characterized by just such a sudden freeze in the market for short-term, asset-backed financing.

J.E.L. Classification: G12, G21, G24, G32, G33, D8.

Keywords: financial crisis, credit risk, liquidation cost, secured borrowing, repo, asset-backed commercial paper.
1 Introduction

1.1 Motivation

One of the many striking features of the crisis of 2007-09 has been the sudden freeze in the market for the rollover of short-term debt. From an institutional perspective, the inability to borrow overnight against high-quality but long-term assets was a market failure that effectively led to the demise of a substantial part of investment banking in the United States. More broadly, it led to the collapse, in the United States, the United Kingdom, and other countries, of banks and other financial institutions that had relied on significant maturity mismatch between assets and liabilities, and, in particular, on the rollover of short-term wholesale debt in the asset-backed commercial paper (ABCP) and overnight sale and repurchase (repo) markets.

In this paper, we are interested in developing a model of a sudden collapse in the ability to borrow short-term against long-lived assets in the absence of obvious problems of asymmetric information or fears about the value of collateral. We refer to this phenomenon as a “market freeze.” More precisely, a market freeze occurs when the debt capacity, the maximum amount of collateralized borrowing that can be supported by an asset, is a small fraction of the fundamental value, the economic value measured by the NPV of the stream of returns. An extreme form of a market freeze occurs when the fundamental value is close to the maximum possible value of the asset and the debt capacity is close to the minimum possible value of the asset. We develop a model of debt capacity and provide sufficient conditions for the occurrence of this extreme form of market freeze.

Three assumptions are crucial for our results:

(i) the debt has a much shorter tenor than the assets and needs to be rolled over frequently;

(ii) in the event of default by the borrower, the collateral is sold by the creditors and there is a (small) liquidation cost;

(iii) a significant fraction of the potential buyers of the collateral also relies on short-term debt finance.

We take these features as given, without attempting to rationalize them as the result of equilibrium behavior. For example, we take the (short) tenor of the debt as exogenous. There is ample empirical evidence that financial institutions relied heavily on short-term finance prior to the crisis, but we do not attempt to explain why this was so.\footnote{Using data on outstanding repurchase agreements of the US primary dealers (source: Federal Reserve Bank of New York), Morris and Shin (2009) document that during 2003-2007, term repo remained steady} We also take
as given the liquidation costs incurred by the sellers of the assets. More precisely, we assume
the costs are either fixed or proportional to debt capacity, but in either case the costs are
exogenous. Finally, it is important to note that the tenor of the debt and the liquidation
costs are assumed to be the same for all market participants. In particular, in our benchmark
model, as the tenor of the debt becomes shorter for the owner of the asset, it also becomes
shorter for the potential buyers. In an extension, we allow for buy-to-hold investors; our
results continue to hold as long as the likelihood of finding such buyers is sufficiently small.

In efficient markets, the debt capacity of an asset is equal to its NPV or “fundamental”
value. If there are liquidation costs and a positive probability of default, the debt capacity
would naturally be expected to be somewhat less than the fundamental value. In our model
we can derive the much stronger result that the gap between the fundamental value and the
debt capacity can be significantly large even if the liquidation cost incurred in the event of
default is tiny. More precisely, when the tenor of the debt is sufficiently short, other things
being equal, the debt capacity can equal the minimum possible future value of the asset.

The intuition for the result is as follows. When the tenor of the debt is short, the
probability of receiving good news about the asset before the next roll-over date is very
small. Then it is very likely that the next refinancing will be undertaken with the same
information as in the current period. The maximum amount that can be borrowed without
a substantial risk of default is equal to the debt capacity at the next roll-over date, assuming
good news does not arrive in the interim. The borrower will find it optimal to avoid a
substantial risk of default because he wants to avoid the liquidation costs. This means that
today’s debt capacity is less than or equal to the debt capacity at the next rollover date.
Applying this argument repeatedly shows that today’s debt capacity must be less than or
equal to the debt capacity at the maturity of the assets, assuming no arrival of good news.

We have described the market freeze as resulting from the lack of arrival of good news
about the fundamental value of the debt when the tenor of the debt is very short and
constant. We can also interpret the market freeze as resulting from a sudden shortening in
the tenor of the debt. If the arrival of bad news, that perhaps signals a small change in the
fundamental value of the assets, also causes lenders to restrict the tenor of the debt they
are willing to hold, the fall in the debt capacity will be substantial as we have characterized.
Thus, it is not necessary to assume that banks choose short-term finance from the outset.
The freeze may result from lenders suddenly shortening the tenor of the paper they are
willing to hold.\footnote{This interpretation was suggested to us by Arvind Krishnamurthy.}

Our model captures some of the elements of the collapse in short-term asset-backed

\footnote{around $1.5 trillion, whereas overnight repo contracts doubled from $1.5 trillion to $3 trillion; both shrank
by over a trillion dollars by 2009. Acharya, Schnabl and Suarez (2009) show that outstanding ABCP had
typically a maturity of less than one week and rose from $650 billion to over $1.2 trillion between 2004 and
2Q 2007, only to revert back to its 2004 level by 1Q 2009.}
financing witnessed during the crisis of 2007-09. The first such collapse occurred in the summer of 2007. While many special purpose vehicles financed by ABCP had purchased liquidity guarantees from third parties, the providers of these guarantees were themselves feared to be under-capitalized. The money market funds that provided ABCP thus faced the risk of liquidating assets, many of which were asset-backed securities that had little trading liquidity. Acharya, Schnabl and Suarez (2009) document a similar phenomenon in the case of the case of bank-sponsored conduits. Goldsmith-Pinkham and Yorulmazer (2010) analyze a similar episode in which financing of long-term mortgages with short-term wholesale debt led to the near failure of Northern Rock in the United Kingdom in September 2007. The failure of Bear Stearns due to sudden fall in its ability to roll over overnight repo financing in mid-March 2008 is another example of a market freeze. In his analysis of the failure of Bear Stearns, the Federal Reserve Chairman Ben Bernanke observed that “repo markets could be severely disrupted when investors believe they might need to sell the underlying collateral in illiquid markets” (Remarks to the Risk Transfer Mechanisms and Financial Stability Workshop at the Bank for International Settlements, May 29, 2008).

In addition to helping us understand this recent past, our model may suggest ways to increase the stability of the financial system. Understanding the causes of market freezes is a necessary step toward creating a more stable and efficient financial system for the future. Following the crisis, the parallel (“shadow”) banking system, consisting of special purpose vehicles such as SIVs and conduits, securities lending, repo financing etc., has shrunk significantly and reduced the financial system’s lending capacity by several trillion dollars. While some of this collapse was driven by concerns regarding the quality of the assets, liquidity issues relating to the heavy reliance of a large part of the financial sector on short-term rollover debt also played an important role. Restoring the parallel banking system is seen by many as an important step in the reconstruction of the financial system to provide credit.

3 There was a reduction in the rollover of ABCP, the cost of rolling over rose from 10 basis points relative to the Federal Funds rate prior to August 7, 2007, to over 150 bps, and many conduits had to be taken back by banks onto their balance-sheets.

4 Northern Rock had a balance-sheet featuring significant maturity mismatch. Soon after Northern Rock’s woes, other UK banks such as HBOS, Alliance and Leicester, and Bradford and Bingley, that had relied primarily on short-term wholesale debt, suffered too.

5 Bear Stearns relied day-to-day on its ability to obtain short-term finance through repo borrowing. At this time, Bear was reported to be financing $85 billion of a pool of assets, mostly mortgage- and asset-backed securities, on the overnight market (Cohan, 2009). Beginning late Monday, March 10, even though Bear Stearns continued to have high quality collateral, counterparties became unwilling to lend on customary terms, likely fearing the cost of liquidating the collateral in an illiquid market. At the end of the week, the Federal Reserve stepped in and helped arrange a takeover bid by J.P. Morgan Chase. (Securities and Exchange Commission, 2008).

6 In addition, many of these assets are now held directly by central banks or by commercial and investment
financing of long-term assets in order to avoid the instability of the past.

The rest of the paper is organized as follows. Section 2 provides an introduction to the model and results in terms of a simple numerical example. Section 3 derives the main result for the special case of the model with two states. It also illustrates, in terms of a numerical example, that market freezes can occur even if the debt maturity is not as “short” as our main result requires. Section 4 provides a complete characterization of the debt capacity for the general model and extends the limit result to an arbitrary number of states. The proof of the limit result is relegated to Appendix A. Section 5 discusses the related literature. Section 6 concludes.

2 Model and results

In this section, we introduce the essential ideas in terms of a numerical example. For concreteness, consider the case of a bank that wishes to repo an asset. The question we ask is: What is the maximum amount of money that the bank can borrow using the asset as collateral? There are two ways to interpret this exercise. We can imagine that a value maximizing bank is trying to maximize its return on equity by minimizing the amount of capital needed to finance the assets it owns. In this case, every bank that purchases the asset is assumed to have the same motive for maximizing leverage. Alternatively, we can simply see our exercise as establishing a bound on the amount that can be borrowed, assuming that other buyers in the market are limited by a similar bound.

Time is represented by the unit interval [0, 1]. The asset is purchased at the initial date \( t = 0 \). The asset has a finite life (e.g., mortgages) which we normalize to one unit. To keep the analysis simple, we assume that the asset has a terminal value at \( t = 1 \), but generates no income at the intermediate dates \( 0 \leq t < 1 \). We also assume that the risk-free interest rate is 0 and that all market participants are risk neutral.

The arrival of information is modeled as a continuous-time stochastic process. For simplicity, let us assume that there are two states, a low state \( L \) and a high state \( H \). At any point in time, the state is publicly observed. Transitions between states are governed by a stationary Markov chain. Transition probabilities depend on the period of time during which the transition occurs, but not the dates. If we are considering a transition during the...
period $[0, t]$, the transition probability matrix is denoted by

$$
P(t) = \begin{bmatrix} P_{LL}(t) & P_{LH}(t) \\ P_{HL}(t) & P_{HH}(t) \end{bmatrix},$$

where $P_{LH}(t)$ is the probability of a transition between the low state at time 0 and the high state at time $t$ and $P_{HL}(t)$ is the probability of the transition from the high state at time 0 and the low state at time $t$. We assume that the transition matrix takes the form

$$P(t) = e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!},$$

where the matrix $A$ is the generator. Since the transition probabilities in any state must sum to 1 the rows of $A$ must sum to 0. The crucial feature of the transition matrix is that the probability of a change of state converges to 0 as $t \to 0$. That is, as $t \to 0$, $P(t) \to I$, where $I$ is the identity matrix.

The terminal value of the asset depends on the state of the economy at the terminal date $t = 1$. The terminal value of the asset is $v^H$ in the high state and $v^L$ in the low state, where $v^H > v^L > 0$.

We assume that the asset will be financed by debt that has to be rolled over repeatedly. The debt is assumed to have a fixed maturity, denoted by $0 < \tau < 1$, so that the debt must be rolled over $N$ times, where

$$\tau = \frac{1}{N + 1}.$$  

The unit interval is divided into intervals of length $\tau$ by a series of dates denoted by $t_n$ and defined by

$$t_n = n\tau, \quad n = 0, 1, ..., N + 1,$$

where $t_0$ is the date the asset is purchased, $t_n$ is the date of the $n$-th rollover (for $n = 1, ..., N$), and $t_{N+1}$ is the final date at which the asset matures and the terminal value is realized. This time-line is illustrated in Figure 1.

If the bank is forced to default, the lenders will seize and liquidate the collateral. In this event, we assume that the lenders incur a small liquidation cost, so that the net amount recovered is a fraction $\lambda \in [0, 1]$ of the sale price. This assumption has several components. In the first place, it implies that the seized collateral is liquidated, i.e., sold to another buyer. Secondly, the new buyers are also finance constrained so that the sale price is equal to the maximum amount of finance that can be raised using the asset as collateral. Thirdly, the process of seizing and disposing of collateral is not costless. For concreteness, we can think
of the liquidation cost $1 - \lambda$ as a transaction cost (legal costs, commissions, fees, time delay, etc.), although other interpretations are possible (see Pedersen, 2009, for a discussion of variety of transactions costs and illiquidity in markets). Note that similar results could be obtained with a fixed liquidation cost (see the online appendix).

As an example, suppose that a bank has borrowed 90 and, when it comes time to refinance, finds that it can only raise 87 using the assets as collateral. The lender, say a Money Market Fund, cannot hold the collateral and is forced to dispose of it. A finance constrained buyer can borrow 87 using the assets as collateral, so this is the maximum that it can pay for the assets. However, the amount received by the lender will be a smaller amount, say, 86, because transaction costs have to be subtracted from the sale price.

It is crucial for our argument that the recovery rate $\lambda$ is applied to the sale price rather than to the fundamental value of the assets. If the buyer of the assets were a wealthy investor who could buy and hold the assets until maturity, the fundamental value would be the relevant benchmark. The investor might well be willing to pay some fraction of the fundamental value, although he would presumably try to get the assets for less, recognizing the lender’s eagerness to dispose of the collateral. What we are assuming here, by contrast, is that the buyer of the assets is another financial institution that must also issue short-term debt in order to finance the purchase. (We discuss an extension to allow for the presence of buy-to-hold investors in Section 2.2. Also see the online appendix). Hence, the buyer is constrained by the same forces that determined the debt capacity in the first place. Note that the buyer’s subjective valuation of the assets might be much greater than the debt capacity, but the finance constraint prevents him from offering to pay his full value.

### 2.1 A numerical example

To illustrate the method of calculating debt capacity in the presence of rollover risk, we use the following parameter values: the recovery rate is $\lambda = 0.90$, the tenor of the repo is $\tau = 0.01$, the values of the asset are $v^H = 100$ and $v^L = 50$ in the high and low states, respectively, and the generator is

$$
A = \begin{bmatrix}
-8.0 & 8.0 \\
0.1 & -0.1
\end{bmatrix}.
$$

The transition probability matrix for an interval of unit length can be calculated to be

$$
P(1) = \begin{bmatrix}
0.01265 & 0.98735 \\
0.01234 & 0.98766
\end{bmatrix}.
$$

At time 1, the fundamental values are 100 in state $H$ and 50 in state $L$ by assumption. So the fundamental values at time 0 can be calculated by using the terminal values and the
transition probabilities in the matrix \( P \) (1). The fundamental value in state \( H \) at time 0 is
\[
V^H_0 = 0.98766 \times 100 + 0.01234 \times 50 = 99.383
\]
since, starting in state \( H \) at time 0, there is a probability 0.98766 of being in state \( H \) and a probability 0.01234 of being in state \( L \) at time 1. Similarly, the fundamental value in state \( L \) at time 0 is
\[
V^L_0 = 0.98735 \times 100 + 0.01265 \times 50 = 99.367.
\]
Note that the fundamental values are nearly identical. In spite of this, we shall find that the debt capacity of the asset, defined to be the maximum amount that can be borrowed using the asset as collateral, can be very different in the two states.

Whereas the fundamental value only depends on the state, debt capacity is determined by equilibrium in the repo market and has to be calculated for every one of the dates, \( t_0, \ldots, t_{99} \), at which repo contracts mature. To do this, we first have to calculate the transition probabilities over an interval of length \( \tau = 0.01 \), that is, the length of the period between rollover dates. We find that
\[
P(0.01) = \begin{bmatrix} 0.92315 & 0.07685 \\ 0.00096 & 0.99904 \end{bmatrix}.
\]
Notice that the initial state has a much larger impact on the transition probabilities in \( P(0.01) \) than it does in \( P \) (1). For example, the probability of ending up in state \( H \) after an interval 0.01 has passed is almost 1 if you start in state \( H \) but is close to 0.077 if you start in state \( L \). This is because the interval is so short that the state is unlikely to change before the next rollover date.

Consider now the debt capacities at the last rollover date \( t_{99} = 0.99 \). In what follows, we let \( D \) denote the face value of the debt issued and denote the optimal value of \( D \) at date \( t_n \) in state \( s \) by \( D_n^s \). It is never optimal to choose \( D > 100 \) because this leads to default in both states, with associated liquidation costs, but without any increase in the payoff. For values of \( D \) between 50 and 100 or less than 50, the expected value of the debt is increasing in \( D \) holding constant the probability of default. Then it is clear that the relevant face values of debt (\( D \)) to consider are 50 and 100. For any other face value we could increase \( D \) without changing the probability of default.

If we set \( D = 50 \), the debt can be paid off at date 1 in both states and the expected value of the payoff is 50. So the market value of the debt with face value 50 is exactly 50.

Now suppose we set \( D = 100 \). There will be default in state \( L \), but not in state \( H \), at time \( t = 1 \). The payoff in state \( H \) will be 100 but the payoff in state \( L \) will be \((0.9) 50 = 45.0\), because the recovery rate after default is 0.90. The market value of the debt at time \( t_{99} \) will depend on the state at time \( t_{99} \), because the transition probabilities depend on the state.
We can easily calculate the expected payoffs in each state:

\[
\text{state } H : \ 0.99904 \times 100 + 0.00096 \times 0.9 \times 50 = 99.947; \\
\text{state } L : \ 0.07685 \times 100 + 0.92315 \times 0.9 \times 50 = 49.226.
\]

For example, if the state is \( H \) at date \( t_{99} \), then with probability 0.99904 the state is \( H \) at date 1 and the debt pays off 100 and with probability 0.00096 the state is \( L \) at date 1, the asset must be liquidated and the creditors only realize 45.

Comparing the market values of the debt with the two different face values, we can see that the optimal face value will depend on the state. In state \( H \), the expected value of the debt when \( D = 100 \) is 99.947 > 50, so it is optimal to set \( D_{99}^H = 100 \). In state \( L \), on the other hand, the expected value of the debt with face value \( D = 100 \) is only 49.226 < 50, so it is optimal to set the face value \( D_{99}^L = 50 \). Thus, if we use the notation \( B_n^s \) to denote the debt capacity in state \( s \) at date \( t_n \), we have shown that \( B_{99}^H = 99.947 \) and \( B_{99}^L = 50 \).

Next, consider the debt capacities at date \( t_{98} \) = 0.98. Now, the relevant face values to consider are 50 and 99.9470 (since these are the maximum amounts that can be repaid in each state at date \( t_{99} \) without defaulting and incurring the associated liquidation costs).

If \( D = 50 \), the expected payoff is 50 too, since the debt capacity at date \( t_{99} \) is greater than or equal to 50 in both states and, hence, the debt can always be rolled over. In contrast, if \( D = 99.947 \), the debt cannot be rolled over in state \( L \) at date \( t_{99} \) and the liquidation cost is incurred. Thus, the expected value of the debt depends on the state at date \( t_{98} \):

\[
\text{state } H : \ 0.99904 \times 99.9470 + 0.00096 \times 0.9 \times 50 = 99.894, \\
\text{state } L : \ 0.07685 \times 99.9470 + 0.92315 \times 0.9 \times 50 = 49.222.
\]

Comparing the expected value corresponding to different face values of the debt, we see that the optimal face value is \( D_{98}^H = 99.947 \) in state \( H \) and \( D_{98}^L = 50 \) in state \( L \), so that the debt capacities are \( B_{98}^H = 99.894 \) and \( B_{98}^L = 50 \). In fact, we did not really need to do the calculation again to realize that \( B_{98}^L = 50 \). The only change from the calculation we did at \( t_{99} \) is that the payoff in state \( H \) has gone down, so the expected payoff from setting \( D = 99.947 \) must have gone down too and, \textit{a fortiori}, the optimal face value of the debt must be 50.

It is clear that we can repeat this argument indefinitely in state \( L \). At each date \( t_n \), the debt capacity in the high state is lower than it was at \( t_{N+1} \) and the debt capacity in the low state is the same as it was at \( t_{N+1} \). These facts tell us that if it is optimal to set \( D_{n+1}^L = 50 \) at \( t_{n+1} \), then \textit{a fortiori} it will be optimal to set \( D_n^L = 50 \) at date \( t_n \). Thus, the debt capacity is equal to 50 at each date \( t_n \), including the first date \( t_0 = 0 \).

What is the debt capacity in state \( H \) at \( t_0 \)? The probability of staying in the high state from date 0 to date 1 is \((0.99904)^{100} = 0.90842 \) and the probability of hitting the low state at some point is \( 1 - 0.90842 = 0.09158 \) so the debt capacity at time 0 is

\[
B_0^H = 0.90842 \times 100 + 0.09158 \times 0.9 \times 50 = 94.9603.
\]
So the fall in debt capacity occasioned by a switch from the high to the low state at time 0 is $94.963 - 50 = 44.963$ compared to a change in the fundamental value of $99.383 - 99.367 = 0.016$. This fall is illustrated sharply in Figure 2, which shows that, while fundamental values in states $H$ and $L$ will diverge sharply at maturity, they are essentially the same at date 0. Nevertheless, debt capacity in state $L$ is simply the terminal value in state $L$. Thus, a switch to state $L$ from state $H$ produces a sudden drop in debt capacity of the asset.

— Figure 2 here —

### 2.2 Discussion

The intuition for the market freeze result can be explained in terms of the tradeoff between the costs of default and the face value of the debt. Suppose we are in the low information state at date $t_n$. If the period length $\tau$ is sufficiently short, it is very likely that the information state at the next rollover date $t_{n+1}$ will be the low state. Choosing a face value of the debt greater than $B^L_{n+1}$, the maximum debt capacity in the same state at date $t_{n+1}$, will increase the payoff to the creditors if the state switches to $H$ at the next date, but it will also lead to default if the state remains $L$. Since there is a liquidation (transaction) cost, issuing debt with face value greater than the debt capacity is always unattractive if the probability of switching to state $H$ is sufficiently small. Then, the best the borrower can do is to issue debt with a face value equal to the debt capacity assuming the state remains $L$. But this implies that the debt capacity in the low state is $v^L$ at every date. In other words, no matter how high the fundamental value is in state $L$, the borrower is forced to act as if the asset is only worth $v^L$ in order to avoid default.

In the remainder of this section, we consider the role of the different assumptions of the model in driving the limit result on market freezes.

**Credit risk** If $v^H = v^L$, the terminal value of the asset is equal to the fundamental with probability one, so we can set the face value of the debt equal to $v^H = v^L$ without any risk of default. In this case, the debt capacity must be equal to the fundamental value regardless of any other assumptions. So one necessary assumption is the existence of credit risk, that is, a positive probability that the terminal value of the asset will be less than the initial fundamental value. However, this credit risk can be arbitrarily small, as we illustrated in the numerical example where, at time 0, the probability that the asset’s terminal value is 50 is less than 0.01. We could obtain the same results for even smaller values of credit risk at the cost of increasing the number of rollovers.

**Liquidation cost** We need a liquidation cost in order to have a market freeze. If the recovery ratio is $\lambda = 1$, then regardless of the credit risk, the debt capacity will equal the
fundamental value. To see this, simply put the face value of the debt equal to 100 at each date. The market value of the debt will equal the fundamental value of the asset, which must equal the debt capacity. So a necessary condition of the market freeze is \( \lambda < 1 \). The liquidation cost does not need to be large, however. In the numerical example, the loss ratio was 0.1 and it could be made even smaller with an appropriate reduction in the maturity of the debt.

**Debt finance** Among the key assumptions of our model, we take as given that asset purchases are entirely debt-financed, not just for the initial owner of the assets but for all potential buyers. In particular, this assumption rules out the presence of any long-term or buy-to-hold investors. However, this assumption can be relaxed.

Suppose that, when assets are being liquidated, the buyer found by the liquidating creditors is, with probability \( 1 - \beta \), short-term debt financed and, with probability \( \beta \), he is financed by long-term debt or equity. We can think of the buyer with long-term finance as a buy-to-hold investor, such as Warren Buffett, who is willing and able to pay a fraction, possibly 100%, of the fundamental value. Hence, \( 1 - \beta \) can be interpreted as a proxy for the leverage of the financial sector as a whole; conversely, \( \beta \) captures the capital in the financial sector that is freely available for arbitrage purposes, including for asset purchases at fire-sale prices. With this modification, we show in the online appendix that a market freeze occurs under the usual assumptions if the probability \( \beta \) is not too large. Intuitively, if liquidating creditors are certain to find a buyer who can pay the fundamental value of the asset, then our backward induction mechanism fails and there can be no market freeze. However, if such buyers are scarce, because the extent of free long-term capital in the financial sector is limited, then most buyers are also short-term debt financed and our mechanism is back at work. We also show in the online appendix that our main result on the sharp drop in the debt capacity of the asset and the market freeze is consistent with a system-wide leverage that is greater than 10, which can be interpreted as the probability of meeting a buy-to-hold investor \( \beta \) being less than 10%.

**Short-term debt** As a practical matter, many financial firms are indeed funded with short-term rollover debt. There exist agency-based explanations in the literature (for example, Flannery, 1986, Diamond, 1989, 1991, 2004, Calomiris and Kahn, 1991, and Diamond and Rajan, 2001a, 2001b) for the existence of short-term debt as optimal financing in such settings. In contrast to this literature, Brunnermeier and Oehmke (2009) consider a model where a financial institution is raising debt from multiple creditors and argue that there may be excessive short-term debt in equilibrium as short-term debt issuance dilutes long-term debt values and creates among various creditors a “maturity rat race.” Other reasons for the use of short-term debt are the attraction of betting on interest rates if bankers have
short-term horizons and choose to shift risk (see, for example, Allen and Gale, 2000, and Acharya, Cooley, Richardson and Walter, 2009).

**Rollover frequency** We have highlighted the role of rollover risk and indeed our main result requires that the rate of refinancing be sufficiently high in order to obtain a market freeze. Figure 3 illustrates the role of rollover frequency on debt capacity in state $L$ by varying the number of rollovers as $N = 10, 50$ and $100$. Debt capacity with just 10 rollovers is over 90, but falls rapidly to just above 60 with 50 rollovers, and 100 rollovers are sufficient to obtain the limiting result that debt capacity is the terminal value of 50 in state $L$.

--- Figure 3 here ---

Even if the period length is longer than our result requires, so that it is optimal to set the face value greater than the debt capacity (in the same state at the next rollover date), it is still possible that a market freeze occurs, as we show with a numerical example in Section 3.

**Information structure** The crucial property of the information structure is that $P(\tau) \to I$ as $\tau \to 0$, that is, the probability of a change in state in any rollover period gets smaller as the period length gets smaller. Since the number of rollovers $N$ determines the period length $\tau$, in fact, $\tau = \frac{1}{N+1}$, as the number of rollovers increases, $\tau$ gets smaller and information arrives slowly relative to rollovers.

Note that we do not make any special assumptions about the generator $A$. In particular, we can impose a substantial amount of symmetry if desired. For example, the information state can be a symmetric random walk with reflecting barriers. The only essential property is that the probability of a change in states converges to zero as the period length converges to zero.

### 3 Debt capacity with two states

In this section we provide a proof for the market freeze result when there are two states. We make the same assumptions as for the numerical example but the parameters are otherwise arbitrary. For the time being, we treat the tenor of the commercial paper $\tau$ and the number of rollovers $N$ as fixed. Later, we will be interested to see what happens when the tenor $\tau$ becomes very small and the number of rollovers $N$ becomes correspondingly large.

There are two states, a “low” state $L$ and a “high” state $H$. Transitions occur between the rollover dates $t_n$ and are governed by a stationary transition probability matrix

$$P(\tau) = \begin{bmatrix} 1 - q(\tau) & q(\tau) \\ p(\tau) & 1 - p(\tau) \end{bmatrix},$$
where \( p(\tau) \) is the probability of a transition from state \( H \) at time \( t_n \) to state \( L \) at time \( t_{n+1} \) and \( q(\tau) \) is the probability of a transition from state \( L \) at time \( t_n \) to state \( H \) at time \( t_{n+1} \). The one requirement we impose on these probabilities is that the shorter the period length \( \tau \), the more likely it is that there is no change in states before the next rollover date:

\[
\lim_{\tau \to 0} p(\tau) = \lim_{\tau \to 0} q(\tau) = 0.
\]

The terminal value of the asset is \( v_H \) if the terminal state is \( H \) and \( v_L \) if the terminal state is \( L \), where \( 0 < v_L < v_H \).

In the numerical example, we saw that it was optimal to choose a low face value of the debt in the low state and a high face value of the debt in the high state. Here we will provide necessary and sufficient conditions under which it will be optimal, in terms of maximizing debt capacity, to choose high and low face values in the high and low states, respectively.

We begin by considering the low state.

**The low state** Suppose that the economy is in the low state at date \( t_N \), which is the last of the rollover dates. Let \( D \) be the face value of the debt issued by the bank. If \( D > v_H \), the bank will default in both states at date \( t_{n+1} \) and the creditors will receive \( \lambda v_H \) in the high state and \( \lambda v_L \) in the low state. Clearly, the market value of the debt at date \( t_N \) would be greater if the face value were \( D = v_H \), so it cannot be optimal to choose \( D > v_H \). Now suppose that the bank issues debt with face value \( D \), where \( v_L < D < v_H \). This will lead to default in the low state at date \( t_{n+1} \) and the creditors will receive \( D \) in the high state and \( \lambda v_L \) in the low state. Clearly, this is dominated by choosing a higher value of \( D \). Thus, either \( D = v_H \) or \( D \leq v_L \). An exactly similar argument shows that it cannot be optimal to choose \( D < v_L \), so we are left with only two possibilities, either \( D = v_H \) or \( D = v_L \). In the first case, the market value of the debt is \( (1 - q(\tau)) \lambda v_L + q(\tau) v_H \) and in the second case it is \( v_L \). A necessary and sufficient condition for the optimal face value \( D_N^L \) to equal \( v_L \) is

\[
(1 - q(\tau)) \lambda v_L + q(\tau) v_H \leq v_L.
\]

(3)

This condition will clearly be satisfied for all \( \tau > 0 \) sufficiently small, but for the time being we will simply assume that (3) is satisfied.

Now suppose that (3) is satisfied and that \( B_{n+1}^L = v_L \) for \( n' = n, \ldots, N \). Consider what happens in the low state at date \( t_n \). By the usual argument, the only candidates for the optimal face value are \( D = v_L \) and \( D = B_{n+1}^H \). If the face value is \( D = v_L \), the creditors will receive \( v_L \) in both states at date \( t_{n+1} \) and the market value of the debt at date \( t_n \) will be \( v_L \). On the other hand, if the face value of the debt is \( D = B_{n+1}^H \), the creditors receive \( B_{n+1}^H \) in

\footnote{To simplify the argument, we are assuming that there is a liquidation cost at date \( t_{N+1} \) even though there is no need to sell the asset at that date. None of the results depend on this.}
the high state and $\lambda v^L$ in the low state, so the market value of the debt at date $t_n$ is

$$(1 - q(\tau)) \lambda v^L + q(\tau) B^H_{n+1} \le (1 - q(\tau)) \lambda v^L + q(\tau) v^H,$$

since $B^H_{n+1} \le v^H$. But (3) implies that $(1 - q(\tau)) \lambda v^L + q(\tau) v^H \le v^L$, so the debt capacity is $B^L_n = v^L$. In fact, this induction argument shows that the debt capacity is $B^L_n = v^L$ for all $n = 1, \ldots, N$.

**The high state** Now consider the high state. Again, our two candidates for the face value of the debt at each date $t_n$ are $B^H_{n+1}$ and $v_L$. Let us assume that at each date $t_n$ the face value of the debt is set equal to the future debt capacity $B^H_{n+1}$, that is, we begin at date $t_N$ by setting $D^H_N = v^H$ and $B^H_N = (1 - p(\tau)) v^H + p(\tau) \lambda v^L$ and then recursively define $D^H_n = B^H_{n+1}$ and

$$B^H_n = (1 - p(\tau)) B^H_{n+1} + p(\tau) \lambda v^L,$$

for $n = 1, \ldots, N - 1$. It can easily be shown by backward induction that $B^H_n \le B^H_{n+1}$ for any $n$, so in order to show that this strategy is optimal, it is necessary and sufficient to show that $B^H_0 \ge v^L$. By repeated substitution we can show that

$$B^H_0 = (1 - p(\tau)) B^H_1 + p(\tau) \lambda v^L$$

$$= (1 - p(\tau)) \{ (1 - p(\tau)) B^H_2 + p(\tau) \lambda v^L \} + p(\tau) \lambda v^L$$

$$= (1 - p(\tau))^2 (B^H_2 - \lambda v^L) + \lambda v^L$$

$$\cdots$$

$$= (1 - p(\tau))^N (v^H - \lambda v^L) + \lambda v^L.$$

Then the optimal face value $D^H_n = B^H_{n+1}$ for all $n$ if and only if

$$(1 - p(\tau))^N (v^H - \lambda v^L) + \lambda v^L \ge v^L$$

or

$$v^H - \lambda v^L \ge \frac{(1 - \lambda) v^L}{(1 - p(\tau))^N}. \quad (4)$$

We have thus proved the following proposition.

**Proposition 1** Define $\{ (B^H_n, D^H_n, B^L_n, D^L_n) \}^N_{n=0}$ by setting

$$D^H_n = B^H_{n+1}, \quad (5)$$

$$B^H_n = (1 - p(\tau)) B^H_{n+1} + p(\tau) \lambda v^L, \quad (6)$$

and

$$D^L_n = B^L_n = v^L, \quad (7)$$

for $n = 1, \ldots, N$. The values defined by (5-7) constitute a solution to the problem of maximizing debt capacity if and only if (3) and (4) are satisfied.
The qualitative properties of the debt capacities characterized in Proposition 1 are the same as in the numerical example in Section 2.1. In the low state, the debt capacity $B_n^L$ is constant and equal to the lowest possible terminal value, $v^L$. The fundamental value of the asset in the low state $V_n^L$ is greater than the debt capacity at every date $t_n$ except at the terminal date, when they are both equal to $v^L$. In the high state, the debt capacity $B_n^H$ is always less than the fundamental value $V_n^H$, except at the terminal date when both are equal to $v^H$. We call this behavior of the debt capacity a "market freeze" since a switch in the information state from high state to the low state can produce a sudden, sharp drop in debt capacity that is much larger than the drop in fundamental value associated with the same switch.

3.1 Satisfying the conditions for a market freeze

In this "calibration," we find ranges of parameter values that will sustain a market freeze in the two-state model we have just been studying. We showed that there were two necessary and sufficient conditions, (3) and (4), for the debt capacity to be minimized by the strategy described in Proposition 1. We begin by considering the constraints imposed by the inequality (4). First, define the parameter $\gamma$ to be the ratio of the liquidation cost in the low state to the difference between the terminal payoffs

$$\gamma = \frac{(1 - \lambda) v^L}{v^H - v^L}.$$ 

We show in the online appendix that a sufficient condition for (4) is

$$\gamma \leq \frac{e^{-a_{HL}}}{1 - e^{-a_{HL}}}$$

where $a_{HL}$ is the $(H, L)$ element from the generator $A$. This relationship is illustrated in the figure below, where the probability $e^{-a_{HL}}$ is measured on the horizontal axis and the upper bound on $\gamma$, call it $\gamma^*$, is on the vertical axis:

--- Figure 4a here ---

Since we are naturally interested in cases where the probability of switching from the high state is low, we focus on the right hand side of the figure, where the range of admissible values of $\gamma$ becomes very large. This figure shows that (4) will be satisfied for a wide range of values.

Next we focus on condition (3) and the range of values that will satisfy it. The analysis of this case is more difficult, because in addition to satisfying (4) we also want to ensure that the fundamental values in the two states are close together at the initial date, $t = 0$, so that
a switch from the high state to the low state has a small effect on the fundamental value, while having a large effect on the debt capacity.

Since we want the probability of remaining in the high state to be high and since condition (4) is easily satisfied in any case, we can simplify the problem by assuming that the high state is an absorbing state, that is, \( a_{HL} = 0 \). This also makes it harder to satisfy (3), so in that sense we are making things difficult for ourselves. With this assumption, the fundamental value in the high state is \( v_H \) and we assume that the fundamental value in the low state at date 0 is (at least) 95% of \( v_H \). We show in the online appendix that a sufficient condition for (3) is

\[
N \geq \left( \frac{1 + \gamma}{\gamma} \right) \ln \left( \frac{v_H - v_L}{v_H - 0.95v_H} \right).
\]

Thus, the minimum number of rollovers needed to sustain a market freeze is a function of two expressions, \( \frac{\gamma}{1 + \gamma} \), a measure of the size of liquidation costs and \( \ln \left( \frac{v_H - v_L}{v_H - 0.95v_H} \right) \), a measure of the difference between the fundamental values in the two states. To illustrate, suppose we let \( v_H = 2v_L \); then

\[
\frac{v_H - v_L}{v_H - 0.95v_H} = 10.
\]

The relationship between \( \gamma \) and the smallest value of \( N \), call it \( N^* \), that will satisfy (8) is graphed in the figure below.

— Figure 4b here —

A few concrete examples will give a better sense of how plausible these values are.

**Example 2** The asset has a maturity of six months and is funded by overnight repos. So the debt must be rolled over approximately 126 times. The fundamental value in the low state is 0.95 of the fundamental value in the high state. In order for the market to freeze in the low state (debt capacity equal to \( v_L \)), the value of \( \gamma \) must be greater than or equal to

\[
\gamma^* = \frac{\ln(10)}{126 - \ln(10)} = 0.018615.
\]

**Example 3** The asset has a maturity of two years and is funded by short term loans that are rolled over weekly. In total the debt must be rolled over 104 times. In order for the market to freeze in the low state, the value of \( \gamma \) must be greater than or equal to

\[
\gamma^* = \frac{\ln(10)}{104 - \ln(10)} = 0.022642.
\]
Example 4 The asset has a maturity of ten years and is funded by one month loans, so the debt must be rolled over 120 times. In order for the market to freeze in the low state, the value of \( \gamma \) must be greater than or equal to

\[
\gamma^* = \frac{\ln(10)}{120 - \ln(10)} = 0.019564.
\]

In other words, in each of these examples, where \( v_H = 2v_L \), it is sufficient to have a liquidation cost that is roughly 1\% of \( v_H \) (2\% of \( v_L \)) in order to support a market freeze in the low state.

3.2 Debt capacity with intermediate rollover risk

We can get similar results even if the period length is not short enough to generate the result stated in Proposition 1. A simple adaptation of the numerical example will illustrate a scenario in which it is optimal to choose a high face value of debt in the low state, with the result that the bank faces a positive probability of default if the economy remains in the low state. Suppose that the value of the asset in the low state is \( v^L = 40 \). All the other parameters remain the same. Now the loss from default in the low state is less than the gain from a high face value in the high state, so it is optimal for the face value of the debt to be set equal to next period’s debt capacity in the high state.

As before, we calculate the debt capacity, beginning with the last rollover date. The last rollover date is \( t_{99} \). The transition probabilities are given by equation (2) as before. If the face value of the debt is set equal to \( v_H = 100 \) in the low state, the market value of the debt issued will be \( 0.07685 \times 100 + 0.92315 \times 0.90 \times 40 = 40.918 \), which is higher than the face value obtained by setting the face value equal to 40. Thus, the optimal face value implies default if the economy remains in the low state. It is still optimal to set the face value of the debt equal to \( D^H_{99} = 100 \) in the high state, and the debt capacity is now \( B^H_{99} = 99.939 \).

As long as the face value of the debt is set equal to \( B^H_{n+1} \) in both states, the debt capacity satisfies

\[
\begin{bmatrix}
B^H_{n} \\
B^L_{n}
\end{bmatrix} =
\begin{bmatrix}
0.99904 & 0.90 \times 0.00096 \\
0.07685 & 0.90 \times 0.92315
\end{bmatrix}
\begin{bmatrix}
B^H_{n+1} \\
B^L_{n+1}
\end{bmatrix}.
\]

However, this assumes that it is optimal to have default in the low state at every rollover date, which is not necessarily true. Starting at the last rollover date, it can be shown that the debt capacity in state \( L \) rises as we go back in time, reaches a maximum at \( t_{80} \), and then declines as we move to earlier and earlier dates (see Figure 5). The problem is that as the debt capacity rises, the liquidation costs (which are proportional to the debt capacity) also rise and eventually outweigh the upside potential of a switch to the high state.

\[\text{It is also possible to extend this example to the case with fixed costs of liquidation. Details are available from authors upon request.}\]

\[\text{At the point}\]
where the maximum is reached, it is optimal to change the face value of the debt from $B_{n+1}^H$ to $B_{n+1}^L$ and avoid default in the low state. Then the debt capacity is given by the formula above for $n = 80, ..., 99$ and is given by $B_n^L = B_{80}^L$ for $n = 0, ..., 80$. We can use the formula in equation (9) to show that $B_{80}^L = 44.918$ and $B_{80}^H = 98.847$. The gap between the debt capacities in the two states is $94.469 - 44.918 = 49.551$, compared to the negligible difference in the fundamental values 99.2596 and 99.241 in the high and the low states, respectively. Thus, even if it is optimal to capture the upside potential of a switch to the high state, the debt capacity in the low state does not rise much above the minimum value of the asset, i.e., it is 44.918 rather than 40.

— Figure 5 here —

In the rest of the paper, we explore the determinants of debt capacity in a richer model with many states and a broad range of parameters.

4 Debt capacity in the general case

We allow for a finite number of information states or signals, denoted by $S = \{s_1, ..., s_I\}$. The current information state is public information. Transitions among the states are governed by a stationary Markov transition probability $P(\tau)$ given as

$$P(\tau) = \begin{bmatrix} p_{11}(\tau) & \cdots & p_{1I}(\tau) \\ \vdots & \ddots & \vdots \\ p_{I1}(\tau) & \cdots & p_{II}(\tau) \end{bmatrix},$$

where $\tau$ is the interval over which the transitions take place. We assume that the transition matrix takes the form

$$P(\tau) = e^{A\tau} = \sum_{k=0}^{\infty} \frac{(A\tau)^k}{k!},$$

where the matrix $A$ is the generator. The crucial feature of the transition matrix is that the probability of a change of state converges to 0 as $\tau \to 0$. That is, $P(\tau) \to I$ as $\tau \to 0$.

The information state is a stochastic process $\{S(t)\}$ but for our purposes all that matters is the value of this process at the rollover dates. We let $S_n$ denote the value of the information state $S(t_n)$ at the rollover date $t_n$.

The terminal value of the assets is a function of the information state at date $t = 1$. We denote by $v_i$ the value of the assets if the terminal state is $S_N = s_i$ and assume that the values $\{v_1, ..., v_I\}$ satisfy

$$0 < v_1 < \ldots < v_I.$$
Let $V_i^n$ denote the fundamental value of the asset at date $t_n$ in state $i$. Then clearly the values $\{V_i^n\}$ are defined by putting $V_i^{n+1} = v_i$, for $i = 1, ..., I$, and

$$V_i^n = \sum_{j=1}^I p_{ij} (1 - t_n) v_j, \text{ for } n = 0, ..., N \text{ and } i = 1, ..., I,$$

where $p_{ij} (1 - t_n)$ is, of course, the $(i,j)$ entry of $P (1 - t_n)$ denoting the probability of a transition from state $i$ at date $t_n$ to state $j$ at date $t_{N+1} = 1$.

Figure 6 illustrates the fundamental values in a setup with $I = 6$ states where terminal values are $v_i = 40 + i 10$, for $i = 1, ..., 6$. The transition matrix $P$ is described in Appendix B. As in our two-state example, the fundamental values in different states are virtually identical at date 0 though they diverge in steps of 10 at maturity.

— Figure 6 here —

Let $B_i^n$ denote the equilibrium debt capacity of the assets in state $s_i$ at date $t_n$. By convention, we set $B_i^{N+1} = v_i$ for all $i$.

**Proposition 5** The equilibrium values of $\{B_i^n\}$ must satisfy

$$B_i^n = \max_{k=1,...,I} \left\{ \sum_{\{j:B_k^n > B_i^n\}} p_{ij} (\tau) \lambda B_j^{n+1} + \sum_{\{j:B_k^n \leq B_i^n\}} p_{ij} (\tau) B_j^{n+1} \right\}$$

for $i = 1, ..., I$ and $n = 0, ..., N$.

The result is immediate once we apply the now familiar backward induction argument to show that it is always optimal to set the face value of the debt $D_i^n$ equal to $B_i^{n+1}$ for some $j$. Although the result amounts to little more than the definition of debt capacity, it is very useful because it allows us to calculate the debt capacities by backward induction.

The main result on the downward bias of debt capacities is contained in the following proposition.

**Proposition 6** There exists $\tau^* > 0$ such that for all $0 < \tau < \tau^*$, for any $n = 0, ..., N$ and any $i = 1, ..., I$, it is optimal to choose the face value $D_i^n \leq B_i^{n+1}$. Thus,

$$B_i^n = \sum_{\{j:B_k^n > B_i^n\}} p_{ij} (\tau) \lambda B_j^{n+1} + \sum_{\{j:B_k^n \leq B_i^n\}} p_{ij} (\tau) B_j^{n+1},$$

for some $k$ such that $B_k^{n+1} \leq B_i^{n+1}$.
Proof. See Appendix A. ■

Several properties follow immediately from Proposition 6 whenever $0 < \tau < \tau^*$. We provide these results in the form of three corollaries. First, in the lowest state, $s_1$, the debt capacity is constant and equal to $v_1$, the lowest possible terminal value.

**Corollary 7** $B^1_n = v_1$ for all $n$.

**Proof.** From the formula in Proposition 6, for some $k$,

$$B^1_n = \sum_{\{j : B^k_{n+1} > B^l_{n+1}\}} p_{ij}(\tau) \lambda B^j_{n+1} + \sum_{\{j : B^k_{n+1} \leq B^l_{n+1}\}} p_{ij}(\tau) B^k_{n+1}$$

$$\leq \sum_{j=1}^I p_{ij}(\tau) B^1_{n+1} = B^1_{n+1},$$

since $B^k_{n+1} \leq B^1_{n+1}$. Since this inequality holds for $n = 0, ..., N$ and, by convention, $B^1_{N+1} = v_1$, it follows that $B^1_n \leq v_1$, for any $n$.

We can also show that $B^1_n \geq v_1$. To see this, note that $B^i_{N+1} = v_1$ for all $i$. Moreover, if the same condition holds for $n + 1$, it must be true that $B^i_n \geq v_1$, because we can always choose $D^i_n = v_1$.

Thus, we have shown that $B^i_n = v_1$ for all $n$. ■

Second, the debt capacity $B^i_n$ is monotonically non-decreasing in $n$, that is, debt capacity increases as the asset matures, holding the state constant. This follows directly from the fact that, if the face value of the debt equals $B^i_{n+1}$, the debt capacity $B^i_n$ cannot be greater than $B^i_{n+1}$.

**Corollary 8** $B^i_n \leq B^i_{n+1}$, for any $i = 1, ..., I$ and $n = 0, ..., N$.

**Proof.** The inequality follows directly from the formula in Proposition 6:

$$B^i_n = \sum_{\{j : B^k_{n+1} > B^l_{n+1}\}} p_{ij}(\tau) \lambda B^j_{n+1} + \sum_{\{j : B^k_{n+1} \leq B^l_{n+1}\}} p_{ij}(\tau) B^k_{n+1}$$

$$\leq \sum_{j=1}^I p_{ij}(\tau) B^i_{n+1} + \sum_{\{j : B^k_{n+1} \leq B^l_{n+1}\}} p_{ij}(\tau) B^i_{n+1}$$

$$= \sum_{j=1}^I p_{ij}(\tau) B^i_{n+1} = B^i_{n+1},$$

since $B^j_{n+1} < B^k_{n+1} \leq B^l_{n+1}$ implies that $\lambda B^j_{n+1} < B^i_{n+1}$. ■

Third, since $B^i_{N+1} = v_i$ by convention, the preceding result immediately implies that the debt capacity $B^i_n$ is less than or equal to $v_i$. 

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Corollary 9 $B^i_n \leq v_i$ for all $i = 1, \ldots, I$ and $n = 0, \ldots, N$.

Finally, we can confirm that the debt capacity in state $s_i$ at any date $t_n$ is less than the fundamental value $V^i_n$. This follows directly from the formula in Proposition 6 for $n = N + 1$ and any $i$, so suppose that it holds for $n + 1, \ldots, N$ and any $i = 1, \ldots, I$. Then the formula in Proposition 6 implies that, if $D^i_n = B^k_{n+1}$, say,

$$B^i_n = \sum_{\{j: B^i_{n+1} > B^k_{n+1}\}} p_{ij}(\tau) \lambda B^j_{n+1} + \sum_{\{j: B^k_{n+1} \leq B^j_{n+1}\}} p_{ij}(\tau) B^k_{n+1}$$

$$\leq \sum_{\{j: B^i_{n+1} > B^j_{n+1}\}} p_{ij}(\tau) B^j_{n+1} + \sum_{\{j: B^k_{n+1} \leq B^j_{n+1}\}} p_{ij}(\tau) B^j_{n+1}$$

$$\leq \sum_{j=1}^{I} p_{ij}(\tau) V^j_{n+1} = V^i_n,$$

for any $i = 1, \ldots, I$, so by induction the claim holds for any $n = 0, \ldots, N$ and any $i = 1, \ldots, I$.

Some of these properties are illustrated in Figures 7a and 7b which show the debt capacities in the six states of our numerical example for $N = 10$ and $N = 100$ rollovers, respectively. For 10 rollovers, $\tau$ is not sufficiently small to obtain our limit result even in the worst state and debt capacity in each state is in fact higher than the terminal value in that state. Nevertheless, it is still the case that there is a drop in debt capacity of between 5 and 10 as the state changes to the next worse one, without much change in the fundamental value. By contrast, with 100 rollovers, the limit result is obtained and in fact debt capacity in the two worst states (states 1 and 2) is (essentially) the minimum possible value of the asset which is 50. Furthermore, as we go from the best state (state 6) to the second-best state (state 5), debt capacity falls roughly by a magnitude of 25 even though the fundamental value (Figure 6) has hardly changed. Thus, the market freeze is substantially worse with 100 rollovers compared to 10.

— Figure 7a and 7b here —

Proposition 6 shows that, when the period length is sufficiently short (the rollover rate is sufficiently high), there is a downward bias in the debt capacity, because the face value of the debt is bounded above by the future debt capacity in the same state. This is an important step toward proving the existence of a market freeze but two further requirements are needed. First, we need to show that the fundamental values are uniformly high and that the debt capacities are high in some states and low in others.

Consider the debt capacities first. The proposition shows that $B^1_n = v_1$ for all $n = 0, \ldots, N$ so it is enough to show that the debt capacity is high in some states. The following proposition does just that.
Proposition 10 The initial debt capacity in the highest state, $I$, satisfies the inequality

$$B^I_0 \geq e^{-\alpha} v_I + (1 - e^{-\alpha}) \lambda v_1,$$

where $\alpha = \sum_{j \neq I} a_{Ij}$.

So, as long as $\alpha$ is sufficiently small, the debt capacity in the high state will be close to $v_I$. We know that the fundamental value $V^I_0$ lies between $B^I_0$ and $v_I$, so the fundamental value will also be close to $v_I$. Then in order to show that a market freeze is possible it is only necessary to ensure that the fundamental value for the lowest state, $V^I_1$, is also close to $v_I$. This will be true as long as the probability of a transition from the lowest to the highest state is high enough. In general, this probability will depend on the entire matrix $A$ so it is hardly worth trying to write down a sufficient condition in terms of the individual parameters, but the numerical examples have illustrated that this is clearly possible.

5 Related research

At a general level, our result on market freezes can be considered a generalization of the Shleifer and Vishny (1992) and Allen and Gale (1994) results that when potential buyers of assets of a defaulted firm are themselves financially constrained, there is a reduction in the ex-ante debt capacity of the industry as a whole. We expand on their insight by considering short-term debt financing of long-term assets with rollovers to be met mostly by new short-term financing or liquidations to other buyers also financed mostly through short-term debt. Our “market freeze” result can be considered as a particularly perverse dynamic arising through the Shleifer and Vishny (1992) and the Allen and Gale (1994) channel at each rollover date, that through backward induction, can in the worst case drive short-term debt capacity of an asset to its minimum possible cash flow.

More specifically, our paper is related to the literature on freezes and runs in financial markets. Rosenthal and Wang (1993) use a model where owners occasionally need to sell their assets for exogenous liquidity reasons through auctions with private information. Because of the informational rents earned by the privately informed bidders, sellers may not be able to extract the full value of the asset and this liquidation cost gets built into the market price of the asset, making the market price systematically lower than the fundamental value. In our model, the reason for the debt capacity being lower than the fundamental value is not the private information of potential buyers, rather it is the rollover risk and the liquidation cost associated with defaults.

He and Xiong (2009) consider a model of dynamic debt runs in which creditors have supplied debt maturing at differing maturities and each creditor faces the risk, at the time of rolling over the debt, that fundamentals may deteriorate before the remaining debt matures,
causing a fire sale of assets. In their model, the volatility of fundamentals plays a key role in driving the runs, even when the average value of fundamentals has not been affected. Our model is complementary to theirs and somewhat different in the sense that both average value and uncertainty about fundamentals are held constant in our model. It is the rate at which information arrives relative to the rollovers that determines whether there is rollover risk in short-term debt.

Huang and Ratnovski (2008) model the behavior of short-term wholesale financiers who prefer to rely on noisy public signals such as market prices and credit ratings, rather than producing costly information about the institutions they lend to. Hence, wholesale financiers run on other institutions based on imprecise public signals, triggering potentially inefficient runs. While their model is about runs in the wholesale market, as is ours, their main focus is to challenge the peer-monitoring role of wholesale financiers, whereas our main focus is the role of rollover and liquidation risk in generating such runs.

An alternative modelling device to generate market freezes is to employ the notion of Knightian uncertainty (see Knight, 1921) and agents’ overcautious behavior towards such uncertainty. Gilboa and Schmeidler (1989) build a model where agents become extremely cautious and consider the worst-case among the possible outcomes, that is, agents are uncertainty averse and use maxmin strategies when faced with Knightian uncertainty. Dow and Werlang (1992) apply the framework of Gilboa and Schmeidler (1989) to the optimal portfolio choice problem and show that there is an interval of prices within which uncertainty-averse agents neither buy nor sell the asset. Routledge and Zin (2004) and Easley and O’Hara (2005, 2008) use Knightian uncertainty and agents that use maxmin strategies to generate widening bid-ask spreads and freeze in financial markets. Caballero and Krishnamurthy (2008) also use the framework of Gilboa and Schmeidler (1989) to develop a model of financial crises: During periods of increased Knightian uncertainty, agents refrain from making risky investments and hoard liquidity, leading to flight to quality and freezes in markets for risky assets. While ambiguity aversion leads to a market freeze in these models, in our model agents maximize expected utility and the main source of the market freeze is rollover and liquidation risk.

We regard our approach as complementary to Knightian uncertainty. Knightian uncertainty is appropriate when investors have very limited information about the nature of the risks they face. We are interested, by contrast, in explaining the drying up of liquidity in the absence of obvious problems of asymmetric information or fears about the value of collateral. For this purpose, it would seem to be an advantage to appeal to standard assumptions about preferences and beliefs.
6 Conclusion

In this paper, we have attempted to provide a simple information-theoretic model for freezes in the market for short-term financing of finitely lived assets. The key ingredients of our model were rollover risk, liquidation risk, rapid rate of refinancing relative to the arrival of news, and similarity of financial institutions in their degree of maturity mismatch. In particular, our model could be interpreted as a micro-foundation for the funding risk arising in capital structures of financial institutions or special purpose vehicles that have extreme maturity mismatch between assets and liabilities.

In future work, it would be interesting to embed an agency-theoretic role for short-term debt, which we assumed as given, and see how the desirability of such rollover finance is affected when information problems can lead to complete freeze in its availability. While we took the release of information about the underlying asset as ordained by nature, it seems worthwhile to reflect on its deeper foundations, and thereby assess whether a strategic disclosure of information by agents in charge of the asset can alleviate (or aggravate) the problem of freezes.

Appendix A: Proofs

We can solve for the equilibrium debt capacities in the model of Section 4 by backward induction. Let $D$ denote the face value of the debt issued in state $s_i$ at date $t_n$. This debt will pay off $D$ in state $s_j$ at date $t_{n+1}$ if $D \leq B^j_{n+1}$ and $\lambda B^j_{n+1}$ otherwise. In other words, the market value of the debt is given by the formula

$$\sum_{B^i_{n+1} < D} p_{ij}(\tau) \lambda B^i_{n+1} + \sum_{B^i_{n+1} \geq D} p_{ij}(\tau) D$$

and the debt capacity is given by

$$B^i_n = \max_D \left\{ \sum_{B^i_{n+1} < D} p_{ij}(\tau) \lambda B^i_{n+1} + \sum_{B^i_{n+1} \geq D} p_{ij}(\tau) D \right\}.$$

Let $D^i_n$ denote the optimal face value of the debt in state $i$ at date $t_n$. It is clear that the market value of the debt is maximized by setting the face value $D = \lambda B^j_{n+1}$, for some value of $j = 1, \ldots, I$. Thus, we can write the equilibrium condition as

$$B^i_n = \max_{k=1,\ldots,I} \left\{ \sum_{\{j:B^k_{n+1} > B^i_{n+1}\}} p_{ij}(\tau) \lambda B^i_{n+1} + \sum_{\{j:B^k_{n+1} \leq B^i_{n+1}\}} p_{ij}(\tau) B^i_{n+1} \right\}.$$
for $i = 1, ..., I$ and $n = 0, ..., N$.

**Proof of Proposition 6:** For a fixed but arbitrary date $t_n$ and state $s_i$, we compare the strategy of setting $D = B^i_{n+1}$ with the strategy of setting $D = B^k_{n+1}$, where $B^k_{n+1} > B^i_{n+1}$. Consider the difference in the expected values of the debt:

$$
\sum_{\{j:B^i_{n+1} > B^j_{n+1}\}} p_{ij}(\tau) \lambda B^j_{n+1} + \sum_{\{j:B^i_{n+1} \leq B^j_{n+1}\}} p_{ij}(\tau) B^i_{n+1}
$$

$$
- \sum_{\{j:B^k_{n+1} > B^j_{n+1}\}} p_{ij}(\tau) \lambda B^j_{n+1} - \sum_{\{j:B^k_{n+1} \leq B^j_{n+1}\}} p_{ij}(\tau) B^k_{n+1}
$$

$$
= \sum_{\{j:B^i_{n+1} \leq B^j_{n+1}, B^k_{n+1}\}} p_{ij}(\tau) (B^i_{n+1} - \lambda B^j_{n+1}) + \sum_{\{j:B^k_{n+1} \leq B^j_{n+1}\}} p_{ij}(\tau) (B^i_{n+1} - B^k_{n+1})
$$

$$
= p_{ii}(\tau)(B^i_{n+1} - \lambda B^i_{n+1}) + \sum_{\{j:B^i_{n+1} < B^k_{n+1}\}} p_{ij}(\tau)(B^i_{n+1} - B^k_{n+1})
$$

$$
\geq p_{ii}(\tau)(1 - \lambda) v_1 + \sum_{\{j:B^i_{n+1} < B^k_{n+1}\}} p_{ij}(\tau)(v_1 - v_I) + \sum_{\{j:B^k_{n+1} \leq B^j_{n+1}\}} p_{ij}(\tau)(v_1 - v_I)
$$

$$
= p_{ii}(\tau)(1 - \lambda) v_1 + \sum_{\{j:B^i_{n+1} < B^j_{n+1}\}} p_{ij}(\tau)(v_1 - v_I).
$$

Since $B^i_{n+1} \geq v_1$, $B^i_{n+1} - \lambda B^j_{n+1} \geq B^j_{n+1} - B^i_{n+1} \geq (v_1 - v_I)$ for $j = i + 1, ..., I$ and $B^i_{n+1} - B^k_{n+1} \geq v_1 - v_I$. Then it is clear that, for $\tau$ sufficiently small (i.e., $p_{ii}(\tau)$ sufficiently close to 1), the last expression above is positive. Since the last expression is independent of $n$, the bound is uniform, i.e., there exists a constant $\tau^* > 0$ such that, for $\tau < \tau^*$, it is optimal to set $D^i_n = B^i_{n+1}$, for all $i$ and $n$.

**References**


Appendix B: Numerical parameters for the example with $I = 6$ states

The terminal values for the 6 states are chosen as $v_i = 10i$, for $i \in \{5, \ldots, 10\}$. The generator matrix $A$ and the unconditional transition matrices $P(N = 10)$ and $P(N = 100)$ that is, the transition matrices with 10 and 100 rollovers, respectively, are given below.

$$A = \begin{bmatrix}
-8 & 8 & 0 & 0 & 0 & 0 \\
1 & -10 & 9 & 0 & 0 & 0 \\
0 & 1 & -10 & 9 & 0 & 0 \\
0 & 0 & 1 & -10 & 9 & 0 \\
0 & 0 & 0 & 1 & -10 & 9 \\
0 & 0 & 0 & 0 & 0.1 & -0.1
\end{bmatrix}$$

$$P(N = 10) = \begin{bmatrix}
0.498446 & 0.328747 & 0.129744 & 0.034727 & 0.007006 & 0.00133 \\
0.041093 & 0.432477 & 0.341745 & 0.138156 & 0.037333 & 0.009196 \\
0.001802 & 0.037972 & 0.433411 & 0.342035 & 0.13748 & 0.0473 \\
5.36 \times 10^{-5} & 0.001706 & 0.038004 & 0.433336 & 0.338412 & 0.188488 \\
1.20 \times 10^{-6} & 5.12 \times 10^{-5} & 0.001697 & 0.037601 & 0.420155 & 0.540494 \\
2.53 \times 10^{-9} & 1.40 \times 10^{-7} & 6.49 \times 10^{-6} & 0.000233 & 0.006005 & 0.993755
\end{bmatrix}$$

$$P(N = 100) = \begin{bmatrix}
0.923483 & 0.073136 & 0.00328 & 9.82 \times 10^{-5} & 2.21 \times 10^{-6} & 4.04 \times 10^{-8} \\
0.009142 & 0.905609 & 0.081471 & 0.003666 & 0.00011 & 2.52 \times 10^{-6} \\
4.56 \times 10^{-5} & 0.009052 & 0.905652 & 0.081472 & 0.003665 & 0.000113 \\
1.52 \times 10^{-7} & 4.53 \times 10^{-5} & 0.009052 & 0.905652 & 0.081461 & 0.003789 \\
3.79 \times 10^{-10} & 1.51 \times 10^{-7} & 4.53 \times 10^{-5} & 0.009051 & 0.905287 & 0.085617 \\
7.69 \times 10^{-14} & 3.85 \times 10^{-11} & 1.55 \times 10^{-8} & 4.68 \times 10^{-6} & 0.000951 & 0.999044
\end{bmatrix}$$
Figures

Figure 1: Timeline (illustrating $N+1$ state transitions and $N$ rollovers).

Figure 2: Fundamental value ($V$) and debt capacity ($B$) in high ($v^H=100$) and low ($v^L=50$) states as a function of time.
Figure 3: Fundamental value ($V$) and debt capacity ($B$) in low ($v^L=50$) state for different number of rollovers ($N$)
Figure 4a: $\gamma^*$ as a function of $e^{-a_{HL}}$.

Figure 4b: $N^*$ as a function of $\gamma$.
Figure 5: Debt capacity ($B$) in high ($v^H = 100$) and low ($v^L = 40$) states as a function of time.

Figure 6: Fundamental values ($V$) as a function of time.
Figure 7a: Debt capacity (B) as a function of time for rollover frequency N=10

Figure 7b: Debt capacity (B) as a function of time for rollover frequency N=100