Rollover Risk and Market Freezes\textsuperscript{1}

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Abstract

We present a model that can explain a sudden collapse in the amount that can be borrowed against assets, even in the absence of asymmetric information or fears about the value of the collateral. Three features of the model are essential: (i) the debt has a much shorter tenor than the assets and needs to rolled over frequently; (ii) in the event of default by the borrower, the collateral is sold by the creditors and there is a (small) liquidation cost; (iii) all potential buyers of the collateral also rely on short-term debt finance. Under these conditions, the debt capacity of the assets (the maximum amount that can be borrowed using the securities as collateral) can be much less than the fundamental value. In fact, it can equal the minimum possible future value of the asset, which may be zero. This is true even if the fundamental value of the assets is currently high. In particular, a small change in the fundamental value of the assets can be associated with a sudden collapse in the debt capacity. The crisis of 2007-09 has been characterized by just such a sudden freeze in the market for short-term, secured borrowing.

J.E.L. Classification: G12, G21, G24, G32, G33, D8.

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1 Introduction

1.1 Motivation

One of the many striking features of the crisis of 2007-09 has been the sudden freeze in the market for the rollover of short-term debt. From an institutional perspective, the inability to borrow overnight against high-quality but long-term assets was a striking market failure that effectively led to the demise of a substantial part of investment banking in the United States. More broadly, it led to the collapse in the United States, the United Kingdom, and other countries of banks and other financial institutions that had relied on significant maturity mismatch between assets and liabilities, and, in particular, on the rollover of short-term wholesale debt in the asset-backed commercial paper (ABCP) and overnight secured repo markets.

The first such collapse occurred in the summer of 2007. On August 7, 2007, after a week of bad news about sub-prime asset-backed securities (ABS), BNP Paribas halted withdrawals from three investment funds and suspended calculation of the net asset values because it could not “fairly” value their holdings. Investors in ABCP, primarily money market funds, shied away from further financing of assets in conduits and other off-balance sheet vehicles such as SIVs. The prospect that banks might have to take these assets back on to their balance sheets raised concerns about counter-party risk amongst banks and caused LIBOR to shoot upwards. The European Central Bank was forced to pump 95 billion Euros in overnight lending into the market that same day in response to the sudden demand for cash from banks. The sub-prime crisis had taken hold.

The failure of Bear Stearns in mid-March 2008 was the next example of a market freeze as an intrinsic part of its business, Bear Stearns relied day-to-day on its ability to obtain short-term finance through secured borrowing. At this time, Bear was reported to be financing $85 billion of assets on the overnight market (Cohan, 2009). Beginning late Monday, March 10, rumors spread about liquidity problems at Bear Stearns. Even though Bear Stearns continued to have high quality collateral, counterparties became unwilling to lend on customary terms. This unwillingness to fund on a secured basis placed enormous stress on the liquidity of Bear Stearns. On Tuesday, March 11, the holding company liquidity pool declined from $18.1 billion to $11.5 billion. On Thursday, March 13, Bear Stearns’ liquidity

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1 Goldsmith-Pinkham and Yorulmazer (2008) analyze a similar episode that led to the near failure of Northern Rock in the United Kingdom in September 2007. Northern Rock also had a balance-sheet featuring significant maturity mismatch (long-term mortgages financed with short-term wholesale debt). Soon after Northern Rock’s woes, other UK banks such as HBOS, Alliance and Leicester, and Bradford and Bingley, that had relied primarily on short-term wholesale debt, suffered too.

pool fell sharply and continued to fall on Friday. In the end, the market rumors about Bear Stearns’ liquidity problems appeared to have become self-fulfilling and led to the near failure of the firm. Bear Stearns was adequately capitalized by regulatory standards at all times during the period from March 10 to March 17, up to and including the time of its agreement to be acquired by J.P. Morgan Chase. In particular, at the time of its sale, the capital ratio of Bear Stearns was well in excess of the 10% level used by the Federal Reserve Board in its well-capitalized standard.

In his analysis of the failure of Bear Stearns, the Federal Reserve Chairman Ben Bernanke observed:

“[U]ntil recently, short-term repos had always been regarded as virtually risk-free instruments and thus largely immune to the type of rollover or withdrawal risks associated with short-term unsecured obligations. In March, rapidly unfolding events demonstrated that even repo markets could be severely disrupted when investors believe they might need to sell the underlying collateral in illiquid markets. Such forced asset sales can set up a particularly adverse dynamic, in which further substantial price declines fan investor concerns about counterparty credit risk, which then feed back in the form of intensifying funding pressures...[F]uture liquidity planning will have to take into account the possibility of a sudden loss of substantial amounts of secured financing.”

In this paper, we are interested in developing a model of this “adverse dynamic.” More specifically, we are interested in explaining the drying up of liquidity in the absence of obvious problems of asymmetric information or fears about the value of collateral. To do so, we develop a model of debt capacity, the maximum amount of collateralized borrowing that can be supported by an asset. Three assumptions are crucial for our results:

(i) the debt has a much shorter tenor than the assets and needs to rolled over frequently;

(ii) in the event of default by the borrower, the collateral is sold by the creditors and there is a (small) liquidation cost;

(iii) all potential buyers of the collateral also rely on short-term debt finance.

These are essentially the features alluded to in the preceding discussion of the conditions surrounding market freezes in the financial crisis of 2007 and 2008. In what follows, we take these features as given, without attempting to rationalize them as the result of equilibrium behavior. For example, we take the (short) tenor of the debt as exogenous. There is ample

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empirical evidence that financial institutions relied heavily on short-term finance prior to the crisis, but we do not attempt to explain why this was so[4]. We also take as given the liquidation costs. More precisely, we assume the costs are either fixed or proportional to debt capacity, but in either case the costs are exogenous. Finally, it is important to note that the tenor of the debt and the liquidation costs are assumed to be the same for all market participants. In particular, as the tenor of the debt becomes shorter for the owner of the asset, it also becomes smaller for the potential buyers.

Our central result is that, under the maintained assumptions, the debt capacity may suddenly collapse, even though the fundamental value of the asset has changed little. In efficient markets, the debt capacity of an asset is equal to its NPV or “fundamental” value. If there are liquidation costs and a positive probability of default, the debt capacity would naturally be expected to be somewhat less than the fundamental value. In our model we can derive the much stronger result that the gap between the fundamental value and the debt capacity can be arbitrarily large even if the liquidation cost incurred in the event of default is tiny. More precisely, when the tenor of the debt is sufficiently short, other things being equal, the debt capacity can equal the minimum possible future value of the asset. For example, if the minimum possible value of the asset is zero, then the debt capacity will be zero in the worst case, even though the fundamental value of the asset is currently high. We call this last phenomenon a “market freeze.” This remarkable and perhaps counter-intuitive result captures the scenario that Bear Stearns experienced during its failure in March 2008.

The intuition for the result is as follows. When the tenor of the debt is short, the probability of news arriving before the next roll-over date is very small. Then it is very likely that the next refinancing will be undertaken with the same information as in the current period. The maximum amount that can be borrowed without a substantial risk of default is equal to the debt capacity at the next roll-over date, assuming no information has arrived by then. The borrower will find it optimal to avoid a substantial risk of default because he wants to avoid the liquidation costs. This means that today’s debt capacity is less than or equal to the debt capacity at the next roll-over date, assuming no information has arrived in the interim. Applying this argument repeatedly shows that today’s debt capacity must be less than or equal to the debt capacity at the maturity of the assets, assuming no information has arrived in the meantime. In other words, the borrower is forced to act as if his condition will never improve. A substantial part of the fundamental value of the assets consists of the arrival of good news in the future, but it may not be reflected in the

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[4] For instance, Acharya, Schnabl and Suarez (2009) show that outstanding ABCP had typically a maturity of less than one week and rose from $650 billion to over $1.2 trillion between 2004 and 2Q 2007, only to revert back to its 2004 level by 1Q 2009. Morris and Shin (2009) document using data on outstanding repurchase agreements of the US primary dealers (source: Federal Reserve Bank of New York) that during 2003–2007, term repo remained steady around $1.5 trillion, whereas overnight repo contracts doubled from $1.5 trillion to $3 trillion, both shrinking by over a trillion dollars by 2009.
debt capacity. This explains why the fundamental value may be much greater than the debt capacity and an information event that has only a small effect on the fundamental value can have a catastrophic effect on the debt capacity.

We have described the market freeze resulting from the arrival of information about the fundamental value of the debt when the tenor of the debt is very short and constant. We can also interpret the market freeze as resulting from a sudden shortening in the tenor of the debt. Prior to the start of the crisis in August 2007, buyers of ABCP became increasingly unwilling to lend at any but the shortest maturities. If the arrival of news, that perhaps signals a small change in the fundamental value of the assets, also causes lenders to restrict the tenor of the debt they are willing to hold, the effect on the debt capacity will be even greater than what we have characterized. Thus, it is not necessary to assume that banks choose short-term finance from the outset. The freeze may result from lenders suddenly shortening the tenor of the paper they are willing to hold.

While we focused on market freezes in the crisis of 2007-09 for motivation, our model is relevant beyond just explaining the past. Understanding the causes of market freezes is crucial to creating a more stable and efficient financial system for the future. Following the crisis, the parallel (“shadow”) banking system, consisting of structured purpose vehicles such as SIVs and conduits, securities lending, etc., has shrunk significantly and reduced the financial system’s lending capacity by several trillion dollars. While some of this collapse was driven by concerns regarding the quality of the assets, our paper highlights that liquidity issues relating to the heavy reliance on short-term rollover debt also played an important role. Restoring the parallel banking system is seen by many as an important step in the reconstruction of the financial system to provide credit.

Our paper highlights the need to address the problem of roll-over risk in order to avoid the instability of the past.

The rest of the paper is organized as follows. Section 2 provides an introduction to the model and results in terms of a simple numerical example. In particular, it illustrates our main result that a market freeze occurs if the number of rollovers is sufficiently high. Section 3 derives the main result for the special case of a model with two states. It also illustrates, in terms of the numerical example, that market freezes can occur even if the debt maturity is not “short.” Section 4 provides a complete characterization of the debt capacity for the general model and extends the limit result to an arbitrary number of states. The proof of the limit result is relegated to Appendix A. Section 5 discusses the related literature. Section 6 ends.

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5 This interpretation was suggested to us by Arvind Krishnamurthy. See footnote 4 for evidence regarding shortening of debt maturities during 2004–2007.
6 In addition, many of the assets once held by this system are now held directly by central banks or by commercial and investment banks relying on lending facilities provided by the central banks. Some day these holdings will have to find another home and the most likely place would be a revitalized and more stable parallel banking system.
2 Model and results

In this section, we introduce the essential ideas in terms of a numerical example. For concreteness, consider the case of a bank that wishes to repo an asset. The question we ask is: What is the maximum amount of money that the bank can borrow using the asset as collateral? There are two ways to interpret this exercise. We can imagine that a value maximizing bank is trying to maximize its return on equity by minimizing the amount of capital needed to finance the assets it owns. In this case, every bank that purchases the asset is assumed to have the same motive for maximizing leverage. This seems consistent with the evidence of Adrian and Shin (2008) that asset growth (shrinkage) of broker-dealers is coincident with equivalent growth (shrinkage) in their leverage, especially so for repo financing. Acharya, Schnabl and Suarez (2009) also describe how conduits had little equity of their own and were largely financed with extremely short-term ABCP. They also explain why the conduit activity is consistent with minimizing capital of sponsor banks. Alternatively, we can simply see our exercise as establishing a bound on the amount that can be borrowed, assuming that other buyers in the market are limited by a similar bound.

Time is represented by the unit interval \([0, 1]\). The asset is purchased at the initial date \(t = 0\). The asset has a finite life (e.g., mortgages) which we normalize to one unit. To keep the analysis simple, we assume that the asset has a terminal value at \(t = 1\), but generates no income at the intermediate dates \(0 \leq t < 1\). We also assume that the risk-free interest rate is 0 and that all market participants are risk neutral.

For simplicity, let us assume that there are two states of nature, a low state \(L\) and a high state \(H\). The terminal value of the asset depends on the state of the economy at the terminal date \(t = 1\). In the high state, the value of the asset is \(v^H\) and in the low state \(v^L\).

Information becomes available continuously according to a Poisson process with parameter \(\alpha > 0\). That is, the probability that an information event occurs in a short time interval \([t, t + \tau]\) is \(\alpha \tau\). When an information event occurs, the state of the economy changes randomly according to a fixed probability transition matrix

\[
P = \begin{bmatrix}
p_{LL} & p_{LH} \\
p_{HL} & p_{HH}
\end{bmatrix},
\]

where \(p_{LH}\) (\(p_{HL}\)) represents the probability of switching from state \(L\) (\(H\)) to state \(H\) (\(L\)) at an information event, whereas \(p_{LL}\) (\(p_{HH}\)) represents the probability of staying in state \(L\) (\(H\)). Note that the occurrence of an information event is itself random, so the number of information events in a given time interval is random and affects the probability of observing a transition from one state to another.\(^7\)

\(^7\)In the special case where there are only two states, we do not need to distinguish between the probability distribution of the information event and the probability of switching conditional on an information event.
We assume that the asset will be financed by debt that has to be rolled over repeatedly. The debt is assumed to have a fixed maturity, denoted by $0 < \tau < 1$, so that the debt must be rolled over $N$ times, where

$$\tau = \frac{1}{N+1}.$$  

The unit interval is divided into intervals of length $\tau$ by a series of dates denoted by $t_n$ and defined by

$$t_n = n\tau, \ n = 0, 1, ..., N + 1,$$

where $t_0$ is the date the asset is purchased, $t_n$ is the date of the $n$-th rollover (for $n = 1, ..., N$), and $t_{N+1}$ is the final date at which the asset matures and the terminal value is realized. This time-line is illustrated in Figure 1.

— Figure 1 here —

If the bank is forced to default, the lenders will seize and liquidate the collateral. In this example, we assume that the amount recovered is a fraction $\lambda \in [0, 1]$ of the maximum amount of finance that could be raised using the asset as collateral. This assumption has several parts. In the first place, it implies that the seized collateral is liquidated, i.e., sold to another buyer. Secondly, the new buyers are also finance constrained. Thirdly, the process of seizing and disposing of collateral is not costless. For concreteness, we can think of the liquidation cost $1 - \lambda$ as a transaction cost (legal costs, commissions, fees, time delay, etc.), although other interpretations are possible (see Pedersen, 2009, for a discussion of variety of transactions costs and illiquidity in markets). Note that similar results could be obtained with a fixed liquidation cost (see Section 4.1).

As an example, suppose that a bank has borrowed 90 and, when it comes time to refinance, finds that it can only raise 87 using the assets as collateral. The lender, say a Money Market Fund, cannot hold the collateral and is forced to dispose of it. A finance constrained buyer can borrow 87 using the assets as collateral, so this is the maximum that it can pay for the assets. However, the amount received by the lender will be a smaller amount, say, 86, because transaction costs have to be subtracted from the sale price.

It is important to note that the recovery rate $\lambda$ is applied to the maximum debt capacity rather than to the fundamental value of the assets. If the buyer of the assets were a wealthy investor who could buy and hold the assets until maturity, the fundamental value would be the relevant benchmark. The investor might well be willing to pay some fraction of the fundamental value although he would presumably try to get the assets for less, recognizing the lender’s eagerness to dispose of the collateral. What we are assuming here, by contrast,
is that the buyer of the assets is another financial institution that must also issue short-term
debt in order to finance the purchase. Hence, the buyer is constrained by the same forces
that determined the debt capacity in the first place. Note that the buyer’s valuation of the
assets might be much greater than the debt capacity, but the finance constraint prevents
him from offering to pay his true value.

2.1 A numerical example

To illustrate the method of calculating debt capacity in the presence of rollover risk, we
use the following parameter values: the Poisson parameter is $\alpha = 10$, the recovery rate is
$\lambda = 0.90$, the tenor of the repo is $\tau = 0.01$, the values of the asset are $v^H = 100$ and $v^L = 50$
in the high and low states, respectively, and the transition probability matrix is

$$
P = \begin{bmatrix}
0.20 & 0.80 \\
0.01 & 0.99
\end{bmatrix}.
$$

The transition probability matrix for an interval of unit length can be calculated to be

$$
P(1) = \begin{bmatrix}
0.01265 & 0.98735 \\
0.01234 & 0.98766
\end{bmatrix}.
$$

At time 1, the fundamental values are 100 in state $H$ and 50 in state $L$ by assumption. So
the fundamental values at time 0 can be calculated by using the terminal values and the
transition probabilities in the matrix $P(1)$. The fundamental value in state $H$ at time 0 is

$$
V^H_0 = 0.98766 \times 100 + 0.01234 \times 50 = 99.383
$$
since, starting in state $H$ at time 0, there is a probability 0.98766 of being in state $H$ and a
probability 0.01234 of being in state $L$ at time 1. Similarly, the fundamental value in state
$L$ at time 0 is

$$
V^L_0 = 0.98735 \times 100 + 0.01265 \times 50 = 99.367.
$$

Note that the fundamental values are nearly identical. Even though the expected number of
information events in the unit interval is only 10 (this follows from the assumption $\alpha = 10$),
the transition probabilities are close to their ergodic or invariant distribution, which means
that the fundamental values are almost independent of the initial state. In spite of this, we

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8In particular, it is $P(1) = \sum_{k=0}^{\infty} \left\{ \left( \frac{e^{-\alpha}(\alpha)^k}{k!} \right) \begin{bmatrix}
 p_{LL} & p_{LH} \\
p_{HL} & p_{HH}
\end{bmatrix}^k \right\}$. For the numerical example, we ap-
proximate this as $P(1) = \sum_{k=0}^{200} \left\{ \left( \frac{e^{-10}(10)^k}{k!} \right) \begin{bmatrix}
 0.20 & 0.80 \\
0.01 & 0.99
\end{bmatrix}^k \right\}$.
shall find that the debt capacity of the asset, defined to be the maximum amount that can be borrowed using the asset as collateral, can be very different in the two states.

Whereas the fundamental value only depends on the state, debt capacity is determined by equilibrium in the repo market and has to be calculated for every one of the dates, \( t_0, \ldots, t_{99} \), at which repo contracts mature. To do this, we first have to calculate the transition probabilities over an interval of length \( \tau = 0.01 \), that is, the length of the period between rollover dates. We find that

\[
P (0.01) = \begin{bmatrix} 0.92315 & 0.07685 \\ 0.00096 & 0.99904 \end{bmatrix}.
\]  

(2)

Notice that the initial state has a much bigger impact on the transition probability. For example, the probability of ending up in state \( H \) after an interval 0.01 has passed is almost 1 if you start in state \( H \) but is close to 0.077 if you start in state \( L \). This is because the interval is so short that an information event is unlikely to occur before the next rollover date.

Consider now the debt capacities at the last rollover date \( t_{99} = 0.99 \). In what follows, we let \( D \) denote the face value of the debt issued and denote the optimal value of \( D \) at date \( t_n \) in state \( s \) by \( D_n^s \). It is never optimal to choose \( D > 100 \) because this leads to default in both states, with associated liquidation costs, but without any increase in the payoff. For values of \( D \) between 50 and 100 or less than 50, the expected value of the debt is increasing in \( D \) holding constant the probability of default. Then it is clear that the relevant face values of debt \( (D) \) to consider are 50 and 100. For any other face value we could increase \( D \) without changing the probability of default.

If we set \( D = 50 \), the debt can be paid off at date 1 in both states and the expected value of the payoff is 50. So the market value of the debt with face value 50 is exactly 50.

Now suppose we set \( D = 100 \), there will be default in state \( L \) but not in state \( H \) at time 1. The payoff in state \( H \) will be 100 but the payoff in state \( L \) will be \((0.9) 50 = 45.0\), because the recovery rate after default is 0.90. Then the market value of the debt at time \( t_{99} \) will depend on the state at time \( t_{99} \), because the transition probabilities depend on the state. We can easily calculate the expected payoffs in each state:

\[
\text{state } H : 0.99904 \times 100 + 0.00096 \times 0.9 \times 50 = 99.947;
\]

\[
\text{state } L : 0.07685 \times 100 + 0.92315 \times 0.9 \times 50 = 49.226.
\]

For example, if the state is \( H \) at date \( t_{99} \), then with probability 0.99904 the state is \( H \) at date 1 and the debt pays off 100 and with probability 0.00096 the state is \( L \) at date 1, the asset must be liquidated and the creditors only realize 45.

\[\text{We use the approximation } P (0.01) = \sum_{k=0}^{200} \left( e^{-0.1(0.1)^k} \right) \left[ \begin{array}{cc} 0.20 & 0.80 \\ 0.01 & 0.99 \end{array} \right]^k, \text{ as } \alpha \tau = 10(0.01) = 0.1.\]
Comparing the market values of the debt with the two different face values, we can see that the optimal face value will depend on the state. In state $H$, the expected value of the debt when $D = 100$ is $99.947 > 50$, so it is optimal to set $D_{99}^H = 100$. In state $L$, on the other hand, the expected value of the debt with face value $D = 100$ is only $49.226 < 50$, so it is optimal to set the face value $D_{99}^L = 50$. Thus, if we use the notation $B^s_n$ to denote the debt capacity in state $s$ at date $t_n$, we have shown that $B_{99}^H = 99.947$ and $B_{99}^L = 50$.

Next, consider the debt capacities at date $t_{98} = 0.98$. Now, the relevant face values to consider are 50 and $99.9470$ (since these are the maximum amounts that can be repaid in each state at date $t_{99}$ without incurring default and the associated liquidation costs).

If $D = 50$, the expected payoff is 50 too, since the debt capacity at date $t_{99}$ is greater than or equal to 50 in both states and, hence, the debt can always be rolled over. In contrast, if $D = 99.947$, the debt cannot be rolled over in state $L$ at date $t_{99}$ and the liquidation cost is incurred. Thus, the expected value of the debt depends on the state at date $t_{98}$:

- state $H$ : $0.99904 \times 99.9470 + 0.00096 \times 0.9 \times 50 = 99.894$,
- state $L$ : $0.07685 \times 99.9470 + 0.92315 \times 0.9 \times 50 = 49.222$.

Comparing the expected value corresponding to different face values of the debt, we see that the optimal face value is $D_{98}^H = 99.947$ in state $H$ and $D_{98}^L = 50$ in state $L$, so that the debt capacities are $B_{98}^H = 99.894$ and $B_{98}^L = 50$. In fact, we did not really need to do the calculation again to realize that $B_{98}^L = 50$. The only change from the calculation we did at $t_{99}$ is that the payoff in state $H$ has gone down, so the expected payoff from setting $D = 99.947$ must have gone down too and, a fortiori, the optimal face value of the debt must be 50.

It is clear that we can repeat this argument indefinitely. At each date $t_n$, the debt capacity in the high state is lower than it was at $t_{n+1}$ and the debt capacity in the low state is the same as it was at $t_{n+1}$. These facts tells us that if it is optimal to set $D_{n+1}^L = 50$ at $t_{n+1}$, then a fortiori it will be optimal to set $D_n^L = 50$ at date $t_n$. Thus, the debt capacity is equal to 50 at each date $t_n$, including the first date $t_0 = 0$.

What is the debt capacity in state $H$ at $t_0$? The probability of staying in the high state from date 0 to date 1 is $(0.99904)^{100} = 0.90842$ and the probability of hitting the low state at some point is $1 - 0.90842 = 0.09158$ so the debt capacity at time 0 is

$$B_0^H = 0.90842 \times 100 + 0.09158 \times 0.9 \times 50 = 94.9603.$$ 

So the fall in debt capacity occasioned by a switch from the high to the low state at time 0 is $94.963 - 50 = 44.963$ compared to a change in the fundamental value of $99.383 - 99.367 = 0.016$. This fall is illustrated sharply in Figure 2 which shows that while fundamental values in states $H$ and $L$ will diverge sharply at maturity, they are essentially the same at date 0. Nevertheless, debt capacity in state $L$ is simply the terminal value in state $L$. Thus, a switch to state $L$ from state $H$ produces a sudden drop in debt capacity of the asset.
Another way to interpret this result is in terms of a change in the tenor of the debt. Prior to the beginning of the crisis in August 2007, buyers of ABCP became increasingly unwilling to lend at any but the shortest maturities. This change in lending behavior is equivalent to a reduction in $\tau$ in our model. A switch from the the high state to the low state, in addition to implying a change in the fundamental value of the asset, may be associated with a shortening of the tenor of the debt. Assuming the tenor in the high state is greater than $\tau = 0.01$, the debt capacity will be greater than what we have calculated. Then the switch to the low state and the tenor $\tau = 0.01$ will have an even greater effect on the debt capacity.

It is also interesting to compare the rollover debt capacity with the debt capacity if the asset were financed using debt with tenor $\tau = 1.0$, so that there is no need to roll over the debt. To be consistent with the way we calculated the rollover debt capacity, we need to allow for liquidation costs at date 1. In each state, it is optimal to choose the face value of the debt to be $D = 100$, so that there is always default if the low state occurs at date 1. To calculate the expected value of the debt, we use the transition probabilities from the matrix $P(1)$, just as we did with the calculation of fundamental values. The expected value of the debt in each state is

$$
B_0^H = 0.98766 \times 100 + 0.9 \times 0.01234 \times 50 = 99.321.
$$

$$
B_0^L = 0.98735 \times 100 + 0.9 \times 0.01265 \times 50 = 99.304.
$$

As with the fundamental values, the debt capacities with long term debt are much higher than in the rollover case and are also much closer together.\(^{10}\)

\section*{2.2 Discussion}

The intuition for the market freeze result can be explained in terms of the tradeoff between the costs of default and the face value of the debt. Suppose we are in the low information state at date $t_n$. If the period length $\tau$ is sufficiently short, it is very likely that the information state at the next rollover date $t_{n+1}$ will be the low state. Choosing a face value of the debt greater than $B_{n+1}^L$, the maximum debt capacity in the same state at date $t_{n+1}$, will increase the payoff to the creditors if good news arrives at the next date (the state switches to $H$), but it will also lead to default if there is no news (the state remains $L$). Since there is a liquidation (transaction) cost, issuing debt with face value greater than the debt capacity is always unattractive if the probability of good news is sufficiently small. Then, the best

\(^{10}\)The comparison of debt capacities with short-term debt ($\tau = 0.01$) and long-term debt ($\tau = 1.0$) assumes the rate of interest is the same for all maturities. Since the yield curve slopes upwards, this is unrealistic. However, even with a higher interest rate for debt with the longer maturity, the basic point remains valid.
the borrower can do is to issue debt with a face value equal to the debt capacity assuming no new information. But this implies that the debt capacity in the low state is \( v^L \) at every date. In other words, no matter how high the fundamental value is in state \( L \), the borrower is forced to act as if the asset is only worth \( v^L \) in order to avoid default.

In the remainder of this section, we consider the role of the different assumptions of the model in driving the limit result on market freezes.

**Credit risk**  If \( v^H = v^L \), the terminal value of the asset is equal to the fundamental with probability one, so we can set the face value of the debt equal to \( v^H = v^L \) without any risk of default. In this case, the debt capacity must be equal to the fundamental value regardless of any other assumptions. So one necessary assumption is the existence of credit risk, that is, a positive probability that the terminal value of the asset will be less than the initial fundamental value. However, this credit risk can be arbitrarily small, as we illustrated in the numerical example where, at time 0, the probability that the asset’s value is 50 is less than 0.01. We could obtain the same results for smaller values of credit risk at the cost of increasing the number of rollovers.

**Liquidation cost**  We need a liquidation cost in order to have a market freeze. If the recovery ratio is \( \lambda = 1 \), then regardless of the credit risk, the debt capacity will equal the fundamental value. To see this, simply put the face value of the debt equal to 100 at each date. The market value of the debt will equal the fundamental value of the asset, which must equal the debt capacity. So a necessary condition of the market freeze is \( \lambda < 1 \). The liquidation cost does not need to be large, however. In the numerical example, the loss ratio was 0.1 and it could be set equal to a smaller value with an appropriate reduction in the maturity of the debt.

**Short-term debt**  Among the key assumptions of our model, we take as given the short-term nature of debt and liquidation costs. That many financial firms are funded with rollover debt and that debt capacity can be higher with short-term debt under some circumstances for many underlying assets, are interesting facts in their own right. Indeed, there exist agency-based explanations in the literature (for example, Flannery, 1986, Diamond, 1989, 1991, 2004, Calomiris and Kahn, 1991, and Diamond and Rajan, 2001a, 2001b) for the existence of short-term debt as optimal financing in such settings. In contrast to this literature, Brunnermeier and Oehmke (2009) consider a model where a financial institution is raising debt from multiple creditors and argue that there may be excessive short-term debt in equilibrium as short-term debt issuance dilutes long-term debt values and creates among various creditors a “maturity rat race.” Other reasons for the use of short-term debt are the attraction of betting on interest rates if bankers have short-term horizons and choose to shift
Our model presents another counter-example to the claim that short-term debt maximizes debt capacity: debt capacity through short-term borrowing may in fact be arbitrarily close to the minimum value of the asset, suggesting that financial institutions (or their prudential regulation) ought to guard against this possibility by arranging sufficient long-term financing. Providing a micro (for example, an agency-theoretic) foundation for debt maturity in a model where the information about the underlying assets’ fundamental value can change as in our model is a fruitful goal for future research, but one that is beyond the scope of this paper.

Finally, the market freeze in our model can also be interpreted as resulting from a sudden shortening in the tenor of the debt. If a news event, that perhaps signals a small change in the fundamental value of the assets, also causes lenders to restrict the tenor of the debt they are willing to hold, then the effect on the debt capacity will be even greater than what we have characterized.

**Rollover frequency** We have highlighted the role of rollover risk and indeed our main result requires that the rate of refinancing be sufficiently high in order to obtain a market freeze. Figure 3 illustrates the role of rollover frequency on debt capacity in state \( L \) by varying the number of rollovers as \( N = 10, 50 \) and 100. Debt capacity with just 10 rollovers is over 90, but falls rapidly to just above 60 with 50 rollovers, and 100 rollovers are sufficient to obtain the limiting result that debt capacity is the terminal value of 50 in state \( L \).

— Figure 3 here —

We focus on this limiting case of sufficiently many rollovers for two reasons. First, it seems realistic in light of the fact that most conduits had ABCP that was rolled over within a week (See Acharya, Schnabl and Suarez, 2009) and banks were essentially relying on overnight repo transactions for secured borrowing (a market whose failure brought Bear Stearns down). Secondly, this assumption allows us to get a clean result that is easy to understand. But even if the period length is longer than our result requires, so that it is optimal to set the face value greater than the debt capacity, it is still possible that a market freeze occurs, as we show with a numerical example in Section 3.

**Information structure** There are two aspects of the information structure that are crucial. First, information arrival is a Poisson process in which the probability of receiving “news” in a certain time period is proportional to the length of the period. Secondly, because we are assuming the period is short, the rollover frequency is high relative to the information flow. Note that for \( N \) rollovers, the period length between each rollover is \( \tau = 1/N \). When information arrives as a Poisson process with parameter \( \alpha \), the expected
number of information events between successive rollovers is ατ. Hence, as the number of rollovers increases, that is, as τ gets smaller, the expected number of information events in each rollover period gets smaller, that is, information arrives slowly relative to rollovers.

It is important to note that we do not make any special assumptions about the transition probabilities P. In particular, we can impose a substantial amount of symmetry if desired. For example, the information state can be a symmetric random walk with reflecting barriers. The only essential property is that the probability of “no news” converges to one as the period length converges to zero. The Poisson process is one way to satisfy this regularity condition, which seems quite natural in this context.

3 Debt capacity with two states

In this section we provide a proof for the market freeze result when there are two states. We make the same assumptions as for the numerical example but the parameters are otherwise arbitrary. For the time being, we treat the tenor of the commercial paper τ and the number of rollovers N as fixed. Later, we will be interested to see what happens when the tenor τ becomes very small and the number of rollovers N becomes correspondingly large.

There are two states, a “low” state L and a “high” state H. Transitions between states occur at the dates t_n and are governed by a stationary transition probability matrix

\[
P(\tau) = \begin{bmatrix}
1 - q(\tau) & q(\tau) \\
p(\tau) & 1 - p(\tau)
\end{bmatrix},
\]

where \(p(\tau)\) is the probability of a transition from state H to state L and \(q(\tau)\) is the probability of a transition from state L to state H when the period length is \(\tau > 0\). So, \(p(\tau)\) is the probability of “bad news” and \(1 - p(\tau)\) is the probability of “no news” in the high state. Similarly, in the low state, the probability of “good news” is \(q(\tau)\) and the probability of “no news” is \(1 - q(\tau)\). The one requirement we impose on these probabilities is that the shorter the period length, the more likely it is that no news arrives before the next rollover date:

\[
\lim_{\tau \to 0} p(\tau) = \lim_{\tau \to 0} q(\tau) = 0.
\]

The terminal value of the asset is \(v^H\) if the terminal state is H and \(v^L\) if the terminal state is L, where \(0 < v^L < v^H\).

The debt capacity of the assets can be determined by backward induction. Suppose that the economy is in the low state at date \(t_N\), which is the last of the rollover dates. Let \(D\) be the face value of the debt issued by the bank. If \(D > v^H\), the bank will default in both states at date \(t_{N+1}\) and the creditors will receive \(\lambda v^H\) in the high state and \(\lambda v^L\) in the low
Clearly, the market value of the debt at date $t_N$ would be greater if the face value were $D = v^H$, so it cannot be optimal to choose $D > v^H$. Now suppose that the bank issues debt with face value $D$, where $v^L < D < v^H$. This will lead to default in the low state at date $t_{N+1}$ and the creditors will receive $D$ in the high state and $\lambda v^L$ in the low state. Clearly, this is dominated by choosing a higher value of $D$. Thus, either $D = v^H$ or $D \leq v^L$. An exactly similar argument shows that it cannot be optimal to choose $D < v^L$, so we are left with only two possibilities, either $D = v^H$ or $D = v^L$. In the first case, the market value of the debt is $(1 - q(\tau)) \lambda v^L + q(\tau) v^H$ and in the second case it is $v^L$. Clearly, for $\tau$ sufficiently small,

$$\tag{3} (1 - q(\tau)) \lambda v^L + q(\tau) v^H < v^L.$$ 

Let $\tau^* > 0$ denote the critical value such that (3) holds if and only if $\tau < \tau^*$. For any $\tau < \tau^*$ it is strictly optimal to choose $D = v^L$. Then we have found the debt capacity in the low state at date $t_N$, which we denote by $B^L_N = v^L$.

Now consider the high state at date $t_N$. It is easy to see, as before, that the only candidates for the face value of the debt are $D = v^H$ and $D = v^L$. If the bank issues debt with face value $v^H$, there will be default in the low state. The creditors will receive $\lambda v^L$ in the low state and $v^H$ in the high state and the market value of the debt at date $t_N$ will be $(1 - p(\tau)) v^H + p(\tau) \lambda v^L$. If the bank issues debt with face value $v^L$, there will be no default, the creditors will receive $v^L$ in both states at date $t_{N+1}$ and the market value of the debt at date $t_N$ will be $v^L$. If the period length $\tau$ is sufficiently short, we can see that

$$\tag{4} (1 - p(\tau)) v^H + p(\tau) \lambda v^L > v^L.$$ 

Let $\tau^{**} > 0$ denote the critical value such that (4) is satisfied if and only if $\tau < \tau^{**}$. So if $\tau < \tau^{**}$, it is strictly optimal to put $D = v^H$. Then we have found the debt capacity in the high state at date $t_N$, which we denote by $B^H_N = (1 - p(\tau)) v^H + p(\tau) \lambda v^L$.

Now suppose that we have calculated the debt capacities $B^H_n$ and $B^L_n$ for $n = k+1, \ldots, N$. We show that

**Proposition 1** For $0 < \tau < \min \{\tau^*, \tau^{**}\}$ and

$$v^H - \lambda v^L > e^\alpha (1 - \lambda) v^L,$$

the debt capacities of the asset in states $H$ and $L$ are given respectively by the formulae

$$B^H_n = (1 - p(\tau))^{N-n} v^H + \left[1 - (1 - p(\tau))^{N-n}\right] \lambda v^L, \text{ for } n = k, \ldots, N, \tag{5}$$

and

$$B^L_n = v^L, \text{ for } n = k, \ldots, N. \tag{6}$$

---

\[11\] To simplify the argument, we are assuming that there is a liquidation cost at date $t_{N+1}$ even though there is no need to sell the asset at that date. None of the results depend on this.
The low state  Consider what happens in the low state at date \( t_k \). If the face value of the debt issued by the bank is \( D \) at date \( t_k \), then the bank will default in the low state if \( v^L < D < B^H_{k+1} \) and the bank will default in both states if \( D > B^H_{k+1} \). By our usual argument, the only candidates for the optimal face value are \( D = v^L \) and \( D = B^H_{k+1} \). If the face value is \( D = v^L \), the creditors will receive \( v^L \) in both states at date \( t_k + 1 \) and the market value of the debt at date \( t_k \) will be \( v^L \). On the other hand, if the face value of the debt is \( D = B^H_{k+1} \), the creditors receive \( B^H_{k+1} \) in the high state and \( \lambda v^L \) in the low state, so the market value of the debt at date \( t_k \) is

\[
(1 - q(\tau)) \lambda v^L + q(\tau) B^H_{k+1} \leq (1 - q(\tau)) \lambda v^L + q(\tau) v^H,
\]

since \( B^H_{k+1} \leq v^H \). But \( \tau < \tau^* \) implies that \( (1 - q(\tau)) \lambda v^L + q(\tau) v^H < v^L \), so the debt capacity is \( B^L_k = v^L \).

The high state  Now consider the high state. Again, our two candidates for the face value of the debt are \( D = B^H_{k+1} \) and \( D = v_r \), which yield market values at date \( t_k \) of

\[
(1 - p(\tau)) B^H_{k+1} + p(\tau) \lambda v^L
\]

and

\[
v^L
\]

respectively. From our induction hypothesis,

\[
(1 - p(\tau)) B^H_{k+1} + p(\tau) \lambda v^L
\]

\[
= (1 - p(\tau)) \left\{ (1 - p(\tau))^{N-k-1} v^H + \left[ 1 - (1 - p(\tau))^{N-k-1} \right] \lambda v^L \right\} + p(\tau) \lambda v^L
\]

\[
= (1 - p(\tau))^{N-k} (v^H - \lambda v^L) + \lambda v^L.
\]

Then \( D^H_k = B^H_{k+1} \) is strictly optimal if

\[
(1 - p(\tau))^{N-k} (v^H - \lambda v^L) + \lambda v^L > v^L
\]

or

\[
v^H - \lambda v^L > \frac{(1 - \lambda) v^L}{(1 - p(\tau))^{N-k}}. \tag{7}
\]

In order for this inequality to be satisfied for all \( n \) it must be satisfied for \( n = 0 \). We can show that \((1 - p(\tau))^N \geq e^{-\alpha}\), because \( e^{-\alpha} \) is the probability of no information events in the unit interval. Hence, a sufficient condition for \((7)\) to be satisfied is

\[
v^H - \lambda v^L > e^\alpha (1 - \lambda) v^L.
\]

By induction, we have proved that the debt capacities are given by the formulae in \((5)\) and \((6)\) for all \( n = 0, ..., N \).

Note that the sufficient condition above is always satisfied for \( v^L \) sufficiently close to zero, which is a reasonable assumption given the two-state modeling of possible future asset values. For a given \( v^L < v^H \), the condition is also met for liquidation cost that is sufficiently
small, that is, \( \lambda \) close to one. Further, this sufficient condition is in fact not necessary and our analysis in Section 4 of the model with an arbitrary number of states does not rely on it.

The qualitative properties of the debt capacities characterized in Proposition 1 are the same as in the numerical example in Section 2.1. In the low state, the debt capacity \( B_n^L \) is constant and equal to the lowest possible terminal value, \( v^L \). The fundamental value of the asset in the low state \( V_n^L \) is greater than the debt capacity at every date \( t_n \) except at the terminal date, when they are both equal to \( v^L \). In the high state, the debt capacity \( B_n^H \) is always less than the fundamental value \( V_n^H \), except at the terminal date when both are equal to \( v^H \). We call this behavior of the debt capacity a “market freeze” since a switch in the information state from high state to the low state can produce a sudden, sharp drop in debt capacity that is much larger than the drop in fundamental value associated with the switch.

Debt capacity with intermediate rollover risk

We can get similar results even if the period length is not short enough to generate the result stated in Proposition 1. A simple adaptation of the numerical example will illustrate a scenario in which it is optimal to choose a high face value of debt in the low state, with the result that the bank faces a positive probability of default if the economy remains in the low state. Suppose that the value of the asset in the low state is \( v^L = 40 \). All the other parameters remain the same. Now the loss from default in the low state is less than the gain from a high face value in the high state, so it is optimal for the face value of the debt to be set equal to next period’s debt capacity in the high state.

As before, we calculate the debt capacity, beginning with the last rollover date. The last rollover date is \( t_{99} \). The transition probabilities are given by equation (2) as before. If the face value of the debt is set equal to \( v_H = 100 \) in the low state, the market value of the debt issued will be \( 0.07685 \times 100 + 0.92315 \times 0.90 \times 40 = 40.918 \), which is higher than the face value obtained by setting the face value equal to 40. Thus, the optimal face value implies default if the economy remains in the low state. It is still optimal to set the face value of the debt equal to \( D_{99}^H = 100 \) in the high state, and the debt capacity is now \( B_{99}^H = 99.939 \).

As long as the face value of the debt is set equal to \( B_{n+1}^H \) in both states, the debt capacity satisfies

\[
\begin{bmatrix}
B_n^H \\
B_n^L
\end{bmatrix} =
\begin{bmatrix}
0.99904 & 0.90 \times 0.00096 \\
0.07685 & 0.90 \times 0.92315
\end{bmatrix}
\begin{bmatrix}
B_{n+1}^H \\
B_{n+1}^L
\end{bmatrix}.
\]

However, this assumes that it is optimal to have default in the low state at every rollover date, which is not necessarily true. Starting at the last rollover date, it can be shown that the debt capacity in state \( L \) rises as we go back in time, reaches a maximum at \( t_{80} \), and then declines as we move to earlier and earlier dates (see Figure 4). The problem is that as
the debt capacity rises, the liquidation costs (which are proportional to the debt capacity) also rise and eventually outweigh the upside potential of a switch to the high state\footnote{At the point where the maximum is reached, it is optimal to change the face value of the debt from $B^H_{n+1}$ to $B^L_{n+1}$ and avoid default in the low state. Then the debt capacity is given by the formula above for $n = 80, \ldots, 99$ and is given by $B^L_n = B^L_{80}$ for $n = 0, \ldots, 80$. We can use the formula in equation (8) to show that $B^L_{80} = 44.918$ and $B^H_{80} = 98.847$. The gap between the debt capacities in the two states is $94.469 - 44.918 = 49.551$, compared to the negligible difference in the fundamental values $99.2596$ and $99.241$ the high and the low states, respectively. Thus, even if it is optimal to capture the upside potential of a switch to the high state, the debt capacity in the low state does not rise much above the minimum value of the asset, i.e., it is $44.918$ rather than $40$.}

In the rest of the paper, we explore the determinants of debt capacity in a richer model with many states and a broad range of parameters. We model the release of information in terms of the arrival of “news.” The two-state example only allows for no news or bad news in state $H$ and for no news or good news in state $L$. The model presented in Section 4 extends the setup and the results to an arbitrary finite number of information states. At each date, one of three things can happen: either there is “good news” (the information state improves), there is “bad news” (the information state gets worse), or there is “no news.” If the period between roll-over dates is sufficiently short, it is most likely that “no news” will have been released by the time the debt has to be refinanced.

4 Debt capacity in the general case

We allow for a finite number of information states or signals, denoted by $S = \{s_1, \ldots, s_I\}$. The current information state is public information. Changes in the information state arrive randomly. The timing of the information events is a homogeneous Poisson process with parameter $\alpha > 0$. The probability of $k$ information events, in an interval of length $\tau$, is

$$\Pr[K(t + \tau) - K(t) = k] = \frac{e^{-\alpha \tau} (\alpha \tau)^k}{k!},$$

where $K(t)$ is the number of information events between 0 and $t$, and the expected number of information events in an interval of length $\tau$ is

$$E[K(t + \tau) - K(t)] = \sum_{k=1}^{\infty} \left( \frac{e^{-\alpha \tau} (\alpha \tau)^k}{k!} \right) = \alpha \tau.$$

It is also possible to extend this example to the case with fixed costs of liquidation. Details are available from authors upon request.
Conditional on an information event occurring at date \( t \), the probability of a transition from state \( s_i \) to state \( s_j \) is denoted by \( p_{ij} \geq 0 \), where \( \sum_{j=1}^{I} p_{ij} = 1 \). These transition probabilities are described by the \( I \times I \) matrix

\[
P = \begin{bmatrix}
p_{11} & \cdots & p_{1I} \\
\vdots & \ddots & \vdots \\
p_{I1} & \cdots & p_{II}
\end{bmatrix}.
\]

The transition probabilities over an interval of length \( \tau \) depend on the number of information events \( k \), a random variable, and the transition matrix \( P \). The transition matrix for an interval \( \tau \) is denoted by \( P(\tau) \) and defined by

\[
P(\tau) = \sum_{k=0}^{\infty} \frac{e^{-\alpha \tau} (\alpha \tau)^k}{k!} P^k.
\]

The information state is a stochastic process \( \{S(t)\} \) but for our purposes all that matters is the value of this process at the rollover dates. We let \( S_n \) denote the value of the information state \( S(t_n) \) at the rollover date \( t_n \) and we say that there is “no news” at date \( t_{n+1} \) if \( S_{n+1} = S_n \). In other words, we regard the current state as the status quo and say that news arrives only if a new information state is observed. Of course, “no news” is also informative and beliefs about the terminal value of the assets will be updated even if the information state remains the same. Again, it is important to note is that, when the period length \( \tau \) is short, the probability of “news” becomes small and the probability of “no news” becomes correspondingly large. In fact,

\[
\lim_{\tau \to 0} P(\tau) = P(0) = I.
\]

In that case, the informational content of “no news” is also small.

The terminal value of the assets is a function of the information state at date \( t = 1 \). We denote by \( v_i \) the value of the assets if the terminal state is \( S_{N+1} = s_i \) and assume that the values \( \{v_1, \ldots, v_I\} \) satisfy

\[
0 < v_1 < \ldots < v_I.
\]

Let \( V_n^i \) denote the fundamental value of the asset at date \( t_n \) in state \( i \). Then clearly the values \( \{V_n^i\} \) are defined by putting \( V_{N+1}^i = v_i \) for \( i = 1, \ldots, I \) and

\[
V_n^i = \sum_{j=1}^{I} p_{ij} (1 - t_n) v_j, \text{ for } n = 0, \ldots, N \text{ and } i = 1, \ldots, I,
\]

where \( p_{ij} (1 - t_n) \) is, of course, the \((i,j)\) entry of \( P(1 - t_n) \) denoting the probability of a transition from state \( i \) at date \( t_n \) to state \( j \) at date \( t_{N+1} = 1 \).
Figure 5 illustrates the fundamental values in a setup with \( I = 6 \) states where terminal values \( v_1 \) through \( v_6 \) are equally spaced from 50 to 100, and \( \alpha = 10 \). The transition matrix \( P \) is described in Appendix B. As in our two-state example, the fundamental values in different states are virtually identical at date 0 though they diverge in steps of 10 at maturity.

Let \( B^i_n \) denote the equilibrium debt capacity of the assets in state \( s_i \) at date \( t_n \). By convention, we set \( B^i_N+1 = v_i \) for all \( i \).

**Proposition 2** The equilibrium values of \( \{B^i_n\} \) must satisfy

\[
B^i_n = \max_{k=1,\ldots,I} \left\{ \frac{1}{\lambda} \sum_{j=1}^{k-1} p_{ij}(\tau) B^j_{n+1} + \sum_{j=k}^{I} p_{ij}(\tau) B^k_{n+1} \right\}
\]

for \( i = 1, \ldots, I \) and \( n = 0, \ldots, N \).

The result is immediate once we apply the now familiar backward induction argument to show that it is always optimal to set the face value of the debt \( D^i_n \) equal to \( B^j_{n+1} \) for some \( j \). Although the result amounts to little more than the definition of debt capacity, it is very useful because it allows us to calculate the debt capacities by backward induction.

We use the formula in Proposition 2 to obtain the limiting value of the debt capacity as \( \tau \to 0 \). An auxiliary assumption is helpful in proving this result: higher information states are assumed to be “better” in the sense that

\[
V_{in} < V_{i+1,n}, \text{ for all } i = 1, \ldots, I - 1 \text{ and } n = 0, \ldots, N + 1. \tag{9}
\]

A sufficient (but, as we show in Appendix A, not necessary) condition for (9) is that \( \{p_{i+1,j}(\tau)\} \) strictly dominates \( \{p_{i,j}(\tau)\} \) in the sense of first-order stochastic dominance. That is, for all \( i = 1, \ldots, I - 1, \)

\[
\sum_{j=1}^{k} p_{ij}(\tau) > \sum_{j=1}^{k} p_{i+1,j}(\tau), \text{ for all } i, k = 1, \ldots, I - 1. \tag{10}
\]

**Proposition 3** Suppose that (10) is satisfied. Then there exists \( \tau^* > 0 \) such that for all \( 0 < \tau < \tau^* \), for any \( n = 0, \ldots, N \) and any \( i = 1, \ldots, I, \)

\[
B^i_n = \sum_{j=1}^{i-1} p_{ij}(\tau) B^j_{n+1} + \sum_{j=i}^{I} p_{ij}(\tau) B^j_{n+1}.
\]
Proof. See Appendix A. ■

Several properties follow immediately from Proposition 3 whenever $0 < \tau < \tau^*$. We provide these results formally in the form of four corollaries. First, in the lowest state, $s_1$, the debt capacity is constant and equal to $v_1$, the worst possible terminal value.

**Corollary 4** $B_n^1 = v_1$ for all $n$.

Proof. From the formula in Proposition 3,

$$B_n^1 = \sum_{j=1}^{I} p_{ij} (\tau) B_{n+1}^1 = B_{n+1}^1$$

for $n = 0, \ldots, N$ so claim follows from our convention that $B_{N+1}^1 = v_1$. ■

Second, the debt capacity $B_n^i$ is monotonically non-decreasing in $n$, that is, debt capacity increases as the asset matures, holding the state constant. This follows directly from the fact that, if the face of the debt equals $B_{n+1}^i$, the debt capacity $B_n^i$ cannot be greater than $B_{n+1}^i$.

**Corollary 5** $B_n^i \leq B_{n+1}^i$ for any $i = 1, \ldots, I$ and $n = 0, \ldots, N$.

Proof. The inequality follows directly from the formula in Proposition 3

$$B_n^i = \sum_{j=1}^{i-1} p_{ij} (\tau) \lambda B_{n+1}^j + \sum_{j=i}^{I} p_{ij} (\tau) B_{n+1}^i$$

$$\leq \sum_{j=1}^{i-1} p_{ij} (\tau) B_{n+1}^i + \sum_{j=i}^{I} p_{ij} (\tau) B_{n+1}^i$$

$$= \sum_{j=1}^{I} p_{ij} (\tau) B_{n+1}^i = B_{n+1}^i,$$

since $\lambda B_{n+1}^j \leq B_{n+1}^i$ for $j = 1, \ldots, i - 1$. ■

Third, since $B_{N+1}^i = v_i$ by convention, the preceding result immediately implies that the debt capacity $B_n^i$ is less than or equal to $v_i$.

**Corollary 6** $B_n^i \leq v_i$ for all $i = 1, \ldots, I$ and $n = 0, \ldots, N$.

Finally, we can confirm that the debt capacity in state $s_i$ at any date $t_n$ is less than the fundamental value $V_{n}^i$. This follows directly from the formula in Proposition 3 for $n = N + 1$
and any $i$, so suppose that it holds for $n, ..., N$ and any $i = 1, ..., I$. Then the formula in Proposition 3 implies that

$$B^i_n = \sum_{j=1}^{i-1} p_{ij}(\tau) \lambda B^j_{n+1} + \sum_{j=1}^{I} p_{ij}(\tau) B^j_{n+1}$$

$$\leq \sum_{j=1}^{i-1} p_{ij}(\tau) \lambda V^j_{n+1} + \sum_{j=1}^{I} p_{ij}(\tau) V^j_{n+1}$$

$$\leq \sum_{j=1}^{I} p_{ij}(\tau) V^j_{n+1} = V^i_n,$$

for any $i = 1, ..., I$, so by induction the claim holds for any $n = 0, ..., N$ and any $i = 1, ..., I$.

Some of these properties are illustrated in Figures 6a and 6b which show the debt capacities in the six states of our numerical example for $N = 10$ and $N = 100$ rollovers, respectively. For 10 rollovers, $\tau$ is not sufficiently small to obtain our limit result even in the worst state and debt capacity in each state is in fact higher than the terminal value in that state. Nevertheless, it is still the case that there is a drop in debt capacity of between 5 and 10 as the state changes to the next worse one, without much change in the fundamental value. By contrast, with 100 rollovers, the limit result is obtained and in fact debt capacity in the two worst states (states 1 and 2) is (essentially) the minimum possible value of the asset which is 50. Furthermore, as we go from the best state (state 6) to the second-best state (state 5), debt capacity falls roughly by a magnitude of 25 even though the fundamental values (Figure 5) has hardly changed. Thus, the market freeze is substantially worse with 100 rollovers compared to 10.

--- Figure 6a and 6b here ---

4.1 Constant liquidation costs

In this section we provide statements of the results for the case of a constant transaction cost $c > 0$. All the other assumptions of the preceding model are maintained. Because the arguments are very similar we will not repeat the proofs of the results.\(^{13}\)

The form of the liquidation cost has no effect on the argument that the optimal face value of the debt $D_n^i$ belongs to the set $\{B^1_{n+1}, ..., B^I_{n+1}\}$. Then it immediately follows that the equilibrium value of $B^i_n$ must satisfy

$$B^i_n = \max_{k=1, ..., I} \left\{ \sum_{j=1}^{k-1} p_{ij}(\tau) (B^j_{n+1} - c) + \sum_{j=k}^{I} p_{ij}(\tau) B^j_{n+1} \right\}$$

\(^{13}\)Proofs are available from the authors upon request.
for $i = 1, \ldots, I$ and $n = 0, \ldots, N$. Using this formula and familiar arguments we can establish the analogue of Proposition 3.

**Proposition 7** Suppose that (10) is satisfied. Then there exists $\tau^* > 0$ such that for all $0 < \tau < \tau^*$, for any $n = 0, \ldots, N$ and any $i = 1, \ldots, I$,

$$B^i_n = \sum_{j=1}^{i-1} p_{ij}(\tau) (B^j_{n+1} - c) + \sum_{j=i}^{I} p_{ij}(\tau) B^i_{n+1}.$$ 

The analogues of the earlier corollaries follow, for any $0 < \tau < \tau^*$. In the lowest state, the debt capacity $B^1_n$ is always equal to the terminal value $v_1$. In any state, the debt capacity is non-decreasing with respect to the rollover date $t_n$, that is, $B^i_n \leq B^i_{n+1}$ for any $i$ and $n \leq N$. This implies that the debt capacity is less than or equal to the terminal value in each state, that is, $B^i_n \leq v_i$ for all $i$ and $n \leq N$. Finally, we confirm that the debt capacity is less than the fundamental value, that is, $B^i_n \leq V^i_n$ for any $i$ and $n \leq N$.

5 Related research

At a general level, our result on market freezes can be considered a generalization of the Shleifer and Vishny (1992) and Allen and Gale (1994) results that when potential buyers of assets of a defaulted firm are themselves financially constrained, there is a reduction in the ex-ante debt capacity of the industry as a whole. We expand on their insight by considering short-term debt financing of long-term assets with rollovers to be met by new short-term financing or liquidations to other buyers also financed through short-term debt. Our “market freeze” result can be considered as a particularly perverse dynamic arising through the Shleifer and Vishny (1992) and the Allen and Gale (1994) channel at each rollover date, that through backward induction, can in the worst case drive short-term debt capacity of an asset to its minimum possible cash flow.

More specifically, our paper is related to the literature on freezes and runs in financial markets. Rosenthal and Wang (1993) use a model where owners occasionally need to sell their assets for exogenous liquidity reasons through auctions with private information. Because of the informational rents earned by the privately informed bidders, sellers may not be able to extract the full value of the asset and this liquidation cost gets built into the market price of the asset, making the market price systematically lower than the fundamental value. In our model, the reason for the debt capacity being lower than the fundamental value is not the private information of potential buyers, rather it is the rollover risk and the liquidation cost associated with defaults.

He and Xiong (2009) consider a model of dynamic debt runs in which creditors have supplied debt maturing at differing maturities and each creditor faces the risk, at the time of
rolling over the debt, that fundamentals may deteriorate before the remaining debt matures, causing a fire sale of assets. In their model, the volatility of fundamentals plays a key role in driving the runs, even when the average value of fundamentals has not been affected. Our model is complementary to theirs and somewhat different in the sense that both average value and uncertainty about fundamentals are held constant in our model. It is the nature of information revelation – whether good news arrives early or bad news arrives early – that determines whether there is rollover risk in short-term debt.

Huang and Ratnovski (2008) model the behavior of short-term wholesale financiers who prefer to rely on noisy public signals such as market prices and credit ratings, rather than producing costly information about the institutions they lend to. Hence, wholesale financiers run on other institutions based on imprecise public signals, triggering potentially inefficient runs. While their model is about runs in the wholesale market, as is ours, their main focus is to challenge the peer-monitoring role of wholesale financiers, whereas our main focus is the role of rollover and liquidation risk in generating such runs.

An alternative modelling device to generate market freezes is to employ the notion of Knightian uncertainty (see Knight, 1921) and agents’ overcautious behavior towards such uncertainty. Gilboa and Schmeidler (1989) build a model where agents become extremely cautious and consider the worst-case among the possible outcomes, that is, agents are uncertainty averse and use maxmin strategies when faced with Knightian uncertainty. Dow and Werlang (1992) apply the framework of Gilboa and Schmeidler (1989) to the optimal portfolio choice problem and show that there is an interval of prices within which uncertainty-averse agents neither buy nor sell the asset. Routledge and Zin (2004) and Easley and O’Hara (2005, 2008) use Knightian uncertainty and agents that use maxmin strategies to generate widening bid-ask spreads and freeze in financial markets. Caballero and Krishnamurthy (2007) also use the framework of Gilboa and Schmeidler (1989) to develop a model of financial crises: During periods of increased Knightian uncertainty, agents refrain from making risky investments and hoard liquidity, leading to flight to quality and freezes in markets for risky assets. While ambiguity aversion leads to a market freeze in these models, in our model agents maximize expected utility and the main source of the market freeze is rollover and liquidation risk.

We regard our approach as complementary to Knightian uncertainty. Knightian uncertainty is appropriate when investors have very limited information about the nature of the risks they face. We are interested, by contrast, in explaining the drying up of liquidity in the absence of obvious problems of asymmetric information or fears about the value of collateral. For this purpose, it would seem to be an advantage to appeal to standard assumptions about preferences and beliefs.
6 Conclusion

In this paper, we have attempted to provide a simple information-theoretic model for freezes in the market for secured borrowing against finitely lived assets. The key ingredients of our model were rollover risk, liquidation risk, rapid rate of refinancing relative to the arrival of news, and similarity of financial institutions in their degree of maturity mismatch. In particular, our model could be interpreted as a micro-foundation for the funding risk arising in capital structures of financial institutions or special purpose vehicles that have extreme maturity mismatch between assets and liabilities.

In future work, it would be interesting to embed an agency-theoretic role for short-term debt, which we assumed as given, and see how the desirability of such rollover finance is affected when information problems can lead to complete freeze in its availability. While we took the release of information about the underlying asset as ordained by nature, it seems worthwhile to reflect on its deeper foundations, and thereby assess whether a strategic disclosure of information by agents in charge of the asset can alleviate (or aggravate) the problem of freezes.

Appendix A: Proofs

We can solve for the equilibrium debt capacities in the model of Section 4 by backward induction. Let $D$ denote the face value of the debt issued in state $s_i$ at date $t_n$. This debt will pay off $D$ in state $s_j$ at date $t_{n+1}$ if $D \leq B_{n+1}^j$ and $\lambda B_{n+1}^j$ otherwise. In other words, the market value of the debt is given by the formula

$$\sum_{B_{n+1}^j < D} p_{ij}(\tau) \lambda B_{n+1}^j + \sum_{B_{n+1}^j \geq D} p_{ij}(\tau) D$$

and the debt capacity is given by

$$B_n^i = \max_D \left\{ \sum_{B_{n+1}^j < D} p_{ij}(\tau) \lambda B_{n+1}^j + \sum_{B_{n+1}^j \geq D} p_{ij}(\tau) D \right\}.$$

**Proposition 8** $B_n^i \leq B_{n+1}^i$, for $i = 1, \ldots, I - 1$ and $n = 0, \ldots, N + 1$.

**Proof.** The claim is clearly true by definition when $n = N + 1$, so suppose it is true for some arbitrary number $n + 1$. Then, for any $D$ and $i = 1, \ldots, I - 1$, it is clear that

$$\sum_{B_{n+1}^j < D} p_{ij}(\tau) \lambda B_{n+1}^j + \sum_{B_{n+1}^j \geq D} p_{ij}(\tau) D \leq \sum_{B_{n+1}^j < D} p_{i+1,j}(\tau) \lambda B_{n+1}^j + \sum_{B_{n+1}^j \geq D} p_{i+1,j}(\tau) D,$$
because \( \{p_{ij}(\tau)\} \) is first-order stochastically dominated by \( \{p_{i+1,j}(\tau)\} \). It follows immediately that \( B^i_n \leq B^{i+1}_n \) for \( i = 1, \ldots, I - 1 \). The claim in the proposition then follows by induction.

Let \( D^i_n \) denote the optimal face value of the debt in state \( i \) at date \( t_n \). It is clear that the market value of the debt is maximized by setting the face value \( D^i = B^{i+1}_n \), for some value of \( j = 1, \ldots, I \). Thus, we can write the equilibrium condition as

\[
B^i_n = \max_{k=1,\ldots,I} \left\{ \sum_{j=1}^{k-1} p_{ij}(\tau) \lambda B^j_{n+1} + \sum_{j=k}^I p_{ij}(\tau) B^k_{n+1} \right\},
\]

for \( i = 1, \ldots, I \) and \( n = 0, \ldots, N \).

Now suppose that it is optimal to set \( D^i_n = B^{i+1}_n \) for every \( i = 1, \ldots, I \) and \( n = 0, \ldots, N \). The following proposition gives a lower bound to the difference between \( B^{i+1}_n \) and \( B^i_n \).

**Proposition 9** Suppose that it is optimal to set \( D^i_n = B^{i+1}_n \) for every \( i = 1, \ldots, I \) and \( n = 0, \ldots, N \). Then \( B^{i+1}_n \geq B^i_n + e^{-\alpha} (v_{i+1} - v_i) \), for \( i = 1, \ldots, I - 1 \) and \( n = 0, \ldots, N + 1 \).

**Proof.** Think of \( B^i_n \) as the average of two quantities, one being the debt capacity conditional on the occurrence of at least one information event after date \( n \), denoted by \( \tilde{B}^i_n \), and the other being the debt capacity conditional on no information events after \( n \), denoted by \( \hat{B}^i_n \). In the first case, the preceding argument suffices to show that \( \tilde{B}^{i+1}_n \geq \tilde{B}^i_n \). In the second case, we have \( \hat{B}^{i+1}_n = \hat{B}^i_n + (v_{i+1} - v_i) \) since the state remains constant in each case. Since the probability of no information events after date \( t_n \) is \( e^{-\alpha(1-t_n)} \geq e^{-\alpha} \), it follows that

\[
B^{i+1}_n - B^i_n \geq e^{-\alpha} (v_{i+1} - v_i)
\]

as claimed.

**Proposition 10** For all \( \tau > 0 \) sufficiently small, \( D^i_n = B^{i+1}_n \) for all \( i = 1, \ldots, I \) and \( n = 0, \ldots, N \).

**Proof.** For a fixed but arbitrary date \( t_n \) and state \( s_i \), we compare the strategy of setting \( D = B^i_{n+1} \) with the strategy of setting \( D = B^k_{n+1} \). First suppose \( k > i \) and consider the difference in the expected values of the debt:
\[
\sum_{j=1}^{i-1} p_{ij}(\tau) \lambda B^j_{n+1} + \sum_{j=i}^I p_{ij}(\tau) B^i_{n+1} - \sum_{j=1}^{k-1} p_{ij}(\tau) \lambda B^j_{n+1} - \sum_{j=k}^I p_{ij}(\tau) B^k_{n+1} \\
= \sum_{j=i}^k p_{ij}(\tau) (B^i_{n+1} - \lambda B^j_{n+1}) + \sum_{j=k}^I p_{ij}(\tau) (B^i_{n+1} - B^k_{n+1}) \\
= p_{ii}(\tau) (B^i_{n+1} - \lambda B^i_{n+1}) + \sum_{j=i+1}^{k-1} p_{ij}(\tau) (B^i_{n+1} - \lambda B^j_{n+1}) + \sum_{j=k}^I p_{ij}(\tau) (B^i_{n+1} - B^k_{n+1}) \\
\geq p_{ii}(\tau) (1 - \lambda) v_1 + \sum_{j=i+1}^{k-1} p_{ij}(\tau) (v_1 - v_I) + \sum_{j=k}^I p_{ij}(\tau) (v_1 - v_I) \\
= p_{ii}(\tau) (1 - \lambda) v_1 + \sum_{j=i+1}^I p_{ij}(\tau) (v_1 - v_I).
\]

since \( B^i_{n+1} \geq v_1 \), \( B^i_{n+1} - \lambda B^i_{n+1} \geq B^i_{n+1} - B^j_{n+1} \geq (v_1 - v_I) \) for \( j = i + 1, \ldots, I \) and \( B^i_{n+1} - B^k_{n+1} \geq v_1 - v_I \). Then it is clear that, for \( \tau \) sufficiently small, the last expression above is positive.

Similarly, for \( k < i \),

\[
\sum_{j=1}^{i-1} p_{ij}(\tau) \lambda B^j_{n+1} + \sum_{j=i}^I p_{ij}(\tau) B^i_{n+1} - \sum_{j=1}^{k-1} p_{ij}(\tau) \lambda B^j_{n+1} - \sum_{j=k}^I p_{ij}(\tau) B^k_{n+1} \\
= \sum_{j=k}^I p_{ij}(\tau) (\lambda B^j_{n+1} - B^k_{n+1}) + \sum_{j=i}^{i-1} p_{ij}(\tau) (B^i_{n+1} - B^k_{n+1}) \\
\geq \sum_{j=k}^I p_{ij}(\tau)(\lambda B^j_{n+1} - B^k_{n+1}) + p_{ii}(\tau)(B^i_{n+1} - B^k_{n+1}) \\
\geq (1 - p_{ii}(\tau))(\lambda v_1 - v_I) + p_{ii}(\tau)(B^i_{n+1} - B^k_{n+1}).
\]

since \( \lambda B^j_{n+1} - B^k_{n+1} \geq \lambda v_1 - v_I \).

At the last rollover date, \( n = N \) and \( B^i_{N+1} - B^k_{N+1} = (v_{i+1} - v_i) > 0 \) by definition. Then the last line above is positive for \( \tau \) sufficiently small and this proves that it is optimal to set \( D^i_N = B^i_{N+1} = v_i \).

Now suppose that we have shown that it is optimal to set \( D^i_n = B^i_{n+1} \) for \( n = \hat{n}, \ldots, N \) and consider the inequalities above for \( n = \hat{n} \). Then Proposition 9 tells us that \( B^i_{\hat{n}+1} - B^k_{\hat{n}+1} \geq e^{-\alpha} (v_i - v_k) > 0 \) and the last line above must be positive for \( \tau \) sufficiently small. This in turn establishes that \( D^i_{\hat{n}+1} = B^i_{\hat{n}+1} \) and, by induction, we have shown that it is optimal to set \( D^i_n = B^i_{n+1} \) for all this proves that it is optimal to set \( D^i_N = B^i_{N+1} = v_i \) for \( i = 1, \ldots, I \) and \( n = 0, \ldots, N \).
Appendix B: Numerical parameters for the example with $I = 6$ states

The terminal values for the 6 states are chosen as $v_i = 10^i$, for $i \in \{5,...,10\}$. As with the two-state example, we choose the Poisson rate to be $\alpha = 10$ and time to maturity as 1. The transition matrix given an information event $P$ is given below. Also shown are the unconditional transition matrices $P(N = 10)$ and $P(N = 100)$ that is, the transition matrices taking account of the information events with 10 and 100 rollovers, respectively.

$$
\begin{bmatrix}
0.2 & 0.8 & 0 & 0 & 0 & 0 \\
0.1 & 0 & 0.9 & 0 & 0 & 0 \\
0 & 0.1 & 0 & 0.9 & 0 & 0 \\
0 & 0 & 0.1 & 0 & 0.9 & 0 \\
0 & 0 & 0 & 0.1 & 0 & 0.9 \\
0 & 0 & 0 & 0 & 0.01 & 0.99 \\
\end{bmatrix}
$$

$$
\begin{bmatrix}
0.498446 & 0.328747 & 0.129744 & 0.034727 & 0.007006 & 0.00133 \\
0.041093 & 0.432477 & 0.341745 & 0.138156 & 0.037333 & 0.009196 \\
0.001802 & 0.037972 & 0.433411 & 0.342035 & 0.13748 & 0.0473 \\
5.36 \times 10^{-5} & 0.001706 & 0.038004 & 0.433336 & 0.338412 & 0.188488 \\
1.20 \times 10^{-6} & 5.12 \times 10^{-5} & 0.001697 & 0.037601 & 0.420155 & 0.540494 \\
2.53 \times 10^{-9} & 1.40 \times 10^{-7} & 6.49 \times 10^{-6} & 0.000233 & 0.006005 & 0.993755 \\
\end{bmatrix}
$$

$$
\begin{bmatrix}
0.923483 & 0.073136 & 0.00328 & 9.82 \times 10^{-5} & 2.21 \times 10^{-6} & 4.04 \times 10^{-8} \\
0.009142 & 0.905609 & 0.081471 & 0.003666 & 0.00011 & 2.52 \times 10^{-6} \\
4.56 \times 10^{-5} & 0.009052 & 0.905652 & 0.081472 & 0.003665 & 0.000113 \\
1.52 \times 10^{-7} & 4.53 \times 10^{-5} & 0.009052 & 0.905652 & 0.081461 & 0.003789 \\
3.79 \times 10^{-10} & 1.51 \times 10^{-7} & 4.53 \times 10^{-5} & 0.009051 & 0.905287 & 0.085617 \\
7.69 \times 10^{-14} & 3.85 \times 10^{-11} & 1.55 \times 10^{-8} & 4.68 \times 10^{-6} & 0.000951 & 0.999044 \\
\end{bmatrix}
$$
7 References


Knight, Frank (1921) Risk, Uncertainty and Profit (Houghton Mifflin, Boston).


Figures

Figure 1: Timeline (illustrating $N+1$ state transitions and $N$ rollovers).

Figure 2: Fundamental value ($V$) and debt capacity ($B$) in high ($v^H=100$) and low ($v^L=50$) states as a function of time.
Figure 3: Fundamental value ($V$) and debt capacity ($B$) in low ($v^l=50$) state for different number of rollovers ($N$).

Figure 4: Debt capacity ($B$) in high ($v^h=100$) and low ($v^l=40$) states as a function of time.
Figure 5: Fundamental values (V) as a function of time

Figure 6a: Debt capacity (B) as a function of time for rollover frequency N=10
Figure 6b: Debt capacity (B) as a function of time for rollover frequency N=100