1 Numerical example with fixed cost

Here, we provide a numerical example with constant liquidation cost, where \( c = 4.25 \) so that each time there is a default and the assets are liquidated, there is a fixed liquidation cost of 4.25. As in the example in the text with proportional cost of liquidation, we use the following parameter values: the Poisson parameter is \( \alpha = 10 \), the maturity of the repo is \( \tau = 0.01 \), the values of the asset are \( v^H = 100 \) and \( v^L = 50 \) in the high and low states, respectively\(^1\) and the transition probability matrix is

\[
\mathbf{P} = \begin{bmatrix} 0.20 & 0.80 \\ 0.01 & 0.99 \end{bmatrix}.
\]

The transition probability matrix for an interval of unit length can be calculated to be

\[
\mathbf{P}(1) = \begin{bmatrix} 0.01265 & 0.98735 \\ 0.01234 & 0.98766 \end{bmatrix}.
\] (1)

Note that, we can calculate the fundamental values as in the numerical example with proportional liquidation cost so that we obtain \( V^H_0 = 99.383 \) and \( V^L_0 = 99.367 \). As in the numerical example in the text, we obtain

\[
\mathbf{P}(0.01) = \begin{bmatrix} 0.92315 & 0.07685 \\ 0.00096 & 0.99904 \end{bmatrix}.
\] (2)

Consider now the debt capacities at the last rollover date \( t_{99} = 0.99 \). If we set \( D = 50 \), the debt can be paid off at date 1 in both states and the expected value of the payoff is 50. So the market value of the debt with face value 50 is exactly 50.

Now suppose we set \( D = 100 \), there will be default in state \( L \) but not in state \( H \) at time 1. The payoff in state \( H \) will be 100 but the payoff in state \( L \) will be \( 50 - 4.25 = 45.75 \). Then the market value of the debt at time \( t_{99} \) will depend on the state at time \( t_{99} \), because the transition probabilities depend on the state. We can easily calculate the expected payoffs in each state:

\[
\begin{align*}
\text{state } H : & \quad 0.99904 \times 100 + 0.00096 \times (50 - 4.25) = 99.948; \\
\text{state } L : & \quad 0.07685 \times 100 + 0.92315 \times (50 - 4.25) = 49.919.
\end{align*}
\]

For example, if the state is \( H \) at date \( t_{99} \), then with probability 0.99904 the state is \( H \) at date 1 and the debt pays off 100 and with probability 0.00096 the state is \( L \) at date 1, the asset must be liquidated and the creditors only realize 45.25.

\(^1\)The numerical example with the constant liquidation for \( v^L = 40 \) is omitted. We obtain qualitatively similar results to the results with proportional liquidation cost as in the paper. However, we provide Figure 4 to illustrate the debt capacities for the case with constant liquidation and \( v^L = 40 \).
Comparing the market values of the debt with the two different face values, we can see that the optimal face value will depend on the state. In state $H$, the expected value of the debt when $D = 100$ is $99.948 > 50$, so it is optimal to set $D^H_{99} = 100$. In state $L$, on the other hand, the expected value of the debt with face value $D = 100$ is only $49.919 < 50$, so it is optimal to set the face value $D^L_{99} = 50$. Thus, if we use the notation $B^s_n$ to denote the debt capacity in state $s$ at date $t_n$, we have shown that $B^H_{99} = 99.948$ and $B^L_{99} = 50$.

Next, consider the debt capacities at date $t_{98} = 0$. Now, the relevant face values to consider are $50$ and $99.948$. If $D = 50$, the expected payoff is $50$ too, since the debt capacity at date $t_{99}$ is greater than or equal to $50$ in both states and, hence, the debt can always be rolled over. In contrast, if $D = 99.948$, the debt cannot be rolled over in state $L$ at date $t_{99}$ and the liquidation cost is incurred. Thus, the expected value of the debt depends on the state at date $t_{98}$:

- State $H$: $0.99904 \times 99.948 + 0.00096 \times (50 - 4.25) = 99.896$,
- State $L$: $0.07685 \times 99.948 + 0.92315 \times (50 - 4.25) = 49.915$.

Comparing the expected value corresponding to different face values of the debt, we see that the optimal face value is $D^H_{98} = 99.948$ in state $H$ and $D^L_{98} = 50$ in state $L$, so that the debt capacities are $B^H_{98} = 99.896$ and $B^L_{98} = 50$.

At each date $t_n$, the debt capacity in the high state is lower than it was at $t_{n+1}$ and the debt capacity in the low state is the same as it was at $t_{n+1}$. These facts tells us that if it is optimal to set $D^L_{n+1} = 50$ at $t_{n+1}$, then a fortiori it will be optimal to set $D^L_n = 50$ at date $t_n$. Thus, the debt capacity is equal to $50$ at each date $t_n$, including the first date $t_0 = 0$.

What is the debt capacity in state $H$ at $t_0$? The probability of staying in the high state from date 0 to date 1 is $(0.99904)^{100} = 0.90842$ and the probability of hitting the low state at some point is $1 - 0.90842 = 0.09158$ so the debt capacity at time 0 is

$$B^H_0 = 0.90842 \times 100 + 0.09158 \times (50 - 4.25) = 95.032.$$ 

So the fall in debt capacity occasioned by a switch from the high to the low state at time 0 is $95.032 - 50 = 45.032$ compared to a change in the fundamental value of $99.383 - 99.367 = 0.016$. This fall is illustrated sharply in Figure 2 which shows that while fundamental values in states $H$ and $L$ will diverge sharply at maturity, they are essentially the same at date 0. Nevertheless, debt capacity in state $L$ is simply the terminal value in state $L$. Thus, a switch to state $L$ from state $H$ produces a sudden drop in debt capacity of the asset.

— Figure 2 here —

It is also interesting to compare the rollover debt capacity with the debt capacity if the asset were financed using debt with maturity $\tau = 1.0$, so that there is no need to roll over
the debt. To be consistent with the way we calculated the rollover debt capacity, we need to allow for liquidation costs at date 1. In each state, it is optimal to choose the face value of the debt to be \( D = 100 \), so that there is always default if the low state occurs at date 1. To calculate the expected value of the debt, we use the transition probabilities from the matrix \( P(1) \), just as we did with the calculation of fundamental values. The expected value of the debt in each state is

\[
B_0^H = 0.98766 \times 100 + 0.01234 \times (50 - 4.25) = 99.331.
\]

\[
B_0^L = 0.98735 \times 100 + 0.01265 \times (50 - 4.25) = 99.313.
\]

As with the fundamental values, the debt capacities with long term debt are much higher than in the rollover case and are also much closer together.

## 2 Debt capacity with two states

In this section we provide a proof for the market freeze result when there two states. We make the same assumptions as for the numerical example but the parameters are otherwise arbitrary. For the time being, we treat the maturity of the commercial paper \( \tau \) and the number of rollovers \( N \) as fixed. Later, we will be interested to see what happens when the maturity of commercial paper \( \tau \) becomes very small and the number of rollovers \( N \) becomes correspondingly large.

There are two states, a “low” state \( L \) and a “high” state \( H \). Transitions between states occur at the dates \( t_n \) and are governed by a stationary transition probability matrix

\[
P(\tau) = \begin{bmatrix}
1 - q(\tau) & q(\tau) \\
p(\tau) & 1 - p(\tau)
\end{bmatrix},
\]

where \( p(\tau) \) is the probability of a transition from state \( H \) to state \( L \) and \( q(\tau) \) is the probability of a transition from state \( L \) to state \( H \) when the period length is \( \tau > 0 \). So, \( p(\tau) \) is the probability of “bad news” and \( 1 - p(\tau) \) is the probability of “no news” in the high state. Similarly, in the low state, the probability of “good news” is \( q(\tau) \) and the probability of “no news” is \( 1 - q(\tau) \). The one requirement we impose on these probabilities is that the shorter the period length, the more likely it is that no news arrives before the next rollover date:

\[
\lim_{\tau \to 0} p(\tau) = \lim_{\tau \to 0} q(\tau) = 0.
\]

The terminal value of the asset is \( v^H \) if the terminal state is \( H \) and \( v^L \) if the terminal state is \( L \), where \( 0 < v^L < v^H \).

The debt capacity of the assets can be determined by backward induction. Suppose that the economy is in the low state at date \( t_N \), which is the last of the rollover dates. Let \( D \)}
be the face value of the debt issued by the bank. If $D > v^H$, the bank will default in both states at date $t_{N+1}$ and the creditors will receive $(v^H - c)$ in the high state and $(v^L - c)$ in the low state. \footnote{To simplify the argument, we are assuming that there is a liquidation cost at date $t_{N+1}$ even though there is no need to sell the asset at that date. None of the results depend on this.} Clearly, the market value of the debt at date $t_N$ would be greater if the face value were $D = v^H$, so it cannot be optimal to choose $D > v^H$. Now suppose that the bank issues debt with face value $D$, where $v^L < D < v^H$. This will lead to default in the low state at date $t_{N+1}$ and the creditors will receive $D$ in the high state and $(v^L - c)$ in the low state. Clearly, this is dominated by choosing a higher value of $D$. Thus, either $D = v^H$ or $D = v^L$. An exactly similar argument shows that it cannot be optimal to choose $D < v^L$, so we are left with only two possibilities, either $D = v^H$ or $D = v^L$. In the first case, the market value of the debt is $(1 - q(\tau)) (v^L - c) + q(\tau) v^H$ and in the second case it is $v^L$. Clearly, for $\tau$ sufficiently small,

$$
(1 - q(\tau)) (v^L - c) + q(\tau) v^H < v^L.
$$

(3)

Let $\tau^* > 0$ denote the critical value such that (3) holds if and only if $\tau < \tau^*$, that is, $q(\tau^*) = \frac{c}{v^H - v^L + c}$. For any $\tau < \tau^*$ it is strictly optimal to put $D = v^L$. Then we have found the debt capacity in the low state at date $t_N$, which we denote by $B^L_N = v^L$.

Now consider the high state at date $t_N$. It is easy to see, as before, that the only candidates for the face value of the debt are $D = v^H$ and $D = v^L$. If the bank issues debt with face value $v^H$, there will be default in the low state. The creditors will receive $(v^L - c)$ in the low state and $v^H$ in the high state and the market value of the debt at date $t_N$ will be $(1 - p(\tau)) v^H + p(\tau) (v^L - c)$. If the bank issues debt with face value $v^L$, there will be no default, the creditors will receive $v^L$ in both states at date $t_{N+1}$ and the market value of the debt at date $t_N$ will be $v^L$. If the period length $\tau$ is sufficiently short, we can see that

$$
(1 - p(\tau)) v^H + p(\tau) (v^L - c) > v_L.
$$

(4)

Let $\tau^{**} > 0$ denote the critical value such that (4) is satisfied if and only if $\tau < \tau^{**}$, that is, $p(\tau^{**}) = \frac{v^H - v^L}{v^H - v^L + c}$. So if $\tau < \tau^{**}$, it is strictly optimal to put $D = v^H$. Then we have found the debt capacity in the high state at date $t_N$, which we denote by $B^H_N = (1 - p(\tau)) v^H + p(\tau) (v^L - c)$.

Now suppose that we have calculated the debt capacities $B^H_n$ and $B^L_n$ for $n = k+1, \ldots, N$. We show that

**Proposition 1** For $0 < \tau < \min \{\tau^*, \tau^{**}\}$ and

$$
v^H - v^L > (e^\alpha - 1) c,
$$

the debt capacities of the asset in states $H$ and $L$ are given respectively by the formulae

$$
B^H_n = (1 - p(\tau))^{N-n} v^H + \left[1 - (1 - p(\tau))^{N-n}\right] (v^L - c), \text{ for } n = k, \ldots, N,
$$

(5)
and

\[ B^L_n = v^L, \text{ for } n = k, \ldots, N. \]  

(6)

**The low state** Consider what happens in the low state at date \( t_k \). If the face value of the debt issued by the bank is \( D \) at date \( t_k \), then the bank will default in the low state if \( v^L < D < B^H_{k+1} \) and the bank will default in both states if \( D > B^H_{k+1} \). By our usual argument, the only candidates for the optimal face value are \( D = v^L \) and \( D = B^H_{k+1} \). If the face value is \( D = v^L \), the creditors will receive \( v^L \) in both states at date \( t_{k+1} \) and the market value of the debt at date \( t_k \) will be \( v^L \). On the other hand, if the face value of the debt is \( D = B^H_{k+1} \), the creditors receive \( B^H_{k+1} \) in the high state and \( v^L \) in the low state, so the market value of the debt at date \( t_k \) is

\[
(1 - q(\tau))(v^L - c) + q(\tau)B^H_{k+1} \leq (1 - q(\tau))(v^L - c) + q(\tau)v^H,
\]

since \( B^H_{k+1} \leq v^H \). But \( \tau < \tau^* \) implies that \( (1 - q(\tau))(v^L - c) + q(\tau)v^H < v^L \), so the debt capacity is \( B^L_k = v^L \).

**The high state** Now consider the high state. Again, our two candidates for the face value of the debt are \( D = B^H_{k+1} \) and \( D = v^L \), which yield market values at date \( t_k \) of

\[
(1 - p(\tau))B^H_{k+1} + p(\tau)(v^L - c)
\]

and \( v^L \), respectively. From our induction hypothesis,

\[
(1 - p(\tau))B^H_{k+1} + p(\tau)(v^L - c) = (1 - p(\tau))\left\{ (1 - p(\tau))^{N-k-1}v^H + \left[ 1 - (1 - p(\tau))^{N-k-1} \right](v^L - c) \right\} + p(\tau)(v^L - c) = (1 - p(\tau))^{N-k}(v^H - (v^L - c)) + (v^L - c).
\]

Then \( D^H_k = B^H_{k+1} \) is strictly optimal if

\[
(1 - p(\tau))^{N-k}(v^H - (v^L - c)) + (v^L - c) > v^L
\]

or

\[
v^H - v^L + c > \frac{c}{(1 - p(\tau))^{N-k}}.
\]

(7)

In order for this inequality to be satisfied for all \( n \) it must be satisfied for \( n = 0 \). We can show that \( (1 - p(\tau))^N \geq e^{-\alpha} \), because \( e^{-\alpha} \) is the probability of no information events in the unit interval. Hence, a sufficient condition for (7) to be satisfied is

\[
v^H - v^L > c(e^\alpha - 1).
\]

By induction, we have proved that the debt capacities are given by the formulae in (5) and (6) for all \( n = 0, \ldots, N \).
3 Debt capacity in the general case

We allow for a finite number of information states or signals, denoted by \( S = \{s_1, ..., s_I\} \). The current information state is public information. Changes in the information state arrive randomly. The timing of the information events is a homogeneous Poisson process with parameter \( \alpha > 0 \). The probability of \( k \) information events, in an interval of length \( \tau \), is

\[
Pr \left[ K(t + \tau) - K(t) = k \right] = \frac{e^{-\alpha \tau} (\alpha \tau)^k}{k!},
\]

where \( K(t) \) is the number of information events between 0 and \( t \), and the expected number of information events in an interval of length \( \tau \) is

\[
E \left[ K(t + \tau) - K(t) \right] = \sum_{k=0}^{\infty} \left( \frac{e^{-\alpha \tau} (\alpha \tau)^k}{k!} \right) k = \alpha \tau.
\]

Conditional on an information event occurring at date \( t \), the probability of a transition from state \( s_i \) to state \( s_j \) is denoted by \( p_{ij} \geq 0 \), where \( \sum_{j=1}^{I} p_{ij} = 1 \). These transition probabilities are described by the \( I \times I \) matrix

\[
P = \begin{bmatrix}
p_{11} & \cdots & p_{1I} \\
\vdots & \ddots & \vdots \\
p_{I1} & \cdots & p_{II}
\end{bmatrix}.
\]

The transition probabilities over an interval of length \( \tau \) depend on the number of information events \( k \), a random variable, and the transition matrix \( P \). The transition matrix for an interval \( \tau \) is denoted by \( P(\tau) \) and defined by

\[
P(\tau) = \sum_{k=0}^{\infty} \frac{e^{-\alpha \tau} (\alpha \tau)^k}{k!} P^k.
\]

The information state is a stochastic process \( \{S(t)\} \) but for our purposes all that matters is the value of this process at the rollover dates. We let \( S_n \) denote the value of the information state \( S(t_n) \) at the rollover date \( t_n \) and we say that there is “no news” at date \( t_{n+1} \) if \( S_{n+1} = S_n \). In other words, we regard the current state as the status quo and say that news arrives only if a new information state is observed. Of course, “no news” is also informative and beliefs about the terminal value of the assets will be updated even if the information state remains the same. Again, it is important to note is that, when the period length \( \tau \) is short, the probability of “news” becomes small and the probability of “no news” becomes correspondingly large. In fact,

\[
\lim_{\tau \to 0} P(\tau) = P(0) = I.
\]
In that case, the informational content of “no news” is also small.

The terminal value of the assets is a function of the information state at date \( t = 1 \). We denote by \( v_i \) the value of the assets if the terminal state is \( S_{N+1} = s_i \) and assume that the values \( \{v_1, ..., v_I\} \) satisfy

\[
0 < v_1 < \ldots < v_I.
\]

Let \( V^i_n \) denote the fundamental value of the asset at date \( t_n \) in state \( i \). Then clearly the values \( \{V^i_n\} \) are defined by putting \( V^i_{N+1} = v_i \) for \( i = 1, ..., I \) and

\[
V^i_n = \sum_{j=1}^{I} p_{ij} (1 - t_n) v_j, \quad \text{for } n = 0, ..., N \text{ and } i = 1, ..., I,
\]

where \( p_{ij} (1 - t_n) \) is, of course, the \((i, j)\) entry of \( \mathbf{P} (1 - t_n) \) denoting the probability of a transition from state \( i \) at date \( t_n \) to state \( j \) at date \( t_{N+1} = 1 \).

Figure 4 illustrates the fundamental values in a setup with \( I = 11 \) states where terminal values \( v_1 \) through \( v_{11} \) are equally spaced from 0 to 100, and \( \alpha = 10 \). The transition matrix \( \mathbf{P} \) is described in Appendix B. As in our two-state example, the fundamental values in different states are virtually identical at date 0 though they diverge in steps of 10 at maturity.

— Figure 4 here —

Let \( B^i_n \) denote the equilibrium debt capacity of the assets in state \( s_i \) at date \( t_n \). By convention, we set \( B^i_{N+1} = v_i \) for all \( i \).

**Proposition 2** The equilibrium values of \( \{B^i_n\} \) must satisfy

\[
B^i_n = \max_{k=1, \ldots, I} \left\{ \sum_{j=1}^{k-1} p_{ij} (\tau) (B^j_{n+1} - c) + \sum_{j=k}^{I} p_{ij} (\tau) B^k_{n+1} \right\}
\]

for \( i = 1, ..., I \) and \( n = 0, ..., N \).

The result is immediate once we apply the now familiar backward induction argument to show that it is always optimal to set the face value of the debt \( D^i_n \) equal to \( B^j_{n+1} \) for some \( j \). Although the result amounts to little more than the definition of debt capacity, it is very useful because it allows us to calculate the debt capacities by backward induction.

We use the formula in Proposition 2 to obtain the limiting value of the debt capacity as \( \tau \to 0 \). An auxiliary assumption is helpful in proving this result: higher information states are assumed to be “better” in the sense that

\[
V_{in} < V_{i+1,n}, \quad \text{for all } i = 1, ..., I - 1 \text{ and } n = 0, ..., N + 1.
\]
A sufficient (but, as we show in Appendix A, not necessary) condition for (8) is that \( f_{p_{i+1,j}}(\tau) \) strictly dominates \( f_{p_{i,j}}(\tau) \) in the sense of first-order stochastic dominance. That is, for all \( i = 1, \ldots, I - 1, \)

\[
\sum_{j=1}^{k} p_{ij}(\tau) > \sum_{j=1}^{k} p_{i+1,j}(\tau), \text{ for all } i, k = 1, \ldots, I - 1. \tag{9}
\]

**Proposition 3** Suppose that (9) is satisfied. Then there exists \( \tau^* > 0 \) such that for all \( 0 < \tau < \tau^* \), for any \( n = 0, \ldots, N \) and any \( i = 1, \ldots, I \),

\[
B_{n}^{i} = \sum_{j=1}^{I} p_{ij}(\tau) (B_{n+1}^{j} - c) + \sum_{j=i}^{I} p_{ij}(\tau) B_{n+1}^{i}. \tag{10}
\]

**Proof.** See Appendix A. ■

Several properties follow immediately from Proposition 3 whenever \( 0 < \tau < \tau^* \). We provide these results formally in the form of four corollaries. First, in the lowest state, \( s_1 \), the debt capacity is constant and equal to \( v_1 \), the worst possible terminal value.

**Corollary 4** \( B_{n}^{1} = v_1 \) for all \( n \).

**Proof.** From the formula in Proposition 3

\[
B_{n}^{1} = \sum_{j=1}^{I} p_{ij}(\tau) B_{n+1}^{1} = B_{n+1}^{1}
\]

for \( n = 0, \ldots, N \) so claim follows from our convention that \( B_{N+1}^{1} = v_1 \). ■

Second, the debt capacity \( B_{n}^{i} \) is monotonically non-decreasing in \( n \), that is, debt capacity increases as the asset matures, holding the state constant. This follows directly from the fact that, if the face of the debt equals \( B_{n+1}^{i} \), the debt capacity \( B_{n}^{i} \) cannot be greater than \( B_{n+1}^{i} \).

**Corollary 5** \( B_{n}^{i} \leq B_{n+1}^{i} \) for any \( i = 1, \ldots, I \) and \( n = 0, \ldots, N \).

**Proof.** The inequality follows directly from the formula in Proposition 3

\[
B_{n}^{i} = \sum_{j=1}^{I-1} p_{ij}(\tau) (B_{n+1}^{j} - c) + \sum_{j=i}^{I} p_{ij}(\tau) B_{n+1}^{i}
\]

\[
\leq \sum_{j=1}^{I} p_{ij}(\tau) B_{n+1}^{i} + \sum_{j=i}^{I} p_{ij}(\tau) B_{n+1}^{i}
\]

\[
= \sum_{j=1}^{I} p_{ij}(\tau) B_{n+1}^{i} = B_{n+1}^{i},
\]

8
since $B^i_{n+1} - c \leq B^i_{n+1}$ for $j = 1, \ldots, i - 1$. ■

Third, since $B^i_{N+1} = v_i$ by convention, the preceding result immediately implies that the
debt capacity $B^i_n$ is less than or equal to $v_i$.

**Corollary 6** $B^i_n \leq v_i$ for all $i = 1, \ldots, I$ and $n = 0, \ldots, N$.

Finally, we can show that the debt capacity in state $s_i$ at any date $t_n$ is less than the
fundamental value $V^i_n$.

**Corollary 7** $B^i_n \leq V^i_n$ for any $n = 0, \ldots, N$ and $i = 1, \ldots, I$.

**Proof.** The inequality follows directly from the formula in Proposition 3 for $n = N + 1$
and any $i$, so suppose that it holds for $n, \ldots, N$ and any $i = 1, \ldots, I$. Then the formula in
Proposition 3 implies that

$$B^i_n = \sum_{j=1}^{i-1} p_{ij}(\tau) (B^j_{n+1} - c) + \sum_{j=i}^{I} p_{ij}(\tau) B^i_{n+1}$$

$$\leq \sum_{j=1}^{i-1} p_{ij}(\tau) (V^j_{n+1} - c) + \sum_{j=i}^{I} p_{ij}(\tau) V^i_{n+1}$$

$$\leq \sum_{j=1}^{I} p_{ij}(\tau) V^j_{n+1} = V^i_n,$$

for any $i = 1, \ldots, I$, so by induction the claim holds for any $n = 0, \ldots, N$ and any $i = 1, \ldots, I$. ■

Some of these properties are illustrated in Figures 5a and 5b which show the debt ca-
pacities in the 11 states of our numerical example for $N = 10$ and $N = 10,000$ rollovers,
respectively. For 10 rollovers, $\tau$ is not sufficiently small and the debt capacity is large even
in the worst state. By contrast, with 10,000 rollovers, the debt capacity in the worst case
is (essentially) zero; furthermore, as we go from good states to worse states, debt capacity
falls roughly by a magnitude of 10 even though the fundamental values (Figure 4) are almost
identical in these states.

— Figure 5a and 5b here —

9
Appendix A: Proofs

We can solve for the equilibrium debt capacities by backward induction. Let $D$ denote the face value of the debt issued in state $s_i$ at date $t_n$. This debt will pay off $D$ in state $s_j$ at date $t_{n+1}$ if $D \leq B_{n+1}^j$ and $(B_{n+1}^j - c)$ otherwise. In other words, the market value of the debt is given by the formula

$$\sum_{B_{n+1}^j < D} p_{ij}(\tau) (B_{n+1}^j - c) + \sum_{B_{n+1}^j \geq D} p_{ij}(\tau) D$$

and the debt capacity is given by

$$B_n^i = \max_D \left\{ \sum_{B_{n+1}^j < D} p_{ij}(\tau) (B_{n+1}^j - c) + \sum_{B_{n+1}^j \geq D} p_{ij}(\tau) D \right\}.$$  

Proposition 8 $B_n^i \leq B_{n+1}^i$, for $i = 1, ..., I - 1$ and $n = 0, ..., N + 1$.

Proof. The claim is clearly true by definition when $n = N + 1$, so suppose it is true for some arbitrary number $n + 1$. Then, for any $D$ and $i = 1, ..., I - 1$, it is clear that

$$\sum_{B_{n+1}^j < D} p_{ij}(\tau) (B_{n+1}^j - c) + \sum_{B_{n+1}^j \geq D} p_{ij}(\tau) D \leq \sum_{B_{n+1}^j < D} p_{i+1,j}(\tau) (B_{n+1}^j - c) + \sum_{B_{n+1}^j \geq D} p_{i+1,j}(\tau) D,$$

because $\{p_{ij}(\tau)\}$ is first-order stochastically dominated by $\{p_{i+1,j}(\tau)\}$. It follows immediately that $B_n^i \leq B_{n+1}^i$ for $i = 1, ..., I - 1$. The claim in the proposition then follows by induction. 

Let $D_n^i$ denote the optimal face value of the debt in state $i$ at date $t_n$. It is clear that the market value of the debt is maximized by setting the face value $D = B_{n+1}^j$, for some value of $j = 1, ..., I$. Thus, we can write the equilibrium condition as

$$B_n^i = \max_{k=1, ..., I} \left\{ \sum_{j=1}^{k-1} p_{ij}(\tau) (B_{n+1}^j - c) + \sum_{j=k}^{l} p_{ij}(\tau) B_{n+1}^k \right\},$$

for $i = 1, ..., I$ and $n = 0, ..., N$.

Now suppose that it is optimal to set $D_n^i = B_{n+1}^i$ for every $i = 1, ..., I$ and $n = 0, ..., N$. The following proposition gives a lower bound to the difference between $B_{n+1}^i$ and $B_n^i$.

Proposition 9 Suppose that it is optimal to set $D_n^i = B_{n+1}^i$ for every $i = 1, ..., I$ and $n = 0, ..., N$. Then $B_{n+1}^i \geq B_n^i + e^{-\alpha}(v_{i+1} - v_i)$, for $i = 1, ..., I - 1$ and $n = 0, ..., N + 1$. 

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Proof. Think of $B_n^i$ as the average of two quantities, one being the debt capacity conditional on the occurrence of at least one information event after date $n$, denoted by $\tilde{B}_n^i$, and the other being the debt capacity conditional on no information events after $n$, denoted by $\hat{B}_n^i$. In the first case, the preceding argument suffices to show that $\tilde{B}_n^{i+1} \geq \tilde{B}_n^i$. In the second case, we have $\hat{B}_n^{i+1} = \hat{B}_n^i + (v_{i+1} - v_i)$ since the state remains constant in each case. Since the probability of no information events after date $t_n$ is $e^{-\alpha(1-t_n)}$, it follows that

$$B_n^{i+1} - B_n^i \geq e^{-\alpha} (v_{i+1} - v_i)$$

as claimed. ■

**Proposition 10** For all $\tau > 0$ sufficiently small, $D_n^i = B_n^{i+1}$ for all $i = 1, \ldots, I$ and $n = 0, \ldots, N$.

Proof. For a fixed but arbitrary date $t_n$ and state $s_i$, we compare the strategy of setting $D = B_{n+1}^i$ with the strategy of setting $D = B_{n+1}^k$. First suppose $k > i$ and consider the difference in the expected values of the debt:

$$\sum_{j=1}^{i-1} p_{ij}(\tau) (B_{n+1}^j - c) + \sum_{j=i}^{I} p_{ij}(\tau) B_{n+1}^i - \sum_{j=1}^{k-1} p_{ij}(\tau) (B_{n+1}^j - c) - \sum_{j=k}^{I} p_{ij}(\tau) B_{n+1}^k$$

$$= \sum_{j=i}^{k-1} p_{ij}(\tau) (B_{n+1}^i - (B_{n+1}^j - c)) + \sum_{j=k}^{I} p_{ij}(\tau) (B_{n+1}^i - B_{n+1}^k)$$

$$= p_{ii}(\tau) (B_{n+1}^i - (B_{n+1}^i - c)) + \sum_{j=i+1}^{k-1} p_{ij}(\tau) (B_{n+1}^i - (B_{n+1}^j - c)) + \sum_{j=k}^{I} p_{ij}(\tau) (B_{n+1}^i - B_{n+1}^k)$$

$$\geq p_{ii}(\tau) c + \sum_{j=i+1}^{k-1} p_{ij}(\tau) (v_1 - v_I) + \sum_{j=k}^{I} p_{ij}(\tau) (v_1 - v_I)$$

$$= p_{ii}(\tau) c + \sum_{j=i+1}^{I} p_{ij}(\tau) (v_1 - v_I).$$

since $B_{n+1}^i \geq v_1$, $B_{n+1}^i - (B_{n+1}^j - c) \geq B_{n+1}^i - B_{n+1}^j \geq (v_1 - v_I)$ for $j = i + 1, \ldots, I$ and $B_{n+1}^i - B_{n+1}^k \geq v_1 - v_I$. Then it is clear that, for $\tau$ sufficiently small, the last expression above is positive.
Similarly, for $k < i$, 

$$\sum_{j=1}^{i-1} p_{ij}(\tau) \left( B_{n+1}^{j} - c \right) + \sum_{j=i}^{I} p_{ij}(\tau) B_{n+1}^{j} - \sum_{j=1}^{k-1} p_{ij}(\tau) \left( B_{n+1}^{j} - c \right) - \sum_{j=k}^{I} p_{ij}(\tau) B_{n+1}^{k}$$

$$= \sum_{j=k}^{i-1} p_{ij}(\tau) \left( (B_{n+1}^{j} - c) - B_{n+1}^{k} \right) + \sum_{j=i}^{I} p_{ij}(\tau) \left( B_{n+1}^{i} - B_{n+1}^{k} \right)$$

$$\geq \sum_{j=k}^{i-1} p_{ij}(\tau) \left( (B_{n+1}^{j} - c) - B_{n+1}^{k} \right) + p_{ii}(\tau) \left( B_{n+1}^{i} - B_{n+1}^{k} \right)$$

$$\geq (1 - p_{ii}(\tau)) \left( (v_{1} - c) - v_{I} \right) + p_{ii}(\tau) \left( B_{n+1}^{i} - B_{n+1}^{k} \right).$$ 

since $(B_{n+1}^{j} - c) - B_{n+1}^{k} \geq (v_{1} - c) - v_{1}$.

At the last roll over date, $n = N$ and $B_{N+1}^{j} - B_{N+1}^{k} \geq (v_{i+1} - v_{i}) > 0$ by definition. Then the last line above is positive for $\tau$ sufficiently small and this proves that it is optimal to set $D_{N}^{i} = B_{N+1}^{i} = v_{i}$.

Now suppose that we have shown that it is optimal to set $D_{\hat{n}+1}^{i} = B_{\hat{n}+1}^{i}$ for $\hat{n} + 1, \ldots, N$ and consider the inequalities above for $n = \hat{n}$. Then Proposition 9 tells us that $B_{\hat{n}+1}^{i} - B_{\hat{n}+1}^{k} \geq e^{-a} (v_{i} - v_{k}) > 0$ and the last line above must be positive for $\tau$ sufficiently small. This in turn establishes that $D_{\hat{n}}^{i} = B_{\hat{n}+1}^{i}$ and, by induction, we have shown that it is optimal to set $D_{n}^{i} = B_{n+1}^{i}$ for all $n$ this proves that it is optimal to set $D_{N}^{i} = B_{N+1}^{i} = v_{i}$ for $i = 1, \ldots, I$ and $n = 0, \ldots, N$. \[\square\]
Figures with constant cost of liquidation

Figure 2: Fundamental value (V) and debt capacity (B) in high ($v^H=100$) and low ($v^L=50$) states as a function of time

Figure 3: Fundamental value (V) and debt capacity (B) in low ($v^L=50$) state for different rollover numbers (N)
Figure 4: Debt capacity (B) in high ($\nu^H=100$) and low ($\nu^L=40$) states as a function of time