Reviewing *Income and Wealth Heterogeneity, Portfolio Choice and Equilibrium Asset Returns* by P. Krussell and A. Smith, JPE 1997

Seminar in Asset Pricing Theory

Presented by Saki Bigio
November 2007
Goals of the Paper

- **Reconciliation** of Consumer Based Asset Pricing and Purely Empirical Asset Pricing Theory (CAPM related literature).
Goals of the Paper

- **Reconciliation** of Consumer Based Asset Pricing and Purely Empirical Asset Pricing Theory (CAPM related literature).
- Finding a **Consistent** description of **Joint Distribution of Asset Returns** and **Consumption** both in homoskedastic and heteroskedastic return processes.
Goals of the Paper

- **Reconciliation** of Consumer Based Asset Pricing and Purely Empirical Asset Pricing Theory (CAPM related literature).
- Finding a **Consistent** description of **Joint Distribution** of **Asset Returns** and **Consumption** both in homoskedastic and heteroskedastic return processes.
- Finding an analytically tractable representation of the consumer’s problem to understand mechanisms underlying asset pricing.
Log-Linearization of Budget Constraint

The consumer’s Budget Constraint in terms of the $m$ portfolio return:

$$W_{t+1} = R_{m,t+1}(W_t - C_t)$$  \hspace{1cm} (1)

Updating the budget constraint

$$W_t = C_t + \sum_{i=1}^{\infty} \frac{C_{t+j}}{\left(\prod_{j=1}^{i} R_{m,t+j}\right)}$$  \hspace{1cm} (2)

Dividing both sides by $W_t$:

$$\frac{W_{t+1}}{W_t} = R_{m,t+1} \left(1 - \frac{C_t}{W_t}\right)$$  \hspace{1cm} (3)

Defining $\log X = x$, and taking logs on both sides we obtain:

$$\Delta w_{t+1} = r_{m,t+1} + \log(1 - \exp(c_t - w_t))$$  \hspace{1cm} (4)
Log-Linearization of BC

First order taylor expansion of $\log(1 - \exp(c_t - w_t))$:

$$
\log(1 - \exp(c_t - w_t)) \approx \log \left( 1 - \exp \left( \frac{c_0}{w_0} \right) \right) \\
- \exp \left( \frac{c_0}{w_0} \right) + \frac{1}{1 - \exp \left( \frac{c_0}{w_0} \right)} \left[ (c_t - w_t) - (c_0 - w_0) \right]
$$

and first order taylor expansion yields the following:

$$
\Delta w_{t+1} \approx r_{m,t+1} + \left( 1 - \frac{1}{\rho} \right) ((c_t - w_t)) + k \tag{5}
$$

where $k$ is defined in the distributed notes.
Log-Linear Consumption Surprise Function

We then can use the trivial Inequality:

\[ \Delta w_{t+1} = \Delta c_{t+1} + (c_t - w_t) - (c_{t+1} - w_{t+1}) \]  

(6)

in lifetime budget:

\[ c_t - w_t = \sum_{i=1}^{\infty} \rho^k (r_{m,t+i} - \Delta c_{t+i}) + \frac{\rho}{1 - \rho} k \]  

(7)

By taking expectations to both sides we obtain:

\[ c_t - w_t = E_t \left[ \sum_{i=1}^{\infty} \rho^k (r_{m,t+i} - \Delta c_{t+i}) \right] + \frac{\rho}{1 - \rho} k \]  

(8)
so we can express the unexpected change in consumption as:

\[
c_{t+1} - E_t [c_{t+1}] = w_{t+1} - E_t [w_{t+1}] \\
+ E_{t+1} \left[ \sum_{i=1}^{\infty} \rho^k (r_{m,t+i} - \Delta c_{t+i}) \right] \\
- E_t \left[ \sum_{i=1}^{\infty} \rho^k (r_{m,t+i} - \Delta c_{t+i}) \right]
\]

and regrouping the expectational terms:

\[
c_{t+1} - E_t [c_{t+1}] = [E_{t+1} - E_t] \sum_{i=0}^{\infty} \rho^k (r_{m,t+i}) - [E_{t+1} - E_t] \sum_{i=1}^{\infty} \Delta c_{t+i} \quad (9)
\]

we use 5 and 6 to get rid of \( w_{t+1} - E_t [w_{t+1}] \).

The goal here is to try to eliminate the terms related to consumption.
Assumptions

1. Log-Normal conditional distribution of Asset Returns and Consumption
2. Epstein-Zin Utility

Reminder Log-Normal Distributed Random Variables

\[ x_t \sim LN(\mu, \sigma^2) \rightarrow \log(x_t) \sim N(\mu, \sigma^2) \]

\[ E_t [x_t] = \exp\left(\mu + \frac{\sigma^2}{2}\right) \]

\[ \text{VAR}[x_t] = (\exp(\sigma^2) - 1) E_t [x_t]^2 = (\exp(\sigma^2) - 1) \exp(2\mu + \sigma^2) \]

\[ \text{Mode}[x_t] = \exp(\mu - \sigma^2) \]

\[ \text{Median} = \exp(\mu) \]
Assumptions

Epstein-Zin Utility

\[ U_t = \left\{ (1 - \beta)C_t^{\frac{1-\gamma}{\theta}} + \beta \left( E_t \left[ U_{t+1}^{1-\gamma} \right]^{\frac{1}{\theta}} \right) \right\}^{\frac{\theta}{1-\gamma}} \]  \hspace{1cm} (10)

where

\[ \theta = \frac{1 - \gamma}{1 - \left( \frac{1}{\sigma} \right)} \]

where \( \gamma \) is the coefficient of relative risk-aversion, and \( \sigma \) is the elasticity of intertemporal substitution.
Properties of Epstein-Zin:

1. Properties of Parameters

<table>
<thead>
<tr>
<th>Value of Parameter</th>
<th>Value of $\theta$</th>
<th>Implication</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma \to 1$</td>
<td>$\theta \to 0$</td>
<td>Irrelevance of Variance Terms</td>
</tr>
<tr>
<td>$\sigma \to 1$</td>
<td>$\theta \to \infty$</td>
<td>Variance term in EE becomes 0</td>
</tr>
<tr>
<td>$\gamma = \frac{1}{\sigma}$</td>
<td>$\theta = 1$</td>
<td>Standard Time Separable</td>
</tr>
<tr>
<td>$\gamma = \sigma = 1$</td>
<td>$\theta = 1$</td>
<td>Log Utility</td>
</tr>
</tbody>
</table>

If one replaces the Optimal Consumption Policy in terms of Wealth

$$V_t = \max \{c_t\}$$

$$U_t = \left[ \left( 1 - \beta \right) - \sigma C_t W_t \right]^{1/1-\sigma} W_t$$

(11)
Properties of **Epstein-Zin:**

1. Properties of Parameters

<table>
<thead>
<tr>
<th>Value of Parameter</th>
<th>Value of $\theta$</th>
<th>Implication</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma \rightarrow 1$</td>
<td>$\theta \rightarrow 0$</td>
<td>Irrelevance of Variance Terms</td>
</tr>
<tr>
<td>$\sigma \rightarrow 1$</td>
<td>$\theta \rightarrow \infty$</td>
<td>Variance term in EE becomes 0</td>
</tr>
<tr>
<td>$\gamma = \frac{1}{\sigma}$</td>
<td>$\theta = 1$</td>
<td>Standard Time Separable</td>
</tr>
<tr>
<td>$\gamma = \sigma = 1$</td>
<td>$\theta = 1$</td>
<td>Log Utility</td>
</tr>
</tbody>
</table>

2. If one replaces the Optimal Consumption Policy in terms of Wealth

$$V_t = \max_{\{c_t\}} U_t = \left[ (1 - \beta)^{-\sigma} \frac{C_t}{W_t} \right]^{\frac{1}{1-\sigma}} W_t$$  \hspace{1cm} (11)
1. The Euler Equation for an asset’s return in terms of a market return $r_{m,t+1}$

$$1 = E_t \left[ \beta \left\{ \frac{C_t}{C_{t+1}} \right\}^{\frac{-1}{\sigma}} \left\{ \frac{1}{R_{m,t+1}} \right\}^{1-\theta} R_{i,t+1} \right]$$

(12)

2. The former in terms only of $R_{m,t+1}$:

$$1 = E_t \left[ \left\{ \beta \left\{ \frac{C_t}{C_{t+1}} \right\}^{\frac{-1}{\sigma}} R_{m,t+1} \right\}^{\theta} \right]$$

(13)

which is a standard euler equation when $\theta = 1$. 
Back in our model, we take a 2nd order taylor expansion of this last equation:

**Reminder 2nd order taylor expansions:**

\[
f(x_t) \approx f(x_o) + f'(x_o)(x - x_o) + \frac{f''(x_o)}{2!}(x - x_o)^2
\]

In the EE:

\[
0 = \theta \log(\beta) - \frac{\theta}{\sigma}E_t \Delta c_{t+1} + \theta E_t r_{m,t+1} + \frac{1}{2} \left( \left( \frac{\theta}{\sigma} \right)^2 V_{cc} + \theta^2 V_{mm} - \frac{2\theta^2}{\sigma} V_{cm} \right)
\]

where

\[
V_{cc} = VAR_t(\Delta c_t) = VAR(\Delta c_{t+1} - E_t \Delta c_{t+1})
\]

and reminding that:

\[
VAR(Ax + By) = A^2 VAR(x) + B^2 VAR(y) + 2ABCOV(x, y)
\]

Possible only when asset returns and consumption are conditionally homoskedastic. Variance and covariance terms will include a time subscript.
By clearing out the change in consumption:

\[ E_t \Delta c_t = \mu_m + \sigma E_t r_{m,t+1} \]  \hspace{1cm} (17)

where

\[ \mu_m = \sigma \log \beta + \frac{1}{2} \left[ \frac{\theta}{\sigma} V_{cc} + \theta \sigma V_{mm} - 2 \theta V_{cm} \right] \]

\[ = \sigma \log \beta + \frac{1}{2} \frac{\theta}{\sigma} \text{Var}_t [\Delta c_{t+1} - \sigma r_{m,t+1}] \]  \hspace{1cm} (18)

where in the Heteroskedastic version this becomes:

\[ \mu_{m,t} = \sigma \log \beta + \frac{1}{2} \left[ \frac{\theta}{\sigma} V_{cc,t} + \theta \sigma V_{mm,t} - 2 \theta V_{cm,t} \right] \]
The log-linear version of the General Euler equation is

\[ 0 = \theta \log \beta - \frac{\theta}{\sigma} E_t \Delta c_{t+1} + (\theta - 1) E_t r_{m,t+1} + E_t r_{i,t+1} + \ldots \]  

(19)

cross variance related terms (in document)
Clearing Out Consumption from Budget Constraint

Using this expression one can derive and expression for the Equity Premium

\[ E_t r_{i,t+1} - r_{f,t+1} = -\frac{V_{ii}}{2} + \theta \frac{V_{ic}}{\sigma} + (1 - \theta) V_{im} \] (20)

In this set-up the risk-premium is constant over time. In Heteroskedastic Variances need to impose some sort of structure on law of motion \( \mu_{m,t} \)
In LT Budget Constraint

We now replace the log-linear portfolio euler equation in the intertemporal budget constraint to get:

\[ c_t - w_t = (1 - \sigma) E_t \sum_{j=1}^{\infty} \rho^j r_{m,t+j} + \frac{\rho (k - \mu_m)}{1 - \rho} \]

so it is clear that when \( \sigma > 1 \) income effect dominates the substitution effect.

The approximation to the Epstein-Zin Value function is given by:

\[ v_t = w_t + \left( \frac{1}{1 - \sigma} \right) (c_t - w_t) = w_t + E_t \sum_{j=1}^{\infty} \rho^j r_{m,t+j} \]

that is, the value function is determined by current wealth and a measure of the longrun investment opportunities.
The clearing out the surprise function

Using the Euler equation and plugging it in the consumption surprise equation we obtain:

\[ c_{t+1} - E_t [c_{t+1}] = r_{m,t+1} - E_t r_{m,t+1} + (1 - \sigma) [E_{t+1} - E_t] \sum_{i=0}^{\infty} \rho^k (r_{m,t+i}) \]

This equation is important for the following reasons:

1. Determines conditions on return process that justify original assumption of joint log-normality.
2. The equation shows consumption smoothing dependent on market information
Digression Heteroskedastic Variances

**Note** for Heteroskedastic version of consumption surprise $\mu_{m,t}$ may not be cleared out:

$$C_{t+1} - E_t [c_{t+1}] = \text{Same} - [E_{t+1} - E_t] \sum_{i=1}^{\infty} \rho^k (\mu_{m,t+i})$$

the new term reflects the influence of changing risk on saving. $\mu_{m,t+i}$ is a function of the second moments of consumption.
Back in Homoskedastic Version

Using this equation we directly obtain the following expression:

$$\text{COV}_t (r_{i,t+1}, \Delta c_{t+1}) = V_{i,m} + (1 - \sigma) V_{i,h}$$

where

$$V_{i,h} = \text{COV}_t \left( r_{i,t+1}, [E_{t+1} - E_t] \sum_{i=0}^{\infty} \rho^k (r_{m,t+i}) \right)$$
Finally Asset Pricing without Consumption

Finally, using this terms we arrive to the core equation of the paper:

\[
E_t r_{i,t+1} - r_{f,t+1} = -\frac{V_{ii}}{2} + \frac{\theta}{\sigma} V_{im} + \frac{\theta (1 - \theta)}{\sigma} V_{ih} + (1 - \theta) V_{im}
\]

\[
= -\frac{V_{ii}}{2} + \gamma V_{im} + (\gamma - 1) V_{ih}
\]

1. Assets can be priced using covariances with returns on (1) invested wealth (2) news regarding future returns

2. The only relevant parameter from the utility function is the Coefficient of Relative Risk Aversion
\[ E_t r_{i,t+1} - r_{f,t+1} = \frac{-V_{ii}}{2} + \gamma V_{im} + (\gamma - 1) V_{ih} \]

1. **Risk Premium (net of Jensen’s Inequality Effect):**
   - **Investment Opportunity Effect:** Asset’s Covariance with the market portfolio weighted by \( \gamma \) (risk aversion)
   - **Hedging Effect:** Assets Covariance with future returns
2. **CAPM models only use covariances with return on market portfolio**
   - If CRA is \( \gamma = 1 \)
   - When \( V_{ih} \), investment opportunities are constant
   - If this is the case one could back out \( \gamma \) to test for reasonable parameterization
   
   Mehra-Prescott (1986)
Applications and Important Points within the Paper that I skipped

- Term Structure of Interest Rates

\[ E_t r_{i,t+1} - r_{f,t+1} = -\frac{V_{ii}}{2} + \gamma V_{im} + (1 - \gamma) V_{ih} \]
Applications and Important Points within the Paper that I skipped

- Term Structure of Interest Rates

\[ E_t r_{i,t+1} - r_{f,t+1} = -\frac{V_{ii}}{2} + \gamma V_{im} + (1 - \gamma) V_{ih} \]

- Heteroskedastic Returns

If we define a law of motion for \( \mu_{m,t} \)

\[ \mu_{m,t} = \mu_0 + \psi E_t r_{m,t+1} \]

which hold if \( V_{mh,t} = V_{mh}^0 + V_{mh}^1 E_t r_{m,t+1} \) in which case:

\[ E_t r_{i,t+1} - r_{f,t+1} = \beta_1 + \beta_2 V_{mm,t} \]

which implies that the equity premium is a GARCH process.
Applications and Important Points within the Paper that I skipped

- Term Structure of Interest Rates

\[ E_t r_{i,t+1} - r_{f,t+1} = -\frac{V_{ii}}{2} + \gamma V_{im} + (1 - \gamma) V_{ih} \]

- Heteroskedastic Returns
  If we define a law of motion for \( \mu_{m,t} \)

\[ \mu_{m,t} = \mu_0 + \psi E_t r_{m,t+1} \]

which hold if \( V_{mh,t} = V_{mh}^0 + V_{mh}^1 E_t r_{m,t+1} \) in which case:

\[ E_t r_{i,t+1} - r_{f,t+1} = \beta_1 + \beta_2 V_{mm,t} \]

which implies that the equity premium is a GARCH process.

- Accuracy of the Representation
Campbell’s Conclusions

1. Expected excess log return is composed by the 3 elements discussed previously.
Campbell's Conclusions

1. Expected excess log return is composed by the 3 elements discussed previously.
2. Restricted Heteroskedasticity can be incorporated in the former analysis as discussed.
Things that I don’t have clear

- General Equilibrium Approach to understand predictability
  How to link this with a setup similar to Lucas’s 78 ”Asset Pricing in an Exchange Economy”
Things that I don’t have clear

- General Equilibrium Approach to understand predictability
  How to link this with a setup similar to Lucas’s 78 ”Asset Pricing in an Exchange Economy”

- More work needs to be done in the Dividend Process (where do this things come from)
Things that I don’t have clear

- General Equilibrium Approach to understand predictability
  How to link this with a setup similar to Lucas’s 78 “Asset Pricing in an Exchange Economy”
- More work needs to be done in the Dividend Process (where do these things come from)
- What is the referential portfolio m?