Reviewing *Price Drift as an Outcome of Differences in Higher Order Beliefs* by S. Banerjee, R Kaniel and I. Kremes

Seminar in Asset Pricing Theory

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Introduction

- Higher Order Beliefs and Aumann’s Theorem
- Keynes (1936) Beauty Contest
- Milgrom and Stokey "No Trade Theorems"
Auction 1: How much money is in the jar?
Auction 2: How much money is in the jar?
Let $V_t \sim N\left(0, \frac{1}{\rho_v}\right)$ the value of an asset.

Let $S^i_t$ be a signal received by agent $i$ where it’s law of motion is:

$$S^i_t = V_t + \epsilon^i_t$$

with $\epsilon_t \sim N\left(0, \frac{1}{\rho_\epsilon}\right)$.

Thus unconditional distribution of the signal is:

$$S^i_t \sim N\left(0, \frac{1}{\rho_s}\right)$$

where $\frac{\rho_\epsilon + \rho_v}{\rho_v \rho_\epsilon} = \frac{1}{\rho_s}$. 
Q: Given an observed signal $S_t$ what is $E(V|S)$?

To find out we need Bayes Theorem:

$$f(V|S) = \frac{f(S|V)f(V)}{f(S)}$$

so given that the problem Gaussian, we can easily substitute the formulas of a Gaussian Process into this rule and find:

$$f(S|V) \propto \sqrt{\rho_e} \exp \left( -\frac{1}{2}(S - V)^2 \rho_v \right)$$

$$f(V) \propto \sqrt{\rho_v} \exp \left( -\frac{1}{2}(V)^2 \rho_v \right)$$

$$f(S) \propto \sqrt{\rho_s} \exp \left( -\frac{1}{2}S^2 \rho_s \right)$$

$$f(V|S) = \sqrt{\rho_e + \rho_v} \exp \left( -\frac{1}{2}[(S - V)^2 \rho_v + V^2 \rho_v - S^2 \rho_s] \right)$$
Now

\[
[(S - V)^2 \rho_e + V^2 \rho_v - S^2 \rho_s] = V^2 (\rho_e + \rho_v) - 2VS \rho_e + S^2 (\rho_e - \rho_s)
\]

\[
= \left( V \sqrt{(\rho_e + \rho_v)} - S \sqrt{(\rho_e - \rho_s)} \right)^2
\]

\[
= \left( V - S \frac{\sqrt{(\rho_e - \rho_s)}}{\sqrt{(\rho_e + \rho_v)}} \right)^2 (\rho_e + \rho_v)
\]

\[
= \left( V - \frac{\rho_e}{(\rho_e + \rho_v)} S \right)^2 (\rho_e + \rho_v)
\]

so the solution to our question is:

\[
V|S^e \sim N \left( \frac{\rho_e}{(\rho_e + \rho_v)} S^i, \frac{1}{(\rho_e + \rho_v)} \right)
\]

so the expected value after the signal is:

\[
E [V|S] = \frac{\rho_e}{(\rho_e + \rho_v)} S^i
\]

or inverting the variances:

\[
E [V|S] = \frac{\sigma_v}{(\sigma_e + \sigma_v)} S^i
\]
With this digression, we are ready to get the intuition behind the paper:

- Time: Two Periods
- **Consumer’s Problem** (Note the mistake in the paper)

\[
\max_x E \left[ -\exp(-\gamma x (V - P_1)) | I^i \right]
\]

assuming that prices are observed, the optimal is obtained from the definition of this expectation:

\[
- E \left[ \exp(\gamma x (V - P_1)) | I^i \right] = - \left[ \exp \left( \gamma x \left( E^i [V] - P_1 \right) + \frac{(\gamma x)^2 \left( \frac{1}{\rho_v} \right)}{2} \right) \right]
\]

so taking the first order condition w.r.t. \( x \):

\[
0 = \left( -\gamma (E^i [V] - P_1) + \gamma^2 x \left( \frac{1}{\rho_v} \right) \right)
\]

\[
x^i = \frac{\rho_v}{\gamma} (E^i [V] - P_1)
\]
**Definition:** Equilibrium in the asset market requires:

1. $x^i$ solves the consumer problem.
2. A prediction for optimal forecasting: $E^i [V]$.
3. Market Clearing: $\int x^i = 0$

The myopic solution implies a clearing price. The BIG assumption here is the **Walrasian tatonnement** mechanics. This implies that the agents don’t recover any information from market prices.
Then, the optimal forecasting rule from the signal extraction problem (2) in the optimal portfolio (1) in the market clearing condition yields the solution:

\[
\frac{\rho V}{\gamma} \left( \int_i E^i [V] - P_1 \right) = 0
\]

\[
\frac{\rho \epsilon}{\rho \epsilon + \rho v} \int_i S^i = P_1
\]

\[
P_1 = \frac{\rho \epsilon}{\rho \epsilon + \rho v} V
\]

assuming an infinite number of agents.
**Definition:** Equilibrium with Noisy Traders:

1. $x^i$ solves the consumer problem.
2. A prediction for optimal forecasting: $E^i [V|S_1, P_1]$.
3. Market Clearing: $\int_i x^i = Z_1$, where $Z_1 \sim N(0, \frac{1}{\rho_z})$

Thus, now we break with the **Walrasian tatonnement** and agents instantaneously observe $P_1$, and change their beliefs automatically. Thus $\int_i E^i [V|S_1, P_1] = V$ as a property of (2). $E^i [V|S_1, P_1] \neq V$ and is a complicated expression depending on the stochastic process $Z_1$ and $S_1$ but the key issue is that agents on average don't commit mistakes. Thus we have the following:
Proposition: (i) Prices under rational expectations assumption (in (2)) imply mean reversion
(ii) Prices under Difference of Opinion Assumption may admit price drift.
Solution [REE Myopic Case]: The Rational Expectations Equilibrium with Noisy Traders is obtained by replacing (1) into (2) and assuming that $\int_i E^i [V|S_1, P_1] = V$:

$$
E[V - P_1|P_1] = E\left[V - \bar{E}[V] + \frac{\gamma}{\rho V} Z_1|P_1\right]
$$

$$
= E\left[V - \bar{E}[V] + \frac{\gamma}{\rho V} Z_1|P_1\right]
$$

$$
= E[V - \int_i E^i [V|S_1, P_1]] + \frac{\gamma}{\rho V} E[Z_1|P_1]
$$

$$
= \int_i E[V - E^i [V|S_1, P_1]] + \frac{\gamma}{\rho V} E[Z_1|P_1]
$$

$$
= \frac{\gamma}{\rho V} E[Z_1|P_1]
$$

so since

$$
Z_1 = \frac{\rho \varepsilon}{\rho V + \rho \varepsilon} \int_i E^i [V|S_1^i, P_1] - P_1
$$

$$
E[V - P_1|P_1] = \frac{\gamma}{\rho V} k [P_1]
$$

for some $k < 0$. So prices imply mean reversion.
Now agents don’t take into account information on prices but just their signal.

Solution [DOO Myopic Case:]
Hence from (1) and (3)

\[ P_1 = \bar{E} [V] - \frac{\gamma}{\rho_v} Z_1 = \frac{\rho_\epsilon}{\rho_{V,0} + \rho_\epsilon} V - \frac{\gamma}{\rho_{V,0} + \rho_\epsilon} Z_1 \]

then \( E[V - P_1|P_1] = E[V|P_1] - P_1 \) so using our formula for signal extraction:

\[
\begin{align*}
&= \frac{\rho_\epsilon}{\rho_{V,0} + \rho_\epsilon} \frac{1}{\rho_{v,0}} \left( \frac{1}{\rho_{v,0}} \right)^2 \frac{1}{\rho_{v,0}} + \left( \frac{\gamma}{\rho_{V,0} + \rho_\epsilon} \right)^2 \frac{1}{\rho_z} \\
&= \left( \frac{\rho_{V,0} (\rho_z \rho_\epsilon - \gamma^2)}{\rho_\epsilon^2 \rho_z + \gamma^2 \rho_{V,0}} \right) P_1 \\
&\text{so the condition is that } \left( \frac{\rho_{V,0} (\rho_z \rho_\epsilon - \gamma^2)}{\rho_\epsilon^2 \rho_z + \gamma^2 \rho_{V,0}} \right) > 1.
\]
- T = 0, 1, 2

- **Assumptions (1):** Agents don’t observe opinion of others.

- **Assumptions (2):** Noisy supply is given by:

  The problem is solved by backward induction:

  \[ Z_t = \sum_{i=0}^{t} u_i \text{ with } u_i \sim \mathcal{N} \left( 0, \frac{1}{\rho_u} \right) \]
Difference of Opinion Structure

\[ x_2 = \rho_v \frac{(E^i - P_2)}{\gamma} \]

Applying the same procedure:

\[ P_2 = E[\bar{V}] - \frac{\gamma}{\rho_v} Z_2 \]

Conjecture: prices in the first period are given by:

\[ P_1 = aE[\bar{V}] + bZ_1 \]

Based on the conjecture:

\[ P_2 \sim N \left( E^i [P_2|P_1, E^i [V]] , \frac{1}{\rho_{P_2}} \right) \]

the conditional mean is:

\[ E^i [P_2|P_1, E^i [V]] = k_1 P_1 + k_2 E^i [V] \]
Proposition (i) A unique linear equilibrium described in which
\[ P_1 = a\bar{E}[V] + bZ_1 \] and
\[ E^i [P_2|P_1, E^i [V]] = k_1 P_1 + k_2 E^i [V] \]
(ii) Optimal demand in the first period is given by:
\[ x^i_1 = \frac{1}{\gamma} \rho_V (E^i [V] - P_1) + \frac{1}{\gamma} \rho_{p_2} (E^i [P_2|P_1, E^i [V] - P_1]) \]
where:
1. \( \frac{1}{\gamma} \rho_{p_2} (E^i [P_2|P_1, E^i [V] - P_1]) \) refers to a speculative position
2. And \( \frac{1}{\gamma} \rho_V (E^i [V] - P_1) \) is the secured Long Term Position.
Proposition Without differences in Higher Order Beliefs, there is no price drift.

Proposition With differences in Higher Order Beliefs \( \exists \) an equilibrium with:

(i) \( P_1 = a \bar{E} [V] + bZ_1 + c \bar{E} [\bar{E} [V]] \)

\[ E^i [P_1 | P_1, E^i [V], E^i [\bar{E} [V]]] = k_1 P_1 + k_2 E^i [V] + k_3 [E^i [\bar{E} [V]]] \]

(ii) With differences in Higher Order Beliefs there is price drift:

\[ E [P_2 - P_1 | P_1] \]
Let’s finish the game...
The Author’s Insight

According to the solution...