In the recent literature of empirical asset pricing there has been considerable evidence of time-varying expected returns. With time-varying expected returns, the investment opportunity set becomes time-varying itself which introduces a new component of demand for risky assets, namely intertemporal hedging demand, in the spirit of Merton (1971). An asset is a good hedge for an investment opportunity set if it tends to pay off well when it is deteriorating.
In the environment that follows, the risky asset will play the role as being a good hedge since its covariance with the return on the risky asset and its expected return is negative.

The authors find that hedging demand can be a large component of risky asset demand and thus, myopic portfolio choice is misleading for an investor with a long horizon.

However, the authors do not take parameter uncertainty into account which may weaken hedging demand drastically, particularly, for investors with a long horizon.

A major contribution of the paper is that it provides approximate analytical solutions to not only the portfolio policy but also the consumption policy.
(A1) Wealth consist of two tradable assets, a risky and a riskless asset, and thus return to wealth is given by

\[ R_{t+1}^p = \alpha_t (R_{t+1} - R^f) + R^f, \tag{1} \]

where \( \alpha_t \) is the proportion of wealth invested in the risky asset.

(A2) The expected log excess return depends on a single state \( x \) so that

\[ E_t r_{t+1} - r^f = x_t \tag{2} \]

where \( x \) follows an AR(1),

\[ x_{t+1} = \mu + \phi (x_t - \mu) + \eta_{t+1} \tag{3} \]

where \( \eta_{t+1} \sim N(0, \sigma^2_{\eta}) \). Note that homoskedasticity rules out that the state variable predicts changes in risk.

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(A3) The unexpected log return on the risky asset $u$ is also conditionally homoskedastic and normally distributed and is correlated with shocks to the state variable

$$\text{var}_t(u_{t+1}) = \sigma_u^2$$  \hspace{1cm} (4)

$$\text{cov}_t(u_{t+1}, \eta_{t+1}) = \sigma_{u\eta}.$$  \hspace{1cm} (5)

Negative correlation is the central statistical property of mean-reversion/time-varying-expected returns/predictability, and thus hedging demand.
(A4) The investor is infinitely lived and has preferences given by

\[ U(C_t, E_t U_{t+1}) = \left[ (1 - \delta) C_t^{(1-\gamma)/\theta} + \delta (E_t U_{t+1}^{1-\gamma})^{1/\theta} \right]^\theta/(1-\gamma) \]  \hspace{1cm} (6)

where \( \delta \) is the discount factor, \( \gamma \) is the coefficient of relative risk aversion, \( \psi \) is the elasticity of IMRS and \( \theta \) is defined as \( \theta = (1 - \gamma)/(1 - \psi^{-1}) \).
The individual’s problem is to maximize (6) subject to the budget constraint

\[ W_{t+1} = R_{t+1}^p (W_t - C_t), \quad (7) \]

for which the Euler equation for asset \( i \) is given by

\[ 1 = E_t \left[ \left\{ \delta \left( \frac{C_{t+1}}{C_t} \right)^{-1/\psi} \right\}^{\theta} R_{p,t+1}^{-\left(1-\theta\right)} R_{t+1}^i \right] \quad (8) \]

where \( i \) is either the risk free asset, the risky asset or the portfolio.

For example, for the portfolio, (8) reduces to

\[ 1 = E_t \left[ \left\{ \delta \left( \frac{C_{t+1}}{C_t} \right)^{-1/\psi} \right\}^{\theta} R_{t+1}^p \right] \quad (9) \]
Dividing (6) by $W_t$ and using the budget constraint (7) we get an expression for utility per unit of wealth

$$V_t = \left\{ (1 - \delta) \left( \frac{C_t}{W_t} \right)^{1 - 1/\psi} + \delta \left( 1 - \frac{C_t}{W_t} \right)^{1 - 1/\psi} \right\} \frac{1}{(1 - \psi)} (E_t[V_{t+1}^{1-\gamma} R_{p,t+1}^{1-\gamma}])^{1/\theta},$$

where $V_t \equiv U_t/W_t$.

The value function per unit of wealth can be written conveniently in terms of $(1 - \delta)$ and the consumption wealth ratio

$$V_t = (1 - \delta)^{-\psi/(1-\psi)} \left( \frac{C_t}{W_t} \right)^{1/(1-\psi)}. \quad (11)$$
Approximate Solution Method

Portfolio Choice

- The goal in solving the model is to find a consumption and portfolio rule which satisfy the Euler equation.
- The approximate solution method starts by log-linearizing the Euler equation (9) to obtain

\[ 0 = \theta \log \delta - \frac{\theta}{\psi} E_t \Delta c_{t+1} + \theta E_t r^p_{t+1} + \frac{1}{2} \text{var}_t \left( \frac{\theta}{\psi} \Delta c_{t+1} - \theta r^p_{t+1} \right) \]

which holds only approximately since consumption growth cannot be assumed conditionally lognormally distributed from the outset due to its endogeneity.
- Reordering terms we get that

\[ E_t \Delta c_{t+1} = \psi \log \delta + \nu_{p,t} + E_t r^p_{t+1}, \quad (12) \]

\[ \nu_{p,t} = \frac{1}{2} \frac{\theta}{\psi} \text{var}_t (\Delta c_{t+1} - \psi r^p_{t+1}). \quad (13) \]
Similarly, we can log-linearize (8) for any given asset and thereby also find the expected log risk premium

\[ E_t r_{t+1} - r^f + \frac{1}{2} \sigma_{r,r,t} = \frac{\theta}{\psi} \sigma_{r,c,t} + (1 - \theta) \sigma_{r,p,t}, \]  

(14)

so that the log risk-premium is a 'weighted average' of the CCAPM factor (covariance between risky asset return and consumption growth) and the CAPM factor (covariance between risky asset return and return of wealth portfolio).

The log-linear budget constraint (7) is given by

\[ \Delta w_{t+1} \approx r^p_{t+1} + (1 - (1/\rho))(c_t - w_t) + k, \]  

(15)

where \( k = \log(\rho) + (1 - \rho)\log(1 - \rho)/\rho \), and \( \rho = 1 - \exp[E(c_t - w_t)] \).
Note that the consumption-wealth ratio is approximately given by

\[ c_t - w_t \approx \sum_{i=1}^{\infty} \rho^i (r_{t+i}^p - \Delta c_{t+i}) \]

so that for a relative large consumption-wealth ratio the growth in wealth is relatively large. This is because either returns increase and/or consumption growth decreases.

Further, use a second-order Taylor expansion to get an approximation to (1) to get

\[ r_{t+1}^p = \alpha_t (r_{t+1} - r^f) + r^f + \frac{1}{2} \alpha_t (1 - \alpha_t) \sigma_{r,r,t}. \]  

(16)
Combining (15) and (16) we get

\[ \Delta w_{t+1} \approx \alpha_t(r_{t+1} - r^f) + r^f + \left(1 - \frac{1}{\rho}\right)(c_t - w_t) + k + \frac{1}{2}\alpha_t(1 - \alpha_t)\sigma_{r,t} \]

which is linear in log returns and log consumption, and quadratic in the portfolio weights. This expression now relates the growth in wealth to the individual assets and not just the entire portfolio as in (15).

Rewrite (14) in terms of the covariance between the risky asset return and the consumption-wealth ratio and the variance of the risky asset (i.e. risky asset return is exogenous and consumption-wealth ratio is stationary).
Approximate Solution Method
Portfolio Choice

- Using an identity for consumption growth as well as equations (16) and (17) we can write the Euler equation (14) in the following way

\[ E_t r_{t+1} - r^f + \frac{1}{2} \sigma_{r,r,t} = \frac{\theta}{\psi} (\sigma_{r,c-w,t} + \alpha_t \sigma_{r,r,t}) + (1 - \theta) \alpha_t \sigma_{r,r,t}, \]

where \( \sigma_{r,c-w,t} = \text{cov}_t(r_{t+1}, c_{t+1} - w_{t+1}). \)

- Hence, an asset is risky when the return next period positively covaries with the consumption-wealth ratio since an increase in the consumption-wealth ratio indicates an improvement in the investment opportunity set.
Approximate Solution Method

Portfolio Choice

Solving for the portfolio weight and using the definition of $\theta$ we get

$$\alpha_t = \frac{1}{\gamma} \frac{E_t r_{t+1} - r_f + \frac{1}{2} \sigma_{r,r,t}}{\sigma_{r,r,t}} - \left( \frac{1}{1 - \psi} \right) \left( \frac{\gamma - 1}{\gamma} \right) \frac{\sigma_{r,c-w,t}}{\sigma_{r,r,t}}$$

which can be described as a myopic part (current risk premium) and an intertemporal hedging component. As is apparent the investor holds a larger proportion of risky assets when the risk premium is high and/or the asset is a good hedge for deteriorations in the investment opportunity set.
Note from (19) that the intertemporal hedging demand is zero when returns are unpredictable or when $\gamma = 1$. Otherwise (19) is not a solution to the model since the portfolio allocation is a function of future portfolio and consumption decisions, which are endogenous to the problem. In particular, the dependence of today’s portfolio decision on future portfolio and consumption decisions operates through the conditional covariance of the risky asset return and the consumption-wealth ratio.
Guess that the optimal portfolio policy is linear in the state variable $x$, and that the consumption-wealth ratio is quadratic in the state variable, so that

$$
\alpha_t = a_0 + a_1 x_t,
$$

$$
c_t - w_t = b_0 + b_1 x_t + b_2 x_t^2,
$$

where the parameters $(a_0, a_1, b_0, b_1, b_2)$ are identified using the method of undetermined coefficients.

Proposition 1 and 2 enable us to solve for the unknown coefficients of the model.
Property 1 shows that the value function per unit wealth is convex in $x$, the risk premium, so that the investor can benefit from a negative risk premium as well as from a positive risk premium:

$$V_t = \exp \left[ \frac{b_0 - \psi \log(1 - \delta)}{1 - \psi} + \frac{b_1}{1 - \psi} x_t + \frac{b_2}{1 - \psi} x_t^2 \right].$$

Property 2, implies that in this setting, the optimal solution to the portfolio problem increases in the risk premium.

This is not true with more general utility functions as the income effect of an increase in the risk premium may outweigh the substitution effect.
Property 3 lists an array of particular preference parameters which lead to exact solution of one or both optimal policy functions.

a) When $\psi \neq 0$ and $\gamma \to 1$ the optimal portfolio rule is myopic, given by $a_0 \to \frac{1}{2\gamma}$ and $a_1 \to \frac{1}{\gamma \sigma_u^2}$, although the consumption policy is not.

b) When $\psi \to 1$ and $\gamma \neq 1$ the optimal consumption rule is myopic, given by $b_1 \to 0$ and $b_2 \to 0$, $\rho \to \delta$, and $b_0 \to \log(1-\delta)$, although the portfolio policy is not.
c) In the case of logarithmic utility, $ψ \to 1$ and $γ \to 1$, both consumption and portfolio policies are known to be myopic. In that case we have both a) and b) with $γ = 1$, and both are exact.

d) When expected returns are constant then we are also back at the myopic solutions for both policies as in a) and b).
Property 4 shows that, given a particular $\rho$, the main preference parameter determining portfolio choice is the coefficient of relative risk aversion since the elasticity of IMRS cancels out.

Property 5 shows that, given a particular $\rho$, the slope of the portfolio policy $a_1$, as well as the curvature of the consumption policy $b_2$, do not depend on the average risk premium $\mu$. However, the average allocation to the risky asset, the average consumption-wealth ratio, and the linear sensitivity of consumption to the state variable do depend on the average risk premium.
Results
Calibration Exercise

- The state variable is the log dividend-price ratio. They estimate the following restricted VAR

\[
\begin{pmatrix}
    r_{t+1} - r_f \\
    d_{t+1} - p_{t+1}
\end{pmatrix} = \begin{pmatrix}
    \theta_0 \\
    \beta_0
\end{pmatrix} + \begin{pmatrix}
    \theta_1 \\
    \beta_1
\end{pmatrix} (d_t - p_t) + \begin{pmatrix}
    \epsilon_{1,t+1} \\
    \epsilon_{2,t+1}
\end{pmatrix}
\]

(20)

where the error terms are jointly normally distributed, which allows, the stochastic structure (3), (4), and (5) to be recovered.

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Consumption and Portfolio Decisions When Expected Returns Are Time Varying
A useful normalization of the policy function for asset demand is

$$\alpha_t = \left( a_0 - a_1 \frac{\sigma_u^2}{2} \right) + a_1 \left( x_t + \frac{\sigma_u^2}{2} \right)$$

$$\alpha_t = a^*_0 + a_1 \left( x_t + \frac{\sigma_u^2}{2} \right)$$

which implies that if $x_t = -\left( \frac{\sigma_u^2}{2} \right)$, in which case the risk premium is zero, and thus a risk-averse investor does not hold the risky asset for myopic reasons, all the demand for the risky asset is due to intertemporal hedging.

The intuition is that a conservative investor with $\gamma > 1$ wants to hedge the risk of a deterioration of investment opportunities. Since stocks have $\sigma_{\eta,u} < 0$ they are a hedge.
The fact that the conditional covariance between the risky asset return and the state variable is negative implies mean reversion, that is, high short-term returns tend to be offset by lower returns over the long term since

\[ \text{cov}_t(r_{t+1}, x_{t+1}) = \text{cov}_t(r_{t+1}, r_{t+2}) = \sigma_{\eta,u} < 0 \]
\[ \text{var}_t(r_{t+1} + r_{t+2}) = 2\text{var}_t(r_{t+1}) + 2\text{cov}_t(r_{t+1}, r_{t+2}) < 4\text{var}_t(r_{t+1}). \]

The intercept for the consumption-wealth ratio is the consumption demand out of wealth when the expected risk premium is zero,

\[ c_t - w_t = b_0^* + b_1^* \left( x_t + \frac{\sigma_u^2}{2} \right) + b_2 \left( x_t + \frac{\sigma_u^2}{2} \right)^2. \] (22)
The optimal portfolio choice for various values of risk aversion and elasticities of IMRS are documented in Table II and Figure I. From the table and the figure we learn that optimal portfolio choice varies much more in response to changing risk aversion than in response to changing elasticities of IMRS. Thus, risk aversion is much more important in determining allocation choice than the elasticity of IMRS. Recall Proposition 4.

From Table II, Panel A, we also learn that for $\gamma > 1$ demand due to hedging is positive. Thus, the responsiveness of the allocation to the risky asset to the state variable is more responsive for an investor who follow strategic asset allocation as opposed to tactical asset allocation.
From Table II, Panel B, we also learn that for $\gamma > 1$ intertemporal hedging demand increases the slope $a_1$ so that the slope of hedging demand is positive. This implies that conservative long-horizon investors are actually more aggressive market timers than conservative short-horizon investors.

As implied by Property 2 the slopes of the $\alpha_t$’s are all positive. Furthermore, as $\gamma$ increases $a_1$ approaches zero so that the optimal portfolio rule is very responsive to changes in expected returns when the investor is close to risk-neutral and almost flat for highly risk-averse individuals.
From Table III, Panel A, we learn that coefficients of relative risk aversion well above 4 are needed in order for the investor to hold less than 100% in the risky asset. Panel B reveals that the demand for the risky asset due to hedging is a substantial fraction of the mean optimal percentage allocation. For risk aversion above 4 more than half of the asset demand is due to hedging.

The optimal consumption-wealth ratio choice for various values of risk aversion and elasticities of IMRS are documented in Table IV and Figure II. In contrast to the optimal portfolio choice, the consumption-wealth ratio is very sensitive to both the level of the elasticity of IMRS and the level of risk aversion.
If the time-preference adjusted return is positive, an individual willing to substitute intertemporally, will have higher savings and thus lower current consumption. An individual, unwilling to substitute intertemporally, will have lower savings and thus higher current consumption. If the time-preference adjusted return is negative, the opposite pattern emerges.

This is demonstrated in Panel A of Table IV. An investor with low risk aversion, chooses a portfolio with high average returns, and thus, an increase in the elasticity of IMRS corresponds to a lower average consumption-wealth ratio. An investor with high risk aversion, on the other hand, chooses a safe portfolio with low average returns, and thus, an increase in the elasticity of IMRS corresponds to a higher average consumption-wealth ratio.
Panel A of Table V shows that investors with low risk aversion, and thus more heavily invested in the risky asset, also have high volatility in consumption growth, since consumption inherits the volatility of their portfolios. At the same time, for these investors the volatility of consumption growth varies depending on their elasticity of IMRS. For low elasticity of IMRS investors, consumption growth is relatively less volatile.

Next, they plot the optimal equity allocation and consumption-wealth ratios over time for a power utility specification with $\psi = \frac{1}{4}$ and $\gamma = 5$ and Epstein-Zin specification with $\psi = \frac{1}{4}$ and $\gamma = 20$. Stock holdings are more volatile than consumption wealth ratios over time, and as expected, the low risk aversion investor holds a larger fraction of the risky asset and has a higher consumption-wealth ratio.
Both types of investors are keen stock market participants. In the postwar U.S. period both investors are always long in stocks and the less risk-averse investor shorts the riskless asset to invest more than 100% in the risky asset, except in periods of low dividend yields such as the early 1970s and the mid 1990s.

They have shown that a long-term investor who optimally responds to the estimated predictability of stock returns will time the stock market as well as use stocks as a hedge against deteriorations in the investment opportunity set. But are the utility costs of a suboptimal portfolio choice large relative to the strategic asset allocation?
In order to answer this question they compare the value functions per unit of wealth implied by the strategic asset allocation approach, which allows for hedging and timing, with three alternative investment rules: hedging/no-timing, no-hedging/timing, and no-hedging/no-timing.

For all parameter values they consider, the failure to time the market causes larger utility losses than the failure to hedge intertemporally. Suboptimal myopic portfolio rules imply large utility losses for investors with $\gamma > 1$. Thus, static models of portfolio choice can be seriously misleading.
There is an error in the empirical results. The authors correct the error and find that originally predictability was actually understated as well as $\sigma_{\eta,u}$ which understates the magnitude of hedging demands.

While static models of portfolio choice can be seriously misleading, dynamic models can be misleading as well if parameter uncertainty is ignored, as we will see from Barberis (2001).
Incorporating predictability in optimal portfolio choice may be useless if past relationships change substantially over the passage of time (i.e. predictive power of dividend-yield has decreased after 1995).

Predictability evidence has to interpreted with caution as researchers such as Stambaugh (1999) and Valkanov (2003) demonstrate.
▶ Even if predictability holds, frequent rebalancing due to strategic reasons may be too costly in a world with transaction costs.

▶ Even if predictability holds, which reduces the risk of stock over the long-run, parameter uncertainty may offset predictability, especially over the long-run.
Another issue that may lead to an offset of the benefits of mean-reversion is that when investors understand that they will learn more about the equity premium over time.

Despite all this, their results demonstrate that there is a horizon effect which may make a myopic investment strategy less than ideal.
Arguably, a passive buy-and-hold strategy based on popular investment advice that stocks are safer in the long-run may not hold, since safety is induced by mean-reversion which at the same time yields benefits for the well-being of the investor. Thus, strategic asset allocation, may be better investment advice.

A buy-and-hold strategy is optimal only if expected returns are constant in which case horizon is irrelevant and, therefore, the popular investment advice is a contradiction in itself.