When is Market Incompleteness Irrelevant for the Price of Aggregate Risk?

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Literature on incomplete markets has few general results

Will show that in a large class of models, market incompleteness will not matter for the equity premium
Key Assumptions

- Infinite number of agents
- CRRA utility
- Distribution of idiosyncratic labor shocks independent of aggregate shock
- Capital share in independent of aggregate shock
- Agents face solvency constraints that are proportional to the aggregate endowment
Idea of Article

- Solve for equilibrium allocation in a Bewley model with no aggregate uncertainty
- Show that this allocation satisfies the conditions for optimality in de-trended economies
- Re-scale allocations to get equilibrium in economies with aggregate uncertainty
Economies Studied

- **Bewley**: 1 bond, 1 stock, idiosyncratic shocks, no aggregate shocks
- **Arrow**: Arrow securities, idiosyncratic shocks, aggregate shocks
- **Bond**: 1 bond, 1 stock, idiosyncratic shocks, aggregate shocks
- **Breeden-Lucas**: Arrow securities, no idiosyncratic shocks, aggregate uncertainty
The Model

- Aggregate shocks $z_t \in \mathbb{Z}$
- Idiosyncratic shock $y_t \in \mathbb{Y}$
- State $s_t = (y_t, z_t)$
- History: $s^t = (s_0, s_1, ... s_t)$
- Conditional probability of $s^t$ is $\pi(s^t|s_0)$
- Shocks are 1st order Markov
The Model

- Endowment $e_t(z^t)$
- Labor Income $\eta_t(s^t) = \eta(y_t, z_t)e_t(z^t)$
- Aggregate endowment growth $\lambda(z_{t+1}) = e_{t+1}(z^{t+1})/e_t(z^t)$
- Capital Share: $\alpha$
- Dividend from capital: $\alpha e_t$
- Labor Share: $1 - \alpha = \sum_{y_t \in Y} \Pi_{z_t}(y_t)\eta(y_t, z_t)$
Utility

\[ U(c)(s_0) = \sum_{t=1}^{\infty} \sum_{s^t|s_0} \beta^t \pi(s^t|s_0) c_t(s^t)^{1-\gamma} \]

Written Recursively

\[ U(c)(s^t) = u(c_t(s^t)) + \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) U(c)(s^t, s_{t+1}) \]
De-trending

\[ \hat{c}_t(s^t) = \frac{c_t(s^t)}{e_t(z^t)} \]

\[ \hat{\pi}(s_{t+1}|s_t) = \frac{\pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\gamma}}{\sum_{s_{t+1}} \pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\gamma}} \]

\[ \hat{\beta}(s^t) = \sum_{s_{t+1}} \pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\gamma} \]

Utility

\[ \hat{U}(\hat{\beta})(s^t) = u(\hat{c}_t(s^t)) + \hat{\beta}(s_t) \sum_{s_{t+1}} \hat{\pi}(s_{t+1}|s_t) \hat{U}(\hat{c})(s^t, s_{t+1}) \]
Key Assumptions

Idiosyncratic shocks independent of aggregate shocks

- $\eta(y_t, z_t) = \eta(y_t)$
- $\pi(z_{t+1}, y_{t+1}|y_t, z_t) = \varphi(y_{t+1}|y_t)\phi(z_{t+1}|z_t)$

Aggregate Endowment growth is iid

- $\phi(z_{t+1}|z_t) = \phi(z_{t+1})$
Bewley

Budget Constraint:

\[ \hat{c}_t(y^t) + \frac{\hat{a}_t(y^t)}{\hat{R}_t} + \hat{\sigma}_t(y^t)\hat{v}_t = \eta(y_t) + \hat{a}_{t-1}(y^{t-1}) + \hat{\sigma}_{t-1}(y^{t-1})(\hat{v}_t + \alpha) \]

Borrowing Constraint:

\[ \frac{\hat{a}_t(y^t)}{\hat{R}_t} + \hat{\sigma}_t(y^t)\hat{v}_t \geq \hat{K}_t(y^t) \forall y^t \]

\[ \hat{a}_t(y^t) + \sigma_t(y^t)(\hat{v}_{t+1} + \alpha) \geq \hat{M}_t(y^t) \forall y^t \]
Budget Constraint:

\[ c_t(s^t) + \sum_{z_{t+1}} a_t(s^t, z_{t+1}) q_t(z^t, z_{t+1}) + \sigma_t(s^t) v_t \leq \theta_t(s^t) \]

\[ \theta_{t+1}(s^{t+1}) = \eta(y_{t+1}) e_{t+1}(z_{t+1}) + a_t(s^t, z_{t+1}) \\
+ \sigma_t(s^t)[v_{t+1}(z^{t+1}) + \alpha e_{t+1}(z_{t+1})] \]

Borrowing Constraint:

\[ \sum_{z_{t+1}} a_t(s^t, z_{t+1}) q_t(z^t, z_{t+1}) + \sigma_t(s^t) v_t(z^t) \geq \hat{K}_t(y^t) e_t(z^t) \]

\[ a_t(s^t, z_{t+1}) + \sigma_t(s^t)(v_{t+1} + \alpha e_{t+1}(z_{t+1})) \geq \hat{M}_t(y^t) e_t(z^t) \forall z_{t+1} \]
Budget Constraint:

\[ \hat{c}_t(s^t) + \sum_{z_{t+1}} \hat{a}_t(s^t, z_{t+1}) \hat{q}_t(z^t, z_{t+1}) + \sigma_t(s^t) \hat{v}_t \leq \eta(y_t) + \hat{a}_{t-1}(s^{t-1}, z_t) + \sigma_{t-1}(s^{t-1})[\hat{v}_t(z^t) + \alpha] \]

Borrowing Constraint:

\[ \sum_{z_{t+1}} \hat{a}_t(s^t, z_{t+1}) \hat{q}_t(z^t, z_{t+1}) + \sigma_t(s^t) \hat{v}_t(z^t) \geq \hat{K}_t(y^t) \]

\[ \hat{a}_t(s^t, z_{t+1}) + \sigma_t(s^t)(\hat{v}_{t+1} + \alpha) \geq \hat{M}_t(y^t) \forall z_{t+1} \]
Budget Constraint:

\[ c_t(s^t) + \frac{b_t(s^t)}{R_t(z^t)} + \sigma_t(s^t)v_t(z^t) = \eta(y_t)e_t(z_t) + b_{t-1}(s^{t-1}) + \sigma_{t-1}(s^{t-1})(v_t(z_t) + \alpha e_t(z_t)) \]

Borrowing Constraint:

\[ \frac{\hat{a}_t(y^t)}{\hat{R}_t} + \hat{\sigma}_t(y^t)\hat{v}_t \geq \hat{K}_t(y^t)e_t(z^t) \]
\[ \hat{a}_t(y^t) + \sigma_t(y^t)(\hat{v}_{t+1} + \alpha) \geq \hat{M}_t(y^t)e_t(z^t) \forall z_{t+1} \]
De-trended Bond

Budget Constraint:

$$\hat{c}_t(s^t) + \frac{\hat{b}_t(s^t)}{R_t(z^t)} + \sigma_t(s^t)\hat{v}_t(z^t) = \eta(y_t) + \frac{\hat{b}_{t-1}(s^{t-1})}{\lambda(z_t)}$$

$$+ \hat{\sigma}_{t-1}(s^{t-1})(\hat{v}_t(z^t) + \alpha)$$

Borrowing Constraint:

$$\frac{\hat{b}_t(s^t)}{R_t(z^t)} + \hat{\sigma}_t(s^t)\hat{v}_t \geq \hat{K}_t(y^t)$$

$$\frac{\hat{b}_t(s^t)}{\lambda(z_{t+1})} + \sigma_t(s^t)(\hat{v}_{t+1}(z^{t+1}) + \alpha) \geq \hat{M}_t(y^t) \forall z_{t+1}$$
An equilibrium in a Bewley model can be made into an equilibrium for the Arrow model with growth with:

\[ c_t(\theta_0, s^t) = \hat{c}_t(\theta_0, y^t)e_t(z^t) \]
\[ \sigma_t(\theta_0, s^t) = \hat{\sigma}_t(\theta_0, y^t) \]
\[ a_t(\theta_0, s^t, z_{t+1}) = \hat{a}_t(\theta_0, y^t)e_{t+1}(z^{t+1}) \]
\[ v_t(z^t) = \hat{v}_te_t(z^t) \]
\[ q_t(z^t, z_{t+1}) = \frac{1}{\hat{R}_t} \frac{\phi(z_{t+1})\lambda(z_{t+1})^{-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1})\lambda(z_{t+1})^{1-\gamma}} \]
\[ R_t^A(z^t) = \frac{\sum_{z_{t+1}} \phi(z_{t+1})\lambda(z_{t+1})^{1-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1})\lambda(z_{t+1})^{-\gamma}} \]
An equilibrium in a Bewley Model can be made into an equilibrium for the Bond model with growth with:

\[ c_t(\theta_0, s^t) = \hat{c}_t(\theta_0, y^t)e_t(z^t) \]

\[ \sigma^B_t(\theta_0, s^t) = \frac{\hat{a}_t(\theta_0, y^t)}{\hat{v}_{t+1} + \alpha} + \hat{\sigma}_t(\theta_0, y^t) \]

\[ v_t(z^t) = \hat{v}_t e_t(z^t) \]

\[ R_t(z^t) = \hat{R}_t \frac{\sum_{z_{t+1}} \phi(z_{t+1}) \lambda(z_{t+1})^{1-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1}) \lambda(z_{t+1})^{-\gamma}} \]

\[ b_t(\theta_0, s^t) = 0 \]
Why is $b_t(\theta_0, s^t) = 0$?

Plug in solution Bewley model into de-trended Bond model:

$$\hat{c}_t(y^t) + \frac{\hat{b}_t(y^t)}{R_t} + \sigma_t(y^t)\hat{v}_t = \eta(y_t) + \frac{\hat{b}_{t-1}(y^{t-1})}{\lambda(z_t)} + \hat{\sigma}_{t-1}(y^{t-1})(\hat{v}_t + \alpha)$$

Thus portfolio choice is independent of the history of shocks

Agents hold only stock

All agents bear same amount of aggregate risk
Excess returns generated from either the Arrow or Bond economies can be priced using the Representative Agent CCAPM

$$E_t[(R^s_{t+1} - R_t)\beta(\lambda_{t+1})^{-\gamma}] = 0$$
Implications for Asset Pricing

- The Multiplicative Risk Premium is equal in the Arrow, Bond, and Representative Agent Models

\[
1 + v_t^A = 1 + v_t^B = 1 + v_t^{RE}
\]

- Where \( v_t \) is defined as:

\[
1 + v_t = \frac{E_t[R_{t,1}[\{\alpha e_{t+k}\}]]}{R_{t,1}[1]}
\]
The representative agent framework of Breeden and Lucas can still be used to price assets, even if there is uninsurable idiosyncratic labor risk.

Risk premia are the same across the different setups even though the risk free rate is lower.

History of idiosyncratic shocks has no effect on portfolio allocations.