An Equilibrium Model with Restricted Stock Market Participation

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Outline

- Introduction

- The Model
  - Economy
  - Unrestricted case
  - Restricted case
  - Example

- Discussion
Limited Participation – Empirical findings

- 72.4% of the households held no stock at all (as of 1984) but accounted for 68% of aggregate food expenditures.

- 52.3% of the HH, holdings other liquid assets in excess of 100,000$, held no stock at all.

- The fraction of HH owning stocks increases with average labor income and education.

- Aggregate consumption of Stock holders is more volatile and correlated with the equity risk premia.
Main Insight

- Two types of investors:
  - Non-participants – trade only bonds
  - Participants – trade both stocks and bonds

- The non-participants have a smooth consumption process → stock holders are left alone to bear the aggregate risk of the equity market → stock holders demand an higher equity premium
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The Economy

- Information Structure
  - Probability Space – \((\Omega, f, F, P)\)
  - One dimensional Brownian Motion – \(W(t), t \in [0, T]\)
  - Information Set determined by \(W(t)\)

- Consumption Space
  - A single perishable good
  - The consumption process \(c(t)\) is nonnegative and progressively measurable
The Economy - Securities

Exogenous positive dividend process

\[ \delta(t) = \delta(0) + \int_0^t \mu_{\delta}(s) \, ds + \int_0^t \sigma_{\delta}(s) \, dw(s) \]

Endogenous bond price process

\[ B(t) = \exp \left( \int_0^t r(s) \, ds \right) \quad dB(t) = r(t)B(t)\,dt \]

Endogenous Stock price – claim to dividend stream

\[ S(t) = S(0) + \int_0^t (S(s)\mu(s) - \delta(s)) \, ds + \int_0^t S(s)\sigma(s) \, dw(s). \]
The Economy – Trading Strategies

- Admissible trading strategy is a vector process \((\alpha(t), \theta(t))\) – amounts invested in (bond, stock)

- Trading strategy is said to finance the consumption plan \(c(t)\):

\[
\begin{align*}
    dW(t) &= (\alpha(t)r(t) + \theta(t)\mu(t) - c(t)) \, dt + \theta(t)\sigma(t) \, dw(t) \\
    W &= \alpha + \theta
\end{align*}
\]
Two agents, each with

Agent 1:
- Has access to both the bond and stock market
- $U(c)$ satisfies the Inada condition
- Endowed with $(\beta \text{ shares of bond}, 1 \text{ share of stock})$

Agent 2:
- Prevented from investing in the stock market
- $U(c) = \log(c)$
- Endowed with $(\beta \text{ shares of bond}, 0 \text{ shares of stock})$

The Economy – Agents
An **equilibrium** for the economy $\mathcal{E}$ is a price process $(B, S)$—or equivalently an interest rate stock price process $(r, S)$—and a set $\{c_i^*, (\alpha_i^*, \theta_i^*)\}_{i=1}^2$ of consumption and admissible trading strategies for the two agents such that

(i) $(\alpha_i^*, \theta_i^*)$ finances $c_i^*$ for $i = 1, 2$;

(ii) $c_1^*$ maximizes $U_1$ over the set of consumption plans $c \in \mathcal{C}$ which are financed by an admissible trading strategy $(\alpha, \theta) \in \Theta$ with $\alpha(0) + \theta(0) = S(0) - \beta$;

(iii) $c_2^*$ maximizes $U_2$ over the set of consumption plans $c \in \mathcal{C}$ which are financed by an admissible trading strategy $(\alpha, \theta) \in \Theta$ with $\alpha(0) + \theta(0) = \beta$ and $\theta \equiv 0$; and

(iv) all markets clear, that is, $c_1^* + c_2^* = \delta$, $\alpha_1^* + \alpha_2^* \equiv 0$ and $\theta_1^* = S$. 
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The Unrestricted case – Representative agent

- The second agent can trade the stock
- Equilibrium is constructed by replacing the two agents with a single representative agent as in Huang (1987)
- The representative agent is
  - Endowed with the aggregate supply of securities
  - Has the following utility function:

\[
U(c; \lambda) = \mathbb{E} \left[ \int_0^T e^{-\rho t} u(c(t), \lambda) \, dt \right],
\]

\[
u(c, \lambda) = \max_{c_1+c_2=c} u_1(c_1) + \lambda u_2(c_2)
\]
Solving the Unrestricted case

The Marginal Rate of Substitution \( \varepsilon(t) \) is the pricing kernel

\[
\varepsilon(t) = e^{-\rho t} \frac{U_c(\delta(t), \lambda)}{U_c(\delta(0), \lambda)}
\]

Let \( X(t) \) be some security:

\[
dX(t) = \mu_X(t) dt + \sigma_X(t) dW_t
\]

\( \forall t < s : X(t) = E_t \left\{ \frac{\varepsilon(s)}{\varepsilon(t)} X(s) \right\} \)

\( \Rightarrow \varepsilon(t)X(t) = E_t \{ \varepsilon(s)X(s) \} \)

\( \Rightarrow \varepsilon(t)X(t) \text{ is MTG} \)

\( \Rightarrow d\{ \varepsilon(t)X(t) \} = \psi(t) \cdot W_t \)
Solving the Unrestricted case

\[ dX(t) = \mu_X(t)dt + \sigma_X(t)dW_t \quad d\varepsilon(t) = \mu_{\varepsilon}(t)dt + \sigma_{\varepsilon}(t)dW_t \]

**ITO**

\[ d[\varepsilon(t)X(t)] = \{\varepsilon(t)\mu_X(t) + X(t)\mu_{\varepsilon}(t) + \sigma_X\sigma_{\varepsilon}\}dt + \{\ldots\}dW_t \]

**MTG**

\[ \varepsilon(t)\mu_X(t) + X(t)\mu_{\varepsilon}(t) + \sigma_X\sigma_{\varepsilon} = 0 \]

**Bond**

\[ dB(t) = r(t)B(t)dt \Rightarrow \mu_{\varepsilon}(t) = -\varepsilon(t) \cdot r(t) \]

**Stock**

\[ dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW_t \]

\[ \Rightarrow \sigma_{\varepsilon}(t) = -\varepsilon(t) \cdot \frac{\mu(t) - r(t)}{\sigma(t)} K(t) \]
Solving the Unrestricted case

\[ \varepsilon(t) = e^{-\rho t} \frac{U_c(\delta(t), \lambda)}{U_c(\delta(0), \lambda)} = f(\delta(t), t). \quad d\delta(t) = \mu_\delta(t)dt + \sigma_\delta(t)dW_t \]

\[ d\varepsilon(t) = \mu_\varepsilon(t)dt + \sigma_\varepsilon(t)dW_t \]

\[
\mu_\varepsilon(t) = \frac{e^{-\rho t} \left\{ U_{cc}(\delta(t)) \mu_d(t) + \frac{1}{2} U_{ccc}(\delta(t)) \sigma_d^2(t) - \rho U_c(\delta(t)) \right\}}{U_c(\delta(0))}
\]

\[
\sigma_\varepsilon(t) = \frac{e^{-\rho t} U_{cc}(\delta(t)) \sigma_d(t)}{U_c(\delta(0))}
\]
Solving the Unrestricted case

**Interest Rate**

\[ r(t) = \rho + A(t)\mu_d(t) - \frac{1}{2} A(t)P(t)\sigma_d^2(t) \]

**Market Price of Risk**

\[ k(t) = A(t)\sigma_d(t) \]

\[ A(t) = -\frac{U_{cc}(\delta(t), \lambda)}{U_c(\delta(t), \lambda)}; \quad P(t) = -\frac{U_{ccc}(\delta(t), \lambda)}{U_{cc}(\delta(t), \lambda)}; \]
Solving the Unrestricted case

**Interest Rate**

\[ r(t) = \rho + A(t)\mu_d(t) - \frac{1}{2} A(t)P(t)\sigma_d^2(t) \]

**Discrete**

\[ r_{t+1}^f = -\lambda n\delta + \gamma \cdot E_t[\Delta c_{t+1}] - \frac{1}{2} \gamma^2 \sigma_t^2[\Delta c_{t+1}] \]

**Market Price of Risk**

\[ k(t) = A(t)\sigma_d(t) \]

**Discrete**

\[ \frac{E_t[r_{t+1}^p] - r_{t+1}^f + \frac{1}{2} \sigma_t^2[r_{t+1}^p]}{\sigma_t[r_{t+1}^p]} = \gamma \rho_t(\Delta c_{t+1}, r_{t+1}^p)\sigma_t[\Delta c_{t+1}] \]
Solving the Unrestricted case – getting individual consumption

\[ u(c, \lambda) = \max_{c_1 + c_2 = c} u_1(c_1) + \lambda u_2(c_2) \]

\[ u'_1(c^*_1(t)) = \lambda u'_2(c^*_2(t)) = u_c(\delta(t), \lambda). \]

\[ c^*_1(t) = f_1(u_c(\delta(t), \lambda)) \]

\[ dc^*_i(t) = \mu_{c^*_i}(t) \, dt + \sigma_{c^*_i}(t) \, dw(t) \]

where

\[ \mu_{c^*_i}(t) = \frac{A(t)}{A_i(t)} \mu_{\delta}(t) - \frac{1}{2} \frac{A(t)}{A_i(t)} \frac{A(t)}{2 A_i(t)} P_i(t) \sigma_{\delta}(t)^2 + \frac{1}{2} \frac{A(t)^2}{A_i(t)^2} P_i(t) \sigma_{\delta}(t)^2, \]

\[ \sigma_{c^*_i}(t) = \frac{A(t)}{A_i(t)} \sigma_{\delta}(t) \]
Solving the Unrestricted case – Choosing $\lambda$

$\lambda$ is chosen such that the budget constraint for agent 2 is satisfied:

$$\beta = E \left[ \int_0^T e^{-\rho t} \frac{u_c(\delta(t), \lambda)}{u_c(\delta(0), \lambda)} c_2^*(t) \, dt \right] = \frac{1 - e^{-\rho T}}{\rho} \frac{\lambda}{u_c(\delta(0), \lambda)}.$$
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The Restricted case – Rep. agent

- Second Agent does not face a complete market
- Equilibrium consumption allocation is not efficient
- Following Cuoco and He (1994) – Aggregation in case of incomplete market

\[
U(c; \lambda) = E \left[ \int_0^T e^{-\rho t} u(c(t), \lambda(t)) \, dt \right],
\]

\[
u(c(t), \lambda) = \max u_1(c_1(t)) + \lambda(t) \cdot u_2(c_2(t))
\]

\[
\lambda(t) = \frac{u_1'(c_1^*(t))}{u_2'(c_2^*(t))}.
\]
Solving the Restricted case

- MRS is the SDF
  \[ \xi(t) = e^{-\rho t} \frac{u_c(\delta(t), \lambda(t))}{u_c(\delta(0), \lambda(0))} \]

- MRS follows
  \[ d\xi(t) = -\xi(t) r(t) \, dt - \xi(t) \kappa(t) \, dw(t), \]

- Agent 1 is facing a complete market
  \[ e^{-\rho t} u_1'(c_1^*(t)) = \psi_1 \xi(t) \]

- Agent 2 invests in bond and has log utility (He and Pearson 1991)
  \[ e^{-\rho t} u_2'(c_2^*(t)) = \psi_2 B(t)^{-1} \]
Solving the Restricted case

- Use the above to get the evolution of lambda

\[ d\lambda(t) = -\lambda(t)\kappa(t) \, dw(t) = \frac{u''[f_1(u_c(\delta(t), \lambda(t)))]}{u_c(\delta(t), \lambda(t))} \lambda(t)\sigma_\delta(t) \, dw(t) \]

- Lambda is negatively correlated with aggregate consumption

- If the given dividend process is Markovian:
  - Unrestricted case – \( \delta(t) \) is the state of the economy
  - Restricted case – \( (\delta(t), \lambda(t)) \) is the state
Solving the Restricted case

**Interest Rate**

\[ r(t) = \rho + A(t)\mu_\delta(t) - \frac{1}{2}A(t)P_1(t)\sigma_\delta(t)^2 \]

**Market Price of Risk**

\[ \kappa(t) = A_1(t)\sigma_\delta(t) \]

\[
A(t) = -\frac{U_{cc}(\delta(t), \lambda(t))}{U_c(\delta(t), \lambda(t))}; \quad P(t) = -\frac{U_{ccc}(\delta(t), \lambda(t))}{U_{cc}(\delta(t), \lambda(t))};
\]

**CRRA**

\[
A(t) = \frac{\gamma}{c}; \quad P(t) = \frac{(\gamma + 1)}{c};
\]
Corollary 2. The optimal consumption policies \( c_1^* \) and \( c_2^* \) satisfy

\[
dc_i^*(t) = \mu_{c_i^*}(t) \, dt + \sigma_{c_i^*}(t) \, dw(t),
\]

where

\[
\mu_{c_1^*}(t) = \frac{A(t)}{A_1(t)} \mu_\delta(t) - \frac{1}{2} \frac{A(t)}{A_1(t)} P_1(t) \sigma_\delta(t)^2 + \frac{1}{2} P_1(t) \sigma_\delta(t)^2,
\]

\[
\mu_{c_2^*}(t) = \frac{A(t)}{A_2(t)} \mu_\delta(t) - \frac{1}{2} \frac{A(t)}{A_2(t)} P_1(t) \sigma_\delta(t)^2,
\]

and

\[
\sigma_{c_1^*}(t) = \sigma_\delta(t)
\]

\[
\sigma_{c_2^*} \equiv 0.
\]
Following Mehra and Prescott (1985), calibrate:

\[ \frac{\mu_d}{\delta(t)} = 0.0183, \quad \frac{\sigma_d}{\delta(t)} = 0.0357 \]

- Stockholders have CRRA utility
- Now, MPR and interest rate depend only on consumption share of non-stockholders \( \Phi \):

\[
k(t) = A_1 \cdot \sigma_d = \frac{\gamma \cdot \sigma_d}{\delta(t)(1 - \phi)} = \frac{\gamma \cdot 0.0357 \delta(t)}{\delta(t)(1 - \phi)} = \frac{0.0357\gamma}{1 - \phi}
\]
Market Price of Risk

\[ k(t) = \frac{0.0357\gamma}{(1 - \phi)} \]

Mehra and Prescott’s estimated MPR

RA of 3.3, real interest rate of 1.3%

\( \Phi \)
Interest Rate
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Example

- Both agents have log utility

- Dividend growth follows

\[ \delta(t) = \delta(0) + \int_0^t \delta(s) \tilde{\mu}_8 \, ds + \int_0^t \delta(s) \tilde{\sigma}_8 \, dw(s) \]

- Now, Lamda represents consumption ratio

\[ \lambda(t) = \frac{u_1'(c_1(t))}{u_2'(c_2(t))} = \frac{c_2(t)}{c_1(t)} \]
## Example - results

<table>
<thead>
<tr>
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<th>Unrestricted</th>
<th>Restricted</th>
</tr>
</thead>
<tbody>
<tr>
<td>r(t)</td>
<td>$\rho + \mu_d - \sigma_d^2$</td>
<td>$\rho + \mu_d - (1 + \lambda(t))\sigma_d^2$</td>
</tr>
<tr>
<td>K(t)</td>
<td>$\sigma_d$</td>
<td>$(1 + \lambda(t))\sigma_d$</td>
</tr>
<tr>
<td>d\lambda(t)</td>
<td>0</td>
<td>$-\left(\lambda(t) + \lambda(t)^2\right) \cdot \sigma_d^2 dW_t$</td>
</tr>
<tr>
<td>C1(t)</td>
<td>$\frac{\delta(t)}{1 + \lambda}$</td>
<td>$\mu_c(t) = \mu_d(t) + \left(\lambda(t) + \lambda(t)^2\right)\sigma_d^2, \sigma_c(t) = (1 + \lambda(t))\sigma_d$</td>
</tr>
<tr>
<td>C2(t)</td>
<td>$\frac{\lambda\delta(t)}{1 + \lambda}$</td>
<td>$\mu_c(t) = \mu_d(t) - \left(\lambda(t) + \lambda(t)^2\right)\sigma_d^2, \sigma_c(t) = 0$</td>
</tr>
</tbody>
</table>
Simulated 1000 years with:

\[ \mu_d = 0.0183 \]
\[ \sigma_d = 0.0357 \]
\[ \lambda_0 = 1 \]
\[ T = 100,000 \]
\[ \rho = 0.001 \]
\[ dt = 0.1 \]
Unrestricted - consumption
### Consumption Growth – Correlations and Stdevs

<table>
<thead>
<tr>
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<th>DDIV</th>
<th>DC1</th>
<th>DC2</th>
<th>DC_U</th>
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<tr>
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<td>1.000000</td>
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<td>0.071014</td>
<td>1.000000</td>
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<table>
<thead>
<tr>
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<th>Stdev</th>
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<td>0.023346</td>
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</tbody>
</table>
Lamda

Graph showing unrestricted and restricted data over a range of values.
Interest Rate
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Discussion - Theoretical

- Participation in the model is determined only by the wealth held by each type of agent, however:
  - As financial markets become more accessible, participation increases (1989→2002: 74% increase)
  - Participation might decrease in bad times

- Model indirectly considers non-participants to be less risk-averse, however, the opposite is more reasonable

- Dependence of the results on utility specification:
  - Agent 1 utility – ARA must decrease with c
  - Agent 2 log utility → no hedging demands → consumption of agent 2 even smoother
Discussion – Empirical

- The following should vary with participation:
  - Equity Premium (negative relationship)
  - Interest Rate (positive)

- Test implications:
  - Time Series – historical data
  - Cross Section – across countries

- Endogeneity issues?