Learning About Predictability: The Effects of Parameter Uncertainty on Dynamic Asset Allocation

Yihong Xia
Presented by: Esben Hedegaard

November 30, 2009
Outline

1 Introduction
2 Model
3 Effect of Learning
4 Discussion
Motivation

How much should a long-horizon investor invest in equities?
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Several studies find evidence of return predictability. Consequently, people have investigated the effect of return predictability on portfolio choice (KS: Kandel and Stambaugh (1996), BSL: Brennan, Schwartz and Lagnado (1997), CV: Campbell and Viceira (1999), B: Babaris (2000)).

- KS: One-period investment horizon with estimation risk
- BSL & CV: Multi period setting with known return predictability
- B: Multi period setting with estimation risk. No learning!
Motivation

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Xia: Continuous time model with unknown return predictability, which the investor tries to learn about over time.
Questions

1. What is the **optimal equity allocation** in the presence of parameter uncertainty and learning?

2. What is the effect of the **investment horizon**?

3. How does parameter uncertainty and learning effect the importance of **market timing**?

Returns are **potentially** predictable—can be iid. The estimated model might indicate predictability when returns are really iid., and vice versa.
Example and Idea

Regress returns on price-dividend ratio

\[ r_{t+1} = a_t + \beta \left( \frac{d}{p} \right)_t + \varepsilon_{t+1} \]

with significant estimate \( b_t \) of \( \beta \).
Example and Idea

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Regression may be misspecified, \( t \)-statistic may be misleading due to small-sample effects.

Derive optimal asset allocation assuming

1. The estimate \( b_t \) of \( \beta \) is not equal to the true value
2. We will learn more about \( \beta \) in the future
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Derive optimal asset allocation assuming

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2. We will learn more about \( \beta \) in the future

Consequence:

1. The estimate of \( \beta \) and its variance becomes state-variables.
2. The investor hedges against changes in these variables.
Summary of Results

Horizon effect:

1. Optimal allocation is horizon dependent, in a non-monotonous way.
2. Not true that long-horizon investors should allocate more to equity.
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Horizon effect:
1. Optimal allocation is horizon dependent, in a non-monotonous way.
2. Not true that long-horizon investors should allocate more to equity.

Effect of predictive variable
1. *Without* learning, allocation is increasing in predictor
2. *With* learning, allocation is *not* monotone in predictor due to hedging demand
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2. Model
3. Effect of Learning
4. Discussion
Model

- Risk-free asset evolves as
  \[ dB(t) = rB(t)dt \Rightarrow B(t) = B(0)e^{rt} \]
Model

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  \[ dB(t) = rB(t)dt \quad \Rightarrow B(t) = B(0)e^{rt} \]

- Stock price evolves as:
  \[ dP(t) = \mu(t)P(t)dt + \sigma_P \ P(t)dZ(t) \]

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\[ \mu(t) \] is predictable by \( S \):
\[ \mu(t) = \alpha(t) + \beta(t)'S(t) \]
where \( \alpha \) and \( \beta \) are unknown.
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  with unknown \[\mu(t)\] and known \[\sigma_P \]

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  where \(\alpha\) and \(\beta\) are unknown.

- \(\beta(t)\) itself may change over time:
  \[ d\beta(t) = (a_0(P, S, t) + a_1(P, S, t)\beta(t))dt + \eta(P, S, t)dZ(t) \]
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  \text{Unknown} & \quad \text{Known}
  \end{align*} \]

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- The predictive variables follow a Markov process
  \[ dS(t) = (A_0(P, S, t) + A_1(P, S, t)\beta)dt + \sigma_S(P, S, t)dZ \]
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- Assume \( \bar{\mu} \) and \( \bar{S} \) are known, so \( \alpha(t) = \bar{\mu} - \beta(t)\bar{S} \).
Model

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  \[ dB(t) = rB(t)dt \quad \Rightarrow \quad B(t) = B(0)e^{rt} \]

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  \( \text{Measurement 2} \)

- Assume \( \bar{\mu} \) and \( \bar{S} \) are known, so
  \[ \alpha(t) = \bar{\mu} - \beta(t)\bar{S}. \]
The Investor’s Problem

Information $\mathcal{F}_t^I$ generated by $(P(t), S(t))$.

Choose consumption $c(t)$ and portfolio $x(t)$ to

$$\max_{x(t), c(t)} E \left( \int_0^T U(c(t), t) dt + B(W(T), T) \bigg| \mathcal{F}_0^I \right)$$

s.t.

$$dW = (rW + xW(\bar{\mu} + \beta(S - \bar{S}) - r) - c) dt + xW \sigma_P dZ$$

where $U(c_t) = e^{-\rho t \frac{c_t^{1-\gamma}}{1-\gamma}}, \quad \gamma \neq 1.$
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where $U(c_t) = e^{-\rho t \frac{c_t^{1-\gamma}}{1-\gamma}}, \quad \gamma \neq 1$.

Separation of investor’s problem

1. Update estimate of $\beta$ using cont. time Kalman filter
   - Prior $\beta \sim N(b_0, \nu_0)$
   - $\beta|\mathbb{F}_t^I \sim N(b_t, \nu_t)$

2. Optimization problem given $b_t, \nu_t$. 
Special Case

Use $\alpha(t) = \bar{\mu} - \beta(t)\bar{S}$ to write $\mu(t) = \bar{\mu} + \beta(t)(s - \bar{s})$ and

$$dP(t) = (\bar{\mu} + \beta(s - \bar{s}))P(t)dt + \sigma_P P(t)dZ_P(t)$$
$$= (\bar{\mu} + b(s - \bar{s}))P(t)dt + (\beta - b)(s - \bar{s})P(t)dt + \sigma_P P(t)dZ_P(t)$$
$$= (\bar{\mu} + b(s - \bar{s}))P(t)dt + \sigma_P P(t)d\hat{Z}_P(t)$$

where $d\hat{Z}_P = dZ_P + \frac{(s - \bar{s})(\beta - b)}{\sigma_P} dt$.

The investor observes $\hat{Z}_P(t)$, but not $\beta$ or $Z_P(t)$. 

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Introduction

Model

Effect of Learning

Discussion
Special Case

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The investor observes $\hat{Z}_P(t)$, but not $\beta$ or $Z_P(t)$.

Now, suppose

1. There is 1 predictor $s$ following an Ornstein-Uhlenbeck process (cont. time AR(1))

$$ds = \kappa(\bar{s} - s)dt + \sigma_s dZ_s$$

2. $\beta$ is constant

3. $s$ and $P$ are uncorrelated
Recall

\[ dP(t) = (\bar{\mu} + b(s - \bar{s})) P(t) dt + \sigma_P P(t) d\hat{Z}_P(t) \]

Suppose \( b > 0 \) and \( s > \bar{s} \).

An unexpected excess return \( d\hat{Z}_P(t) > 0 \) \( \Rightarrow \) increase estimate of \( \beta \).

By how much?
Kalman Filter Updating: Intuition

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Put more weight on new info when

- you are less confident in your current estimate: high \( \nu \)
- when volatility of the stock is small: small \( \sigma_P \)
- when signal is strong: large \( s - \bar{s} \)
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Equation (14)-(16) in Xia simplify to

\[
\begin{align*}
    db &= \frac{\nu(s - \bar{s})}{\sigma_P} d\hat{Z}_P \\
    d\nu &= -\frac{\nu^2(s - \bar{s})^2}{\sigma_P^2} dt
\end{align*}
\] (1)
Solving for Optimal Allocation

The budget constraint becomes

$$dW = (rW + xW(\bar{\mu} - b(s - \bar{s}) - r) - c) dt + xW\sigma_P d\hat{Z}_P.$$  \hspace{1cm} (2)

The investor’s value function is

$$J(W, b, \nu, s, t) = \max_{c(\tau), x(\tau)} \mathbb{E}\left( \int_t^T e^{-\rho \tau} \left( \frac{c^{1-\gamma}}{1-\gamma} + \frac{e^{-\rho \tau} W_1^{1-\gamma}}{1-\gamma} \right) \bigg| \mathcal{F}_t \right)$$

subject to (2) and Kalman filter formulae for $b$ and $\nu.$
Solving for Optimal Allocation

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\]

subject to (2) and Kalman filter formulae for \(b\) and \(\nu\).

Solution is

\[ J(W, b, \nu, s, t) = e^{-\rho t} \frac{W^{1-\gamma}}{1-\gamma} \phi(b, \nu, s, t) \]

where \(\phi\) solves a 2nd order PDE in \((b, \nu, s, t)\)
Optimal Strategy

Assuming $\beta$ follows an O-U process,

$$x^* = \bar{\mu} + b(s - \bar{s}) - r \underbrace{\frac{\phi_b}{\gamma \sigma^2_P} \nu(s - \bar{s})}_{\text{Hedge due to learning}} + \frac{\phi_b}{\gamma \sigma^2_P} \nu(s - \bar{s})$$

Myopic

$$+ \frac{\phi_s}{\gamma \sigma^2_P} \sigma_P \sigma_s \rho s P + \frac{\phi_b}{\gamma \sigma^2_P} \sigma_P \sigma_b \rho \beta P$$

Hedge changes in $s$

Hedge changes in $\beta$

Two effects of parameter uncertainty

1. **Direct effect**: Effect of learning (0 when $\beta$ is known, i.e. $\nu = 0$)
2. **Indirect effect**: Need to hedge against changes in the investment opportunity set driven by changes in $s$ and $\beta$. 
\( \beta \) is Assumed Constant

\( \beta \) is assumed constant, but is unknown. Then

\[
\text{cov} \left( db, \frac{dP}{P} \right) = \nu (s - \bar{s})
\]

Thus,

1. If \( s - \bar{s} > 0 \), a positive unexpected return leads to an increase in \( b \)
2. If \( s - \bar{s} < 0 \), a positive unexpected return leads to a decrease in \( b \)
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Also

\[
x^* = \frac{\bar{\mu} + b(s - \bar{s}) - r}{\gamma \sigma^2_P} + \frac{\phi b}{\gamma \sigma^2_P \phi} \nu(s - \bar{s}) + \frac{\phi s}{\gamma \sigma^2_P \phi} \sigma_P \sigma_s \rho_{sP}
\]

- Myopic
- Hedge due to learning
- Hedge changes in \( s \)
## Table I

<table>
<thead>
<tr>
<th>Parameter Descriptions</th>
<th>Notation</th>
<th>Parameter Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Volatility (standard deviation) of the dividend yield</td>
<td>$\sigma_s$</td>
<td>0.6%</td>
</tr>
<tr>
<td>2. Historical long run mean of the stock return</td>
<td>$\bar{\mu}$</td>
<td>9.1%</td>
</tr>
<tr>
<td>3. Volatility (standard deviation) of the stock return</td>
<td>$\sigma_p$</td>
<td>14.4%</td>
</tr>
<tr>
<td>4. Correlation coefficient between the dividend yield and the stock return processes</td>
<td>$\rho_{sp}$</td>
<td>-0.93</td>
</tr>
<tr>
<td>5. Mean reversion coefficient for the dividend yield process</td>
<td>$\kappa$</td>
<td>0.19</td>
</tr>
<tr>
<td>6. Historical long run mean of the dividend yield</td>
<td>$\bar{s}$</td>
<td>4.0%</td>
</tr>
<tr>
<td>7. Coefficient of risk aversion</td>
<td>$\gamma$</td>
<td>5.0</td>
</tr>
<tr>
<td>8. Subjective discount factor</td>
<td>$\rho$</td>
<td>1%</td>
</tr>
<tr>
<td>9. Real interest rate</td>
<td>$r$</td>
<td>3.4%</td>
</tr>
<tr>
<td>10. Current estimate of $\beta$</td>
<td>$b_0$</td>
<td>4.54</td>
</tr>
<tr>
<td>(January 1950–December 1997)</td>
<td></td>
<td></td>
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<tr>
<td>11. Current estimate variance</td>
<td>$v_0$</td>
<td>4.33</td>
</tr>
<tr>
<td>(January 1950–December 1997)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12. Current estimate of $\beta$</td>
<td>$b_0$</td>
<td>6.76</td>
</tr>
<tr>
<td>(January 1950–December 1977)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>13. Current estimate variance</td>
<td>$v_0$</td>
<td>6.38</td>
</tr>
<tr>
<td>(January 1950–December 1977)</td>
<td></td>
<td></td>
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</table>
Horizon Effect: Myopic Allocation

Independent of $T$. 

<table>
<thead>
<tr>
<th>$d/p$</th>
<th>1m</th>
<th>1y</th>
<th>5y</th>
<th>10y</th>
<th>20y</th>
</tr>
</thead>
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<tr>
<td>2.0%</td>
<td>-0.32</td>
<td>-0.32</td>
<td>-0.32</td>
<td>-0.32</td>
<td>-0.32</td>
</tr>
<tr>
<td>3.0</td>
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<td>0.12</td>
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<td>0.12</td>
<td>0.12</td>
</tr>
<tr>
<td>4.0</td>
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<td>0.55</td>
<td>0.55</td>
<td>0.55</td>
<td>0.55</td>
</tr>
<tr>
<td>5.0</td>
<td>0.98</td>
<td>0.98</td>
<td>0.98</td>
<td>0.98</td>
<td>0.98</td>
</tr>
<tr>
<td>6.0</td>
<td>1.42</td>
<td>1.42</td>
<td>1.42</td>
<td>1.42</td>
<td>1.42</td>
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</tbody>
</table>

Panel B: Myopic Stock Allocation: $x_1^*$

<table>
<thead>
<tr>
<th>1m</th>
<th>1y</th>
<th>5y</th>
<th>10y</th>
<th>20y</th>
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<tr>
<td>-0.32</td>
<td>-0.32</td>
<td>-0.32</td>
<td>-0.32</td>
<td>-0.32</td>
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<tr>
<td>0.12</td>
<td>0.12</td>
<td>0.12</td>
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<tr>
<td>0.55</td>
<td>0.55</td>
<td>0.55</td>
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</tr>
<tr>
<td>0.98</td>
<td>0.98</td>
<td>0.98</td>
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<td>0.98</td>
</tr>
<tr>
<td>1.42</td>
<td>1.42</td>
<td>1.42</td>
<td>1.42</td>
<td>1.42</td>
</tr>
</tbody>
</table>

Table II ($\gamma = 5.0$, $b_0 = 4.5$, and $v_0 = 4.0$)
Horizon Effect: Direct Effect of Param Uncertainty

Hedge for param uncertainty $\frac{\phi b}{\gamma \sigma_p^2} \nu (s - \bar{s})$:

- Positive if $s < \bar{s}$, negative if $s > \bar{s}$.
- Increasing in $T$
- Suppose $b_0 > 0$ and $s > \bar{s}$: A high unexpected return (good) leads to upward revision of $b$ (good). So there is a positive correlation between stock returns and good news. Sell stocks to hedge.
- $s < \bar{s}$: high unexpected return (good) leads to downward revision of $b$ (bad for the future). Negative correlation between stock returns and good news. Hedge by holding the stock.
Horizon Effect: Indirect Effect of Param Uncertainty

2nd, indirect, effect to hedge changes in $s$: \[ \frac{\phi_s}{\gamma \sigma_P^2 \phi} \sigma_P \sigma_s \rho_{sp}. \] Note $\rho_{sp} = -0.93$.

- Hump shape
- If $\beta$ is known ($\nu_0 = 0$), hedge demand is increasing in $T$ (App)
- If $\nu_0 > 0$, uncertainty about the predictive power of $s$ becomes larger as $T$ grows. Less need to hedge.
The Horizon Effect: Total

Inherits hump shape.
Interim consumption reduces the effective horizon.

Table II (\(\gamma = 5.0, b_0 = 4.5, \text{ and } v_0 = 4.0\))

<table>
<thead>
<tr>
<th>(d/p)</th>
<th>1m</th>
<th>1y</th>
<th>5y</th>
<th>10y</th>
<th>20y</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0%</td>
<td>-0.34</td>
<td>-0.34</td>
<td>-0.32</td>
<td>-0.27</td>
<td>-0.18</td>
</tr>
<tr>
<td>3.0</td>
<td>0.12</td>
<td>0.13</td>
<td>0.20</td>
<td>0.27</td>
<td>0.36</td>
</tr>
<tr>
<td>4.0</td>
<td>0.56</td>
<td>0.61</td>
<td>0.75</td>
<td>0.83</td>
<td>0.86</td>
</tr>
<tr>
<td>5.0</td>
<td>0.99</td>
<td>1.07</td>
<td>1.19</td>
<td>1.22</td>
<td>1.19</td>
</tr>
<tr>
<td>6.0</td>
<td>1.43</td>
<td>1.49</td>
<td>1.49</td>
<td>1.42</td>
<td>1.32</td>
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Panel A: Optimal Stock Allocation

<table>
<thead>
<tr>
<th>1m</th>
<th>1y</th>
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<tr>
<td>-0.32</td>
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<td>-0.14</td>
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<td>0.27</td>
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<td>1.42</td>
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<td>1.46</td>
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</table>
**Horizon Effect: Comparison of Strategies**

**Solid**: Without param uncert. **Dashed**: With Higher $d/p \Rightarrow$ higher stock allocation

---

**Figure 1.**

- **Figure a**: $d/p = 2\%$, without interim consumption

- **Figure c**: $d/p = 4\%$, without interim consumption

- **Figure e**: $d/p = 6\%$, without interim consumption

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Horizon Effect: Comparison of Strategies

**Solid**: Without param uncert. **Dashed**: With Higher $d/p \Rightarrow$ higher stock allocation

**Without** param uncertainty:
- Hedge demand positive, increasing in $T$

**With** param uncertainty:
- High current return could imply improved future investment opportunities ($d\hat{Z}_P > 0 \Rightarrow b \uparrow$)
- The longer the horizon, the lower the initial equity allocation

Important: Long-horizon investors do not always have higher equity allocation than short-horizon investors.
### Effect of Prior Uncertainty $\nu_0$

**Table VI**

<table>
<thead>
<tr>
<th>Prior $b_0$</th>
<th>$\nu_0 = 0.0$</th>
<th>$\nu_0 = 1.0$</th>
<th>$\nu_0 = 2.0$</th>
<th>$\nu_0 = 3.0$</th>
<th>$\nu_0 = 4.0$</th>
<th>$\nu_0 = 6.0$</th>
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<tr>
<td>4.5</td>
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<td>0.02</td>
<td>0.06</td>
<td>0.09</td>
<td>0.16</td>
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</tbody>
</table>

**Panel A: $d/p = 2\%$**

**Panel B: $d/p = 4\%$**

1. **Direct effect:** Increasing in $\nu_0$:
   - More uncertainty about $b \Rightarrow$ larger hedge
Effect of Prior Uncertainty $\nu_0$

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<tr>
<td>Panel B: $d/p = 4%$</td>
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1. **Direct effect**: Increasing in $\nu_0$:
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2. **Indirect effect**: Decreasing in $\nu_0$:
   - Investor reacts more conservatively to changes in $s \Rightarrow$ smaller hedge
Effect of Prior Uncertainty $\nu_0$

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### Panel A: $d/p = 2\%$

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### Panel B: $d/p = 4\%$

1. **Direct effect**: Increasing in $\nu_0$:
   - More uncertainty about $b \Rightarrow$ larger hedge

2. **Indirect effect**: Decreasing in $\nu_0$:
   - Investor reacts more conservatively to changes in $s \Rightarrow$ smaller hedge

3. **Larger $b$** $\Rightarrow$ predictability more important $\Rightarrow$ each component increases
Market Timing

Figure a: $b_0 = 0.0$, without interim consumption

$0.6$
$0.5$
$0.4$
$0.3$
$0.2$
$0.1$
$0$

$0.02$ $0.04$ $0.06$ $0.08$ $0.1$ $0.12$ $0.14$

Figure c: $b_0 = 1.5$, without interim consumption

Dynamic w/o learning
Myopic
Optimal

Figure 2. Optimal allocation
Market Timing

$b_0 = 0$ indicates no predictability

1. Myopic and dynamic strategy w/o learning hold constant amount in stocks (treat returns as truly iid)

2. Optimal strategy allows for learning:
   - Unexpected high ret (good) $\Rightarrow$ new $b > 0$ (good when $d/p$ is high).
   - Good news pos corr with future investment opportunities
   - Short stock to hedge

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   - Short stock to hedge

\[ b_0 > 0 \] indicates predictability

1. Allocation for myopic and dynamic strategy w/o learning is linear in predictive variable

2. For the optimal strategy, the hedge demand offsets the effect of higher expected return
Historical Optimal Portfolios

Proportion of wealth in stocks, given historical data.

Figure 3. Simulated optimal portfolio allocation using historical stock return and dividend yield ($\gamma = 5.0$, $b_n = 6.76$, $r_n = 6.38$, and $T = 20$). This figure plots the optimal
Historical Optimal Portfolios

1. For iid strategy, allocation is constant
2. For BSL (dynamic without parameter uncertainty), allocation is highly volatile
3. For KS (myopic with parameter uncertainty), allocation is less aggressive
4. Optimal strategy is more smooth than BSL, and timing is different from KS.
Historical Optimal Portfolios

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2. For BSL (dynamic without parameter uncertainty), allocation is highly volatile
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4. Optimal strategy is more smooth than BSL, and timing is different from KS.

Late 90’s:
1. Increasing stock prices lead to low $p/d$ ratios
2. Investor revises $b$ downwards
3. $b$ remains positive, implying negative expected returns
Certainty Equivalent

$CEV^i$: The investor is indifferent btw. receiving

1. 1 USD now, and trade using strategy $i$
2. $CEV^i$ at time $T$

![Graph showing optimal vs myopic certainty equivalent wealth over time.]

Figure 4. Economic value of uncertain predictability: certainty equivalent wealth (CEW) comparison using historical stock return and dividend yield ($\gamma = 5.0$, $T = 20$ years, $b_o = 6.76$, and $\nu_o = 6.38$). This figure compares the economic value of the optimal and KS invest-

$CEV$ is higher for the optimal strategy.
Cheap Shots... 

1. Relate explanations in empirical section more closely to Kalman filtering and parameter estimates.
2. Show CEV for the dynamic strategy without learning.
   - To show that learning is important, compare to strategy without learning, not myopic.
3. Show optimal consumption as well (or consumption-wealth ratio).
4. What happens with EZ-preferences?
5. Unrealistic assumption that parameters of all processes are known.

<table>
<thead>
<tr>
<th></th>
<th>Xia</th>
<th>PS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Param. uncertainty</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Predictors are</td>
<td>Perfect</td>
<td>Imperfect</td>
</tr>
<tr>
<td>Dynamic learning</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Interim cons.</td>
<td>Yes</td>
<td>No</td>
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</table>
Predictive Regression vs. Predictive System

**Predictive regression** (the standard):

\[ E_t(r_{t+1}) = a + bx_t \]

Leads to highly volatile conditional expected returns.
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Leads to highly volatile conditional expected returns.

**Predictive system:**

\[ E_t(r_{t+1}) \neq a + bx_t \]

but \( E_t(r_{t+1}) \) and \( x_t \) are correlated.

Past values of returns and predictors also matter for \( E_t(r_{t+1}) \).

Leads to more smooth conditional expected returns.
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Leads to more smooth conditional expected returns.

**Advantage:** PS
Figure 3. Simulated optimal portfolio allocation using historical stock return and dividend yield ($\gamma = 5.0$, $b_0 = 6.76$, $r_0 = 6.38$, and $T = 20$). This figure plots the optimal
Quarterly EP in PS

Panel A. Predictor: Dividend Yield

Panel B. Predictor: CAY

Panel C. Predictors: Dividend Yield, CAY, and Bond Yield

Figure 9. The equity premium: Regression vs. system with prior information. This figure plots the time...
Xia has dynamic learning:

1. The investor continuously updates his estimate of $\beta$.
2. He anticipates this when choosing his equity allocation.
3. Investor continuously updates his portfolio.
Xia has dynamic learning:

1. The investor continuously updates his estimate of $\beta$
2. He anticipates this when choosing his equity allocation
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PS have ‘one-time’ learning:

1. Use historical data to form a posterior over unknown parameters using Bayesian updating
2. Equity allocation evolves linearly from initial to final allocation
3. Base equity allocation on this posterior without assuming that you will learn more in the future
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Advantage: Xia
Price-dividend ratio must predict future returns or future dividend growth. It’s not a restriction, it’s an identity!

- Including dividends in the model would mean including more information
- Could potentially improve results
- Most likely through lower standard errors on parameters/faster learning
Other Issues

Interim consumption: Xia has it, PS do not. Allowing for interim consumption is similar to shortening the horizon.

Neither Xia nor PS
- Model seasonality in dividends
- Include time-variation/predictability in 2nd moments
Budget Constraint

Let $\theta^B_t, \theta^P_t$ be the number of bonds and stocks.

$$W_t = \theta^B_t B_t + \theta^P_t P_t = W_0 + \int_0^t \theta^B_s dB_s + \int_0^t \theta^P_s dP_s - \int_0^t c(t) dt$$
Let $\theta_t^B, \theta_t^P$ be the number of bonds and stocks.

$$W_t = \theta_t^B B_t + \theta_t^P P_t = W_0 + \int_0^t \theta_s^B dB_s + \int_0^t \theta_s^P dP_s - \int_0^t c(t) dt$$

Then

$$dW_t = \theta_t^B dB_t + \theta_t^P dP_t - c_t dt$$

$$= \theta_t^B rB_t dt + \theta_t^P (\mu_t P_t dt + \sigma_P P_t dZ_P) - c_t dt$$
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Let $x_t$ be the fraction of wealth in stocks:

$$x_t = \theta_t^P P_t / W_t, 1 - x_t = \theta_t^B B_t / W_t$$
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Let $x_t$ be the fraction of wealth in stocks:

$$x_t = \theta_t^P P_t / W_t, 1 - x_t = \theta_t^B B_t / W_t$$

$$= (1 - x_t) W_t r dt + \mu_t x_t W_t dt + x_t W_t \sigma_P dZ_P - c_t dt$$

$$= (rW_t + x_t(\mu_t - r) W_t - c_t) dt + x_t W_t \sigma_P dZ_P$$

$$= (rW_t + x_t(\bar{\mu} + \beta (S_t - \bar{S}) - r) W_t - c_t) dt + x_t W_t \sigma_P dZ_P$$

using $\mu_t = \alpha + \beta S_t = \bar{\mu} - \beta \bar{S} + \beta S_t = \bar{\mu} + \beta (S_t - \bar{S})$ as $\alpha = \bar{\mu} - \beta \bar{S}$.