Long-Run Risk, the Wealth-Consumption Ratio, and the Temporal Pricing of Risk

By Ralph Koijen, Hanno Lustig, Stijn Van Nieuwerburgh and Adrien Verdelhan *

Representative agent consumption-based asset pricing models have made great strides in accounting for many important features of asset returns. The long-run risk (LRR) models of Ravi Bansal and Amir Yaron (2004) is a prime example of this progress. Yet, several other representative agent models, such as the external habit model of John Y. Campbell and John H. Cochrane (1999) and the variable rare disasters model of Xavier Gabaix (2008) seem to be able to match a similar set of asset pricing moments. Additional moments would be useful to help distinguish between these models. Hanno Lustig, Stijn Van Nieuwerburgh and Adrien Verdelhan (2009) argue that the wealth-consumption ratio is such a moment. A comparison of the wealth-consumption ratio in the LRR model and in the data is favorable to the LRR model. This is no small feat because the wealth-consumption ratio is not a target in the usual calibrations of the model, and the LRR is – so far – the sole model able to reproduce both the equity premium and the wealth-consumption ratio. The LRR model matches the properties of the wealth-consumption ratio despite the fact that it implies a negative real bond risk premium. This is because it generates quite a bit of consumption cash-flow risk to offset the negative discount rate risk. This can be seen in long-horizon variance ratios for consumption. So relative to the data, the consumption cash-flow risk is too high and the discount rate (which is close to the long-horizon real bond risk premium) seems too low.

Because of a lack of data, it is hard to assess directly whether a negative real bond risk premium is counter-factual. Yet, we know that the bond risk premium at long-horizons contains crucial information about the properties of the pricing kernel. In particular, Fernando Alvarez and Urban Jermann (2005) show that the ratio of the infinite bond risk premium to the maximum risk premium is linked to the fraction of the variance of the pricing kernel that arises from the permanent component. The decomposition in permanent versus transitory components of the pricing kernel is model-free. Like the Lars P. Hansen and Ravi Jagannathan (1991) bound, this moment directly describes a property of the pricing kernel and links it to observable asset return characteristics. The low (nominal) bond risk premium and high equity risk premium in the data suggest that most of the shocks to the pricing kernel are permanent.

Since the bond market data are nominal in nature, we augment the LRR model for an inflation process and study the properties of the long-horizon nominal bond risk premium. We show that the long-run risk model, which is successful at matching the wealth-consumption ratio, high equity risk premium and the nominal yields at short maturities implies too much (little) transitory variation in the nominal (real) pricing kernel. This is because the nominal bond risk premium at infinite horizon is too high, or in other words because the real bond risk premium at infinite horizon is too low and thus the inflation risk premium too high. We conclude that the

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I. Stock and Bond Risk Premia in the Long Run Risk Model

The long-run risk literature works off the class of preferences due to Larry Epstein and Stan Zin (1989). Let \( U_t(C_t) \) denote the utility derived from consuming \( C_t \). The value function of the representative agent takes the following recursive form:

\[
U_t(C_t) = \left( 1 - \delta \right) C_t^{\frac{1}{1-\gamma}} + \delta \left( E_t U_{t+1}^{1-\gamma} \right)^{\frac{1}{1-\gamma}}.
\]

The time discount factor is \( \delta \), the risk aversion parameter is \( \gamma \geq 0 \), and the inter-temporal elasticity of substitution (IES) is \( \psi > 0 \). The parameter \( \theta \) is defined by \( \theta \equiv (1 - \gamma)/(1 - \frac{1}{\psi}) \).

When \( \psi > 1 \) and \( \gamma > 1 \), then \( \theta < 0 \) and agents prefer early resolution of uncertainty.

On the technology side, we adopt the specification of Bansal and Ivan Shaliastovich (2008) for consumption growth, dividend growth, and inflation:

\[
\begin{align*}
\Delta c_{t+1} &= \mu_g + x_t + \sigma_{gt} \eta_{t+1} \\
x_{t+1} &= \mu_x + \sigma_{xt} \xi_{t+1} \\
\sigma_{g,t+1}^2 &= \sigma_g^2 + \nu_g (\sigma_{gt}^2 - \sigma_g^2) + \sigma_{gw} \omega_{g,t+1} \\
\sigma_{x,t+1}^2 &= \sigma_x^2 + \nu_x (\sigma_{xt}^2 - \sigma_x^2) + \sigma_{wx} \omega_{x,t+1} \\
\Delta d_{t+1} &= \mu_d + \phi_d x_t + \varphi_d \sigma_{gt} \eta_{t+1} \\
\pi_{t+1} &= \pi_t + \varphi_g \sigma_{gt} \eta_{t+1} + \varphi_{\pi g} \sigma_{gt} \xi_{t+1} + \sigma_{\pi g} \xi_{t+1} + \kappa_0 \omega_{g,t+1} \\
\tau_{t+1} &= \mu_e + \alpha_e (\pi_t - \mu_e) + \alpha_e x_t \\
&+ \varphi_{\pi g} \sigma_{gt} \eta_{t+1} + \varphi_{x} \sigma_{xt} \xi_{t+1} + \sigma_{x} \xi_{t+1} + \kappa_0 \omega_{g,t+1}.
\end{align*}
\]

All shocks are i.i.d standard normal, except \( \text{Corr}(\eta_{t+1}, \xi_{t+1}) = \tau_{g,t+1} \). This specification builds on Bansal and Yaron (2004) and delivers empirically plausible stock and nominal bond prices. Real consumption growth contains a persistent long-run expected growth component \( x_t \). Shocks to (short-run) consumption growth have a stochastic volatility \( \sigma_{x,t+1}^2 \). As in Bansal and Shaliastovich (2008), this volatility differs from the conditional variance of the long-run component \( x_t \), which is denoted \( \sigma_{x,t+1}^2 \). The inflation process is similar to that in Jessica Wachter (2006) and Monika Piazzesi and Martin Schneider (2006).

For our numerical results, we use the calibration of Bansal and Shaliastovich (2008), repeated in Table A.1 in the appendix. Table A.2 summarizes the model loadings on state variables. The model is simulated for 60,000 months and aggregated up to quarterly frequency for comparison with our quarterly data. Negative values for \( \sigma_{g,t+1}^2 \) and \( \sigma_{x,t+1}^2 \) are replaced by very small positive values in simulation. The model matches several key features of aggregate consumption and dividend growth, as well as inflation. More details are available in the appendix.

A central object in the LRR model is the log wealth-consumption ratio, \( wc_t = w_t / c_t \). It is the price-dividend ratio of a claim to aggregate consumption. It is affine in the state variables \( x_t, \sigma_{g,t}^2 \) and \( \sigma_{x,t}^2 \):

\[
wc_t = \mu_{wc} + W_x x_t + W_{gs} (\sigma_{gt}^2 - \sigma_g^2) + W_{xs} (\sigma_{xt}^2 - \sigma_x^2).
\]

The appendix derives the coefficients \( W_x \), \( W_{gs} \) and \( W_{xs} \) as functions of the structural parameters. When the IES exceeds 1, an increase in expected consumption growth and a decrease in short-run or long-run consumption volatility increase the wealth-consumption ratio. The log real stochastic discount factor (SDF) can now be written as a function of log consumption growth and the change in the log wealth-consumption ratio:

\[
sdf_{t+1} = \left[ \theta \log \delta + (\theta - 1) \kappa_0^2 \right] - \gamma \Delta c_{t+1} + (\theta - 1) (wc_{t+1} - \kappa_1^2 wc_t),
\]

where \( \kappa_0^2 \) and \( \kappa_1^2 \) are linearization constants, which are a function of the long-run average log wealth-consumption ratio \( \mu_{wc} \). Note that when \( \theta = 1 \) (\( \gamma = \frac{1}{\psi} \)), the above recursive preferences collapse to the standard power utility preferences, and changes in the wealth-consumption ratio are not priced. The only priced shocks are short-run consumption growth shocks \( \eta_{t+1} \). Hence, the empirical failures of the power utility model and the successes of the LRR model must
be attributable to their respective implications for the wealth-consumption ratio. Lustig, Van Nieuwerburgh and Verdelhan (2009) estimate the wealth-consumption ratio in the data, using a preference-free no-arbitrage approach. Table A.3 shows that the LRR model’s implications are broadly consistent with the data. In particular, the LRR model implies that the claim to aggregate consumption is not very risky, resulting in a high mean wealth-consumption ratio of 50 and a low consumption risk premium.

Next, we turn to stock prices. Like the wealth-consumption ratio, the price-dividend ratio of the claim to aggregate dividends is affine in the same three state variables. As in the power utility model, the equity risk premium reflects compensation for short-run consumption growth risk. The bulk of the risk premium, however, is compensation for long-run consumption risk, and short-run, and long-run consumption growth volatility risk. Table A.3 shows that the model matches the properties of the price-dividend ratio and the equity risk premium well. Because the dividend claim has more exposure to long-run risk \( \phi_x > 1 \), its ends up being much riskier than the consumption claim. This is reflected in a low price-dividend ratio of 22 and a high equity risk premium of 6.25 percent per year.

Finally, the log price of an \( n \)-period nominal bond is affine in the same three state variables, as well as in expected inflation \( \pi_t \). Expected inflation (short-run volatility) unambiguously increases (decreases) nominal bond yields. The effect of long-run growth (long-run volatility) on nominal yields is positive (negative) at short maturities, but negative (positive) at long maturities. These sign reversals at long maturities do not arise for real yields; they result from a negative correlation between expected inflation and long-run growth. Consistent with the findings of Bansal and Shaliastovich (2008), Table A.3 shows that the LRR model matches the one-year to five-year nominal bond yields well. The yield levels are close to the average yields in the Fama-Bliss data for 1952-2008, and the five-minus-one year yield spread of 1.18 percent is reasonably close to the historical 0.56 percent spread.

However, the same table shows that nominal yields on longer horizon bonds are very high in the model. For example, the difference between the 30-year and the 5-year bond yield is 6.44 percent per year. The same spread between constant maturity Treasury yields in the 1952-2008 data is only 0.33 percent. Hypothetical 200-year nominal yields are 20 percent per year in the model. Likewise, the nominal bond risk premium increases sharply with maturity. Table A.3 shows that the five-year nominal bond risk premium is 2.97 percent, which is substantially higher than the 0.92 percent premium we estimate in the data. Table A.4 shows that the one-year risk premium on a 200-year bond is as high as 24.4 percent. In the next section, we connect the high nominal bond risk premium at very long maturities to the transitory component of the SDF. The bottom panel of Table A.4 which is for real yields, is informative about the origins of the high nominal yields and risk premia. It shows that the real yield curve is downward sloping. Real bond risk premia are negative at all horizons and are as low as -16 percent at the 200-year maturity. Real bonds are a hedge in the LRR model because their returns are high in those states of the world where the representative agent’s inter-temporal marginal rate of substitution is high (long-run growth is low or economic uncertainty is high). To generate an upward sloping nominal yield curve, inflation risk must more than offset this hedging effect. Current and future inflation are unexpectedly high exactly when long-run growth is unexpectedly low \( \phi_{x,t} < 0 \) and \( \phi_{z,t} < 0 \), generating a capital loss on the bond in high marginal utility states of the world. When the inflation risk is calibrated to match nominal yield data for maturities of one through five years, it also implies a very high nominal bond risk premium at very long horizons.

II. Permanent and Transitory Components of the SDF

Let the SDF be the growth rate of the pricing kernel: \( SDF_{t+1} = M_{t+1}/M_t \). Alvarez and Jermann (2005) show that -under mild regularity conditions- any pricing kernel \( M \) has a decomposition into a permanent component \( M_t^P \) and a transitory component \( M_t^T \):

\[
M_t = M_t^P M_t^T,
\]
where the permanent component is a martingale $E_1 [M_{t+1}^p] = M^p_t$ and the transitory component is defined as:

$$M^p_t = \lim_{\tau \to \infty} \frac{\beta^{t+\tau}}{P_t(\tau)},$$

for some number $\beta \in (0,1)$. Ralph Koijen, Lustig and Van Nieuwerburgh (2009) derive this decomposition for affine models. The appendix applies their insights to the LRR model, which belongs to this class. We obtain expressions for the permanent ($SDF_{t+1}^p$) and the transitory ($SDF_{t+1}^T$) components of the SDF, as well as for their logs. We do this decomposition both for the nominal and for the real SDF, where the nominal log SDF is $sdft_{t+1} = sdf_{t+1} - \pi_{t+1}$. We focus on the nominal decomposition here.

We define the conditional variance ratio $\omega_t$ as the ratio of the conditional variance of the permanent component of the nominal log SDF to the conditional variance of the entire nominal log SDF:

$$\omega_t = \frac{\text{Var}_t[\text{sdft}_{t+1}^p]}{\text{Var}_t[\text{sdft}_{t+1}^T]} = 1 - \frac{\eta + \gamma \sigma_t^2}{\max \{ \sigma_t^2, \frac{1}{2} \text{Var}_t[\text{sdft}_{t+1}^p(n)] \}}$$

We show in the appendix that $\omega_t$ equals one minus the ratio of the log bond risk premium on a nominal infinite maturity bond (without Jensen adjustment) to the maximum nominal risk premium in the economy (without Jensen adjustment).

Alvarez and Jermann (2005) show that, in a model with only permanent shocks to the pricing kernel, the infinite horizon bond is the highest risk premium in the economy. Conversely, in a model with only permanent shocks, bond risk premia of all maturities are zero and the yield curve is flat. Hence, to have realistic term structure implications, the SDF must have some transitory component, but its variation must not be too large. In the data, long-horizon nominal bond risk premia are low compared to, say, equity risk premia. Hence, the data discipline $\omega_t$ to be close to one on average. Alvarez and Jermann (2005) argue that this conclusion holds both for nominal and real bonds.

Table A.3 reports the mean, standard deviation, and autocorrelation of the stochastic discount factor (SDF) and its permanent ($SDF^p$) and transitory ($SDF^T$) components for the benchmark LRR calibration. Unsurprisingly, the permanent component of the SDF is more volatile than the transitory component. Our key finding is that the nominal variance ratio $\omega_t$ is very low: only 0.37 on average. The reason is that in the LRR model the long-horizon nominal bond risk premium is very high, relative to the maximum nominal risk premium in the economy. Because the real bond risk premium is highly negative, the real variance ratio is much higher than one: 1.66 on average. Hence, the LRR model fails to generate a conditional variance ratio which is close to 1. Inflation introduces too large a transitory component to the nominal SDF.

This conditional variance ratio is tightly linked to the dynamics of the wealth-consumption ratio. With power utility, e.g. constant relative risk-aversion preferences (CRRA), the change in the log wealth-consumption ratio is no longer a priced factor in the log SDF. The real SDF now only features permanent shocks: $sdft_{t+1} - \text{Var}_t[\text{sdft}_{t+1}^p] = -\gamma \sigma_{t+1}$. When $\theta = 1$, the real bond risk premium is zero at all maturities. The nominal bond risk premium is still only 0.07 percent at the 200-year horizon, while the maximum risk premium is 0.62 percent. While the average variance ratio $\omega$ is closer to the data, the power utility model generates an equity risk premium puzzle and a nominal interest rate of 20 percent per year for the one- through five-year yields, both of which are highly counterfactual.

Our analysis raises the question of whether a change in the calibration of the LRR model may solve these issues. We considered both changes on the real and on the nominal side of the economy. On the real side, a slight decrease in the persistence of the long-run component in consumption growth $\rho_c$ could decrease the long-horizon consumption variance ratios and the real variance ratio significantly, and increase the long term real yield from negative to positive values. As a result, the model would need to rely less on a large inflation risk premium in order to match.
the nominal yield curve, thus lowering the transitory variation in the nominal pricing kernel. However, if all the other parameters are maintained at their previous values, the model would then imply too much volatility of the wealth-consumption ratio and an equity risk premium that is much too low.

On the nominal side, we first check the robustness of our results to a slightly different calibration of the inflation dynamics. In the appendix, we vary each inflation parameter by twenty-five percent in either direction, and find that \( \alpha \) only changes noticeably with \( \sigma_x \) and \( \sigma_e \). To further investigate the sensitivity to these two parameters, Figure 1 in the appendix plots \( \omega_t \) (left axis) and the five-year nominal bond risk premium (right axis) against \( \alpha_e \) (horizontal axis). As we vary \( \alpha_x \) away from its benchmark value of -0.35, we simultaneously vary \( \alpha_e \) to match the observed persistence of quarterly inflation. We also choose \( \mu_x \) and \( \sigma_x \) to keep the mean and volatility of inflation at their benchmark values. The figure shows that \( \omega_t \) is essentially unchanged over a wide range of values for \( \alpha_x \) and never comes close to the desired value of one. Next, we consider a calibration that matches the observed mean, variance, and persistence of inflation, the 5-1-year yield spread, and the persistence of the 5-year nominal bond risk premium. This calibration delivers a nominal variance ratio \( \omega_t \) that is much too high. Finally, we ask whether we can find inflation parameters that deliver a nominal variance ratio of 1. We find that we can, while matching the mean inflation, the slope of the nominal term structure, and the persistence of the nominal BRP, but inflation ends up being 2.5 times too volatile and not persistent enough.

**III. Conclusion**

Matching the wealth-consumption ratio and the \( \omega \) ratio is a challenge for dynamic asset pricing models. This challenge is not unique to the LRR model, but equally applies to the habit and the rare disasters model. Future research should investigate how these models can be modified to match the variance ratio. One fruitful avenue to discipline the parameters of the inflation process seems to be to add a monetary policy rule to the LRR framework.

**REFERENCES**


We first compute the different components of equation 2:

\[ R_{t+1}^c = \frac{W_{t+1}}{W_t - C_t} = \frac{C_{t+1} - W_{t+1}}{C_t - W_{t+1}}. \]

We use the Campbell (1991) approximation of the log total wealth return on a claim to aggregate consumption, the \( \text{consumption risk premium} \):

\[ r_{t+1}^c = \kappa_0 + \Delta c_{t+1} + w_{Ct+1} - \kappa_1 w_{ct}, \]

where the linearization constants \( \kappa_0 \) and \( \kappa_1 \) are non-linear functions of the unconditional mean log wealth-consumption ratio \( \mu_{wc} \):

\[ \kappa_1 = \frac{e^{\mu_{wc}}}{e^{\mu_{wc}} - 1} > 1 \text{ and } \kappa_0 = - \log(e^{\mu_{wc}} - 1) + \frac{e^{\mu_{wc}}}{e^{\mu_{wc}} - 1} \mu_{wc}. \]

Throughout the paper, we use lower letters to denote logs.

The Euler equation for any asset \( i \) with lognormal return \( R_t^i \) implies:

\[ 0 = E_t[sdf_t+1] + E_t[r_t^i] + \frac{1}{2} \text{Var}_t[sdf_t+1] + \frac{1}{2} \text{Var}_t[r_t^i] + \text{Cov}_t[sdf_t+1, r_t^i] \]

We conjecture that the wealth-consumption ratio is linear in the state variables \( x_t, \sigma^2_{gt} \) and \( \sigma^2_{st} \):

\[ wc_t = \mu_{wc} + W_x x_t + W_{gs} (\sigma^2_{gt} - \sigma^2_g) + W_{zs} (\sigma^2_{st} - \sigma^2_z) \]

We first compute the different components of equation 2:

\[ r_{t+1}^c = r_0^c + [1 + W_x (\rho - \kappa_1)] x_t + W_{gs} (\sigma^2_{gt} - \sigma^2_g) + W_{zs} (\sigma^2_{st} - \sigma^2_z) + \sigma_{gt} \sigma_{t+1} + W_z \sigma_{st} \sigma_{t+1} + W_{gs} \sigma_{gw} \sigma_{g,t+1} + W_{zs} \sigma_{zw} \sigma_{z,t+1} \]

\[ E_t[r_{t+1}^c] = r_0 + [1 + W_x (\rho - \kappa_1)] x_t + W_{gs} (\sigma^2_{gt} - \sigma^2_g) + W_{zs} (\sigma^2_{st} - \sigma^2_z) + \sigma_{gt} \eta_{t+1} + W_z \sigma_{st} \eta_{t+1} + W_{gs} \sigma_{gw} \eta_{g,t+1} + W_{zs} \sigma_{zw} \eta_{z,t+1} \]

\[ r_{t+1}^c - E_t[r_{t+1}^c] = \sigma_{gt} \eta_{t+1} + W_z \sigma_{st} \eta_{t+1} + W_{gs} \sigma_{gw} \eta_{g,t+1} + W_{zs} \sigma_{zw} \eta_{z,t+1} \]

\[ \text{Var}_t[r_{t+1}^c] = \sigma^2_{gt} + W^2 z \sigma^2_{zt} + W_{gs}^2 \sigma^2_{gw} + W_{zs}^2 \sigma^2_{zw} \]

\[ \kappa_0^c = \mu_g + (1 - \kappa_1) \mu_{wc} \]
Epstein and Zin (1989) show that the log real stochastic discount factor is

\[
sdf_{t+1} = \theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1) r_{c_t+1}^c
\]

\[
= \mu_s + \left\{ \frac{\theta}{\psi} + (\theta - 1) [1 + W_x (\rho - \kappa_1^x)] \right\} x_t
\]

\[
+ \{W_{gs} (\nu_g - \kappa_1^x) (\theta - 1) \} (\sigma_g^2 - \sigma_g^2) + \{W_{xs} (\nu_x - \kappa_1^x) (\theta - 1) \} (\sigma_{xt}^2 - \sigma_x^2)
\]

\[
+ \left\{ \theta \left( \frac{1 - 1}{\psi} \right) - 1 \right\} \sigma_g \eta_{t+1} + (\theta - 1) \left\{ W_x \sigma_{xt} e_{t+1} + W_{gs} \sigma_{gw} w_{g,t+1} + W_{xs} \sigma_{xw} w_{x,t+1} \right\}
\]

\[
sdf_{t+1} - \mathbb{E}_t [sdf_{t+1}] = \left\{ \theta \left( 1 - \frac{1}{\psi} \right) - 1 \right\} \sigma_g \eta_{t+1} + (\theta - 1) \left\{ W_x \sigma_{xt} e_{t+1} + W_{gs} \sigma_{gw} w_{g,t+1} + W_{xs} \sigma_{xw} w_{x,t+1} \right\}
\]

\[
\mathbb{E}_t [sdf_{t+1}] = \mu_s + \left\{ \frac{\theta}{\psi} + (\theta - 1) [1 + W_x (\rho - \kappa_1^x)] \right\} x_t
\]

\[
+ \{W_{gs} (\nu_g - \kappa_1^x) (\theta - 1) \} (\sigma_g^2 - \sigma_g^2) + \{W_{xs} (\nu_x - \kappa_1^x) (\theta - 1) \} (\sigma_{xt}^2 - \sigma_x^2)
\]

\[
\mathbb{V}_t [sdf_{t+1}] = \left\{ \theta \left( 1 - \frac{1}{\psi} \right) - 1 \right\}^2 \sigma_g^2 + (\theta - 1)^2 \left\{ W_x^2 \sigma_{xt}^2 + W_{gs}^2 \sigma_{gw}^2 + W_{xs}^2 \sigma_{xw}^2 \right\}
\]

\[
\mu_s = \theta \log \delta - \frac{\theta}{\psi} \mu_g + (\theta - 1) r_0^c
\]

\[
\text{Cov}_t [r_{c_{t+1}}^c, sdf_{t+1}] = \mathbb{E}_t [(r_{c_{t+1}}^c - \mathbb{E}_t [r_{c_{t+1}}^c]) (sdf_{t+1} - \mathbb{E}_t [sdf_{t+1}])]
\]

\[
= \left\{ \theta \left( 1 - \frac{1}{\psi} \right) - 1 \right\} \sigma_g^2 + W_x^2 (\theta - 1) \sigma_{xt}^2 + W_{gs}^2 (\theta - 1) \sigma_{gw}^2 + W_{xs}^2 (\theta - 1) \sigma_{xw}^2
\]

Plugging these different components into equation (2) evaluated at \( i = c \) yields:

\[
0 = r_0^c + \mu_s + \frac{\theta^2}{2} \left\{ \left( 1 - \frac{1}{\psi} \right)^2 \sigma_g^2 + W_x^2 \sigma_{xt}^2 + W_{gs}^2 \sigma_{gw}^2 + W_{xs}^2 \sigma_{xw}^2 \right\}
\]

\[
+ \theta \left\{ - \frac{1}{\psi} + [1 + W_x (\rho - \kappa_1^x)] \right\} x_t
\]

\[
+ \frac{\theta}{2} \left\{ 2 W_{gs} (\nu_g - \kappa_1^x) + \theta \left( \frac{1 - 1}{\psi} \right)^2 \right\} (\sigma_g^2 - \sigma_g^2)
\]

\[
+ \frac{\theta}{2} \left\{ 2 W_{xs} (\nu_x - \kappa_1^x) + \theta W_x^2 \right\} (\sigma_{xt}^2 - \sigma_x^2)
\]

Then setting all coefficients equal to zero we obtain:

\[
\Rightarrow \quad W_x = \frac{1 - \frac{1}{\psi}}{\kappa_1^x - \rho}
\]

\[
\Rightarrow \quad W_{gs} = \frac{\theta \left( 1 - \frac{1}{\psi} \right)^2}{2 (\kappa_1^x - \nu_g)}
\]

\[
\Rightarrow \quad W_{xs} = \frac{\theta}{2 (\kappa_1^x - \nu_x) \left( 1 - \frac{1}{\psi} \right)^2}
\]
As we did for the return on the consumption claim, we compute innovations in the dividend claim return, and its conditional mean and variance:

\[ D_{x} \]

with the market price of risk vector \( \Lambda = [\lambda_{\eta}, \lambda_{c}, \lambda_{gw}, \lambda_{xw}] \) given by:

\[
\begin{align*}
\lambda_{\eta} &= -\left\{ \theta \left( \frac{1}{\psi} - 1 \right) \right\} = \gamma > 0 \\
\lambda_{c} &= (1 - \theta) W_{s} = \frac{\gamma - \frac{1}{\psi}}{\kappa_{s} - \rho} \\
\lambda_{gw} &= (1 - \theta) W_{gs} = -\frac{(\gamma - 1) (\gamma - \frac{1}{\psi})}{2 (\kappa_{g} - \nu_{g})} \\
\lambda_{xw} &= (1 - \theta) W_{xs} = -\frac{(\gamma - 1)(\gamma - \frac{1}{\psi})}{2 (\kappa_{x} - \nu_{x}) (\kappa_{s} - \rho)^{2}}
\end{align*}
\]

If the IES is sufficiently large (\( \gamma > 1/\psi \)), then \( \lambda_{c} > 0, \lambda_{gw} < 0, \) and \( \lambda_{xw} < 0. \)

**Equity Risk Premium**

We log-linearize return on portfolio: \( r_{t+1} = \kappa_{0} + \Delta d_{t+1} + pd_{t+1} - \kappa_{1} pd_{t} \), and conjecture that the price-dividend ratio is linear in the state variables: \( pd_{t} = \mu_{pd} + D_{x} x_{t} + D_{gs} (\sigma_{x}^{2} - \sigma_{g}^{2}) + D_{xs} (\sigma_{x}^{2} - \sigma_{x}^{2}) \)

As we did for the return on the consumption claim, we compute innovations in the dividend claim return, and its conditional mean and variance:

\[
\begin{align*}
\mu_{pd} &= \mu_{pd} \\
\varrho_{d} &= \varrho_{d} \\
\varrho_{x} &= \varrho_{x} \\
\varrho_{gs} &= \varrho_{gs} \\
\varrho_{gw} &= \varrho_{gw} \\
\varrho_{xw} &= \varrho_{xw} \\
\epsilon_{x} &= \epsilon_{x} \\
\epsilon_{gs} &= \epsilon_{gs} \\
\epsilon_{gw} &= \epsilon_{gw} \\
\epsilon_{xw} &= \epsilon_{xw} \\
\theta_{x} &= \theta_{x} \\
\theta_{gs} &= \theta_{gs} \\
\theta_{gw} &= \theta_{gw} \\
\theta_{xw} &= \theta_{xw} \\
\gamma_{x} &= \gamma_{x} \\
\gamma_{gs} &= \gamma_{gs} \\
\gamma_{gw} &= \gamma_{gw} \\
\gamma_{xw} &= \gamma_{xw} \\
\kappa_{0} &= \kappa_{0} + \mu_{pd} (1 - \kappa_{1}) + \mu_{d}
\end{align*}
\]

\[ \frac{\sigma_{g}^{2} + \sigma_{x}^{2} + \sigma_{gw}^{2} + \sigma_{xw}^{2}}{\sigma_{g}^{2} + \sigma_{x}^{2} + \sigma_{gw}^{2} + \sigma_{xw}^{2}} \]

\[ \frac{\gamma_{x} \varrho_{x} \varrho_{gw}^{2} + (\theta_{x} - 1) \varrho_{gw}^{2}}{\gamma_{x} \varrho_{x} \varrho_{gw}^{2} + (\theta_{x} - 1) \varrho_{gw}^{2}} \]

\[ \text{Var}_{t} [r_{t+1}] = (\theta_{x} - 1) \left[ W_{gs} D_{g} \sigma_{gw}^{2} + W_{xs} D_{x} \sigma_{xw}^{2} \right] - \gamma_{x} \varrho_{x} \varrho_{gw}^{2} + (\theta_{x} - 1) W_{x} D_{x} \sigma_{g}^{2} \]

\[ \text{Cov}_{t} [r_{t+1}, s {d f}_{t+1}] = (\theta_{x} - 1) \left[ W_{gs} D_{g} \sigma_{gw}^{2} + W_{xs} D_{x} \sigma_{xw}^{2} \right] - \gamma_{x} \varrho_{x} \varrho_{gw}^{2} + (\theta_{x} - 1) W_{x} D_{x} \sigma_{g}^{2} \]

\[ \text{Recall that the log riskfree rate is } y(1) = -\mathbb{E}_{t} [s {d f}_{t+1}] - \frac{1}{2} \text{Var}_{t} [s {d f}_{t+1}] \]

\[ ^{2} \text{Recall that the log riskfree rate is } y(1) = -\mathbb{E}_{t} [s {d f}_{t+1}] - \frac{1}{2} \text{Var}_{t} [s {d f}_{t+1}] \]
Plug these different components into equation (2):

\[ 0 = \mu_s + r_0 + \frac{1}{2} \left[ \gamma^2 - 2\gamma \varphi_d \tau_d g + \varphi_d^2 \right] \sigma_g^2 + \frac{1}{2} \left[ W_x (\theta - 1) + D_x \right] \sigma_x^2 + \frac{1}{2} \left[ W_{gs} (\theta - 1) + D_{gs} \right] \sigma_w^2 \]

\[ + \frac{1}{2} \left[ W_{xs} (\theta - 1) + D_{xs} \right] \sigma_{sw}^2 \]

\[ + \left\{ -\frac{1}{\psi} + [\phi_x + D_x (\rho - \kappa_1)] \right\} x_t \]

\[ + \left\{ \frac{1}{2} \left[ \gamma^2 - 2\gamma \varphi_d \tau_d g + \varphi_d^2 \right] + W_{gs} (\kappa_1^c - \nu_0) (1 - \theta) + D_{gs} (\nu_0 - \kappa_1) \right\} (\sigma_{gt}^2 - \sigma_g^2) \]

\[ + \left\{ \frac{1}{2} \left[ W_x (\theta - 1) + D_x \right]^2 + W_{xs} (\kappa_1^c - \nu_x) (1 - \theta) + D_{xs} (\nu_x - \kappa_1) \right\} (\sigma_{xt}^2 - \sigma_x^2) \]

Then setting all coefficients equal to zero we get:

\[ \Rightarrow D_s = \frac{\phi_x - \frac{1}{\gamma}}{\kappa_1 - \rho} \]

\[ \Rightarrow D_{gs} = \frac{1}{2} \left[ \gamma^2 - 2\gamma \varphi_d \tau_d g + \varphi_d^2 \right] - \frac{1}{2} \left( \gamma - \frac{1}{\kappa_1} \right) (\gamma - 1) \]

\[ \Rightarrow D_{xs} = \frac{1}{2} \left[ \frac{\phi_x - \frac{1}{\gamma}}{\kappa_1 - \rho} - \frac{\gamma}{\kappa_1 - \rho} \right]^2 - \frac{1}{2} \left( \gamma - 1 \right) \left( \gamma - \frac{1}{\kappa_1} \right) \]

Plugging these into (7) implicitly defines a nonlinear equation in one unknown (i.e., \(\mu_{pd}\)), which can be solved for numerically, characterizing the mean price-dividend ratio.

The \(D\) coefficients are the betas of the equity market portfolio with respect to the four fundamental consumption growth shocks.

The equity risk premium is equal to:

\[ \mathbb{E}_t \left[ r^e_{t+1} \right] = -\text{Cov}_t \left[ r_{t+1}, \text{sdf}_{t+1} \right] \]

\[ = \left( \varphi \tau_d g \right) \lambda_d \sigma_{gd}^2 + D_s \lambda_c \sigma_{xd}^2 + D_{gs} \lambda_{gw} \sigma_{gw}^2 + D_{xs} \lambda_{xw} \sigma_{xw}^2 \]

\[ + \left( G_0 + G_{gs} \sigma_g^2 + G_{gs} \sigma_{gw}^2 \right) + G_{gs} (\sigma_{gt}^2 - \sigma_g^2) + G_{xs} (\sigma_{xt}^2 - \sigma_x^2) \]

\[ \Rightarrow G_0 = D_{gs} \lambda_{gw} \sigma_{gw}^2 + D_{xs} \lambda_{xw} \sigma_{xw}^2 \]

\[ \Rightarrow G_{gs} = \varphi \tau_d g \gamma \]

\[ \Rightarrow G_{xs} = D_s \lambda_c \]

**Real Bond Returns and Risk Premium**

We start off the expression for the real stochastic discount factor derived in the first sub-section above. Let define the following three parameters: \(s_x \equiv -\frac{1}{\lambda_c}, s_{gs} \equiv -\frac{1}{\lambda_c} (\gamma - 1)(\gamma - \frac{1}{\kappa_1}), \) and \(s_{xs} \equiv -\frac{1}{2} (\gamma - 1)(\gamma - \frac{1}{\kappa_1}) \left( \frac{1}{\kappa_1} - \rho \right)^2.\) Using notation defined above and in the previous sub-sections, the real stochastic discount factor is:

\[ \text{sdf}_{t+1} = \mu_s + s_x x_{t+1} + s_{gs} \left( \sigma_{gt}^2 - \sigma_g^2 \right) + s_{xs} \left( \sigma_{xt}^2 - \sigma_x^2 \right) \]

\[ - \lambda_t \sigma_{gt} \eta_{t+1} - \lambda_c \sigma_{xt} \xi_{t+1} - \lambda_{gw} \sigma_{gw} w_{g,t+1} - \lambda_{xw} \sigma_{xw} w_{x,t+1} \]
Let \( p^b_t(n) = \log (P_t^b(n)) \) be the log price and \( y^b_t(n) = -\frac{1}{n} p^b_t(n) \) the yield of an \( n \)-period real bond. We conjecture that the log prices of real bonds are linear in the state variables: 
\[
p_t(n) = -B_0(n) - B_x(n) \varepsilon_t - B_{gs}(n) \left( \sigma_{zt}^2 - \sigma_{z_t}^2 \right) - B_{xs} \left( \sigma_{xt}^2 - \sigma_{x_t}^2 \right)
\]

The coefficients are initialized at zero and satisfy the following recursions:
\[
\begin{align*}
B_0(n) &= B_0(n-1) - \mu_s - \frac{1}{2} \left[ \lambda_{gw} + B_{gs}(n-1) \right]^2 \sigma_{gw}^2 \\
&\quad - \frac{1}{2} \left( \lambda_{x_w} + B_{xs}(n-1) \right)^2 \sigma_{x_w}^2 + \lambda_s \sigma_g^2 \\
&\quad - \frac{1}{2} \left[ \lambda_e + B_x(n-1) \right]^2 \sigma_x^2 \\
B_s(n) &= \rho B_s(n-1) + \frac{1}{\psi} \\
B_{gs}(n) &= \nu_g B_{gs}(n-1) + \frac{1}{2} (\gamma - 1) \left( \gamma - \frac{1}{\psi} \right) - \frac{1}{2} \psi^2 \\
B_{xs}(n) &= \nu_x B_{xs}(n-1) + \frac{1}{2} (\gamma - 1) \left( \gamma - \frac{1}{\psi} \right) \left( \frac{1}{\kappa_1^2 - \rho} \right) - \frac{1}{2} \left[ \frac{1}{\kappa_1^2 - \rho} B_x(n-1) \right]^2
\end{align*}
\]

These recursions imply the following limit values:
\[
\begin{align*}
B_0(\infty) &= \frac{1}{\psi(1 - \rho)} \\
B_{gs}(\infty) &= \frac{\frac{1}{2} (\gamma - 1) \left( \gamma - \frac{1}{\psi} \right) - \frac{1}{2} \psi^2}{1 - \nu_g} \\
B_{xs}(\infty) &= \frac{\frac{1}{2} (\gamma - 1) \left( \gamma - \frac{1}{\psi} \right) \left( \frac{1}{\kappa_1^2 - \rho} \right) - \frac{1}{2} \left[ \frac{1}{\kappa_1^2 - \rho} B_x(\infty) \right]^2}{1 - \nu_x}
\end{align*}
\]

We define \( B(\infty) \equiv [B_0(\infty), B_{gs}(\infty), B_{xs}(\infty)]' \).

The real bond risk premium on monthly holding period returns is equal to:
\[
\begin{align*}
\nu_{r_t+1}(n) &= ny_t^b(n) - (n - 1) y_{t+1}^b(n-1) \\
r_t(n) &= -B_x(n-1) \sigma_{xt} \varepsilon_{t+1} - B_{gs}(n-1) \sigma_{gw} \varepsilon_{t+1} - B_{xs}(n-1) \sigma_{xw} \varepsilon_{t+1} \\
E_x \left[ r_{t+1}^b(n) \right] &= -\text{Cov}_t \left[ \nu_t, \varepsilon_{t+1} \right] \\
E_x \left[ r_{t+1}^{b,e}(n) \right] &= -\text{Cov}_t \left[ \nu_t, s_{t+1} \right] \\
F_0(n) &= F_{gs}(n) \sigma_{zt}^2 + F_{xs}(n) \sigma_{xt}^2 + F_{gw}(n) \left( \sigma_{gt}^2 - \sigma_g^2 \right) + F_{xs}(n) \left( \sigma_{zt}^2 - \sigma_{x_t}^2 \right) \\
F_{gs}(n) &= -B_{gs}(n-1) \lambda_{gw} \sigma_{gw}^2 - B_{xs}(n-1) \lambda_{xw} \sigma_{xw}^2 \\
F_{xs}(n) &= -B_{xs}(n-1) \lambda_e \\
\end{align*}
\]

We now define some vectors and matrices to present results in a more compact way. Let the vector \( X_t \) summarize all real state variables: \( X_t \equiv [\varepsilon_t, \sigma_{zt}^2 - \sigma_g^2, \sigma_{zt}^2 - \sigma_{z_t}^2]' \). Let \( \varepsilon_{t+1} \) denote the corresponding gaussian, i.i.d shocks: \( \varepsilon_{t+1} \equiv [\varepsilon_{t+1}, \omega_{t+1}, w_{x_t+1}, w_{z_t+1}]' \). We define \( \Sigma_t \equiv \text{diag} [\sigma_{zt}^2, \sigma_g^2, \sigma_{z_t}^2] \). The law of motion of the state vector \( X_t \) is \( X_{t+1} = \Gamma X_t + \Sigma_t \varepsilon_{t+1} \), where \( \Gamma \) is a 3 by 3 diagonal matrix with \( \rho, \sigma_{zt}, \sigma_{z_t} \) on the diagonal. Let \( B(n) \) denote all the \( n \)-period real bond parameters: \( B(n) \equiv [B_x(n), B_{gs}(n), B_{xs}(n)]' \). Using this notation, we can rewrite the real bond risk premium
Let \( p \) be the log price and \( y^*(n) = -\frac{1}{n} \log p^*(n) \) the yield of an \( n \)-period nominal bond.

We conjecture that the log prices of nominal bonds are linear in the state variables: 

\[
B^g_t(n) = -B^g_0(n) - B^g_1(n)x_t - B^g_2(n)(\alpha - \sigma^2_\xi) - B^g_3(n)(\alpha - \sigma^2_\xi) - B^g_4(n)(\pi_t - \mu_x)
\]

We start off the expression for the real stochastic discount factor derived above. We use a \(^s\) superscript to denote nominal variables. The nominal stochastic discount factor is then:

\[
sdf^s_t(n) = sdf^s_t(n - 1) = \mu_s + \mu_x + s_x x_t + s_g g_t (\sigma_{\pi g}^2 - \sigma_g^2) + s_z z_t (\sigma_{\pi z}^2 - \sigma_z^2) - (\pi_t - \mu_x) - (\lambda_\pi + \varphi_\pi)(\sigma_{\pi g}^2 + (\lambda_\pi + \varphi_\pi)(\sigma_{\pi z}^2 - \sigma_z^2)) \]

As:

\[
\mathbb{E}_t \left[ r^b_{t+1}(n) \right] = -B(n - 1)\Sigma_t A.
\]

Nominal Bond Returns and Risk Premium

The coefficients are initialized at zero and satisfy the following recursions:

\[
B^g_0(n) = B^g_0(n - 1) - \mu_s + \mu_x - \frac{1}{2} \left[ \left[ \sigma_\pi + B^s_x(n - 1)\sigma_z \right]^2 + \left[ \lambda_{\pi w} + B^s_{g x}(n - 1)\right]^2 \right] \sigma_{\pi w}^2
\]

\[
-\frac{1}{2} \left[ \lambda_{\pi x} + B^s_{z x}(n - 1)\right]^2 \sigma_{\pi x}^2 + \left[ \lambda_{\pi g} + \varphi_\pi g B^s_{x g}(n - 1)\right]^2 \sigma_{\pi g}^2
\]

\[
-\frac{1}{2} \left[ \varphi_\pi \lambda + B^s_{x g}(n - 1) + \varphi_{x g} B^s_{x g}(n - 1)\right]^2 \sigma_{\pi g}^2
\]

\[
B^g_1(n) = \rho B^s_1(n - 1) + \alpha_\pi B^s_0(n - 1) - s_x
\]

\[
B^g_2(n) = \nu_g B^s_2(n - 1) - s_{x g} - \frac{1}{2} \left[ \lambda_{\pi g} + \varphi_{\pi g} + \varphi_{x g} B^s_{x g}(n - 1)\right]^2
\]

\[
B^g_{x z}(n) = \nu_z B^s_{x z}(n - 1) - s_{x z} - \frac{1}{2} \left[ \lambda_{x g} + \varphi_{\pi x} + B^s_{x g}(n - 1) + \varphi_{x x} B^s_{x x}(n - 1)\right]^2
\]

\[
B^g_3(n) = \alpha_\pi B^s_3(n - 1) + 1
\]

These recursions imply the following limit values:

\[
B^g_0(\infty) = \frac{\lambda_{\pi w} B^s_{b x}(\infty) - s_x}{1 - \rho}
\]

\[
B^g_1(\infty) = \frac{-s_{x g} - \frac{1}{2} \left[ \lambda_{\pi g} + \varphi_{\pi g} + \varphi_{x g} B^s_{x g}(\infty)\right]^2}{1 - \nu_g}
\]

\[
B^g_2(\infty) = \frac{-s_{x z} - \frac{1}{2} \left[ \lambda_{x g} + \varphi_{\pi x} + B^s_{x g}(\infty) + \varphi_{x x} B^s_{x x}(\infty)\right]^2}{1 - \nu_z}
\]

\[
B^g_3(\infty) = \frac{1}{1 - \alpha_\pi}
\]

We define \( B^g(\infty) \equiv \left[ B^s_0(\infty), B^s_1(\infty), B^s_{x z}(\infty), B^s_3(\infty) \right] \).
The nominal bond risk premium on monthly holding period returns is equal to:

\[
\begin{align*}
\rho_{t+1}^B(n) & \equiv \eta_{t+1}(n) - (n-1)\eta_{t+1}(n-1) \\
\rho_{t+1}^R(n) - \mathbb{E}_t \left[ \rho_{t+1}^B(n) \right] & = - \left( B_s^g(n-1) + B_s^w(n-1) \varphi_{xx} \right) \sigma_{z,t+1} - B_{g,t+1}^g(n-1) \sigma_{g,t+1} w_{g,t+1} \\
& \quad - B_{s,t+1}^g(n-1) \sigma_{x,t+1} - B_{s,t+1}^w(n-1) (\varphi_{g,t+1} \eta_{t+1} + \sigma_{z,t+1} \xi_{t+1}) \\
\mathbb{E}_t \left[ \rho_{t+1}^{b,s,x}(n) \right] & = \mathbb{E}_t \left[ \rho_{t+1}^B(n) \right] = - \operatorname{Cov}_t \left[ \rho_{t+1}^B(n), \text{SDF}_{t+1}^R \right] \\
& = \left[ F_0^g(n) + F_0^w(n) \sigma_{g}^2 + F_x^g(n) \sigma_{x}^2 \right] + F_y^g(n) (\sigma_{g}^2 - \sigma_{y}^2) + F_x^w(n) (\sigma_{x}^2 - \sigma_{y}^2) \\
F_0^g(n) & = - \left( \lambda_{gw} B_{s}^g(n-1) \sigma_{g}^2 + \lambda_{xw} B_{s}^w(n-1) \sigma_{x}^2 + \sigma_{x} \sigma_{z} B_{s}^w(n-1) \right) \\
F_0^w(n) & = - (\lambda_{g} + \varphi_{xx}) \left( B_{s}^w(n-1) + B_{g}^w(n-1) \varphi_{xx} \right) \\
\end{align*}
\]

Define the following vector and matrix objects:

\[
\begin{align*}
\hat{\Lambda}^g & \equiv [\lambda_{g}, \varphi_{g}, \lambda_{x}, \varphi_{xx}, \lambda_{gw}, \lambda_{bx}, \sigma_{x}] \\
\hat{B}^g(n) & \equiv [B_s^g(n) \varphi_{g}, B_s^w(n) \varphi_{x}, B_s^w(n), B_s^w(n), B_s^w(n) \sigma_{x}] \\
\hat{\Sigma}_t & \equiv \operatorname{diag} \left[ \sigma_{g}^2, \sigma_{x}^2, \sigma_{g}^2, \sigma_{x}^2, 1 \right] \\
\hat{\epsilon}_{t+1} & \equiv [\epsilon_{t+1}, w_{g,t+1}, w_{x,t+1}, \xi_{t+1}] \\
\end{align*}
\]

Then we can write the nominal bond risk premium compactly as:

\[
\mathbb{E}_t \left[ \rho_{t+1}^{b,s,x}(n) \right] = - \hat{B}^g(n-1) \hat{\Sigma}_t \hat{\Lambda}^g.
\]

**Alvarez-Jermann Decomposition: Real SDF**

The following proposition shows how to decompose the SDF of the long-run risk model into a permanent and a transitory component.

**Proposition 1.** The stochastic discount factor of the long-run risk model can be decomposed into a permanent and a transitory component

\[
\begin{align*}
\frac{M_{t+1}^T}{M_t^T} & = \beta \exp \left( - B_\infty' \left( I - \Gamma \right) X_t + B_\infty' \Sigma_T^2 \hat{\epsilon}_{t+1} \right), \\
\frac{M_{t+1}^T}{M_t^T} & = \beta^{-1} \exp \left( \mu_0 + \left[ S' + B_\infty' \left( I - \Gamma \right) \right] X_t - \left( \lambda'/ + B_\infty' \right) \Sigma_T^2 \hat{\epsilon}_{t+1} - \lambda \eta \sigma \eta \xi_{t+1} \right). \\
\end{align*}
\]

To show this, we start from the definition of the transient component of the pricing kernel:

\[
M_t^T = \lim_{n \to \infty} \frac{\beta^{t+n}}{P_t^R(n)}.
\]

Recall that log real bond prices are affine in the state vector:

\[
\begin{align*}
p_t^R(n) & = - B_0(n) - B_x(n) x_t - B_{g,t}(n) (\sigma_{g,t}^2 - \sigma_{y}^2) - B_{x,t}(\sigma_{x,t}^2 - \sigma_{y}^2) \\
& = - B_0(n) - B(n)' X_t.
\end{align*}
\]
We can then write the transitory component of the SDF as:

\[ M_t^T = \lim_{n \to \infty} \beta^{t+n} \exp \left( B_0(n) + B(n)X_t \right). \]

The constant \( \beta \) is chosen in order to satisfy Assumption 1 in Alvarez and Jermann (2005):

\[ 0 < \lim_{n \to \infty} \frac{P_h(n)}{\beta^n} < \infty. \]

Recall that \( B_0(n) \) is defined recursively:

\[ B_0(n) = B_0(n-1) - \mu_s - \frac{1}{2} \left\{ [\lambda_g + B_g(n-1)]^2 \sigma_g^2 \right\} \]

\[ - \frac{1}{2} \left\{ [\lambda_w + B_w(n-1)]^2 \sigma_w^2 + \lambda_s^2 \sigma_g^2 + [\lambda_c + B_s(n-1)]^2 \sigma_s^2 \right\} \]

Because of the affine term structure of the model and the stationarity of the state vector \( X \), the limit \( \lim_{n \to \infty} B(n) = B(\infty) \) is finite. Taking limits on both sides of the equation above leads to:

\[ \lim_{n \to \infty} B_0(n) - B_0(n-1) = -\mu_s - \frac{1}{2} \left\{ [\lambda_g + B_g(\infty)]^2 \sigma_g^2 \right\} \]

\[ - \frac{1}{2} \left\{ [\lambda_w + B_w(\infty)]^2 \sigma_w^2 + \lambda_s^2 \sigma_g^2 + [\lambda_c + B_s(\infty)]^2 \sigma_s^2 \right\} \]

The limit of \( B_0(n) - B_0(n-1) \) is finite, so that \( B_0(n) \) grows at a linear rate in the limit. We choose the constant \( \beta \) to offset the growth in \( B_0(n) \) as \( n \) becomes very large. Setting

\[ \beta = \exp \left( \mu_s + \frac{1}{2} \left\{ [\lambda_g + B_g(\infty)]^2 \sigma_g^2 + [\lambda_w + B_w(\infty)]^2 \sigma_w^2 + \lambda_s^2 \sigma_g^2 + [\lambda_c + B_s(\infty)]^2 \sigma_s^2 \right\} \right) \]

guarantees that Assumption 1 in Alvarez and Jermann (2005) is satisfied.

We can now write the transient component of the SDF as:

\[ \frac{M_{t+1}^T}{M_t^T} = \beta \exp \left( -B'_\infty (I - \Gamma) X_t + B'_\infty \Sigma \xi_{t+1} \right). \]

To derive the permanent component of the SDF, let us go back to the SDF itself. Let \( S \) and \( \Lambda \) denote the parameters of the real SDF: \( S \equiv [s_x, s_{gs}, s_{xs}] \), \( \Lambda \equiv [\lambda_c, \lambda_gw, \lambda_{sxw}] \). Then the real SDF is:

\[ SDF_{t+1} = \frac{M_{t+1}}{M_t} = \exp \left( \mu_s + S'X_t - \Lambda' \Sigma \xi_{t+1} - \lambda_s \sigma_g \eta_{t+1} \right). \]

As a result, the permanent component of the SDF is:

\[ \frac{M_{t+1}^P}{M_t^P} = \left( \frac{M_{t+1}^T}{M_t^T} \right)^{-1} \]

\[ = \beta^{-1} \exp \left( \mu_s + [S' + B'_\infty (I - \Gamma)] X_t - (\Lambda' + B'_s) \Sigma \xi_{t+1} - \lambda_s \sigma_g \eta_{t+1} \right). \]

We need to verify that the permanent component is a martingale, i.e that \( E_t[M_{t+1}^P/M_t^P] = 1 \).
To do this, recall that the bond parameters evolve as:

\[
\begin{align*}
B_s(n) &= \rho B_s(n-1) - s_x \\
B_gy(n) &= \nu_y B_gy(n-1) - s_gy - \frac{1}{2} \lambda^2
\end{align*}
\]

\[
B_{xy}(n) = \nu_x B_{xy}(n-1) - s_{xy} - \frac{1}{2} [\lambda_e + B_x(n-1)]^2.
\]

Taking limits as \(n \to \infty\) leads to:

\[
B(\infty)'(I - \Gamma) = -S' + [0, \ -\frac{1}{2} \lambda^2, \ -\frac{1}{2} [\lambda_e + B_x(\infty)]^2]'.
\]

To check the martingale condition, plug the definition of \(\beta\) in the following expression:

\[
E_t \left[ \frac{M_{t+1}^P}{M_t^P} \right] = \beta^{-1} \exp \left( \mu_s + \left[ S' + B'_n (I - \Gamma) \right] X_t + \frac{1}{2} \left( \Lambda' + B'_\infty \right) \Sigma_t (\Lambda + B_\infty) + \frac{1}{2} \lambda^2 \right).
\]

The term in front of \(X_t\) is equal to \([0, \ -\frac{1}{2} \lambda^2, \ -\frac{1}{2} [\lambda_e + B_x(\infty)]^2]'\). Terms in \(\sigma^2_{gt}\) and \(\sigma^2_{xt}\) cancel out. We next plug in the expression for \(\beta\) and check that \(E_t \left[ \frac{M_{t+1}^P}{M_t^P} \right] = 1\).

We now turn to the conditional variances of the log SDF and its transient and permanent components, \(\text{Var}_t[sdf_{t+1}], \text{Var}_t[sdf^p_{t+1}]\) and \(\text{Var}_t[sdf^{T}_{t+1}]\).

\[
\begin{align*}
\text{Var}_t[sdf_{t+1}] &= \Lambda' \Sigma_t \Lambda + \lambda^2_{\gamma} \sigma^2_{gt} \\
\text{Var}_t[sdf^p_{t+1}] &= B'_\infty \Sigma_t B_\infty \\
\text{Var}_t[sdf^{T}_{t+1}] &= (\Lambda' + B'_\infty) \Sigma_t (\Lambda + B_\infty) + \lambda^2_{\gamma} \sigma^2_{gt}.
\end{align*}
\]

The conditional variance ratio \(\text{Var}_t[sdf^p_{t+1}] / \text{Var}_t[sdf^{T}_{t+1}]\) equals

\[
\frac{V_t[sdf^p_{t+1}]}{V_t[sdf^{T}_{t+1}]} = 1 - \frac{-B'_\infty \Sigma_t \Lambda - \frac{1}{2} \lambda^2_{\gamma} \Sigma_t B_\infty}{\frac{1}{2} \Lambda' \Sigma_t \Lambda + \frac{1}{2} \lambda^2_{\gamma} \sigma^2_{gt}}
\]

The first term in the numerator corresponds to the bond risk premium \((-B'_\infty \Sigma_t \Lambda)\). It includes the Jensen term \((\frac{1}{2} B'_\infty \Sigma_t B_\infty)\). As a result, the numerator corresponds to the bond risk premium without the Jensen term. The denominator corresponds to the maximum risk premium (also without the Jensen term).

Note that the maximal Sharpe ratio in the model is:

\[
\text{MaxSR}_t = \sigma_t \left( \log \text{SDF}_{t+1} \right)
\]

\[
= \sqrt{\lambda^2 \sigma^2_x + \lambda^2_\omega \sigma^2_{gw} + \lambda^2_\omega \sigma^2_{gw} + \lambda^2_{\eta} \sigma^2_{gt}}
\]

\[
= (\Lambda' \Sigma_t \Lambda + \lambda^2_{\gamma} \sigma^2_{gt})^{1/2}
\]

Alvarez-Jermann Decomposition: Nominal SDF

The following proposition shows how to decompose the nominal SDF of the long-run risk model into a permanent and a transitory component. To avoid confusion, we use \(MN\) to denote the nominal pricing kernel.

**Proposition 2.** The stochastic discount factor of the long-run risk model can be decomposed into a
The constant $\tilde{\beta}$ is chosen in order to satisfy Assumption 1 in Alvarez and Jermann (2005):

$$0 < \lim_{n \to \infty} \frac{P_{t+1}^B(n)}{\tilde{\beta}^n} < \infty.$$
choose the constant $\tilde{\beta}$ to offset the growth in $B^s_0(n)$ as $n$ becomes very large. Setting

$$\tilde{\beta} = \exp \left( \mu_s - \mu_x + \frac{1}{2} \left\{ [\sigma_x + B^s_0(\infty)\sigma_x]^2 + [\lambda_{yw} + B^s_{y_0}(\infty)]^2 \sigma_{yw}^2 \right\} + \frac{1}{2} \left\{ [\lambda_{zw} + B^s_{x_0}(\infty)]^2 \sigma_{zw}^2 + [\varphi_{xg} + \lambda_x + \varphi_{xg} B^s_0(\infty)]^2 \sigma_g^2 \right\} + \frac{1}{2} \left[ \varphi_{px} + \lambda_c + B^s_x(\infty) + \varphi_{xx} B^s_x(\infty) \right]^2 \sigma^2_x \right)$$

ensures that Assumption 1 in Alvarez and Jermann (2005) is satisfied.

We can now write the transient component of the SDF as:

$$\frac{MN^T_{t+1}}{MN^T_t} = \tilde{\beta} \exp \left( -B^s_{\infty}(I - \tilde{\Gamma}) \tilde{X}_t + B^s_{\infty}(\lambda_{x} + \varphi_{xg}) \sigma_{gt} \xi_{t+1} + B^s_{\infty}(\infty) \sigma_{z \xi_{t+1}} \right) + \left[ B^s_{\infty}(\infty) + B^s_{\infty}(\varphi_{xg}) \sigma_{xt} \xi_{t+1} + B^s_{\infty}(\infty) \sigma_{gw} w_{g,t+1} + B^s_{\infty}(\infty) \sigma_{wx} w_{x,t+1} \right] $$

$$= \tilde{\beta} \exp \left( -B^s_{\infty}(I - \tilde{\Gamma}) \tilde{X}_t + \hat{B}^s_{\infty} \tilde{\xi}_{t+1} \right),$$

where

$$\tilde{\Gamma} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & \nu_g & 0 & 0 \\ 0 & 0 & \nu_x & 0 \\ \alpha_x & 0 & 0 & \alpha_x \end{bmatrix}$$

To derive the permanent component of the SDF, let us go back to the SDF itself. Let $\tilde{S} \equiv [s_x, s_{gs}, s_{sx}, -1]'$. Then the nominal SDF is:

$$\frac{MN_{t+1}^P}{MN_t^P} = \exp \left( \mu_s - \mu_x + \tilde{S}' \tilde{X}_t - (\lambda_{x} + \varphi_{xg}) \sigma_{gt} \xi_{t+1} - (\lambda_c + \varphi_{px}) \sigma_{xt} \xi_{t+1} - \lambda_{yw} \sigma_{gw} w_{g,t+1} - \lambda_{xw} \sigma_{sx} w_{x,t+1} - \sigma_{x} \xi_{t+1} \right)$$

As a result, the permanent component of the SDF is:

$$\frac{MN_{t+1}^P}{MN_t^P} = \frac{MN_{t+1}^T}{MN_t^T} \left( \frac{MN_{t+1}^T}{MN_t^T} \right) \tilde{\beta}^{-1} \exp \left( \mu_s - \mu_x + \tilde{S}' + B^s_{\infty}(I - \tilde{\Gamma}) \right) \tilde{X}_t - [\lambda_{x} + \varphi_{xg} + B^s_{\infty}(\infty) \varphi_{xg} \sigma_{gt} \xi_{t+1} - [\lambda_c + \varphi_{px} + B^s_x(\infty) + B^s_x(\infty) \varphi_{px} \sigma_{xt} \xi_{t+1} - [\lambda_{yw} + B^s_{y_0}(\infty)] \sigma_{gw} w_{g,t+1} - [\lambda_{xw} + B^s_{x_0}(\infty)] \sigma_{sx} w_{x,t+1} - \sigma_{x} \xi_{t+1} \right)$$

$$= \tilde{\beta}^{-1} \exp \left( \mu_s - \mu_x + \tilde{S}' + B^s_{\infty}(I - \tilde{\Gamma}) \right) \tilde{X}_t - (\tilde{\Lambda}^s + \hat{B}^s_{\infty}) \tilde{\xi}_{t+1}.$$
do so, recall that the bond parameters evolve as:

\[
\begin{align*}
B^g_s(n) &= \rho B^g_s(n-1) + \alpha_s B^g_s(n-1) - s_x \\
B^g_p(n) &= \nu_s B^g_p(n-1) - s_{gs} - \frac{1}{2} \left[ \lambda_e + \varphi_{eg} + \varphi_{pzg} B^g_p(n-1) \right]^2 \\
B^g_{sz}(n) &= \nu_s B^g_{sz}(n-1) - s_{gs} - \frac{1}{2} \left[ \lambda_e + \varphi_{pz} + \varphi_{sz} B^g_{sz}(n-1) \right]^2 \\
B^g(n) &= \alpha_s B^g(n-1) + 1.
\end{align*}
\]

Taking limits as \( n \to \infty \) leads to:

\[
B(\infty)^g(I - \hat{\Gamma}) + \tilde{S'} = \begin{bmatrix} 0, & -\frac{1}{2} \left[ \lambda_e + \varphi_{eg} + \varphi_{pzg} B^g_s(\infty) \right]^2, & -\frac{1}{2} \left[ \lambda_e + \varphi_{pz} + \varphi_{sz} B^g_{sz}(\infty) \right]^2, 0 \end{bmatrix}.
\]

To check the martingale condition, we plug in the definition of \( \hat{\beta} \) in the expression for the permanent component of the nominal SDF, and use the above equation for \( B(\infty)^g(I - \hat{\Gamma}) + \tilde{S'} \). After some algebra, we indeed find that

\[
\mathbb{E} t \begin{bmatrix} MN_{t+1}^P \\
MN_{t+1}^f \end{bmatrix} = 1.
\]

We now turn to the conditional variances of the log SDF and its transient and permanent components, \( \text{Var}_t[\text{sdf}_{t+1}], \text{Var}_t[\text{sdf}_{t+1}^p] \) and \( \text{Var}_t[\text{sdf}_{t+1}^s] \):

\[
\begin{align*}
\text{Var}_t[\text{sdf}_{t+1}^g] &= (\lambda_e + \varphi_{eg})^2 \sigma_{glt}^2 + (\lambda_e + \varphi_{pz})^2 \sigma_{zt}^2 + \lambda_{gw}^2 \sigma_{gw}^2 + \lambda_{zw}^2 \sigma_{zw}^2 + \sigma_z^2 \\
&= \hat{\Lambda}_\infty \hat{\Sigma}_1 \hat{\Lambda}_\infty^g \\
\text{Var}_t[\text{sdf}_{t+1}^s, \text{sdf}_{t+1}^p] &= B^s_\infty(\infty)^2 \varphi_{pzg}^2 \sigma_{glt}^2 + \left[ B^s_\infty(\infty) + B^s_\infty(\infty) \varphi_{gzt} \right]^2 \sigma_{zt}^2 \\
&+ \left[ \lambda_e + \varphi_{pz} + \varphi_{sz} B^g_{sz}(\infty) \right]^2 \sigma_{ew}^2 + \left[ \lambda_e + \varphi_{eg} + \varphi_{pzg} B^g_s(\infty) \right]^2 \sigma_{e1}^2 \\
&= \hat{\Lambda}_\infty \hat{\Sigma}_1 \hat{\Lambda}_\infty^s \hat{\Lambda}_\infty^p \\
\text{Var}_t[\text{sdf}_{t+1}^s, \text{sdf}_{t+1}^p] &= \left[ \lambda_e + \varphi_{eg} + B^g_s(\infty) \varphi_{pzg} \right]^2 \sigma_{glt}^2 + \left[ \lambda_e + \varphi_{pz} + B^g_{sz}(\infty) \right]^2 \sigma_{zt}^2 \\
&+ \left[ \lambda_{gw} + B^s_{gs}(\infty) \right]^2 \sigma_{gw}^2 + \left[ \lambda_{zw} + B^s_{gs}(\infty) \right]^2 \sigma_{zw}^2 + \left[ \sigma_e + B^s_e(\infty) \right]^2 \sigma_{e1}^2 \\
&= \left( \hat{\Lambda}_\infty + \hat{\beta}^s_{\infty} \hat{\Sigma}_1 \hat{\Lambda}_\infty \right)^2.
\end{align*}
\]

The conditional variance ratio \( \text{Var}_t[\text{sdf}_{t+1}^s, \text{sdf}_{t+1}^p]/\text{Var}_t[\text{sdf}_{t+1}^p] \) equals

\[
\frac{V_t[\text{sdf}_{t+1}^s, \text{sdf}_{t+1}^p]}{V_t[\text{sdf}_{t+1}^p]} = 1 - \frac{-\hat{B}^s_{\infty} \hat{\Sigma}_1 \hat{\Lambda}_\infty - \frac{1}{2} \hat{B}^s_{\infty} \hat{\Sigma}_1 \hat{B}^s_\infty}{\frac{1}{2} \hat{\Lambda}_\infty \hat{\Sigma}_1 \hat{\Lambda}_\infty^s \hat{\Lambda}_\infty^p}.
\]

The first term in the numerator corresponds to the nominal bond risk premium of an infinite horizon bond, which includes a Jensen term. The second term in the numerator is that Jensen term. As a result, the numerator corresponds to the nominal bond risk premium without the Jensen term. The denominator corresponds to the maximum nominal risk premium, also without the Jensen term.

**Calibration**

Table A.4 reports the model parameter values we use; they are the ones proposed in Bansal and Shaliastovich (2008). Table A.2 reports the model loadings on state variables. Table A.3 presents the mean, standard deviation and autocorrelation of the annualized wealth-consumption ratio (WC),
price-dividend ratio \((PD)\), one- and five-year nominal yields \((y^S(1)\) and \(y^S(5)\)), equity risk premium \((ERP)\), and the 5-year nominal bond risk premium \((BRP^P)\) in the model and in the data. Table A.5 reports the mean, standard deviation and autocorrelation of the stochastic discount factor \((SDF)\), its permanent \((SDF^F)\) and transitory \((SDF^T)\) components, the conditional variance ratio \(\omega\), the maximum risk premium without Jensen adjustment \((Max\ RP)\) and the risk premium of an infinite maturity bond without Jensen adjustment \((BRP(\infty))\). Table A.4 reports the mean and standard deviations of the real and nominal yields and bond risk premia in the model and compare them to the same moments in the actual nominal data. Table A.6 reports moments of quarterly inflation in the model and in the data. Quarterly inflation is obtained as the sum of three consecutive monthly inflation rates.

The Bansal and Shaliastovich (2008) calibration generates an annual consumption growth rate of 2.12 percent with a standard deviation of 3.52 percent. It generates an annual inflation rate of 3.52 percent with a standard deviation of 2.49 percent.

As a robustness check, we report in Table A.7 the mean maximum risk premium \((MRP)\), the mean bond risk premium \((BRP_J)\) (including the Jensen term) and the mean variance ratio \(\omega\) for different values of the inflation parameters. We vary one parameter at a time, and simulate the model for a low and a high value of each parameter (25 percent above and below the benchmark value reported in Table A.1). The only exception is the parameter \(\alpha_\pi\), which we cannot increase by 25 percent without running into stationarity issues. The high value is a 10 percent increase for that parameter. Figure A.1 plots \(\omega_t\) (left axis) and the five-year nominal bond risk premium (right axis) against \(\alpha_x\) (horizontal axis). As we vary \(\alpha_x\) away from its benchmark value of -0.35, we simultaneously vary \(\alpha_\pi\) to match the observed persistence of quarterly inflation. We also choose \(\mu_\pi\) and \(\sigma_\pi\) to keep the mean and volatility of inflation at their benchmark values.

**Figure A.1. Variance Ratio and Nominal Bond Risk Premium: Sensitivity Analysis**

The figure plots the conditional variance ratio \(\omega_t\) (against the left axis) and the five-year nominal bond risk premium (against the right axis) for different values of the parameter \(\alpha_x\) (on the horizontal axis). As we vary \(\alpha_x\), away from its benchmark value of -0.35, we simultaneously vary \(\alpha_\pi\), to match the observed persistence of quarterly inflation. We also choose \(\mu_\pi\), and \(\sigma_\pi\), to keep the mean and volatility of inflation at their benchmark values.

**Empirical Variance Ratios**

Alvarez and Jermann (2005) show that – assuming that the process \(X_t\) satisfies the same regularity conditions as above and that \(X_{t+1}/X_t\) is strictly stationary and \(\lim_{k \to \infty} \frac{1}{k} \text{Var}(E_{t+k}X_t) = 0\) – then

\[
\text{Var} \left( \frac{X_{t+k}^P}{X_t^P} \right) = \lim_{k \to \infty} \frac{1}{k} \text{Var} \left( \frac{X_{t+k}}{X_t} \right),
\]

Note: The figure plots the conditional variance ratio \(\omega_t\) (against the left axis) and the five-year nominal bond risk premium (against the right axis) for different values of the parameter \(\alpha_x\) (on the horizontal axis). As we vary \(\alpha_x\), away from its benchmark value of -0.35, we simultaneously vary \(\alpha_\pi\), to match the observed persistence of quarterly inflation. We also choose \(\mu_\pi\), and \(\sigma_\pi\), to keep the mean and volatility of inflation at their benchmark values.
Note that the entropy measure used by Alvarez and Jermann (2005) collapses to the half-variance since all variables are conditionally normal. This result implies that long-horizon variance ratios are informative about the variance of the permanent component. We now turn to the empirical variance ratios of the two components of the SDF, e.g., consumption growth and the wealth consumption ratio.

If changes in log consumption or changes in the log wealth-consumption ratio are i.i.d., then the variance of long-horizon changes in each variable should grow with the horizon. We compute variance ratios at horizon $h$ as $VR(h) = \frac{\text{Var}\left[\sum_{j=0}^{h} \Delta x_{t+j}\right]}{h \text{Var}(\Delta x_t)}$, for $x = c$ and $x = wc$. We simulate the model at monthly frequency. Table A.1 in the appendix reports the model parameters. We start from the parameter values in Bansal and Shaliastovich (2008).

Figure A.2 reports these variance ratios for consumption growth, the change in the wealth-consumption ratio, and inflation. The left panel corresponds to actual data; the right panel uses simulated series. Let us first focus on actual data. The variance ratio of the wealth-consumption ratio clearly decreases with the horizon. It is below 0.6 within five years. Consumption growth exhibits a very different pattern: its variance ratio first increases for horizons up to 5 years; it then decreases, but even after 15 years, the variance ratio is still above one. As a result, there is strong evidence of persistence and mean-reversion in the wealth-consumption ratio, but not in consumption growth.

Let us now turn to simulated data. The variance ratios of the wealth-consumption ratio are in line with the data. They decrease linearly with the horizon, from 1 to approximately 0.5 at the 30-year horizon. In the data, the variance ratio decreases from 1 to 0.6. Consumption growth, however, exhibits a very different pattern. At long horizons, it displays more persistence in the model than in the data. The bottom panel shows that the inflation persistence is similar in model and data, with a slight divergence maybe at longer horizons.
Figure A.2. Variance Ratios for Consumption Growth, the Change in the Log Wealth to Consumption Ratio and Inflation in the Data and in the Model.

The variance ratio of $\Delta x_t$ is equal to $VR(h) = \frac{\text{Var}[\sum_{j=0}^{h-1} \Delta x_{t+j}]}{h \text{Var}(\Delta x_t)}$. The left panel corresponds to actual data. The right panel corresponds to simulated data. Data are quarterly. Actual data come from Lustig et al. (2009). The sample is 1952 II-2008 IV.
Table A.1—Model Parameter Values

<table>
<thead>
<tr>
<th>Parameter</th>
<th>BS(2008)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Preference Parameters:</strong></td>
<td></td>
</tr>
<tr>
<td>Subjective discount factor</td>
<td>$\delta$ 0.9987</td>
</tr>
<tr>
<td>Intertemporal elasticity of substitution</td>
<td>$\psi$ 1.5</td>
</tr>
<tr>
<td>Risk aversion coefficient</td>
<td>$\gamma$ 8</td>
</tr>
<tr>
<td><strong>Consumption Growth Parameters:</strong></td>
<td></td>
</tr>
<tr>
<td>Mean of consumption growth</td>
<td>$\mu_g$ 0.0016</td>
</tr>
<tr>
<td>Long-run risk persistence</td>
<td>$\rho$ 0.991</td>
</tr>
<tr>
<td>News volatility level</td>
<td>$\sigma_g$ 0.004</td>
</tr>
<tr>
<td>News volatility persistence</td>
<td>$\nu'_g$ 0.85</td>
</tr>
<tr>
<td>News volatility of volatility</td>
<td>$\sigma_{gw}$ 1.15e−6</td>
</tr>
<tr>
<td>Long run-risk volatility level</td>
<td>$\sigma_x$ 0.004$\sigma_g$</td>
</tr>
<tr>
<td>Long run-risk volatility persistence</td>
<td>$\nu_x$ 0.996</td>
</tr>
<tr>
<td>Long run-risk volatility of volatility</td>
<td>$\sigma_{xw}$ 0.06$^2\sigma_{gw}$</td>
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<tr>
<td><strong>Dividend Growth Parameters:</strong></td>
<td></td>
</tr>
<tr>
<td>Mean of dividend growth</td>
<td>$\mu_d$ 0.0015</td>
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<tr>
<td>Dividend leverage</td>
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<tr>
<td>Dividend loading on news volatility</td>
<td>$\phi_{qs}$ 0</td>
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<tr>
<td>Dividend loading on long-run risk volatility</td>
<td>$\phi_{xs}$ 0</td>
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<tr>
<td>Volatility loading of dividend growth</td>
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<td><strong>Inflation Parameters:</strong></td>
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<tr>
<td>Mean of inflation rate</td>
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<tr>
<td>Inflation leverage on news</td>
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<tr>
<td>Inflation leverage on long-run news</td>
<td>$\varphi_{\pi x}$ −2.0</td>
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<tr>
<td>Inflation shock volatility</td>
<td>$\sigma_\pi$ 0.0035</td>
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<tr>
<td>Expected inflation AR coefficient</td>
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<td>Expected inflation loading on long-run risk</td>
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<tr>
<td>Expected inflation leverage on news</td>
<td>$\varphi_{\pi g}$ 0</td>
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<tr>
<td>Expected inflation leverage on long-run news</td>
<td>$\varphi_{xx}$ −1.0</td>
</tr>
<tr>
<td>Expected inflation shock volatility</td>
<td>$\sigma_x$ 4.0e−6</td>
</tr>
</tbody>
</table>

This table reports the calibrated parameters values for our simulation. We take them from Table IV and Table C.I in Bansal and Shaliastovich (2008).
\[
\begin{align*}
\text{constant} & \quad x & \quad \sigma_{gt}^2 - \sigma_{g}^2 & \quad \sigma_{xt}^2 - \sigma_{x}^2 \\
\text{wc} & \quad \mu_{wc} & \quad W_x = \frac{1}{\kappa_1 - \rho} & \quad W_{gs} = \frac{\phi_x}{2(n - \kappa_1)} & \quad W_{xs} = \frac{\phi_x}{2 \nu_x} \\
 & & \quad 6.4 & \quad 31 & \quad -7.7 & \quad -1.8 \times 10^6 \\
\text{pd} & \quad \mu_{pd} & \quad D_x = \frac{\phi_x - 2}{\kappa_1 - \rho} & \quad D_{gs} = \frac{\gamma^2 - 2 \gamma \varphi_x \tau_d + \gamma \varphi_x}{\kappa_1 - \rho} & \quad D_{xs} = \frac{2}{2 \nu_x} \\
 & & \quad 5.6 & \quad 66 & \quad 1.3 \times 10^2 & \quad -4.3 \times 10^6 \\
\text{ERP} & & \quad (1 - \theta) W_{gs} D_{gs} \sigma_{gs}^2 & \quad G_{gs} = \gamma \varphi_d \tau_d & \quad G_{xs} = (1 - \theta) W_x D_x \\
 & & \quad + (1 - \theta) W_{xs} D_{xs} \sigma_{xs}^2 & \quad + \varphi_d \tau_d \gamma \sigma_{gs}^2 & \quad + \varphi_x - (1 - \theta) W_{x} \sigma_{x}^2 \\
& & \quad 0.003 & \quad 0 & \quad 4.8 & \quad 4.6 \times 10^4 \\
\text{BRP} & & \quad (\theta - 1) W_{gs} B_{gs}(n - 1) \sigma_{gs}^2 & \quad F_x(n) = 0 & \quad F_{gs}(n) = 0 & \quad F_{xs}(n) = (\theta - 1) W_x B_x(n - 1) \\
\text{(Real)} & & \quad + (\theta - 1) W_{xs} B_{xs}(n - 1) \sigma_{xs}^2 & \quad + (\theta - 1) W_x B_x(n - 1) \sigma_{x}^2 \\
& & \quad -0.0014 & \quad 0 & \quad 0 & \quad -2.1 \times 10^4 \\
\text{BRP} & & \quad (\theta - 1) W_{gs} B_{gs}^2(n - 1) \sigma_{gs}^2 & \quad F_{gs}^2(n) = 0 & \quad F_{gs}(n) = - (\gamma + \varphi_x g) & \quad F_{xs}(n) = [(\theta - 1) W_x - \varphi_x x] \\
\text{(Nominal)} & & \quad + (\theta - 1) W_{xs} B_{xs}^2(n - 1) \sigma_{xs}^2 & \quad + \varphi_x g B_{gs}(n - 1) & \quad \times \varphi_x g B_{gs}^2(n - 1) & \quad \times \left(B_{gs}^2(n - 1) + B_{gs}^2(n - 1) \varphi_x x \right) \\
& & \quad + \sigma_x \sigma_x B_{gs}(n - 1) & \quad + \sigma_x \sigma_x B_{gs}^2(n - 1) & \quad \sigma_x \sigma_x B_{gs}^2(n - 1) & \quad \sigma_x \sigma_x B_{gs}^2(n - 1) \varphi_x x \\
& & \quad + \gamma (\gamma + \varphi_x g) \varphi_x x B_{gs}^2(n - 1) \sigma_x^2 & \quad + \gamma (\gamma + \varphi_x g) \varphi_x x B_{gs}^2(n - 1) \sigma_x^2 & \quad + ((\theta - 1) W_x - \varphi_x x) \left(B_{gs}^2(n - 1) + B_{gs}^2(n - 1) \varphi_x x \right) \sigma_x^2 \\
& & \quad 0.0015 & \quad 0 & \quad -0 & \quad 4.3 \times 10^4 \\
\end{align*}
\]
This table reports the mean, standard deviation and autocorrelation of the annualized wealth-consumption ratio (\(WC\)), price-dividend ratio (\(PD\)), one- and five-year nominal yields (\(y^8(1)\) and \(y^8(5)\)), equity risk premium (\(ERP\)), and the 5-year nominal bond risk premium (\(BRP^8\)). The mean and standard deviation are annualized (multiplied by 12 and \(\sqrt{12}\) for \(ERP\) and \(BRP^8\)) and reported in percentage points. The moments from the data are in the upper panel and are taken from Lustig et al. (2009). They pertain to the period 1953-2008. The lower panel reports the moments obtained from model simulations.

<table>
<thead>
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<th></th>
<th>Mean</th>
<th>Std</th>
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<tr>
<td><strong>Data</strong></td>
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<tr>
<td>WC</td>
<td>88.59</td>
<td>14.11</td>
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<tr>
<td>PD</td>
<td>27.53</td>
<td>7.20</td>
<td>0.95</td>
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<tr>
<td>(y^8(1))</td>
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<td>(y^8(5))</td>
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<td>ERP</td>
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<td>(BRP^8)</td>
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<td>PD</td>
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<td>(y^8(1))</td>
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<td>0.99</td>
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<td>(y^8(5))</td>
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<td>0.99</td>
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<tr>
<td>(BRP^8)</td>
<td>2.97</td>
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<td>0.99</td>
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### Table A.4—Real and Nominal Yield Curves

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<thead>
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<td><strong>Nominal Bonds - Data</strong></td>
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<tr>
<td>Mean Yields</td>
<td>5.33</td>
<td>5.52</td>
<td>5.69</td>
<td>5.80</td>
<td>5.89</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Std</td>
<td>2.81</td>
<td>2.77</td>
<td>2.70</td>
<td>2.69</td>
<td>2.65</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Nominal Bonds - Model</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean Yields</td>
<td>5.19</td>
<td>5.46</td>
<td>5.75</td>
<td>6.06</td>
<td>6.38</td>
<td>12.82</td>
<td>20.02</td>
</tr>
<tr>
<td>Std</td>
<td>2.92</td>
<td>2.79</td>
<td>2.65</td>
<td>2.53</td>
<td>2.43</td>
<td>1.60</td>
<td>0.36</td>
</tr>
<tr>
<td>Mean BRP</td>
<td>0.33</td>
<td>0.93</td>
<td>1.59</td>
<td>2.27</td>
<td>2.97</td>
<td>16.81</td>
<td>24.43</td>
</tr>
<tr>
<td>Std</td>
<td>0.07</td>
<td>0.18</td>
<td>0.28</td>
<td>0.38</td>
<td>0.46</td>
<td>1.13</td>
<td>1.18</td>
</tr>
<tr>
<td><strong>Real Bonds - Model</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean Yields</td>
<td>1.26</td>
<td>1.05</td>
<td>0.83</td>
<td>0.61</td>
<td>0.39</td>
<td>−4.71</td>
<td>−13.63</td>
</tr>
<tr>
<td>Std</td>
<td>1.39</td>
<td>1.35</td>
<td>1.32</td>
<td>1.30</td>
<td>1.29</td>
<td>1.10</td>
<td>0.25</td>
</tr>
<tr>
<td>Mean BRP</td>
<td>−0.39</td>
<td>−0.83</td>
<td>−1.28</td>
<td>−1.73</td>
<td>−2.19</td>
<td>−11.14</td>
<td>−16.21</td>
</tr>
<tr>
<td>Std</td>
<td>0.05</td>
<td>0.10</td>
<td>0.15</td>
<td>0.19</td>
<td>0.23</td>
<td>0.52</td>
<td>0.55</td>
</tr>
</tbody>
</table>

The top panel reports the mean and standard deviation of nominal bond yields in the Fama-Bliss data. The data are for 1952 until 2008, and only bond yields of maturities one through five years are available. The maturity is in years. The yields and returns are annualized and reported in percentage points. The middle panel does the same for nominal bond yields for a 60,000 month simulation of the LRR model. It also reports the mean and standard deviation of the nominal bond risk premia. The bottom panel reports the same model-implied moments for real bonds.

### Table A.5—Conditional Variance Ratio

<table>
<thead>
<tr>
<th>Mean</th>
<th>Std</th>
<th>AR(1)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Nominal SDF</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SDF$^g$</td>
<td>0.99</td>
<td>0.23</td>
</tr>
<tr>
<td>SDF$^{g,P}$</td>
<td>1.00</td>
<td>0.14</td>
</tr>
<tr>
<td>SDF$^{g,T}$</td>
<td>0.98</td>
<td>0.10</td>
</tr>
<tr>
<td>ω$^g_t$</td>
<td>0.37</td>
<td>0.06</td>
</tr>
<tr>
<td>Max RP</td>
<td>30.62</td>
<td>2.52</td>
</tr>
<tr>
<td>BRP($\infty$)</td>
<td>18.72</td>
<td>1.04</td>
</tr>
<tr>
<td><strong>Real SDF</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SDF$^R$</td>
<td>1.00</td>
<td>0.23</td>
</tr>
<tr>
<td>SDF$^{R,P}$</td>
<td>1.00</td>
<td>0.30</td>
</tr>
<tr>
<td>SDF$^{R,T}$</td>
<td>1.02</td>
<td>0.07</td>
</tr>
<tr>
<td>ω$^R_t$</td>
<td>1.65</td>
<td>0.11</td>
</tr>
<tr>
<td>Max RP</td>
<td>30.69</td>
<td>2.54</td>
</tr>
<tr>
<td>BRP($\infty$)</td>
<td>−19.05</td>
<td>0.58</td>
</tr>
</tbody>
</table>

This table reports the mean, standard deviation and autocorrelation of the stochastic discount factor (SDF), its permanent ($SDF^P$) and transitory ($SDF^T$) components, the conditional variance ratio $\omega$, the maximum risk premium without Jensen adjustment ($Max\ RP$) and the risk premium of an infinite maturity bond without Jensen adjustment ($BRP(\infty)$). The table reports the autocorrelation of each monthly variable in logs. The top (bottom) panel focuses on the nominal (real) stochastic discount factor.

The numbers are computed from a 60,000 month simulation.
### Table A.6—Inflation: Model vs Data

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th></th>
<th>Model</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Std</td>
<td>AR(1)</td>
<td>Mean</td>
</tr>
<tr>
<td>$\pi_t$</td>
<td>0.85</td>
<td>0.62</td>
<td>0.86</td>
<td>0.88</td>
</tr>
</tbody>
</table>

This table reports the mean, standard deviation and autocorrelation of the quarterly inflation rate. The left panel corresponds to actual data, from Lustig et al. (2009). The right panel corresponds to simulated data, from the model. The mean and standard deviation are in percentage.

### Table A.7—Sensitivity to Inflation Specification

<table>
<thead>
<tr>
<th></th>
<th>Max RP</th>
<th>BRP($\infty$)</th>
<th>$\omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Low</td>
<td>High</td>
<td>Low</td>
</tr>
<tr>
<td>$\mu_p$</td>
<td>30.62</td>
<td>30.62</td>
<td>18.72</td>
</tr>
<tr>
<td>$\varphi_{\pi g}$</td>
<td>30.62</td>
<td>30.62</td>
<td>18.72</td>
</tr>
<tr>
<td>$\varphi_{\pi x}$</td>
<td>30.64</td>
<td>30.60</td>
<td>18.70</td>
</tr>
<tr>
<td>$\sigma_{\pi}$</td>
<td>30.62</td>
<td>30.62</td>
<td>18.72</td>
</tr>
<tr>
<td>$\alpha_{\pi}$</td>
<td>30.62</td>
<td>30.62</td>
<td>5.61</td>
</tr>
<tr>
<td>$\alpha_x$</td>
<td>30.62</td>
<td>30.62</td>
<td>14.54</td>
</tr>
<tr>
<td>$\varphi_{z g}$</td>
<td>30.62</td>
<td>30.62</td>
<td>18.72</td>
</tr>
<tr>
<td>$\varphi_{z x}$</td>
<td>30.62</td>
<td>30.62</td>
<td>18.63</td>
</tr>
<tr>
<td>$\sigma_z$</td>
<td>30.62</td>
<td>30.62</td>
<td>18.72</td>
</tr>
</tbody>
</table>

This table reports the mean maximum risk premium ($\text{Max RP}$), the mean bond risk premium $\text{BRP}(\infty)$ (including the Jensen term) and the mean variance ratio $\omega$. We vary one parameter at a time, and simulate the model for a low and a high value of each parameter (25 percent above and below the benchmark value reported in the first column of Table A.1). The only exception is the parameter $\alpha_{\pi}$, which we cannot increase by 25 percent without running into stationarity issues. The high value is a 10 percent increase for that parameter.