Instructor: Victor F. Araman — April 27, 2005

Stochastic Simulation

Homework 5

due on May 5, 2005

Exercise 1 This problem involves the down and in (knock-in) option discussed in class. We use the notation contained in the paper "Monte Carlo Methods for Security Pricing" by Boyle, Broadie and Glasserman (BBG).
Let $C(S_0, H, K, T, \sigma^2, r)$ denote the "fair value of this option". Assume we use the risk-neutral measure to price this option. Read the paper carefully and try to answer the questions below.

1. Write $C(S_0, H, K, T, \sigma^2, r)$ as an expectation of a particular random variable and write an expression for the density associated with that expectation.

2. Assume now that importance sampling has been applied as in BBG. Repeat part 1. for this new setup. Make your expression as simple as possible, In particular, use $\mu = (2b + c)/T$ and $L_{-\mu,\mu}(X_1, X_2, ...X_m)$ for the likelihood ratio.

3. Next we are interested in estimating the value of the partial derivative of C with respect to the risk-free interest rate r. Write the partial derivative $\partial C/\partial r$ in the form of an expectation of a particular random variable. In writing $C(r)$, we suppress other variables that are held fixed. Don't bother to write out explicit expressions for any partial derivatives with respect to r.

4. How would you use simulation to estimate $\partial C(r)/\partial r$? Be sure to specify the random variables simulated and how you would form a point estimate and confidence interval.

Exercise 2

This problem involves the estimation of the derivative with respect to the arrival rate λ of the expected waiting time of job 2, W_2 in any $M/M/1$ queue. Let $\alpha(\lambda) = \mathbb{E}_\lambda W_2$. Recall that W_0 and $W_{n+1} = [W_n + X_{n+1}]^+$ for all $n \geq 0$ and where $X_k = V_{k-1} - U_k$ with $V_k \sim \exp(\mu)$ and $U_{k-1} \sim \exp(\lambda)$ for all k.

1. Let U_k^* be an exponential random variable with parameter λ_0 and U_k be an exponential random variable with parameter $\lambda = \lambda_0 + h$, $h > 0$. Show that

$$\frac{\lambda_0}{\lambda_0 + h} U_k^* \stackrel{D}{=} U_k,$$

where $\stackrel{D}{=}$ denotes equality in distribution.

2. Let $W_n(\lambda)$ denote the waiting time of job n when the arrival rate is λ. Show that the following recursions holds:

$$W'_{n+1}(\lambda_0) = \begin{cases} W'_{n+1}(\lambda_0) & , \; W_{n+1}(\lambda_0) > 0 \\ 0 & , \; W_{n+1}(\lambda_0) = 0. \end{cases} \tag{1}$$

3. The random numbers in table 1 were generated using $\lambda = 1$ and $\mu = 2$. Numbers are routed to two decimal places. From this table estimate the value of $\alpha'(1)$ using three observations of the appropriate random number. Also give an argument for whether $\alpha'(1) > 0$ or $\alpha'(1) < 0$.

i	V_{i-1}	U_i	W_i
1	0.34	0.11	0.23
2	0.38	0.02	0.58
3	0.45	1.73	0.00
4	0.05	2.52	0.00
5	0.82	0.21	0.62
6	0.67	0.90	0.38
7	0.81	2.22	0.00
8	0.17	1.07	0.00
9	0.09	1.88	0.00
10	2.04	1.06	0.98
11	0.66	0.33	1.31
12	0.03	0.02	1.32
13	0.43	2.95	0.80
14	0.12	2.01	0.00

Table 1: Random numbers generated.