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Estimating Production Uncertainty in Stochastic Frontier Production Function Models

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Abstract

One of the main purposes of the frontier literature is to estimate inefficiency. Given this objective, it is unfortunate that the issue of estimating "firm-specific" inefficiency in cross sectional context has not received much attention. To estimate firm-specific (technical) inefficiency, the standard procedure is to use the *mean* of the inefficiency term conditional on the entire composed error as suggested by Jondrow, Lovell, Materov and Schmidt (1982). This conditional mean could be viewed as the average loss of output (return). It is also quite natural to consider the conditional *variance* which could provide a measure of production uncertainty or risk. Once we have the conditional mean and variance, we can report standard errors and construct confidence intervals for firm level technical inefficiency. Moreover, we can also perform hypothesis tests. We postulate that when a firm attempts to move towards the frontier it not only increases its efficiency, but it also reduces its production uncertainty and this will lead to shorter confidence intervals. Analytical expressions for production uncertainty under different distributional assumptions are provided, and it is shown that the technical inefficiency as defined by Jondrow et al. (1982) and the production uncertainty are monotonic functions of the entire composed error term. It is very interesting to note that this monotonicity result is valid under different distributional assumptions of the inefficiency term. Furthermore, some alternative measures of production uncertainty are also proposed, and the concept of production uncertainty is generalized to the panel data models. Finally, our theoretical results are illustrated with an empirical example.

1. Introduction

A production frontier refers to the maximum output attainable by a given technology and an input bundle, while a cost frontier refers to the minimum cost to produce a given level of output. The distance by which a firm lies below its production frontier or above its cost frontier is a measure of the firm's inefficiency. For this purpose, the stochastic frontier model pioneered by Aigner, Lovell and Schmidt (1977) and Meeusen and van den Broeck (1977) has attracted a great deal of attention in the literature since its introduction. For the ith firm (or unit) the stochastic frontier model can be written as

$$y_i = f(x_i, \beta) + \epsilon_i, \quad i = 1, 2, \ldots, n, \tag{1}$$

where y_i is the output, $f(\cdot)$ is the production function, x_i is a vector of nonstochastic inputs, β is the vector of unknown parameters, and ϵ_i is the stochastic error term. They proposed that the error terms ϵ_i is composed of two components, i.e.,

$$\epsilon_i = v_i - u_i, u_i \geq 0, \tag{2}$$

where v_i and u_i are independent and unobservable components of ϵ_i. The v_i are assumed to be a two-sided error term representing the statistical noise and are assumed to be normally distributed with mean 0 and variance σ_v^2, and u_i is a one sided error term representing technical inefficiency.

Since $u_i \geq 0$, the production by each firm (or unit) is bounded above by a stochastic frontier, (SF_i),

$$SF_i = f(x_i, \beta) + v_i. \tag{3}$$

The inclusion of random error v_i in (3) indicates that SF_i is stochastic and expresses maximal output of the ith firm given vector of inputs x_i. The nonnegative component u_i allows firms to be technically inefficient relative to their own frontier, i.e.,

$$y_i = SF_i - u_i, u_i \geq 0. \tag{4}$$

Thus, as Jondrow, Lovell, Materov and Schmidt (1982, p. 234) noted, "u_i measures technical inefficiency in the sense that it measures the shortfall of output y_i from its maximal possible value given by the stochastic frontier."

The residuals $\hat{\epsilon}_i$ can be easily obtained from model (1), however, the problem of decomposing $\hat{\epsilon}_i$ into its components $\hat{v}_i$ and $\hat{u}_i$ for cross sectional data had remained unsolved for some time. Aigner et al. (1977) and Schmidt and Lovell (1979) showed that the average technical inefficiency can be estimated by the mean of the distribution of u_i. However, how to estimate the technical inefficiency for each firm was still unresolved. Jondrow et al. (1982) suggested a solution to this problem by proposing to estimate the mean or mode of the conditional distribution of u_i given ϵ_i, which can be used as a point estimate of u_i.

In this paper, we carry the idea of Jondrow et al. (1982) a step further, and hypothesize that under the "standard" framework, when a firm attempts to move towards the frontier it not only increases its technical efficiency (TE) but also reduces its production uncertainty (PU). We propose to measure the production uncertainty by the conditional variance of u_i given ϵ_i. Using the expressions for the conditional mean and variance, we construct confidence intervals for firm specific inefficiency. Moreover, using the standard errors one can also perform hypothesis tests. Furthermore, interpretations of various measures are also provided.

The paper is organized as follows. In section 2, we propose a measure of production uncertainty and the analytical expressions for various distributions of u_i are obtained. The interpretations of technical inefficiency and production uncertainty are discussed in Section 3. In section 4, an alternative measure of production uncertainty is proposed, and the results are extended to the panel data model. Construction of confidence intervals and the procedure for hypothesis tests are discussed in Section 5. Our results are illustrated with an example in Section 6. And finally, some concluding remarks are made in Section 7.

The main contributions of this paper are as follows. First, a "new" concept called production uncertainty, is introduced, and its analytical expressions are derived under different distributional assumptions on the error term. Production uncertainty is defined as the conditional variance of the inefficiency term conditional on the entire composed error. Knowing conditional mean (inefficiency) and the conditional variance, one can obtain the confidence interval, CI, for inefficiency measure and can perform hypothesis tests. The confidence intervals as obtained by this straightforward method are identical to those obtained by Horrace and Schmidt (1996). Thus, this study also provides an alternative view and derivation of confidence interval for the inefficiency term.

2. Measures of Production Uncertainty

Consider again the stochastic frontier model given by (1), i.e.,

$$y_i = f(x_i, \beta) + \epsilon_i. \tag{5}$$

In model (5), Aigner et al. (1977) assumed that v_i is distributed as normal with mean zero and variance σ_v^2, and u_i is distributed as half normal, $u_i \sim| N(0, \sigma_u^2)\}$, or exponential. Both of these distributions of u have a mode at $u = 0$. Stevenson (1980) considered that u is assumed to be distributed as a truncated normal with mode μ, and v is assumed to be distributed as normal with mean zero and variance σ_v^2.

We assume that $v_i \sim (N, \sigma_v^2)$ can consider the following three cases for u_i.

Case I: $u_i \sim |N(0, \sigma_u^2)|$, *(Aigner et al., 1977).*

Here, the probability density function $(p \cdot d \cdot f)$ of u_i is

$$k(u_i) = \frac{2}{\sqrt{2\pi}} \frac{1}{\sigma_u} \exp\left\{-\frac{u_i^2}{2\sigma_u^2}\right\}, u_i > 0, \tag{6}$$

Case II: $u_i \sim |N(\mu, \sigma_u^2)|$, *(Stevenson, 1980).*

For this case, the $p \cdot d \cdot f$ of u_i is

$$k(u_i) = \frac{1}{\{1 - \Phi(-\mu/\sigma_u)\}} \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma_u} \exp\left\{-\frac{(u_i - \mu)^2}{2\sigma_u^2}\right\}, u_i > 0, \tag{7}$$

where $\Phi(\cdot)$ is the distribution function of the standard normal distribution.

Case III: u_i*'s are exponentially distributed, (Aigner et al., 1977).*

Here the $p \cdot d \cdot f$ is

$$k(u_i) = \frac{1}{\sigma_u} \exp\left\{-\frac{u_i}{\sigma_u}\right\}, u_i \geq 0. \tag{8}$$

Jondrow et al. (1982) obtained the expressions for $E(u_i \mid \epsilon_i)$, i.e., the expressions for technical inefficiency (TIE) in cases I and III and Greene (1990) reported $E(u_i \mid \epsilon_i)$ for the Stevenson's case. Given that $E(u_i \mid \epsilon_i)$ is now accepted as a relevant indicator for technical inefficiency, we propose to measure the production uncertainty (PU) due to inefficiency by the conditional variance, $Var(u_i \mid \epsilon_i)$. Of course, there could be production uncertainty due to other factors beside technical inefficiency.

2.1. *Technical Inefficiency and Production Uncertainty*

Case I: $u_i \sim |N(0, \sigma_u^2)|$. For this case, the conditional distribution of u_i given ϵ_i is truncated normal with mean μ_{i^*} and variance σ_*^2, i.e., the $p{\cdot}d{\cdot}f$ is given by

$$f(u_i \mid \epsilon_i) = \frac{1}{\{1 - \Phi(r_i)\}} \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma_*} \exp\left\{-\frac{(u_i - \mu_{i^*})^2}{2\sigma_*^2}\right\}, \qquad u_i \geq 0, \tag{9}$$

where

$$\mu_{i^*} = \frac{-\epsilon_i \sigma_u^2}{\sigma^2}, \qquad \sigma_*^2 = \frac{\sigma_u^2 \sigma_v^2}{\sigma^2},$$
$$\sigma^2 = \sigma_v^2 + \sigma_u^2, \quad r_i = -\frac{\mu_{i^*}}{\sigma_*} = \frac{\epsilon_i \lambda}{\sigma}, \quad \text{and } \lambda = \frac{\sigma_u}{\sigma_v}.$$

From (9) one can obtain,

$$E(u_i \mid \epsilon_i) = \mu_{i^*} + \sigma_* h(r_i), \tag{10}$$

and

$$Var(u_i \mid \epsilon_i) = \sigma_*^2 \left\{1 + r_i h(r_i) - h^2(r_i)\right\}, \tag{11}$$

where

$$h(z) = \frac{\Phi(z)}{1 - \Phi(z)}$$

is the hazard (or failure) rate for a standard normal random variable whose $p{\cdot}d{\cdot}f$ is denoted by $\phi(\cdot)$.

Case II: $u_i \sim |N(\mu, \sigma_u^2)|$. Stevenson (1980) considered the cost-minimization problem, where he considered $\epsilon_i = v_i + u_i$. For this case the expressions for $f(u_i \mid \epsilon_i)$, $E(u_i \mid \epsilon_i)$, and $Var(u_i \mid \epsilon_i)$ remain the same as in case I, but now

$$\mu_{i^*} = \frac{\mu \sigma_v^2 - \epsilon_i \sigma_u^2}{\sigma^2}, \qquad \sigma_*^2 = \frac{\sigma_u^2 \sigma_v^2}{\sigma^2},$$
$$\sigma^2 = \sigma_v^2 + \sigma_{u,}^2 \qquad r_i = -\frac{\mu_{i^*}}{\sigma_*} = \frac{\mu}{\sigma\lambda} + \frac{\epsilon_i \lambda}{\sigma}, \qquad \text{and } \lambda = \frac{\sigma_u}{\sigma_v}.$$

Case III: u_i's are exponential. For this case, again, the expressions for $f(u_i \mid \epsilon_i)$, $E(u_i \mid \epsilon_i)$, and $Var(u_i \mid \epsilon_i)$ remain the same as in case I. However, μ_{i^*} and σ_* are now defined as

$$\mu_{i^*} = -\left(\epsilon_i + \frac{\sigma_v^2}{\sigma_u}\right), \qquad \sigma_* = \sigma_v,$$

and

$$r_i = -\frac{\mu_{i^*}}{\sigma_*} = \left(\frac{\epsilon_i}{\sigma_v} + \frac{\sigma_v}{\sigma_u}\right).$$

Finally, we introduce another measure, called the Coefficient of Production Uncertainty (CPU), which is defined as

$$CPU = \frac{\sqrt{Var(u_i \mid \epsilon_i)}}{1 - E(u_i \mid \epsilon_i)}. \tag{12}$$

CPU measures production uncertainty per unit of efficiency. This is similar to the definition of the coefficient of variation. It is unit free and ranges from 0 to ∞.

3. Interpretation and Monotonicity of Inefficiency and Uncertainty Measures

3.1. Interpretation of $E(u_i \mid \epsilon_i)$

After estimating model (5), we can only recover (an estimate of) ϵ_i, and this can be viewed as a "sufficient statistic" for u_i. From this point of view, $E(u_i \mid \epsilon_i)$ is a Rao-Blackwellization step [see, for example, Lehmann (1983, p. 50, Theorem 6.4)] to estimate firm-specific inefficiency given the composite error term. Since, $E[E(u_i \mid \epsilon_i)] = E(u_i)$, and $Var[E(u_i \mid \epsilon_i)] \leq Var(u_i)$, it seems, in using $E(u_i \mid \epsilon_i)$ as an estimator for u_i, we would be doing better than even using the actual values of u_i. Of course, in practice we do not attain the above properties as we replace ϵ_i and other parameters by their estimates.

From equation (10), we have

$$E(u_i \mid \epsilon_i) = \sigma_*(h(r_i) - r_i).$$

Many of the properties of this inefficiency and of the uncertainty measure, as we will see later, can be derived by analyzing the hazard function $h(r_i)$. Properties of $h(r_i)$ have been studied extensively in the statistics literature [for example, Sampford (1953) and Barrow and Cohen (1954)].

Using the standard definition of hazard function, we can write [see, for example, Lancaster (1990, p. 7)]

$$h(r_i) = \lim_{dr_i \to 0} \frac{P_r(r_i \leq R \leq r_i + dr_i \mid R \geq r_i)}{dr_i},$$

where R is a standard normal random variable. Roughly speaking, $h(r_i)dr_i$ gives the probability that R will not exceed r_i too far after it has reached $r_i = \frac{\epsilon_i \lambda}{\sigma}$. As expected, for

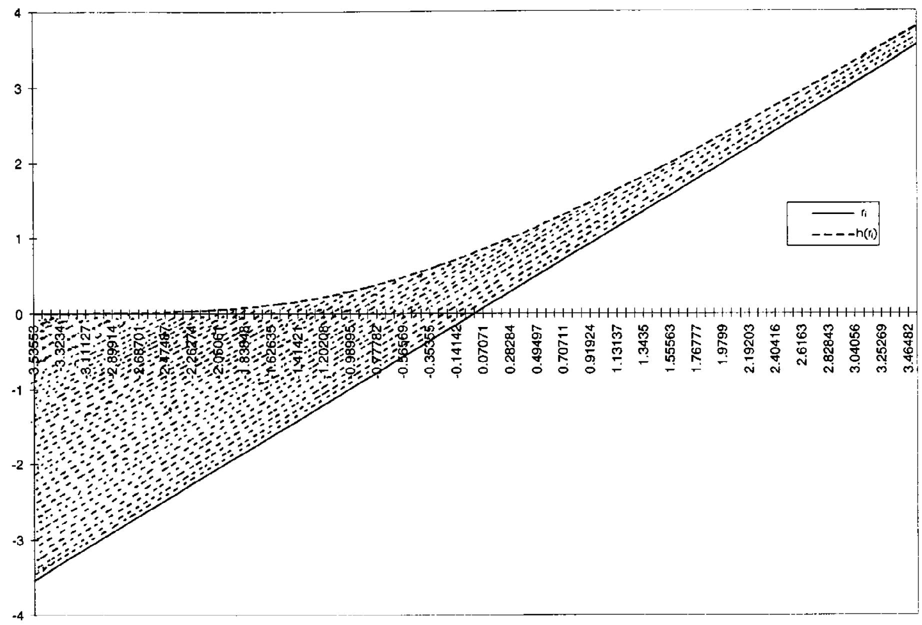

Figure 1. Plot of $h(r_i)$ against r_i

fixed λ and σ, when ϵ_i is small (large), $h(r_i)$ is also small (large). Since here (for production function) $\epsilon_i = v_i - u_i$, smaller values of ϵ_i (keeping v_i fixed) means more inefficiency. Therefore, $E(u_i \mid \epsilon_i) = \sigma_*(h(r_i) - r_i)$ is the relative value of $h(r_i)$ to $r_i = \frac{\epsilon_i \lambda}{\sigma}$. The lower is this value, the lower is the inefficiency. Note that the function $h(r_i)$ has the following properties:

i) $h(r_i) \geq r_i, r_i \in (-\infty, \infty)$,

ii) $h(r_i) \to 0$, as $r_i \to -\infty$,

iii) $h(r_i) \to r_i$, as $r_i \to \infty$.

iv) $h(r_i) \leq 1$, when $r_i \leq 0$, and

v) $h(r_i) \leq 1$, when $r_i \geq 0$.

A much clearer picture is obtained by plotting $h(r_i)$ against r_i, for $-\infty < r_i < \infty$. This can be done independent of any data or model. Plot of $h(r_i)$ against r_i is presented in Figure 1. In Figure 1, the shaded region $(h(r_i) - r_i)$ is essentially the inefficiency measure, $E(u_i \mid \epsilon_i)$. It is clear that

$$E(u_i \mid \epsilon_i) \to \infty \text{ as } \epsilon_i \text{ or } r_i \to -\infty, \text{ and}$$

$$E(u_i \mid \epsilon_i) \to 0 \text{ as } \epsilon_i \text{ or } r_i \to \infty.$$

3.2. Monotonicities of $E(u_i \mid \epsilon_i)$ and $Var(u_i \mid \epsilon_i)$

For Case I, Jondrow et al. (1982, p. 235) noted (without proof) that $E(u_i \mid \epsilon_i)$ is monotonic in ϵ_i. We have not seen any explicit proof of this in the literature. Therefore, we provide a simple proof below. Note that for the production function it will be monotonically decreasing, and will be monotonically increasing for the cost function.

RESULT: *For the model given in (1) and (2), $E(u_i \mid \epsilon_i)$ and $Var(u_i \mid \epsilon_i)$ are monotonically decreasing functions of ϵ_i.*

Proof. For simplicity let us consider Case I. To show that $E(u_i \mid \epsilon_i)$ is monotonically decreasicreasing in ϵ_i, we note

$$\frac{dE(u_i \mid \epsilon_i)}{d\epsilon_i} = \frac{dE(u_i \mid \epsilon_i)}{dr_i}\frac{dr_i}{d\epsilon_i}.$$

But $\frac{dr_i}{d\epsilon_i} = \frac{\lambda}{\sigma} \geq 0$. Therefore, let us only consider

$$\begin{aligned} \frac{d}{dr_i}E(u_i \mid \epsilon_i) &= \frac{d}{dr_i}\sigma_*[h(r_i) - r_i] \\ &= \sigma_*\left[\frac{dh(r_i)}{dr_i} - 1\right]. \end{aligned} \tag{13}$$

Now

$$\begin{aligned} \frac{dh(r_i)}{dr_i} &= \frac{d}{dr_i}\left[\frac{\Phi(r_i)}{1-\Phi(r_i)}\right] \\ &= \frac{(\Phi(r_i))^2 - r_i\Phi(r_i)(1-\phi(r_i))}{(1-\Phi(r_i))^2} \\ &= (h(r_i))^2 - r_i h(r_i). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{d}{dr_i}E(u_i \mid \epsilon_i) &= \sigma_*\left[\{h(r_i)\}^2 - r_i h(r_i) - 1\right] \\ &= -\frac{1}{\sigma_*}Var(u_i \mid \epsilon_i). \end{aligned} \tag{14}$$

Thus,

$$\frac{d}{dr_i}E(u_i \mid \epsilon_i) < 0, \text{ and, hence, } \frac{d}{d\epsilon_i}E(u_i \mid \epsilon_i) < 0.$$

To prove that $Var(u_i \mid \epsilon_i)$ is monotonically decreasing in ϵ_i, it is sufficient to show that

$\frac{d}{dr_i} Var(u_i \mid \epsilon_i) \leq 0$. From equation (11)

$$\begin{aligned}\frac{d}{dr_i} Var(u_i \mid \epsilon_i) &= \sigma_*^2 \left[h(r_i) + r_i \frac{dh(r_i)}{dr_i} - 2h(r_i) \frac{dh(r_i)}{dr_i} \right] \\ &= \sigma_*^2 \left[h(r_i) + r_i h(r_i)(h(r_i) - r_i) - 2(h(r_i))^2 (h(r_i) - r_i) \right] \\ &= \sigma_*^2 h(r_i) \left[\left\{ 1 + r_i h(r_i) - h^2(r_i) \right\} - (h(r_i) - r_i)^2 \right] \\ &= h(r_i) \left[Var(u_i \mid \epsilon_i) - \{E(u_i \mid \epsilon)\}^2 \right]. \end{aligned} \tag{15}$$

From Barrow and Cohen (1954, p. 405), equation (2), it follows that

$$\left[Var(u_i \mid \epsilon_i) - \{E(u_i \mid \epsilon_i)\}^2 \right] < 0. \tag{16}$$

Thus, for a production function,

$$\frac{d}{dr_i} Var(u_i \mid \epsilon_i) < 0. \qquad \blacksquare$$

Since both conditional mean and variance decrease monotonically with ϵ_i, the most efficient firm will have the least production uncertainty. This is what we should expect since when a firm is moving close to the frontier it can allow only for a limited variation in its production. However, note that from (14), $-\frac{d}{dr_i} E(u_i \mid \epsilon_i) = \frac{1}{\sigma_*} Var(u_i \mid \epsilon_i)$, which means the rate at which it can decrease its efficiency will be proportional to the production uncertainty. In other words, at a higher level of uncertainty, there is an opportunity for larger improvement.

As a by-product, equation (14) provides some further interesting results. We have

$$\begin{aligned}\frac{d}{dr_i} Var(u_i \mid \epsilon_i) &= \frac{d}{dr_i} \left[-\sigma^* \frac{d}{dr_i} E(u_i \mid \epsilon_i) \right] \\ &= -\sigma^* \frac{d^2}{dr_i^2} [E(u_i \mid \epsilon_i] \\ &= -\sigma^* \frac{d^2}{dr_i^2} \left[\sigma^* \{h(r_i) - r_i\} \right] \\ &= \sigma^{*^2} \frac{d}{dr_i} \left[\frac{dh(r_i)}{dr_i} - 1 \right] \\ &= -\sigma^{*^2} \frac{d^2}{dr_i^2} h(r_i). \end{aligned} \tag{17}$$

Since the left hand side is ≤ 0, we get a new result, that $\frac{d^2 h(r_i)}{dr_i^2} \geq 0$, i.e., "the normal" hazard function increases at a nondecreasing rate. This also shows that the rate at which a firm can decrease its production uncertainty is proportional to the curvature of the hazard function, i.e., the rate of the rate of decrease in inefficiency. The above result also holds for Cases II and III. This follows obviously, since r_i is a function of ϵ_i and $\frac{dr_i}{d\epsilon_i} = \lambda/\sigma$ in cases I and II, and $\frac{dr_i}{d\epsilon_i} = \frac{1}{\sigma_v}$ in case III. Note that this has a larger implication since the above results and interpretations are "free" of distributional assumptions about u_i.

4. Further Extensions and Panel Data Models

4.1. Further Extensions

Battese and Coelli (1988) argued that since the production function is generally defined for the logarithm of the production, the technical efficiency for the ith firm should be defined as $E[\exp(-u_i) \mid \epsilon_i]$. They also extended the Jondrow et al. (1982) results to the case of cross sectional and time series model under the assumption that the firm effects are non-negative, time invariant and follow a truncated normal distribution. Following Battese and Coelli (1988) definition of technical efficiency we define the production uncertainty by the conditional variance, $Var[\exp(-u_i) \mid \epsilon_i]$. For the Cases I, II and III considered in section 2, expressions for these measures are

$$E[\exp(-u_i) \mid \epsilon_i)] = \frac{1-\Phi(\sigma_* + r_i)}{1-\Phi(r_i)} \bar{e}^{\mu_{i*}+1/2\sigma_*^2}, \tag{18}$$

and

$$Var[\exp(-u_i) \mid \epsilon_i] = \frac{\bar{e}^{2\mu_{i*}+\sigma_*^2}}{1-\Phi(r_i)} \left[\{1-\Phi(2\sigma_* + r_i)\} e^{\sigma_*^2} - \frac{\{1-\Phi(\sigma_* + r_i)\}^2}{1-\Phi(r_i)} \right], \tag{19}$$

where μ_{i*}, σ_*^2 and r_i are defined in section 2 for each case. It is interesting to note that $E[\exp(-u_i) \mid \epsilon_i]$ is monotonic whereas $Var[\exp(-u_i) \mid \epsilon_i]$ is not monotonic in ϵ_i.

4.2. Panel Data Model

For the cross sectional time series data, the stochastic frontier model given in (1) can be written as

$$y_{it} = f(x_{it}\beta) + \epsilon_{it}, \tag{20}$$

where $\epsilon_{it} = v_{it} - u_i$, y_{it} is the output for the ith firm ($i = 1, 2, \ldots, M$) at time t, ($t = 1, 2, \ldots, T$), x_{it} is the nonstochastic vector of inputs andβ is the vector of coefficients corresponding to the inputs. The random variables v_{it} are assumed to be independent and identically distributed (iid) as $N(0, \sigma_v^2)$ and u_i's are non negative, iid random variables following a truncated distribution. Furthermore, it is assumed that the v_{it} are independent of u_i, and v_{it} and u_i are also independent of the input variables in the model.

For model (20) when $u_i \sim |N(0, \sigma_u^2)|$, Battese and Coelli (1998) derived $E(u_i \mid \epsilon_i)$ and $E[\exp(-u_i) \mid \epsilon_i]$ where ϵ_i is now a vector $(\epsilon_{i1}, \epsilon_{i2} \ldots \epsilon_{iT})'$. Since $f(u_i \mid \epsilon_{i1}, \epsilon_{i2}, \ldots, \epsilon_{iT}) = f(u_i \mid \epsilon_{i0})$ where $\epsilon_{i0} = \sum_{t=1}^{T} \epsilon_{it}/T$, the expressions for $E(u_i \mid \epsilon_i)$, $Var(u_i \mid \epsilon_i)$, $E[\exp(-u_i) \mid \epsilon_i]$, and $Var[\exp(-u_i) \mid \epsilon_i]$ remain the same as given in section 2 and 4.1. However, now for case I, when $u_i \sim |N(0, \sigma_u^2)|$,

$$\mu_{i^*} = -\frac{T\epsilon_{i0}\sigma_u^2}{\sigma_v^2 + T\sigma_u^2}, \ \sigma_*^2 = \frac{\sigma_u^2\sigma_v^2}{\sigma_v^2 + T\sigma_u^2}, \quad \text{and} \quad r_i = -\frac{\mu_{i*}}{\sigma_*}. \tag{21-a}$$

For case II, when $u_i \sim |N(\mu, \sigma_u^2)|$,

$$\mu_{i*} = \frac{\mu\sigma_v^2 - T\epsilon_{i0}\sigma_u^2}{\sigma_v^2 + T\sigma_u^2}, \quad \sigma_*^2 = \frac{\sigma_u^2\sigma_v^2}{\sigma_v^2 + T\sigma_u^2}, \quad \text{and } r_i = -\frac{\mu_{i*}}{\sigma_*}. \tag{21-b}$$

And finally, for case III, when the u_i are exponential

$$\mu_{i*} = -\left(\epsilon_{i0} + \frac{\sigma_v^2}{T\sigma_u^2}\right), \quad \sigma_* = \sigma_v \quad \text{and } r_i = -\frac{\mu_{i*}}{\sigma_*}. \tag{21-c}$$

Since the expressions are essentially the same as before, for the panel data model the earlier monotonicity and other results will also be valid.

5. Construction of Confidence Intervals

Once we have the conditional mean and variances, $E(u_i \mid \epsilon_i)$, and $Var(u_i \mid \epsilon_i)$, we can easily construct confidence intervals (CI) for $u_i \mid \epsilon_i$. Let us denote $E(u_i \mid \epsilon_i) = \bar{\mu}_i$, $\sqrt{Var(u_i \mid \epsilon_i)} = \bar{\sigma}_i$, and $w_i = \frac{u_i - \bar{\mu}_i}{\bar{\sigma}_i}$. Therefore, the range of w_i is $-\frac{\bar{\mu}_i}{\bar{\sigma}_i} \leq w_i < \infty$. Then, $(1-\alpha)100\%$ confidence interval for the inefficiency, $u_i \mid \epsilon_i$, is given by

$$\bar{\mu}_i + c_\ell\bar{\sigma}_i \leq u_i \mid \epsilon_i \leq \bar{\mu}_i + c_u\bar{\sigma}_i, \tag{22}$$

where c_ℓ and c_u are such that

$$\int_{-\frac{\bar{\mu}_i}{\bar{\sigma}_i}}^{c_\ell} f(w_i)dw_i = \int_{c_u}^{\infty} f(w_i)dw_i = \frac{\alpha}{2}, \tag{23}$$

and $f(w_i)$ is the $p{\cdot}d{\cdot}f$ of w_i. It is clear that $E(w_i) = 0$, $Var(w_i) = 1$, and given ϵ_i, the $p{\cdot}d{\cdot}f$ can be derived as

$$f(w_i) = \frac{1}{1 - \Phi\left(-\frac{\mu_{i*}}{\sigma_*}\right)} \frac{1}{\sqrt{2\pi}\left(\frac{\sigma_*}{\bar{\sigma}_i}\right)} \exp\left\{-\frac{\left(w_i - \left(\frac{\mu_{i*} - \bar{\mu}_i}{\bar{\sigma}_i}\right)\right)^2}{2(\sigma_*/\bar{\sigma}_i)^2}\right\}. \tag{24}$$

Using $f(w_i)$, one can find that

$$c_\ell = \frac{\mu_{i*} - \bar{\mu}_i}{\bar{\sigma}_i} + \Phi^{-1}\left[\Phi\left(\frac{-\mu_{i*}}{\sigma_*}\right) + \frac{\alpha}{2}\left\{1 - \Phi\left(\frac{-\mu_{i*}}{\sigma_*}\right)\right\}\right]\frac{\sigma_*}{\bar{\sigma}_i} \tag{25}$$

and

$$c_u = \frac{\mu_{i*} - \bar{\mu}_i}{\bar{\sigma}_i} + \Phi^{-1}\left[1 - \frac{\alpha}{2}\left\{1 - \Phi\left(\frac{-\mu_{i*}}{\sigma_*}\right)\right\}\right]\frac{\sigma_*}{\bar{\sigma}_i}. \tag{26}$$

Using (25) and (26), the lower confidence bound (LCB) and the upper confidence bound

(UCB) of (22) can be simplified as

$$\begin{aligned} LCB &= \bar{\mu}_i + c_\ell \bar{\sigma}_i \\ &= \mu_{i*} + \Phi^{-1}\left[1 - \left(1 - \frac{\alpha}{2}\right)\left\{1 - \Phi\left(\frac{-\mu_{i*}}{\sigma_*}\right)\right\}\right]\sigma_* \\ &= \mu_{i*} + \Phi^{-1}\left[\frac{\alpha}{2} + \left(1 - \frac{\alpha}{2}\right)\Phi\left(\frac{-\mu_{i*}}{\sigma_*}\right)\right]\sigma_*, \end{aligned} \tag{27}$$

and

$$\begin{aligned} UCB &= \bar{\mu}_i + c_u \bar{\sigma}_i \\ &= \mu_{i*} + \Phi^{-1}\left[1 - \frac{\alpha}{2}\left\{1 - \Phi\left(\frac{-\mu_{i*}}{\sigma_*}\right)\right\}\right]\sigma_*. \end{aligned} \tag{28}$$

In a recent paper, Horrace and Schmidt (1996) suggested a method of constructing confidence intervals for estimates of technical efficiency. Hjalmarsson et al. (1996) used those intervals for their data set from the 15 Colombian cement plants. Horrace and Schmidt (1996) results could be adapted for $u_i \mid \epsilon_i$, and their CI can be stated as

$$\mu_{i*} + z_\ell \sigma_* \le u_i \mid \epsilon_i \le \mu_{i*} + z_u \sigma_*, \tag{29}$$

where

$$z_\ell = \Phi^{-1}\left[\frac{\alpha}{2} + \left(1 - \frac{\alpha}{2}\right)\Phi\left(\frac{-\mu_{i*}}{\sigma_*}\right)\right]$$

and

$$z_u = \Phi^{-1}\left[1 - \frac{\alpha}{2}\left\{1 - \Phi\left(\frac{-\mu_{i*}}{\sigma_*}\right)\right\}\right].$$

As expected, (27) and (28) are exactly the same as the lower and upper bounds of the CI given in (29). Since our method and that of Horrace and Schmidt (1996) give the same CI, it does not matter which formula one uses. Their intervals are based on the sample mean and variance from the underlying $N(\mu_{i*}, \sigma_*^2)$ random variable. However, since we want to find the confidence interval for $u_i \mid \epsilon_i$, our formulation (22), using $E(u_i \mid \epsilon_i)$ and $\sqrt{Var(u_i \mid \epsilon_i)}$ seems to be more natural. Empirical researchers can now report "standard errors" for firm level technical (in)efficiency estimates. Moreover, they can also perform hypothesis tests. One way to carry out tests for the significance for the ith firm level inefficiency would be to use $\frac{\triangle}{\mu_i}/\frac{\triangle}{\sigma_i}$, and compare it with the appropriate critical values c_ℓ and c_u as defined in (23). For an one-sided test, $H_0 : \bar{\mu}_i = 0$ against $H_a : \bar{\mu}_i > 0$, only the upper critical value, c_u, defined as

$$\int_{c_u}^{\infty} f(w_i)dw_i = \alpha$$

should be used. To have a closer look at c_ℓ and c_u in (23), let us consider

$$\begin{aligned} c_l &= \frac{\mu_{i*} - \bar{\mu}_i}{\bar{\sigma}_i} + \Phi^{-1}\left[\Phi\left(\frac{-\mu_{i*}}{\sigma_*}\right) + \frac{\alpha}{2}\left\{1 - \Phi\left(\frac{-\mu_{i*}}{\sigma_*}\right)\right\}\right]\frac{\sigma_*}{\bar{\sigma}_i} \\ &= -\frac{\sigma_*}{\bar{\sigma}_i}h(r_i) + \Phi^{-1}\left[\Phi(r_i) + \frac{\alpha}{2}\{1 - \Phi(r_i)\}\right]\frac{\sigma_*}{\bar{\sigma}_i}, \end{aligned} \tag{30}$$

and similarly,

$$c_u = -\frac{\sigma_*}{\bar{\sigma}_i}h(r_i) + \Phi^{-1}\left[1 - \frac{\alpha}{2}\{1 - \Phi(r_i)\}\right]\frac{\sigma_*}{\bar{\sigma}_i}. \tag{31}$$

Unlike in the standard CI cases, here c_ℓ and c_u depend on "i", in particular on $\bar{\sigma}_i$. We note that both c_u and c_ℓ are proportional to $\frac{1}{\bar{\sigma}_i}$, as expected. However, the final confidence bounds do not depend on $\bar{\sigma}_i$ as in (27) and (28).

Following the above approach, we can also obtain the confidence interval for $\exp(-u_i) \mid \epsilon_i$ by using $E[\exp(-u_i) \mid \epsilon_i]$ and $Var[\exp(-u_i) \mid \epsilon_i]$. For simplicity, again let us denote $E[\exp(-u_i) \mid \epsilon_i] = \underline{\mu_i}$, $\sqrt{Var[\exp(-u_i) \mid \epsilon_i]} = \underline{\sigma_i}$, and $w_i = \frac{\exp(-u_i)|\epsilon_i - \underline{\mu_i}}{\underline{\sigma_i}}$. It is clear that $\frac{-\underline{\mu_i}}{\underline{\sigma_i}} \leq w_i \leq \frac{1-\underline{\mu_i}}{\underline{\sigma_i}}$.

Thus, one can obtain the result that the $(1-\alpha)100\%$ confidence interval for $\exp(-u_i) \mid \epsilon_i$ is

$$\underline{\mu_i} + c'_\ell\underline{\sigma_i} \leq \exp(-u_i) \mid \epsilon_i \leq \underline{\mu_i} + c'_u\underline{\sigma_i}, \tag{32}$$

where, now c'_ℓ and c'_u are such that

$$\int_{\frac{-\underline{\mu_i}}{\underline{\sigma_i}}}^{c'_\ell} f(w_i)dw_i = \int_{c'_u}^{\frac{1-\underline{\mu_i}}{\underline{\sigma_i}}} f(w_i)dw_i = \alpha/2, \tag{33}$$

and

$$\begin{aligned} f(w_i) &= \frac{1}{1 - \Phi\left(\frac{-\mu_{i*}}{\sigma_*}\right)}\frac{1}{\sqrt{2\pi}\sigma_*}\left(\frac{\underline{\sigma_i}}{\underline{\mu_i} + \underline{\sigma_i}w_i}\right) \\ &\quad \times \exp\left\{-\frac{-\log(\underline{\mu_i} + \underline{\sigma_i}w_i) - \mu_{i_*}}{2\sigma_*^2}\right\}, \\ &\quad -\frac{\underline{\mu_i}}{\underline{\sigma_i}} \leq w_i \leq \frac{1 - \underline{\mu_i}}{\underline{\sigma_i}}. \end{aligned} \tag{34}$$

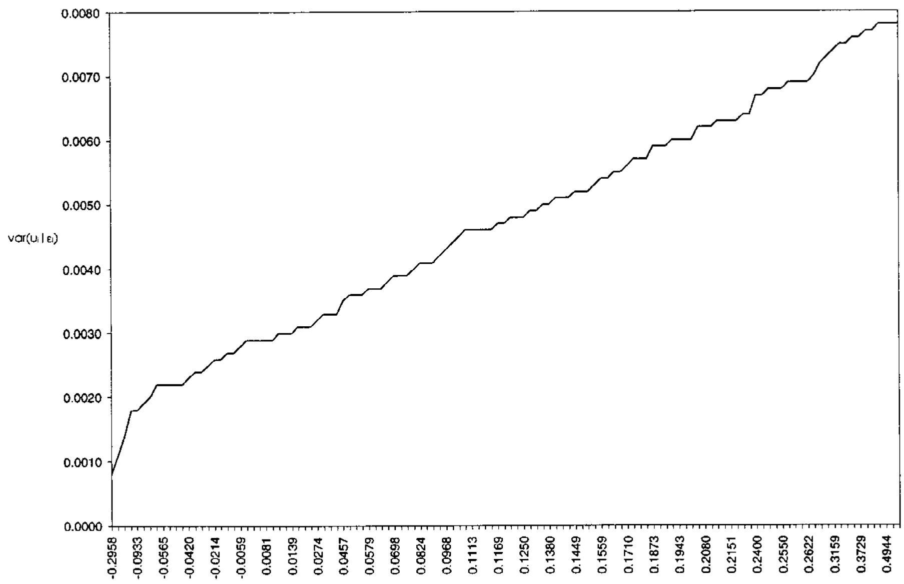

Figure 2. Plot of $Var(u_i \mid E_i)$ against E_i

From the above $p \cdot d \cdot f$, one can obtain c'_ℓ and c'_u, as

$$c'_\ell = \frac{\exp\{-(\mu_{i_*} + z_u \sigma_*)\} - \underline{\underline{\mu_i}}}{\underline{\sigma_i}}, \tag{35}$$

$$z_u = \Phi^{-1}\left[1 - \frac{\alpha}{2}\left\{1 - \Phi\left(\frac{-\mu_{i_*}}{\sigma_*}\right)\right\}\right],$$

and

$$c'_u = \frac{\exp\{-(\mu_{i_*} + z_\ell \sigma_*)\} - \underline{\underline{\mu_i}}}{\underline{\sigma_i}} \tag{36}$$

$$z_\ell = \Phi^{-1}\left[\frac{\alpha}{2} + \left(1 - \frac{\alpha}{2}\right)\Phi\left(\frac{-\mu_{i_*}}{\sigma_*}\right)\right].$$

From (32), after simplification the lower confidence bound (LCB) and the upper confidence

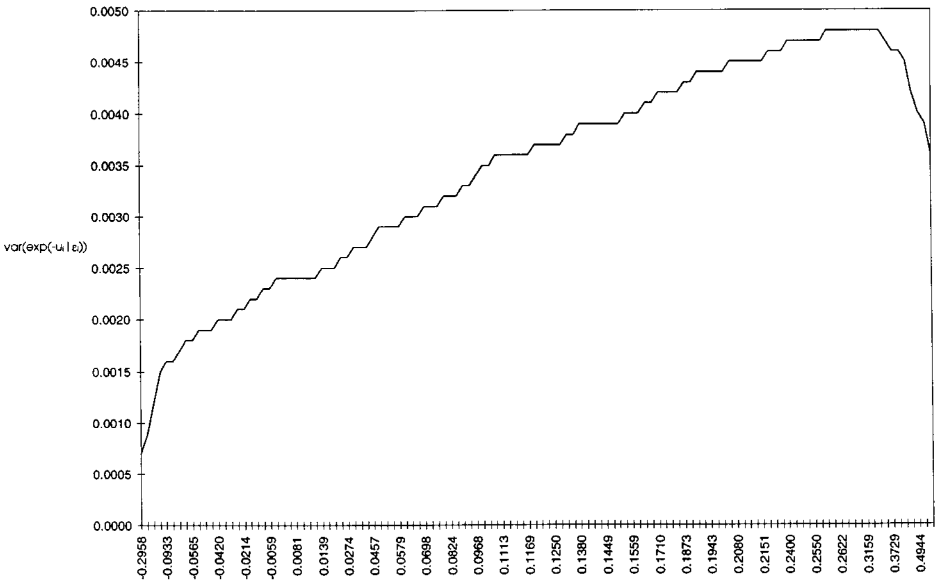

Figure 3. Plot of $Var[\exp(-u_i) \mid \epsilon_i]$ against ϵ_i

bound (UCB) are

$$
\begin{aligned}
LCB &= \underline{\mu_i} + c'_\ell + \underline{\sigma_i} \\
&= \exp\{-(\mu_{i_*} + z_u \sigma_*\} \\
&= \exp\left\{-\left(\mu_{i_*} + \Phi^{-1}\left(1 - \frac{\alpha}{2}\left(1 - \Phi\left(\frac{-\mu_{i_*}}{\sigma_*}\right)\right)\right)\sigma_*\right)\right\} \\
&= \exp\{-\,UCB \text{ of } u_i \mid \epsilon_i \text{ in equation (28)}\}
\end{aligned}
\tag{37}
$$

and

$$
\begin{aligned}
UCB &= \underline{\mu_i} + c'_u \underline{\sigma_i} \\
&= \exp\{-(\mu_{i_*} + z_\ell \sigma_*)\} \\
&= \exp\left\{-\left(\mu_{i_*} + \Phi^{-1}\left\{\frac{\alpha}{2} + \left(1 - \frac{\alpha}{2}\right)\Phi\left(\frac{-\mu_{i_*}}{\sigma_*}\right)\right\}\sigma_*\right)\right\} \\
&= \exp\{-\,LCB \text{ of } u_i \mid \epsilon_i \text{ in equation (27)}\}.
\end{aligned}
\tag{38}
$$

The lower and upper bounds in (37) and (38) are the same as Horrace and Schmidt (1996, eq. (5)). In fact, (37) and (38) directly follow from (27) and (28) due to the monotonicity of $\exp(-u_i)$ as a function of u_i.

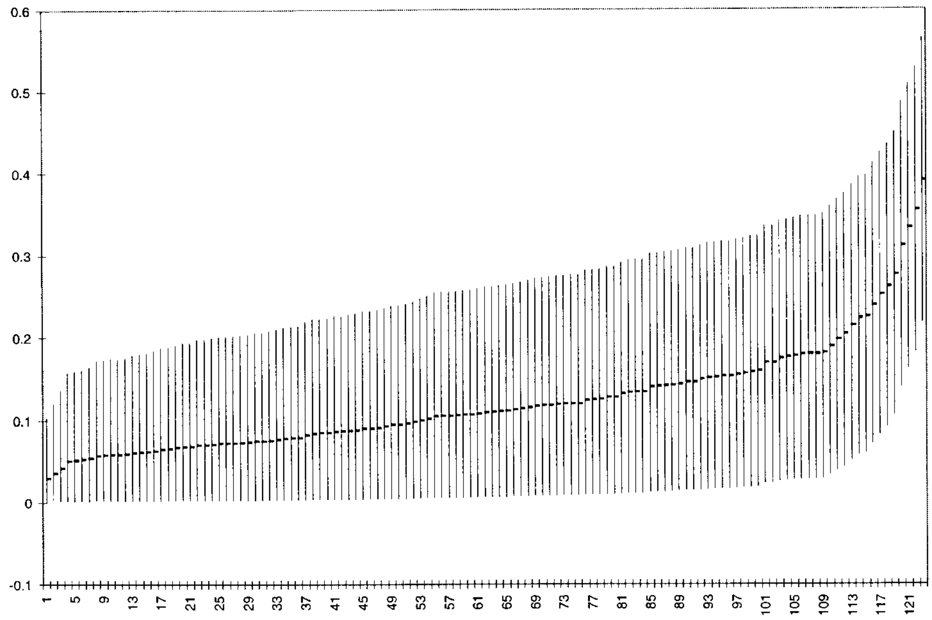

Figure 4. Confidence interval for technical inefficiency: u_i/ϵ_i

6. Empirical Illustration

To illustrate our hypothesis that when a firm attempts to move towards its frontier it not only increases its technical efficiency (TE) but also reduces its production uncertainty (PU), we use the data set of the U.S. electric utility industry, first used by Christensen and Greene (1976) and later by Greene (1990). This data set consists of 123 firms. We use the model and data set given in Greene (1990, p. 154 and appendix). Consider the restricted specification of the cost function

$$\begin{aligned}\ln(cost/P_f)_i &= \beta_0 + \beta_1 \ln Q_i + \beta_2(\ln Q_i)^2 \\ &\quad + \beta_3 \ln(P_l/P_f)_i + \beta_4 \ln(P_k/P_f)_i + \epsilon_i, \quad i = 1, 2, \ldots, n, \qquad (39)\end{aligned}$$

where Q is the output that is a function of labor (l), capital (k), and fuel (f), and P_l, P_k and P_f denote the factor prices of labor, capital, and fuel and n is the number of firms. Since (39) is a cost function, $\epsilon_i = v_i + u_i$, $u_i \geq 0$, in contrast to our earlier definition in (2).

Using estimates of β's, σ's and λ's from Table 1 of Greene (1990, p. 150), first we obtained $\hat{\epsilon}_i$ for each case, (i.e., for cases I, II and III). Then, we estimated the technical inefficiency (TIE), $E(u_i \mid \epsilon_i)$, and the production uncertainty (PU), $Var(u_i \mid \epsilon_i)$; and the technical efficiency (TE), $E[\exp(-u_i) \mid \epsilon_i]$ and the corresponding production uncertainty, i.e., $Var[\exp(-u_i) \mid \epsilon_i]$. The results are reported in Tables 1 and 2. To save space we report

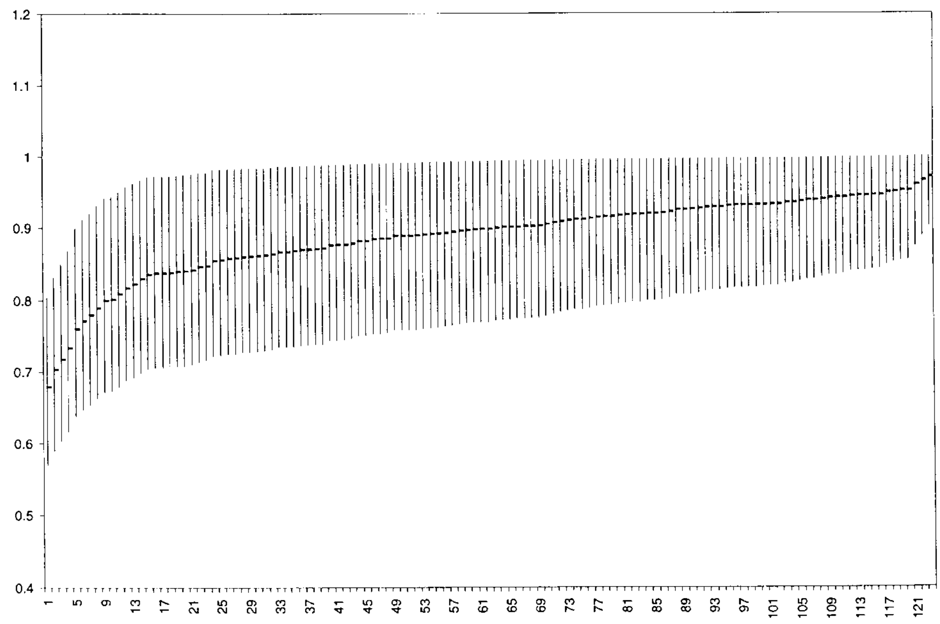

Figure 5. Confidence interval for technical efficiency: $\exp(-u_i/\epsilon_i)$

only the results for Case I. The results for Cases II and III are similar. We observe that irrespective of the distribution of u_i, New Mexico Electric Services (No. 91 in the data set) is the most efficient, Montana Power (No. 2 in the data set) is the second most efficient, and Maine Public Service (No. 8) is the least efficient.

Production uncertainties corresponding to two different definitions are plotted in Figures 2 and 3. From Figure 2, we observe that $Var(u_i \mid \epsilon_i)$ is a monotonic function of ϵ_i, but Figure 3 reveals that $Var[\exp(-u_i) \mid \epsilon_i]$ is not monotonic. From Table 1, we notice that the most efficient firm, No. 91 ($TIE = 0.0303$) has the least production uncertainty, $PU = 0.0008$ and the least efficient firm, No. 8, having technical inefficiency, $TIE = 0.3916$ has the highest production uncertainty, $PU = 0.0078$ along with three other firms. By using the definition of technical efficiency given by Battese and Coelli (1988), again, we note that the most efficient firm, No. 91, with TE of 97.05% has the least production uncertainty 0.0007. However, the least efficient firm in this case, again No. 8, with TE of 67.86% does not have the highest production uncertainty. This is due to the non-monotonicity of $Var[\exp(-u_i) \mid \epsilon_i]$. It is interesting to note that for the first fifteen efficient firms, the magnitudes of PU are almost the same irrespective of the definition of production uncertainty used. However, for the fifteen least efficient firms there are significant differences in the production uncertainty defined by $Var(u_i \mid \epsilon_i)$ and $Var[\exp(-u_i) \mid \epsilon_i]$. Theoretically, if we extend the Figure 3 for higher values of ϵ_i, it can be seen that $Var[\exp(-u_i) \mid \epsilon_i]$ will further decrease monotonically. Therefore, according to this definition, production

Table 1. Technical inefficiency, production uncertainty and the 95% confidence bounds for technical inefficiency: Case I.

Firm No.	EP	TIE	PU	LCB	UCB
91	−0.2958	0.0303	0.0008	0.0009	0.1032
2	−0.2200	0.0363	0.0011	0.0011	0.1201
116	−0.1609	0.0424	0.0014	0.0013	0.1363
5	−0.0970	0.0511	0.0018	0.0017	0.1574
11	−0.0933	0.0517	0.0018	0.0017	0.1588
86	−0.0871	0.0527	0.0019	0.0018	0.1611
6	−0.0783	0.0542	0.0020	0.0018	0.1644
66	−0.0588	0.0577	0.0022	0.0020	0.1720
48	−0.0565	0.0581	0.0022	0.0020	0.1730
79	−0.0535	0.0587	0.0022	0.0021	0.1742
7	−0.0527	0.0588	0.0022	0.0021	0.1745
22	−0.0508	0.0592	0.0022	0.0021	0.1753
123	−0.0420	0.0610	0.0023	0.0022	0.1790
47	−0.0387	0.0616	0.0024	0.0022	0.1804
50	−0.0361	0.0622	0.0024	0.0023	0.1815
88	−0.0288	0.0637	0.0025	0.0023	0.1846
92	−0.0214	0.0653	0.0026	0.0024	0.1879
113	−0.0192	0.0658	0.0026	0.0025	0.1889
96	−0.0137	0.0671	0.0027	0.0026	0.1914
82	−0.0086	0.0683	0.0027	0.0026	0.1937
20	−0.0059	0.0689	0.0028	0.0027	0.1949
52	0.0019	0.0708	0.0029	0.0028	0.1986
119	0.0021	0.0709	0.0029	0.0028	0.1987
58	0.0052	0.0717	0.0029	0.0029	0.2001
63	0.0081	0.0724	0.0029	0.0029	0.2015
87	0.0084	0.0725	0.0029	0.0029	0.2017
122	0.0103	0.0729	0.0030	0.0029	0.2026
13	0.0109	0.0731	0.0030	0.0030	0.2029
27	0.0139	0.0739	0.0030	0.0030	0.2043
75	0.0189	0.0752	0.0031	0.0031	0.2068
95	0.0190	0.0753	0.0031	0.0031	0.2068
23	0.0223	0.0761	0.0031	0.0032	0.2084
77	0.0274	0.0775	0.0032	0.0033	0.2109
55	0.0321	0.0789	0.0033	0.0034	0.2133
112	0.0333	0.0792	0.0033	0.0034	0.2139
32	0.0337	0.0793	0.0033	0.0034	0.2141
59	0.0457	0.0829	0.0035	0.0037	0.2202
19	0.0518	0.0847	0.0036	0.0039	0.2234
80	0.0527	0.0850	0.0036	0.0039	0.2239
12	0.0539	0.0854	0.0036	0.0039	0.2246
41	0.0579	0.0867	0.0037	0.0041	0.2267
70	0.0588	0.0869	0.0037	0.0041	0.2271
34	0.0624	0.0881	0.0037	0.0042	0.2290
31	0.0646	0.0888	0.0038	0.0043	0.2302
73	0.0698	0.0906	0.0039	0.0044	0.2330
49	0.0705	0.0908	0.0039	0.0045	0.2334
3	0.0723	0.0914	0.0039	0.0045	0.2344
57	0.0774	0.0932	0.0040	0.0047	0.2372
46	0.0824	0.0949	0.0041	0.0049	0.2400
101	0.0826	0.0950	0.0041	0.0049	0.2401
25	0.0866	0.0964	0.0041	0.0051	0.2423

Table 1. Continued.

Firm No.	EP	TIE	PU	LCB	UCB
121	0.0914	0.0982	0.0042	0.0053	0.2450
69	0.0968	0.1002	0.0043	0.0055	0.2480
26	0.1028	0.1025	0.0044	0.0058	0.2515
67	0.1099	0.1052	0.0045	0.0061	0.2556
54	0.1107	0.1055	0.0046	0.0062	0.2560
51	0.1113	0.1058	0.0046	0.0062	0.2564
37	0.1125	0.1063	0.0046	0.0063	0.2571
62	0.1137	0.1067	0.0046	0.0063	0.2578
118	0.1144	0.1070	0.0046	0.0064	0.2582
99	0.1169	0.1080	0.0047	0.0065	0.2596
104	0.1215	0.1099	0.0047	0.0068	0.2623
94	0.1219	0.1101	0.0048	0.0068	0.2626
83	0.1241	0.1110	0.0048	0.0070	0.2639
4	0.1250	0.1114	0.0048	0.0070	0.2644
64	0.1281	0.1127	0.0049	0.0072	0.2663
56	0.1320	0.1144	0.0049	0.0075	0.2686
43	0.1349	0.1156	0.0050	0.0077	0.2704
30	0.1380	0.1169	0.0050	0.0079	0.2722
111	0.1404	0.1180	0.0051	0.0081	0.2737
21	0.1411	0.1183	0.0051	0.0082	0.2741
107	0.1441	0.1197	0.0051	0.0084	0.2759
40	0.1449	0.1200	0.0052	0.0085	0.2764
115	0.1456	0.1203	0.0052	0.0085	0.2768
65	0.1458	0.1204	0.0052	0.0085	0.2769
106	0.1543	0.1243	0.0053	0.0093	0.2821
28	0.1559	0.1250	0.0054	0.0094	0.2831
36	0.1567	0.1254	0.0054	0.0095	0.2836
38	0.1617	0.1278	0.0055	0.0100	0.2867
78	0.1621	0.1280	0.0055	0.0100	0.2869
117	0.1710	0.1323	0.0056	0.0110	0.2925
120	0.1746	0.1340	0.0057	0.0114	0.2947
105	0.1759	0.1347	0.0057	0.0115	0.2955
84	0.1764	0.1349	0.0057	0.0116	0.2958
53	0.1873	0.1404	0.0059	0.0130	0.3027
72	0.1885	0.1410	0.0059	0.0131	0.3034
102	0.1902	0.1419	0.0059	0.0134	0.3046
97	0.1915	0.1426	0.0060	0.0136	0.3054
45	0.1943	0.1440	0.0060	0.0140	0.3072
42	0.1973	0.1456	0.0060	0.0144	0.3091
71	0.1974	0.1457	0.0060	0.0145	0.3091
103	0.2045	0.1494	0.0062	0.0156	0.3136
74	0.2080	0.1513	0.0062	0.0162	0.3159
17	0.2094	0.1521	0.0062	0.0164	0.3168
60	0.2109	0.1529	0.0063	0.0167	0.3178
109	0.2115	0.1532	0.0063	0.0168	0.3182
93	0.2151	0.1552	0.0063	0.0175	0.3205
44	0.2170	0.1562	0.0063	0.0179	0.3217
10	0.2212	0.1586	0.0064	0.0187	0.3245
68	0.2232	0.1597	0.0064	0.0191	0.3258
98	0.2401	0.1693	0.0067	0.0231	0.3368
110	0.2400	0.1693	0.0067	0.0231	0.3367
108	0.2501	0.1751	0.0068	0.0258	0.3433

Table 1. Continued.

Firm No.	EP	TIE	PU	LCB	UCB
81	0.2529	0.1768	0.0068	0.0266	0.3452
89	0.2550	0.1781	0.0068	0.0272	0.3466
35	0.2581	0.1799	0.0069	0.0282	0.3487
14	0.2599	0.1810	0.0069	0.0287	0.3498
18	0.2599	0.1810	0.0069	0.0287	0.3498
39	0.2622	0.1824	0.0069	0.0295	0.3514
61	0.2749	0.1901	0.0070	0.0338	0.3598
114	0.2886	0.1986	0.0072	0.0390	0.3689
15	0.2986	0.2049	0.0073	0.0432	0.3756
90	0.3159	0.2159	0.0074	0.0512	0.3872
33	0.3299	0.2250	0.0075	0.0583	0.3966
24	0.3329	0.2269	0.0075	0.0598	0.3986
85	0.3538	0.2406	0.0076	0.0714	0.4127
29	0.3729	0.2532	0.0076	0.0827	0.4256
76	0.3881	0.2634	0.0077	0.0921	0.4359
100	0.4098	0.2779	0.0077	0.1060	0.4506
1	0.4621	0.3132	0.0078	0.1405	0.4860
16	0.4944	0.3350	0.0078	0.1622	0.5078
9	0.5253	0.3559	0.0078	0.1831	0.5288
8	0.5779	0.3916	0.0078	0.2187	0.5644

EP: $\hat{\epsilon}_i$; TIE: Technical Inefficiency $= E(u_i \mid \epsilon_i)$; PU: Production Uncertainty $= Var(u_i \mid \epsilon_i)$; LCB: 95% lower confidence bound; UCB: 95% upper confidence bound.

uncertainties are smaller for the most and least efficient firms. As explained earlier, when a firm operates at its most efficient level, we can expect least uncertainty, and this is true for either definition of production uncertainty. When a firm is least efficient, perhaps the relative production is at such a low level that there is a little scope for variation in output. It is at the middle level of efficiency (which we can call the experimental stage) where firms can be expected to have greater production uncertainty, i.e., a higher variation in output. From this point of view, the non-monotonicity of $Var[\exp(-u_i) \mid \epsilon_i]$ does not seem surprising. Also, since the cost function (39) is in logarithm form, Battese and Coelli (1988) definition of technical efficiency $E[\exp(-u_i) \mid \epsilon_i]$, and hence the conditional variance, $Var[\exp(-u_i) \mid \epsilon_i]$ are more appropriate for our case. We are, however, unable to explain the monotonicity differences between $Var(u_i \mid \epsilon_i)$ and $Var[\exp(-u_i) \mid \epsilon_i]$ for higher values of ϵ_i. A possible explanation is that $(1 - u_i)$ is only the linear approximation part of $\exp(-u_i)$. This issue requires further investigation.

Next, using expressions (22) and (32), we obtained the confidence intervals for technical inefficiency, $u_i \mid \epsilon_i$, and for the technical efficiency, $\exp(-u_i) \mid \epsilon_i$. These confidence intervals are plotted in Figures 4 and 5, and the lower and upper confidence bounds are also reported in Tables 1 and 2. We also observe that in accordance with our hypothesis, the confidence interval is smallest for the most efficient firm. For example, for the Jondrow et al. (1982) definition of technical inefficiency, the most efficient firm, No. 91 gives the smallest confidence interval and the most inefficient firm, No. 8 gives the widest confidence interval. For firm No. 91, the lower and upper bounds are 0.0009 and 0.1032, giving

Table 2. Technical efficiency, production uncertainty and the 95% confidence bounds for technical efficiency: Case I.

Firm No.	EP	TE	PU	LCB	UCB
8	0.5779	0.6786	0.0036	0.5687	0.8036
9	0.5253	0.7032	0.0039	0.5893	0.8327
16	0.4944	0.7181	0.0040	0.6018	0.8503
1	0.4621	0.7340	0.0042	0.6151	0.8689
100	0.4098	0.7603	0.0045	0.6373	0.8994
76	0.3881	0.7714	0.0046	0.6467	0.9120
29	0.3729	0.7793	0.0046	0.6534	0.9206
85	0.3538	0.7892	0.0047	0.6619	0.9311
24	0.3329	0.8000	0.0048	0.6713	0.9419
33	0.3299	0.8015	0.0048	0.6726	0.9434
90	0.3159	0.8088	0.0048	0.6790	0.9501
15	0.2986	0.8177	0.0048	0.6869	0.9577
114	0.2886	0.8228	0.0048	0.6915	0.9617
61	0.2749	0.8298	0.0048	0.6978	0.9668
39	0.2622	0.8361	0.0048	0.7037	0.9710
14	0.2599	0.8373	0.0048	0.7048	0.9717
18	0.2599	0.8373	0.0048	0.7048	0.9717
35	0.2581	0.8382	0.0047	0.7056	0.9722
89	0.2550	0.8397	0.0047	0.7071	0.9732
81	0.2529	0.8408	0.0047	0.7081	0.9738
108	0.2501	0.8422	0.0047	0.7094	0.9746
98	0.2401	0.8470	0.0047	0.7140	0.9772
110	0.2400	0.8471	0.0047	0.7141	0.9772
68	0.2232	0.8551	0.0046	0.7220	0.9810
10	0.2212	0.8561	0.0046	0.7229	0.9814
44	0.2170	0.8581	0.0046	0.7249	0.9823
93	0.2151	0.8590	0.0045	0.7258	0.9826
109	0.2115	0.8606	0.0045	0.7275	0.9833
60	0.2109	0.8609	0.0045	0.7278	0.9834
17	0.2094	0.8616	0.0045	0.7285	0.9837
74	0.2080	0.8622	0.0045	0.7291	0.9839
103	0.2045	0.8638	0.0045	0.7308	0.9845
71	0.1974	0.8670	0.0044	0.7341	0.9857
42	0.1973	0.8671	0.0044	0.7341	0.9857
45	0.1943	0.8684	0.0044	0.7355	0.9861
97	0.1915	0.8697	0.0044	0.7369	0.9865
102	0.1902	0.8702	0.0044	0.7374	0.9867
72	0.1885	0.8710	0.0043	0.7383	0.9870
53	0.1873	0.8715	0.0043	0.7388	0.9871
84	0.1764	0.8762	0.0042	0.7439	0.9885
105	0.1759	0.8765	0.0042	0.7441	0.9885
120	0.1746	0.8770	0.0042	0.7448	0.9887
117	0.1710	0.8785	0.0042	0.7464	0.9891
78	0.1621	0.8823	0.0041	0.7506	0.9900
38	0.1617	0.8824	0.0041	0.7508	0.9901
36	0.1567	0.8845	0.0040	0.7531	0.9906
28	0.1559	0.8848	0.0040	0.7535	0.9906
106	0.1543	0.8854	0.0040	0.7542	0.9908
65	0.1458	0.8888	0.0039	0.7581	0.9915
115	0.1456	0.8889	0.0039	0.7582	0.9915
40	0.1449	0.8892	0.0039	0.7585	0.9916

Table 2. Continued.

Firm No.	EP	TE	PU	LCB	UCB
107	0.1441	0.8895	0.0039	0.7589	0.9916
21	0.1411	0.8906	0.0039	0.7603	0.9919
111	0.1404	0.8909	0.0039	0.7606	0.9919
30	0.1380	0.8918	0.0039	0.7617	0.9921
43	0.1349	0.8930	0.0038	0.7631	0.9923
56	0.1320	0.8941	0.0038	0.7644	0.9925
64	0.1281	0.8956	0.0037	0.7662	0.9928
4	0.1250	0.8967	0.0037	0.7676	0.9930
83	0.1241	0.8970	0.0037	0.7680	0.9931
94	0.1219	0.8978	0.0037	0.7690	0.9932
104	0.1215	0.8980	0.0037	0.7692	0.9932
99	0.1169	0.8997	0.0036	0.7713	0.9935
118	0.1144	0.9006	0.0036	0.7725	0.9936
62	0.1137	0.9008	0.0036	0.7728	0.9937
37	0.1125	0.9012	0.0036	0.7733	0.9937
51	0.1113	0.9016	0.0036	0.7738	0.9938
54	0.1107	0.9019	0.0036	0.7741	0.9938
67	0.1099	0.9021	0.0035	0.7745	0.9939
26	0.1028	0.9046	0.0035	0.7776	0.9942
69	0.0968	0.9066	0.0034	0.7803	0.9945
121	0.0914	0.9084	0.0033	0.7827	0.9948
25	0.0866	0.9099	0.0033	0.7848	0.9950
46	0.0824	0.9112	0.0032	0.7866	0.9951
101	0.0826	0.9112	0.0032	0.7866	0.9951
57	0.0774	0.9128	0.0032	0.7888	0.9953
3	0.0723	0.9144	0.0031	0.7911	0.9955
49	0.0705	0.9149	0.0031	0.7918	0.9956
73	0.0698	0.9151	0.0031	0.7921	0.9956
31	0.0646	0.9167	0.0030	0.7944	0.9957
34	0.0624	0.9173	0.0030	0.7953	0.9958
70	0.0588	0.9184	0.0030	0.7968	0.9959
41	0.0579	0.9186	0.0029	0.7972	0.9959
12	0.0539	0.9198	0.0029	0.7989	0.9961
80	0.0527	0.9201	0.0029	0.7994	0.9961
19	0.0518	0.9204	0.0029	0.7998	0.9961
59	0.0457	0.9220	0.0028	0.8023	0.9963
32	0.0337	0.9252	0.0027	0.8072	0.9966
112	0.0333	0.9253	0.0027	0.8074	0.9966
55	0.0321	0.9256	0.0027	0.8079	0.9966
77	0.0274	0.9268	0.0026	0.8098	0.9967
23	0.0223	0.9281	0.0026	0.8119	0.9968
75	0.0189	0.9289	0.0025	0.8132	0.9969
95	0.0190	0.9289	0.0025	0.8132	0.9969
27	0.0139	0.9301	0.0025	0.8152	0.9970
13	0.0109	0.9308	0.0024	0.8164	0.9970
122	0.0103	0.9310	0.0024	0.8166	0.9971
87	0.0084	0.9314	0.0024	0.8174	0.9971
63	0.0081	0.9315	0.0024	0.8175	0.9971
58	0.0052	0.9322	0.0024	0.8186	0.9972
52	0.0019	0.9329	0.0024	0.8199	0.9972
119	0.0021	0.9329	0.0024	0.8198	0.9972
20	−0.0059	0.9347	0.0023	0.8229	0.9973

Table 2. Continued.

Firm No.	EP	TE	PU	LCB	UCB
82	−0.0086	0.9352	0.0023	0.8239	0.9974
96	−0.0137	0.9363	0.0022	0.8258	0.9975
113	−0.0192	0.9375	0.0022	0.8279	0.9975
92	−0.0214	0.9379	0.0021	0.8287	0.9976
88	−0.0288	0.9394	0.0021	0.8314	0.9977
50	−0.0361	0.9408	0.0020	0.8340	0.9977
47	−0.0387	0.9413	0.0020	0.8350	0.9978
123	−0.0420	0.9419	0.0020	0.8361	0.9978
22	−0.0508	0.9435	0.0019	0.8392	0.9979
7	−0.0527	0.9439	0.0019	0.8398	0.9979
79	−0.0535	0.9440	0.0019	0.8401	0.9979
48	−0.0565	0.9445	0.0018	0.8411	0.9980
66	−0.0588	0.9449	0.0018	0.8419	0.9980
6	−0.0783	0.9481	0.0017	0.8484	0.9982
86	−0.0871	0.9495	0.0016	0.8512	0.9982
11	−0.0933	0.9504	0.0016	0.8532	0.9983
5	−0.0970	0.9510	0.0015	0.8543	0.9983
116	−0.1609	0.9591	0.0012	0.8726	0.9987
2	−0.2200	0.9648	0.0009	0.8868	0.9989
91	−0.2958	0.9705	0.0007	0.9019	0.9991

EP: $\hat{\epsilon}_i$; TE: Technical Efficiency $= E(\exp(-u_i \mid \epsilon_i)$; PU: Production Uncertainty $= Var(\exp(-u_i) \mid \epsilon_i)$; LCB: 95% lower confidence bound; UCB: 95% upper confidence bound.

the width of the confidence interval 0.1023. For firm No. 8, the lower and upper bounds are 0.2187 and 0.5644, which gives for this interval a width of 0.3457. By using the definition of technical efficiency, $E[\exp(-u_i) \mid \epsilon_i]$ again the most efficient firm, No. 91, has the smallest confidence interval, i.e., $CI = (0.9019, 0.9991)$, which gives the confidence width of 0.0972. However, on the other end, the least efficient firm, No. 8, does not have the largest confidence interval, but rather firm No. 90, with a confidence width equal to 0.2711.

7. Conclusion

In this paper, we have introduced the new concept of "production uncertainty," defined as $Var(u_i \mid \epsilon_i)$. We have shown that when a firm moves towards its frontier it not only increases its technical efficiency but also reduces its production uncertainty. Jondrow et al. (1982) noted that the technical inefficiency, $E(u_i \mid \epsilon_i)$ is a monotonic function of ϵ_i. We have proved that both the technical inefficiency and production uncertainty are monotonic functions of ϵ_i. Thus, the ranking of the firms in terms of technical inefficiency and production uncertainty will be the same as those that can be obtained from the estimated values of ϵ_i. Also, we have shown that the results are also valid for different distributional assumptions of u_i. The most interesting result is that when a firm reaches its most efficient level it also has the least production uncertainty. Production uncertainty is also defined as $Var[\exp(-u_i) \mid \epsilon_i]$, corresponding to the technical efficiency, $E[\exp(-u_i) \mid \epsilon_i]$ introduced by Battese and Coelli (1988). Furthermore, we have also extended our results to the panel data models. Using $E(u_i \mid \epsilon_i)$, $Var(u_i \mid \epsilon_i)$, $E[\exp(-u_i) \mid \epsilon_i]$, and $Var[\exp(-u_i) \mid \epsilon_i]$ we have derived expressions for the confidence interval of $u_i \mid \epsilon_i$ and $\exp(-u_i) \mid \epsilon_i$. As expected, the most efficient firms yield the shortest confidence interval. However, for the least efficient firms, the results using two definitions of production uncertainty are different. We have illustrated our concepts and theoretical results using the U.S. Electric Utility industry data set used earlier by Greene (1990). As an extension of our work, it is possible to find the higher order conditional moments of u_i or $\exp(-u_i)$ given ϵ_i and obtain conditional skewness and kurtosis measures. These might shed further light on the behavior of firm-specific (in)efficiency measures.

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