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# ON THE ESTIMATION OF TECHNICAL INEFFICIENCY IN THE STOCHASTIC FRONTIER PRODUCTION FUNCTION MODEL*

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The error term in the stochastic frontier model is of the form $(v-u)$, where $v$ is a normal error term representing pure randomness, and $u$ is a non-negative error term representing technical inefficiency. The entire $(v-u)$ is easily estimated for each observation, but a previously unsolved problem is how to separate it into its two components, $v$ and $u$. This paper suggests a solution to this problem, by considering the expected value of $u$, conditional on $(v-u)$. An explicit formula is given for the half-normal and exponential cases.

## 1. Introduction

Consider a production function $y_i = g(x_i, \beta) + \varepsilon_i$ $(i = 1, 2, \ldots, N)$, where $y_i =$ output for observation $i$, $x_i =$ vector of inputs for observation $i$, $\beta =$ vector of parameters, $\varepsilon_i =$ error term for observation $i$. The 'stochastic frontier' (also called 'composed error') model, introduced by Aigner, Lovell and Schmidt (1977) and Meeusen and van den Broeck (1977), postulates that the error term $\varepsilon_i$ is made up of two independent components,

$$\varepsilon_i = v_i - u_i, \tag{1}$$

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where $v_i \sim N(0,\sigma_v^2)$ is a two-sided error term representing the usual statistical noise found in any relationship, and $u_i \geqq 0$ is a one-sided error term representing technical inefficiency. Note that $u_i$ measures technical inefficiency in the sense that it measures the shortfall of output $(y_i)$ from its maximal possible value given by the stochastic frontier $[g(x_i,\beta)+v_i]$.

When a model of this form is estimated, one readily obtains residuals $\hat{\varepsilon}_i = y_i - g(x_i,\hat{\beta})$, which can be regarded as estimates of the error terms $\varepsilon_i$. However the problem of decomposing these estimates into separate estimates of the components $v_i$ and $u_i$ has remained unsolved for some time. Of course, the *average* technical inefficiency — the mean of the distribution of the $u_i$ — is easily calculated. For example, in the half-normal case [$u_i$ distributed as the absolute value of a $N(0,\sigma_u^2)$ variable], the mean technical inefficiency is $\sigma_u\sqrt{(2/\pi)}$, and this can be evaluated given one's estimate of $\sigma_u$, as in Aigner, Lovell and Schmidt (1977) or Schmidt and Lovell (1979). Or average technical inefficiency can be estimated by the average of the $\hat{\varepsilon}_i$. But it is also clearly desirable to be able to estimate the technical inefficiency $u_i$ *for each observation.* Indeed this was Farrell's (1957) original motivation for introducing production frontiers, and the ability to compare levels of efficiency across observations remains the most compelling reason for estimating frontiers.

Intuitively, this should be possible because $\varepsilon_i = v_i - u_i$ can be estimated and it obviously contains information on $u_i$. In this paper, we proceed by considering the conditional distribution of $u_i$ given $\varepsilon_i$. This distribution contains whatever information $\varepsilon_i$ yields about $u_i$. Either the mean or the mode of this distribution can be used as a point estimate of $u_i$. For the commonly assumed cases of half-normal and exponential $u_i$, these expressions are easily evaluated.

## 2. The half-normal case

We consider the two-part disturbance given in (1) above, with $v_i \sim N(0,\sigma_v^2)$ and $u_i \sim |N(0,\sigma_u^2)|$. For notational simplicity, we drop the observation subscript ($i$) in this and the following section. We define

$$\sigma^2 = \sigma_u^2 + \sigma_v^2, \qquad u_* = -\sigma_u^2\,\varepsilon/\sigma^2, \qquad \sigma_*^2 = \sigma_u^2\,\sigma_v^2/\sigma^2.$$

Then our main result (proved in the appendix) is the following:

*Theorem 1. The conditional distribution of u given ε is that of a* $N(\mu_*,\sigma_*^2)$ *variable truncated at zero.*

We can use this distribution to draw inferences about $u$. For example, confidence intervals for $u$ are easily constructed. As a point estimate of $u$, we can use either the mean or the mode of its conditional distribution. The mean is

$$E(u\mid\varepsilon)=\mu_*+\sigma_*\frac{f(-\mu_*/\sigma_*)}{1-F(-\mu_*/\sigma_*)}, \tag{2}$$

where $f$ and $F$ represent the standard normal density and cdf, respectively. We can also note that $-\mu_*/\sigma_*=\varepsilon\lambda/\sigma$, where $\lambda=\sigma_u/\sigma_v$; this is the same point at which $f$ and $F$ are evaluated in calculating the likelihood function. Thus we obtain

$$E(u\mid\varepsilon)=\sigma_*\left[\frac{f(\varepsilon\lambda/\sigma)}{1-F(\varepsilon\lambda/\sigma)}-\left(\frac{\varepsilon\lambda}{\sigma}\right)\right]. \tag{3}$$

The second point estimator for $u$, the mode of the conditional distribution, is the minimum of $\mu_*$ and zero, which we can write as

$$\begin{aligned} M(u\mid\varepsilon)&=-\varepsilon(\sigma_u^2/\sigma^2) \quad \text{if} \quad \varepsilon\leqq 0,\\ &=0 \qquad\qquad\quad\;\; \text{if} \quad \varepsilon>0. \end{aligned} \tag{4}$$

The mode $M(u\mid\varepsilon)$ can be given an appealing interpretation as a maximum likelihood estimator; it can be derived by maximizing the joint density of $u$ and $v$ with respect to $u$ and $v$, subject to the constraint that $v-u=\varepsilon$, as in Materov (1981).

Incidentally, it is easily verified that the expressions in (3) and (4) are non-negative, and monotonic in $\varepsilon$. Also, the more general truncated normal case of Stevenson (1980) yields similar results, with minor algebraic complications.

Of course, $\mu_*$ and $\sigma_*$ are unknown, and thus in using any of the above results we will have to replace $\mu_*$ and $\sigma_*$ by their estimates, say $\hat{\mu}_*$ and $\hat{\sigma}_*$. [For example, in place of $E(u\mid\varepsilon)$ we must use $\hat{E}(u\mid\varepsilon)$, the difference being evaluation at $\hat{\mu}_*$, $\hat{\sigma}_*$ in place of $\mu_*$, $\sigma_*$; and so forth.] In principle, the variability due to this sampling error should be taken into account. However, this would be very difficult to do. Furthermore, it is clear that the sampling error disappears asymptotically, and thus can be ignored for large enough samples. This is in contrast to the variability intrinsic to the conditional distribution of $u$ given $\varepsilon$, which is independent of sample size, being just a reflection of the obvious fact that $\varepsilon$ contains only imperfect information about $u$.

## 3. The exponential case

This case is identical to the half-normal case, except that now the technical inefficiency error term $u$ is assumed to follow the one-parameter exponential distribution with density $f(u)=\exp(-u/\sigma_u)/\sigma_u$. Our results are similar to those for the half-normal case. Define $A=\varepsilon/\sigma_v+\sigma_v/\sigma_u$. Then we have the following result:

*Theorem 2. The conditional distribution of $u$ given $\varepsilon$ is that of a $N(-\sigma_v A, \sigma_v^2)$ variable truncated at zero.*

The mean and mode of this distribution are

$$E(u\,|\,\varepsilon)=\sigma_v\left[\frac{f(A)}{1-F(A)}-A\right], \tag{5}$$

$$\begin{aligned} M(u\,|\,\varepsilon) &= -\varepsilon-\sigma_v^2/\sigma_u \quad \text{if} \quad \varepsilon \leqq -\sigma_v^2/\sigma_u, \\ &= 0 \qquad\qquad\quad\;\; \text{if} \quad \varepsilon > -\sigma_v^2/\sigma_u. \end{aligned} \tag{6}$$

## 4. An example

Schmidt and Lovell (1980) estimated a system consisting of a stochastic frontier production function and first-order conditions for cost minimization, based on a sample of 111 steam-electric generating plants. The estimates on which our calculations are based are those reported in the first column of table 1 of Schmidt and Lovell. In particular, note that $\hat{\sigma}_u^2=0.01445$, $\hat{\sigma}_v^2=0.00326$, and that the estimated average technical inefficiency (mean of $u$) is 0.0959, indicating about 9.6 percent technical inefficiency.

We have calculated (our estimate of) the conditional distribution of $u$ given $\varepsilon$, for each observation, based on the results of section 2 since estimation assumed half-normal $u$. We do not present results for all 111 observations, but rather point out some interesting aspects of these results.

(1) The mean of $\hat{E}(u\,|\,\varepsilon)$ is 0.0939, which is in the same ballpark as the 0.0959 reported above, and as the mean of 0.0943 of the $-\hat{\varepsilon}$. The mean of $\hat{M}(u\,|\,\varepsilon)$ is 0.0687.

(2) The most positive $\hat{\varepsilon}$ (most technically efficient observation) in the sample is 0.1589 (a modest outlier, about 2.75 standard deviations from the mean of the $\hat{\varepsilon}$). This yields $\hat{\mu}_*=-0.1296$, $\hat{\sigma}_*=0.0516$, so that the conditional distribution of $u$ given $\varepsilon$ is the extreme right tail of a normal — only about 0.006 of the area of $N(-0.1296, 0.0515^2)$ lies to the right of zero. We have $\hat{M}(u\,|\,\varepsilon)=0$, $\hat{E}(u\,|\,\varepsilon)=0.0166$; a 95% confidence interval for $u$ is [0.00046, 0.0568].

(3) Twenty observations (including the one just cited) have $\hat{M}(u \mid \varepsilon)=0$; each of these also has a fairly small value of $\hat{E}(u \mid \varepsilon)$. The most technically efficient observations can be characterized as having relatively high outputs, low capital stocks, and high levels of fuel consumption and labor usage. They also represent plants of fairly recent vintage, the mean year of plant installation being 1959. Their level of allocative inefficiency [see Schmidt and Lovell (1979)] is below average, though not strongly so.

(4) The most technically inefficient observation in the sample had $\hat{\varepsilon}=-0.4554$ (a large outlier, about 4 standard deviations from the mean). This yields $\hat{\mu}_*=0.3716$, $\hat{\sigma}_*=0.0516$, so that the conditional distribution of $u$ given $\varepsilon$ is basically an untruncated normal distribution. We have $\hat{M}(u \mid \varepsilon)=\hat{E}(u \mid \varepsilon)=0.3716$, with a 95% confidence interval for $u$ being [0.2705, 0.4727].

(5) Five observations (including the one just cited) have estimated rates of technical inefficiency in excess of 20%. These five most technically inefficient observations are more difficult to characterize than are the 20 most technically efficient observations. They have rather average outputs and (naturally) above average input usage, and they also have slightly above average levels of allocative inefficiency. They represent plants of relatively early vintage, their mean year of plant installation being 1951.

## 5. Conclusions

In this paper, we have proposed a method of separating the error term of the stochastic frontier model into its two components for each observation. This enables one to estimate the level of technically inefficiency for each observation in the sample, and largely removes what had been viewed as a considerable disadvantage of the stochastic frontier model relative to other models (so-called deterministic frontiers) for which technical inefficiency is readily measured for each observation.

## Appendix

In the half-normal case, $v \sim N(0,\sigma_v^2)$, $u$ is distributed as the absolute value of $N(0,\sigma_u^2)$, $v$ and $u$ are independent, and $\varepsilon=v-u$. We wish to find the distribution of $u$ conditional on $\varepsilon$.

The joint density of $u$ and $v$ is the product of their individual densities; since they are independent,

$$f(u,v)=\frac{1}{\pi\sigma_u\sigma_v}\exp\left[\frac{-1}{2\sigma_u^2}u^2-\frac{1}{2\sigma_v^2}v^2\right], \quad u \geqq 0. \tag{A.1}$$

Making the transformation $\varepsilon = v - u$, the joint density of $u$ and $\varepsilon$ is

$$f(u,\varepsilon)=\frac{1}{\pi\sigma_u\sigma_v}\exp\left[-\frac{1}{2\sigma_u^2}u^2-\frac{1}{2\sigma_v^2}(u^2+\varepsilon^2+2u\varepsilon)\right]. \tag{A.2}$$

The density of $\varepsilon$ is given by eq. (8) of Aigner, Lovell and Schmidt (1977),

$$f(\varepsilon)=\frac{2}{\sqrt{2\pi}\sigma}(1-F)\exp\left[-\frac{1}{2\sigma^2}\varepsilon^2\right], \tag{A.3}$$

where $\sigma^2=\sigma_u^2+\sigma_v^2, \lambda=\sigma_u/\sigma_v$, and $F$ is the standard normal cdf, evaluated at $\varepsilon\lambda/\sigma$. Therefore, the conditional density of $u$ given $\varepsilon$ is the ratio of (A.2) to (A.3), which we can write as

$$f(u\,|\,\varepsilon)=\frac{1}{\sqrt{2\pi}\sigma_*}\frac{1}{1-F}\exp\left[\frac{-1}{2\sigma_*^2}u^2-\frac{1}{\sigma_v^2}u\varepsilon-\frac{\lambda^2}{2\sigma^2}\varepsilon^2\right], \quad u\geqq 0, \tag{A.4}$$

where $\sigma_*^2=\sigma_u^2\,\sigma_v^2/\sigma^2$. With a little algebra, this simplifies to

$$f(u\,|\,\varepsilon)=\frac{1}{1-F}\frac{1}{\sqrt{2\pi}\sigma_*}\exp\left[\frac{-1}{2\sigma_*^2}(u+\sigma_u^2\,\varepsilon/\sigma^2)^2\right], \quad u\geqq 0. \tag{A.5}$$

Except for the term involving $1-F$, this looks like the density of $N(\mu_*,\sigma_*^2)$, with $\mu_*=-\sigma_u^2\,\varepsilon/\sigma^2$. Finally, note that $F$ is evluated at $\varepsilon\lambda/\sigma=-\mu_*/\sigma_*$, and thus $(1-F)$ is just the probability that a $N(\mu_*,\sigma_*^2)$ variable be positive. Thus, (A.5) is indeed the density of a $N(\mu_*,\sigma_*^2)$ variable truncated at zero.

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