

# OPTIMAL INVESTMENT IN A DEFAULTABLE BOND

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## Abstract

The present paper analyzes the optimal investment strategy in a defaultable (corporate) bond and a money market account in a continuous time model. The treatment of information on the firm's asset value is based on an approach unifying the structural model and the reduced-form model. Specifically, the asset value will be assumed to be observable only at finitely many time points before the maturity of the bond. The optimal investment process will be worked out first for a small time-horizon with a general risk-averse utility function, then a multi-period optimal strategy with logarithmic and power utility will be presented using backward induction. The optimal investment strategy is analyzed numerically for the logarithmic utility.

KEY WORDS: corporate bond, default risk, utility maximization, optimal investment

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# 1 Introduction

This paper addresses a utility maximization problem of an investor who can invest in a defaultable (corporate) bond or shortsell such bond continuously over a finite time-horizon. In order to pose this problem we need to adopt a model for the evolution of the price of the bond, the default time, and the recovery value in case of a default. There is an abundance of literature available on this subject. The first work dates back to Black and Scholes (1973) and Merton (1974). These authors proposed to price a discount corporate bond as a contingent-claim on the firm's asset value, which is often assumed to be a (jump) diffusion process. A default is triggered when the firm's asset value process first hits a pre-defined threshold. This approach, called "structural", has been extended and enriched by other researchers. For example, Black and Cox (1976) considered the default barriers as a safety covenant, Geske (1977) developed a model for pricing coupon bonds. In addition, Longstaff and Schwartz (1995) provided closed form expressions for the price of coupon bonds with a stochastic interest rate and complex capital structures; Collin-Dufresne and Goldstein (2001) proposed a model that provided stationary leverage ratios and generalized the Longstaff and Schwartz (1995) model to a multi-factor case. However, the structural approach is criticized for its dependence on the complete observability of the firm's asset value process, which is arguable from an investor's point of view. In addition it yields a predictable default time which is counterintuitive. In practice default always has a bit of a surprise effect, especially to those who heavily invested in the bond.

Another approach, called "reduced form" was developed in the 1990's. In these models the

default time has an intensity and thus is not predictable. Typically the intensity is modeled as a function of the state variables, and a default occurs when the cumulative intensity reaches a unit exponential random variable that is independent of all state variables. The reader can find several of papers on the reduced form models, including Jarrow and Turnbull (1995), Jarrow et al. (1997), Madan and Unal (1998), Lando (1998), Duffie and Singleton (1999). The reduced-form approach yields a totally inaccessible default time, and integrates very well the techniques developed for the default-free term-structure models. It also yields relatively simple pricing formulas. However, the introduction of an exogenous exponential random variable to determine the default lacks economic justification.

Despite the apparent difference between the structural and reduced-form models, researchers noticed the connection between them and tried to unify the two models by means of information reduction. An incomplete list of references consists of Duffie and Lando (2001), Collin-Dufresne, Goldstein and Helwege (2003), Cetin, Jarrow, Protter and Yildirim (2004), Guo, Jarrow and Zeng (2005a, 2005b). Jarrow and Protter (2004) comment that the difference between the two approaches is whether the information regarding the firm asset value can be observed by the market or not, and that a structural model with a predictable default time can be transformed into a reduced-form model with a default intensity by information reduction. Guo et al. (2005a) actually calculate the default intensity in various reduced information models.

The purpose of this paper is to investigate the optimal investment in a defaultable bond. While there is a sizable literature on the valuation of defaultable bonds, little work has been

done on optimal investment in such bonds. In our model an investor's portfolio consists of a bank account and a discount corporate bond. The investor observes the firm's asset value process only at discrete time points, i.e., we adopt the reduced information model. As pointed out in Guo, Jarrow and Zeng (2005a) this seems to be a realistic assumption, since investors usually observe the asset value when the firm provides its quarterly reports. The default will have an intensity process in this case, as pointed out by the same authors. The asset value process is assumed to follow a geometric Brownian motion, and the bond contract provides a constant default boundary and a constant recovery rate. Default occurs if the asset value drops to the level of the default boundary. The bond market is assumed to be frictionless with no transaction costs, and investors can trade continuously in time. The interest rate is assumed to be fixed. Meindl and Primbs (2006) consider this problem through the reduced-form approach, using receding horizon control and a simple binomial optimization technique.

In this paper we derive an optimal investment strategy to maximize the investor's expected utility of wealth at a terminal time. The optimal investment process turns out to be continuous when there is no corporate news coming in, but it has jumps at times of corporate news release.

The remainder of this paper is organized in the following way. In Section 2 we present the model of the firm's asset value process and derive the bond price. In Section 3 we define the optimization problem of an investor. In section 4 we solve the problem assuming that the terminal time for the optimization problem is small enough so that after the initial news release at time zero there is no additional news release before the terminal time. We solve

this optimization problem for general utility functions. In Section 5 we study the cases of logarithmic, power, and negative exponential utility functions. In Section 6 we discuss the numerical properties of the logarithmic utility case. In section 7 we generalize our solution to a multiple period setting allowing an arbitrary number of news releases between the initial and the terminal times. We use here backwards induction, and present closed-form solutions for the cases of logarithmic and power utility functions. Finally, we summarize and give concluding remarks in Section 8.

## 2 Model for the defaultable bond

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space and  $\{\mathcal{F}_t, t \in [0, T]\}$  be an augmented filtration, where  $T$  is the maturity time of the defaultable bond issued by a firm. All processes appearing in this paper will be adapted to the above filtration (though some will be adapted to a smaller filtration as well). Let  $\{X_t, t \leq T\}$  be the asset value process of the firm and  $\{p_t, \mathcal{F}_t, t \leq T\}$  the price process of the bond. We assume that the wealth of the firm can be observed only at times  $0 = t_0 < t_1 < \dots < t_n = T$  and at the default time  $\tau$  if such default happens before (or at) the maturity of the bond. Suppose that  $X_0 = x$  is a known constant. We model the asset value of the firm with a geometric Brownian motion

$$(2.1) \quad dX_t = \mu X_t dt + \sigma X_t dW_t, \quad t \leq T$$

where  $\{W_t, \mathcal{F}_t; t \leq T\}$  is a standard Brownian motion,  $\mu$  and  $\sigma > 0$  are known constants. Let  $l > 0$  be a constant representing the liabilities of the form. The default time  $\tau$  is defined as  $\inf\{t \geq 0 : X_t \leq l\}$ , i.e., the bond defaults when the asset value reaches the liabilities.

The risk-free interest rate  $r$  assumed to be a fixed constant.

The discounted asset value  $X_t^* = e^{-rt} X_t$  satisfies

$$dX_t^* = (\mu - r)X_t^* dt + \sigma X_t^* dW_t.$$

We define

$$(2.2) \quad d\widetilde{W}_t = dW_t + \frac{\mu - r}{\sigma} dt, \quad t \leq T$$

and note that by Girsanov's theorem (Karatzas & Shreve, (1998), Theorem 3.5.1)  $\{\widetilde{W}_t, t \leq T\}$  is a Brownian motion under the probability measure  $Q$  that is given by

$$\frac{dQ}{dP} = Z_T,$$

where

$$(2.3) \quad Z_t = \exp \left\{ -\frac{\mu - r}{\sigma} W_t - \frac{(\mu - r)^2}{2\sigma^2} t \right\}, \quad t \leq T.$$

We cast the equations for the asset value and the discounted asset value in the form

$$dX_t = rX_t dt + \sigma X_t d\widetilde{W}_t$$

$$dX_t^* = \sigma X_t^* d\widetilde{W}_t$$

and express  $X_t$  as

$$(2.4) \quad X_t = x \exp \left\{ \sigma \widetilde{W}_t + \left( r - \frac{\sigma^2}{2} \right) t \right\}.$$

The information available to the market is represented by the filtration  $\{\mathcal{G}_t, t \leq T\}$  where the sigma algebra  $\mathcal{G}_t$  at time  $t \in [t_i, t_{i+1})$  is given by

$$\mathcal{G}_t = \sigma \{X_0, X_1, \dots, X(t_i), H_u, u \leq t\},$$

with  $H_t$  being the indicator function

$$H_t = 1_{\{t \geq \tau\}}, \quad t \leq T.$$

In order to compute the arbitrage-free price for the bond we need the distribution of  $\tau$  under

$Q$ . We express  $\tau$  as

$$\begin{aligned} \tau &= \inf\{t \geq 0 : X_t \leq l\} = \inf\left\{t \geq 0 : \sigma \widetilde{W}_t + \left(r - \frac{\sigma^2}{2}\right)t \leq \log \frac{l}{x}\right\} \\ &= \inf\left\{t \geq 0 : \widetilde{W}_t + \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)t \leq \frac{1}{\sigma} \log \frac{l}{x}\right\}, \end{aligned}$$

and from Karatzas & Shreve (1998), Section 3.5c follows that

$$Q(\tau > t) = \psi(t, \tilde{r}, b(x))$$

where

$$(2.5) \quad \tilde{r} = \frac{r}{\sigma} - \frac{\sigma}{2},$$

$$(2.6) \quad b(x) = \frac{1}{\sigma} \log \frac{l}{x} < 0,$$

and the function  $\psi(\cdot, \cdot, \cdot)$  is defined as

$$(2.7) \quad \psi(t, \lambda, \kappa) = 1 - \int_0^t \frac{|\kappa|}{\sqrt{2\pi u^3}} \exp\left\{-\frac{(\kappa - \lambda u)^2}{2u}\right\} du.$$

Clearly under the probability measure  $P$  we have

$$P(\tau > t) = \psi(t, \tilde{\mu}, b(x))$$

with

$$(2.8) \quad \tilde{\mu} = \frac{\mu}{\sigma} - \frac{\sigma}{2}.$$

We end this section with the computation of the price of the defaultable bond. Suppose that the face value of the bond (the amount paid at maturity if default did not occur) is one unit, and  $F < 1$  is the predetermined recovery value paid at time  $T$  if  $\tau \leq T$ . We shall apply the general arbitrage-free pricing formula

$$e^{-rt} p_t = e^{-rT} \mathbb{E}_Q [1_{\{\tau > T\}} + F \cdot 1_{\{\tau \leq T\}} \mid \mathcal{G}_t].$$

Clearly we have

$$e^{-rt} p_t 1_{\{\tau \leq t\}} = e^{-rT} F 1_{\{\tau \leq t\}}$$

so it suffices to compute the bond price on  $\{t < \tau\}$ . Suppose that  $t \in [t_i, t_{i+1})$  for some  $i = 0, \dots, n-1$  and let

$$(2.9) \quad \tau_i = \inf \{t \geq t_i; X_t \leq l\}.$$

On the event  $\{t < \tau\}$  the discounted bond  $e^{-rt} p_t$  is given by

$$\begin{aligned} & e^{-rT} \mathbb{E}_Q [1_{\{\tau > T\}} + F 1_{\{\tau \leq T\}} \mid X_0, \dots, X(t_i), t < \tau] \\ &= e^{-rT} \mathbb{E}_Q [1_{\{\tau > T\}} + F 1_{\{\tau \leq T\}} \mid X_0, \dots, X(t_i), t < \tau, t_i < \tau] \\ &= e^{-rT} \frac{Q[\tau_i > T \mid X_0, \dots, X(t_i), t_i < \tau] + F Q[t < \tau_i \leq T \mid X_0, \dots, X(t_i), t_i < \tau]}{Q[\tau > t \mid \tau > t_i, X_0, \dots, X(t_i)]} \\ &= e^{-rT} \left[ (1 - F) \frac{\psi(T - t_i, \tilde{r}, b(X(t_i)))}{\psi(t - t_i, \tilde{r}, b(X(t_i)))} + F \right]. \end{aligned}$$

For brevity we introduce

$$(2.10) \quad \gamma(u_1, u_2, z) = (1 - F) \frac{\psi(u_2, \tilde{r}, b(z))}{\psi(u_1, \tilde{r}, b(z))} + F,$$

and summarize our result: for  $t \in [t_i, t_{i+1})$

$$(2.11) \quad e^{-rt} p_t = e^{-rT} \gamma(t - t_i, T - t_i, X(t_i)) 1_{\{\tau > t\}} + e^{-rT} F 1_{\{\tau \leq t\}}.$$

This formula holds for  $t = t_n = T$  as well. There is a jump of the price at  $\tau$  and  $p_\tau < p_{\tau-}$ . The bond price has additional jumps at times  $t_1, \dots, t_{n-1}$ . However, it is left-continuous at the maturity time  $T$ .

### 3 The optimization problem

Let  $s \leq T$  be the terminal time of the investor and  $\pi_t$  be the number of bonds held by her or him at time  $t$ . All available funds not invested in the bond will be put in a bank account continuously earning interests at rate  $r$ . The number of bonds  $\pi_t$  may be negative corresponding to shorting the bond. Borrowing at the rate  $r$  is also permissible, but we shall require that the overall wealth of the investor is almost surely non-negative at any time  $t \in [0, s]$ . Here are the rigorous details of our requirements concerning investment processes.

**DEFINITION 3.1.** A process  $\{\pi_t; t \in [0, s]\}$  is called an investment process if it is predictable with respect to the filtration  $\{\mathcal{G}_t; t \in [0, T]\}$  and  $\pi_t = \pi_\tau$  on  $\{t \geq \tau\}$ .

Predictability of the investment process implies that it is also adapted to  $\{\mathcal{G}_t; t \in [0, s]\}$ , so the investor can base her or his decisions only on the available information. On the other hand predictability prevents arbitrage at times  $t = \tau$  or  $t = t_i$  when the bond price has jumps. The assumption that  $\pi_t$  is flat on  $[\tau, T]$  (whenever  $\tau < T$ ) is only a formality. In that case, formally assuming that the investor holds the defaulted bond is identical to assuming that all the wealth is put in a bank account earning the continuously compounded risk-free

rate.

Let  $V_t$  be the wealth of the investor at time  $t$ . We postulate that the investor is self-financed, so

$$dV_t = \pi_t dp_t + (V_t - \pi_t p_t) r dt.$$

This holds on both  $\{\tau > t\}$  and  $\{\tau \leq t\}$ . Obviously, it simplifies to  $dV_t = rV_t dt$  if  $\tau \leq t$ .

We can write the discounted wealth  $V_t^* = e^{-rt} V_t$  (by Ito's formula) as

$$(3.1) \quad dV_t^* = e^{-rt} \pi_t (dp_t - r p_t dt) = \pi_t dp_t^*,$$

where  $p_t^* = e^{-rt} p_t$  is the discounted bond price. In particular, on  $\{\tau \leq t\}$  we have  $V_t = V_\tau e^{r(t-\tau)}$ .

**DEFINITION 3.2.** A function as  $U : [0, \infty) \rightarrow \mathbb{R} \cup \{-\infty\}$  will be called a utility function if it is concave and strictly increasing on  $[0, \infty)$ , twice continuously differentiable on  $(0, \infty)$ , the derivative function  $U'(\cdot)$  is strictly decreasing on  $(0, \infty)$ , and the following additional requirements hold:

- (i)  $\lim_{z \rightarrow \infty} U'(z) = 0$
- (ii) the pseudo-inverse of  $U'(\cdot)$  defined as  $I(z) = \inf\{u \geq 0 : U'(u) \leq z\}$  satisfies  $I(z) \leq (z^{-m} + 1)K_1$  for some  $m \in \mathbb{N}$ ,  $K_1 > 0$  and all  $z \geq 0$ .
- (iii) If  $U(0) = -\infty$  then  $-U(I(x)) \leq K_2(x^m + 1)$  for some  $m \in \mathbb{N}$ ,  $K_2 > 0$ , and all  $x > 0$ .
- (iv) If  $U'(0) = \lim_{z \downarrow 0} U'(z) < \infty$  then  $\lim_{\vartheta \downarrow 0} U''(\vartheta)$  exists and is finite.

The function  $I(\cdot)$  becomes the inverse function of  $U'(\cdot)$  if  $U'(0) = \infty$ . Otherwise it is strictly decreasing on  $(0, U'(0))$  and zero on  $[U'(0), \infty)$ .

The notation  $V^\pi$  will be used for the wealth process corresponding to the investment process  $\pi$  whenever we want to emphasize this correspondence. We shall drop the superscript  $\pi$  whenever it is possible without causing confusion.

**DEFINITION 3.3.** An investment process will be called admissible if the corresponding wealth process satisfies  $V_t^\pi \geq 0$ , almost surely, for all  $t \in [0, s]$ .

We shall study the following optimization problem

$$\max \{ \mathbb{E}[U(V_s^\pi)]; \pi \in \mathcal{A} \}$$

where  $\mathcal{A}$  is the class of admissible investment processes satisfying  $\mathbb{E}(U(V_s))^- < \infty$ .

The last requirement is included in order to guarantee that  $\mathbb{E}[U(V_s^\pi)]$  exists.

From (3.1) follows that for every investment process the corresponding discounted wealth process  $\{V_t^*, \mathcal{G}_t; t \leq s\}$  is a  $Q$ -local martingale (Protter (2004), Chapter III, Theorem 29). If the investment process is admissible, then it is a non-negative local martingale hence also a supermartingale under  $Q$  with respect to the filtration  $\{\mathcal{G}_t; t \leq s\}$ . Let  $V_0 = v$  be the fixed initial wealth at time zero. The supermartingale property implies that for every admissible investment process the corresponding terminal wealth satisfies the *budget constraint*

$$(3.2) \quad \mathbb{E}_Q(V_s^*) \leq v$$

## 4 Solution to the optimization problem for a small terminal time

In this section we shall assume that  $s < t_1$  (the general case will be discussed in section 7).

The random variable

$$\zeta_s = \mathbb{E}[Z_s | \mathcal{G}_s]$$

will play an important role in our analysis. Note that if  $Y$  is a  $\mathcal{G}_s$ -measurable r.v., then  $\mathbb{E}_Q(Y) = \mathbb{E}(\zeta_s Y)$ .

**THEOREM 4.1.** *Suppose that there exists an admissible investment process  $\{\hat{\pi}_t, t \leq s\}$  such that the corresponding wealth process  $\{\hat{V}_t, t \leq s\}$  satisfies  $\hat{V}_s = I(y\zeta_s)$  for some constant  $y = y(s, x, v) > 0$  and*

$$(4.1) \quad \mathbb{E}_Q(e^{-rs}\hat{V}_s) = v.$$

*Then the investment process  $\hat{\pi}$  is optimal.*

*Proof.* First we show that  $E \left[ \left( U \left( \hat{V}_s \right) \right)^- \right] < \infty$ . This is obvious if  $U(0) > -\infty$  so we assume that  $U(0) = -\infty$ , in which case by item (iii) of Definition 3.2

$$E \left[ \left( U \left( \hat{V}_s \right) \right)^- \right] = -E \left[ U \left( I \left( y\zeta_s \right) \right) 1_{\{U \left( I \left( y\zeta_s \right) \right) \leq 0\}} \right] \leq E \left[ K_2 \left( y^m \zeta_s^m + 1 \right) \right].$$

This last expression is finite since by Jensen's inequality

$$(4.2) \quad E \left[ \zeta_s^m \right] = E \left[ \left( E \left[ Z_s | \mathcal{G}_s \right] \right)^m \right] \leq E \left[ Z_s^m \right] = \exp \left\{ m(m-1) \frac{(\mu-r)^2}{2\sigma^2} s \right\} < \infty.$$

Next we are going to show that  $\hat{\pi}$  is indeed optimal. By the concavity of  $U(\cdot)$

$$U(I(c)) \geq U(a) + [I(c) - a]c \quad \text{for all } a \geq 0, c > 0$$

thus

$$U(I(y\zeta_s)) \geq U(V_s) + [I(y\zeta_s) - V_s]y\zeta_s,$$

where  $V_s$  is the terminal wealth corresponding to an arbitrary admissible investment process.

After taking expectations we get

$$\mathbb{E} \left[ U \left( \widehat{V}_s \right) \right] \geq \mathbb{E} [U(V_s)] + \mathbb{E} \left[ y\zeta_s(\widehat{V}_s - V_s) \right].$$

Both  $\widehat{V}_s$  and  $V_s$  are  $\mathcal{G}_s$ -measurable, hence by (4.1) and (3.2) we have

$$\mathbb{E}[U(\widehat{V}_s)] - \mathbb{E}[U(V_s)] \geq y\mathbb{E}[\zeta_s(\widehat{V}_s - V_s)] = y\mathbb{E}_Q(\widehat{V}_s - V_s) = ye^{rs}[v - \mathbb{E}_Q(V_s^*)] \geq 0. \quad \square$$

In the rest of the section we shall identify the investment process  $\hat{\pi}$  characterized in the above theorem.

**LEMMA 4.2.** *There exists a unique constant  $y = y(s, x, v) > 0$  such that  $\mathbb{E}_Q[e^{-rs}I(y\zeta_s)] = v$ .*

*Proof.* The function  $\xi(y) = e^{-rs}I(y\zeta_s)$  is continuous and decreasing on  $(0, \infty)$ , satisfying  $\lim_{y \rightarrow 0} \xi(y) = \infty$  and  $\lim_{y \rightarrow \infty} \xi(y) = 0$ . The Monotone Convergence Theorem implies that  $\lim_{y \rightarrow 0} \mathbb{E}_Q[e^{-rs}I(y\zeta_s)] = \infty$ . Next we show that  $\lim_{y \rightarrow \infty} \mathbb{E}_Q[e^{-rs}I(y\zeta_s)] = 0$  and the function  $y \mapsto \mathbb{E}_Q[e^{-rs}I(y\zeta_s)]$  is continuous. Both properties will follow from the Dominated Convergence Theorem once we established  $\mathbb{E}_Q[e^{-rs}I(y\zeta_s)] < \infty$  for every  $y \geq 0$ . However, by item

(ii) of Definition 3.2

$$\mathbb{E}_Q[e^{-rs}I(y\zeta_s)] \leq e^{-rs}\mathbb{E}_Q[(y^{-m}\zeta_s^{-m} + 1)K_1] = e^{-rs}K_1(y^{-m}\mathbb{E}[\zeta_s^{1-m}] + 1),$$

and (4.2) demonstrates that this expression is finite (we have to replace  $m$  in (4.2) with  $(1 - m)$ ).  $\square$

Next we are going to represent the optimal terminal wealth in the form

$$(4.3) \quad I(y\zeta_s) = \alpha \cdot 1_{\{s < \tau\}} + \beta(\tau) \cdot 1_{\{s \geq \tau\}}$$

for some deterministic function  $\beta(t) = \beta(t, s, x, v)$  and non-negative constant  $\alpha = \alpha(s, x, v)$ . In order to simplify the notation we shall write  $b$  instead of  $b(x)$ .

**LEMMA 4.3.** *In the previous expression for  $I(y\zeta_s)$ , we have*

$$(4.4) \quad \beta(t) = I\left(ye^{Kt}\left(\frac{l}{x}\right)^L\right)$$

where  $L = (r - \mu)/\sigma^2$ ,  $K = (\mu + r - \sigma^2)(\mu - r)/(2\sigma^2)$ , and

$$(4.5) \quad \alpha = I\left(y\frac{\psi(s, \tilde{r}, b)}{\psi(s, \tilde{\mu}, b)}\right).$$

The constants  $\tilde{\mu}$ ,  $\tilde{r}$ ,  $b = b(x)$  and  $\psi$  have been defined in (2.8), (2.5), (2.6), and (2.7). The constant  $y = y(s, x, v)$  is the one implicitly determined by Lemma (4.2).

*Proof.* From (2.2), (2.3), and (2.4) follows that

$$(4.6) \quad Z_s = e^{Ks}\left(\frac{X_s}{x}\right)^L.$$

$X_t$  is a Markov process, hence for all  $t \leq s$

$$\begin{aligned}
\mathbb{E}[Z_s \mid \tau = t] &= \mathbb{E} \left[ e^{Ks} \left( \frac{X_s}{x} \right)^L \mid X_u > l, \forall u < t, X_t = l \right] \\
&= \mathbb{E} \left[ e^{Ks} \left( \frac{X_s}{x} \right)^L \mid X_t = l \right] \\
&= \mathbb{E} \left[ e^{Ks} \left( \frac{X_s}{x} \right)^L \mid \mathcal{F}_t^X \right] \Big|_{X_t=l} \\
&= e^{Kt} \left( \frac{l}{x} \right)^L,
\end{aligned}$$

where the last identity follows from the fact that the process  $\left\{ e^{Kt} \left( \frac{X_t}{x} \right)^L, t \leq T \right\}$  is a martingale (see (4.6)).

If  $\tau > s$  then

$$\mathbb{E}[Z_s \mid \tau > s] = \frac{\mathbb{E}[Z_s \cdot 1_{\{\tau > s\}}]}{P(\tau > s)} = \frac{Q(\tau > s)}{P(\tau > s)} = \frac{\psi(s, \tilde{r}, b)}{\psi(s, \tilde{\mu}, b)}.$$

It follows that

$$\zeta_s = e^{K\tau} \left( \frac{l}{x} \right)^L \cdot 1_{\{\tau \leq s\}} + \frac{\psi(s, \tilde{r}, b)}{\psi(s, \tilde{\mu}, b)} \cdot 1_{\{\tau > s\}},$$

which completes the proof. □

**LEMMA 4.4.** *The constant  $y = y(s, x, v)$  is determined by*

$$e^{-rs} \left[ \alpha \psi(s, \tilde{r}, b) - \int_0^s \beta(t) \psi'(t, \tilde{r}, b) dt \right] = v,$$

where  $\psi'$  is the derivative of  $\psi$  with respect to the variable  $t$ .

*Proof.* We have

$$\begin{aligned}
v &= \mathbb{E}_Q[e^{-rs}I(y\zeta_s)] \\
&= e^{-rs}\mathbb{E}_Q[\alpha \cdot 1_{\{\tau>s\}} + \beta(\tau) \cdot 1_{\{\tau\leq s\}}] \\
&= e^{-rs}[\alpha Q(\tau > s) + \int_0^s \beta(t)Q(\tau \in dt)] \\
&= e^{-rs}[\alpha\psi(s, \tilde{r}, b) - \int_0^s \beta(t)\psi'(t, \tilde{r}, b)dt],
\end{aligned}$$

hence the lemma follows. □

We are going to search for the optimal investment process in the form

$$(4.7) \quad \hat{\pi}_t = h(t)1_{\{t\leq\tau\}} + h(\tau)1_{\{\tau<t\}}$$

for some continuous function  $h(t) = h(t, s, T, x, v)$ . In the following theorem we shall identify this function. Note that if  $U'(0) = \infty$  then  $\beta(\cdot)$  is continuously differentiable on  $(0, \infty)$  since  $U(\cdot)$  was required to be twice continuously differentiable on  $(0, \infty)$ . However, if  $U'(0) < \infty$  then there may exist a point  $u_0$  such that  $\beta(\cdot)$  is continuously differentiable on  $(0, u_0) \cup (u_0, \infty)$ , but has a breakpoint in  $u_0$ . Let  $\beta'(\cdot)$  be the derivative of  $\beta(\cdot)$  with the understanding that if  $\beta(\cdot)$  has a breakpoint in  $u_0$  then  $\beta'(u_0)$  is arbitrary.

**THEOREM 4.5.** *The optimal investment process is determined by (4.7) where  $h(t)$  is continuous on  $[0, s]$  and is given by  $h(t) = h(0) + \int_0^t h'(u)du$ ,*

$$(4.8) \quad h(0) = \frac{e^{rT}}{1-F} \cdot \frac{v - e^{-rs}\beta(0)}{\psi(T, \tilde{r}, b)} = \frac{v - e^{-rs}\beta(0)}{p_0 - e^{-rT}F},$$

$$(4.9) \quad h'(t) = -\frac{e^{r(T-s)}}{1-F} \cdot \frac{\psi(t, \tilde{r}, b)}{\psi(T, \tilde{r}, b)}\beta'(t) = -\frac{\beta'(t)}{p_t - e^{-r(T-s)}F}.$$

*Proof.* It is clear that the above defined function  $h(\cdot)$  is either included in  $C^1((0, \infty))$  or in  $C^1((0, u_0) \cup (u_0, \infty))$ , similarly to  $\beta(\cdot)$ . We are going to show that even in the second case  $h(\cdot)$  is continuous in  $u_0$ . Suppose that indeed we are in the second case, so  $U'(0) < \infty$ . We need to show that the expression defining  $h'(t)$  on the right-hand side of (4.9) has finite left and right limits in  $t = u_0$  which amounts to showing that both  $\lim_{t \uparrow u_0} \beta'(t)$  and  $\lim_{t \downarrow u_0} \beta'(t)$  are finite. By (4.4) breakpoint  $u_0$  is determined by

$$y \exp\{K u_0\} \left(\frac{l}{x}\right)^L = U'(0).$$

If the above equation leads to a nonpositive value for  $u_0$  then we are in the case of  $h(\cdot)$  being in  $C^1((0, \infty))$ , so we assume that this equation yields a positive  $u_0$ . In the following calculation we shall assume that  $K > 0$ ; the complementary case can be covered in a very similar fashion. For the left limit

$$\lim_{t \uparrow u_0} \beta'(t) = \lim_{z \uparrow U'(0)} K I'(z) z = \lim_{z \uparrow U'(0)} \frac{K z}{U''(I(z))} = \lim_{\vartheta \downarrow 0} \frac{K U'(\vartheta)}{U''(\vartheta)},$$

and this is finite by the last requirement of Definition 3.2. For the right limit

$$\lim_{t \downarrow u_0} \beta'(t) = \lim_{z \downarrow U'(0)} K I'(z) z = 0,$$

so we established the continuity of  $h(\cdot)$  on  $[0, s)$ .

In the remaining part of the proof we are going to show that  $\hat{\pi}$  satisfies

$$(4.10) \quad e^{-rs} \widehat{V}_s = v + \int_0^s \hat{\pi}_t dp_t^*.$$

Formula (2.11) for the case of  $t \in [0, t_1)$  gives

$$(4.11) \quad e^{-rt} p_t = e^{-rT} \gamma(t) 1_{\{\tau > t\}} + e^{-rT} F 1_{\{\tau \leq t\}},$$

where we use  $\gamma(t) = \gamma(t, T, x)$  for the sake of brevity.

We separate two cases:

Case 1:  $\tau > s$ . In this case (4.10) is equivalent to

$$(4.12) \quad e^{-rs}\alpha = v + e^{-rT} \int_0^s h(t)\gamma'(t)dt$$

Case 2:  $\tau = u \leq s$ . Now (4.10) is equivalent to

$$(4.13) \quad e^{-rs}\beta(u) = v + e^{-rT} \int_0^u h(t)\gamma'(t)dt + h(u)[e^{-rT}F - e^{-rT}\gamma(u)]$$

We shall work on case 2 first. We already established that  $h(\cdot)$  is continuous, so (4.13) holds if it is true for  $u = 0$ , and the derivatives of both sides in every point  $u \neq u_0$  are identical. Naturally if both sides are differentiable on  $(0, \infty)$  then we disregard the  $u \neq u_0$  condition in the previous sentence. Taking derivatives of (4.13) with respect to  $u$  we get

$$e^{-rs}\beta'(u) = h'(u)[e^{-rT}F - e^{-rT}\gamma(u)] = -h'(u)e^{-rT}(1 - F) \frac{\psi(T, \tilde{r}, b)}{\psi(u, \tilde{r}, b)},$$

which follows from (4.9). In addition, (4.13) with  $u = 0$  follows from (4.8) and the definition of  $\gamma(\cdot)$ .

Next we consider case 1. Let  $\gamma_1(t) = e^{-rT}\gamma(t) - e^{-rT}F$ . By integration by parts, we have

$$e^{-rT} \int_0^s h(t)\gamma'(t)dt = h(s)\gamma_1(s) - h(0)\gamma_1(0) - \int_0^s h'(t)\gamma_1(t)dt.$$

In order to show that (4.12) holds for  $h'(t)$  specified in (4.9), we substitute  $\gamma_1(t)$ , (4.8) and (4.9) into the right-hand side of the above equation after which it can be simplified to

$$e^{-rT} \int_0^s h(t)\gamma'(t)dt = \frac{1}{\psi(s)} \left[ v + e^{-rs} \int_0^s \psi'(u)\beta(u)du \right] - v.$$

We apply Lemma 4.4 here, and it is straightforward to see (using another partial integration) that the right-hand side of the above expression is exactly  $e^{-rs}\alpha - v$ , which implies (4.12). Note that the partial integrations above are correct even if  $h(\cdot)$  and  $\beta(\cdot)$  have a breakpoint, because both functions are continuous.  $\square$

**REMARK 4.6.** Notice that  $h(\cdot)$  is linear in  $1/(1 - F)$ . Another interesting feature of the optimal investment process is that the terminal wealth  $\hat{V}_s = I(y\zeta_s)$  does not depend on the maturity of the bond  $T$ .

## 5 Examples

In this section we shall investigate some special utility functions.

### A. Power and logarithmic utilities.

Suppose the utility function is

$$(5.1) \quad U_\delta(c) = \begin{cases} \frac{1}{\delta}c^\delta, & \text{if } \delta < 1 \text{ and } \delta \neq 0; \\ \log c, & \text{if } \delta = 0. \end{cases}$$

In this case  $U'_\delta(c) = c^{\delta-1}$  and  $I(u) = u^{\frac{1}{\delta-1}}$  for all  $\delta < 1$ .

By straightforward substitution

$$\begin{aligned}
\alpha &= y^{1/(\delta-1)} \cdot \left[ \frac{\psi(s, \tilde{r}, b)}{\psi(s, \tilde{\mu}, b)} \right]^{1/(\delta-1)} \\
\beta(t) &= y^{1/(\delta-1)} \cdot \exp \left\{ \frac{K}{\delta-1} t \right\} \cdot \left( \frac{l}{x} \right)^{L/(\delta-1)} \\
h(0) &= \frac{e^{rT}}{1-F} \cdot \frac{v - e^{-rs} y^{1/(\delta-1)} \left( \frac{l}{x} \right)^{L/(\delta-1)}}{\psi(T, \tilde{r}, b)} \\
h'(t) &= y^{1/(\delta-1)} \cdot \frac{e^{r(T-s)}}{1-F} \cdot \frac{\psi(t, \tilde{r}, b)}{\psi(T, \tilde{r}, b)} \cdot \frac{K}{1-\delta} \cdot \exp \left\{ \frac{K}{\delta-1} t \right\} \cdot \left( \frac{l}{x} \right)^{L/(\delta-1)}
\end{aligned}$$

The optimal terminal wealth is determined by the above formulas for  $\alpha$ ,  $\beta(t)$ , and (4.3). By

Lemma 4.4 we have

$$y^{1/(\delta-1)} = e^{rs} v \left[ (\psi(s, \tilde{r}, b))^{\frac{\delta}{\delta-1}} (\psi(s, \tilde{\mu}, b))^{-\frac{1}{1-\delta}} - \int_0^s \exp \left\{ \frac{K}{\delta-1} t \right\} \left( \frac{l}{x} \right)^{\frac{L}{\delta-1}} \psi'(t, \tilde{r}, b) dt \right]^{-1}.$$

For the utility function given in (5.1) we use the notation  $h_\delta(\cdot)$  instead of  $h(\cdot)$ , and with an eye on future developments we also write  $h_\delta(t, s, T, v, x)$ . We summarize the above results in the following theorem.

**THEOREM 5.1.** *The optimal investment process for the above utility function is determined by*

$$(5.2) \quad h_\delta(0) = h_\delta(0, s, T, v, x) = \frac{v}{1-F} A(s, T, x)$$

$$(5.3) \quad h'_\delta(t) = \frac{\partial}{\partial t} h_\delta(t, s, T, v, x) = \frac{v}{1-F} B(t, s, T, x)$$

where  $A(s, T, x)$  and  $B(t, s, T, x)$  do not depend on  $v$  or  $F$ . These functions are given by the following formulae:

$$A(s, T, x) = C_1(T, x) \left\{ 1 - \phi(s, x) \left( \frac{l}{x} \right)^{\frac{L}{\delta-1}} \right\}$$

$$B(t, s, T, x) = C_1(T, x)\phi(s)\psi(t, \tilde{r}, b)\frac{K}{1-\delta}C_2(t, x)$$

$$C_1(T, x) = \frac{e^{rT}}{\psi(T, \tilde{r}, b(x))}$$

$$C_2(t, x) = \exp\left\{\frac{K}{\delta-1}t\right\}\left(\frac{l}{x}\right)^{\frac{L}{\delta-1}}$$

and

$$\phi(s, x) = \left\{ (\psi(s, \tilde{r}, b))^{\frac{\delta}{\delta-1}} (\psi(s, \tilde{\mu}, b))^{-\frac{1}{\delta-1}} - \int_0^s C_2(t, x)\psi'(t, \tilde{r}, b)dt \right\}^{-1}.$$

An important feature of this optimal investment process is that it is linear in the initial wealth  $v$ .

In the case of log utility ( $\delta = 0$ ),  $h_0(0)$  and  $h'_0(t)$  can be greatly simplified because  $\mathbb{E}_Q[e^{-rs}I(y\zeta_s)] = v$  implies  $e^{-rs}/y = v$ . Thus,

$$(5.4) \quad h_0(0) = h_0(0, T, v, x) = \frac{v}{1-F} \cdot \frac{e^{rT}}{\psi(T, \tilde{r}, b(x))} \left[ 1 - \left(\frac{l}{x}\right)^{-L} \right]$$

$$(5.5) \quad h'_0(t) = \frac{\partial}{\partial t} h_0(t, T, v, x) = \frac{v}{1-F} \cdot \frac{e^{rT}}{\psi(T, \tilde{r}, b(x))} \psi(t, \tilde{r}, b(x)) K e^{-Kt} \left(\frac{l}{x}\right)^{-L}$$

It is worth noting that in this case the optimal investment process does not depend on  $s$ .

Next we calculate the *value function* of our optimization problem for the power and logarithmic utility function, i.e., the maximal expected utility from terminal wealth

$$W_\delta(v, x, s) = E \left[ U_\delta \left( \hat{V}_s \right) \right].$$

**THEOREM 5.2.** For all  $\delta < 1$ ,  $\delta \neq 0$  the value function is

$$(5.6) \quad W_\delta(v, x, s) = D_\delta(x, s) \cdot \frac{1}{\delta} v^\delta,$$

and for  $\delta = 0$  it is

$$(5.7) \quad W_0(v, x, s) = D_0(x, s) + \log v.$$

where

$$(5.8) \quad D_\delta(x, s) = e^{\delta rs} (\phi(s, x))^\delta \left[ (\psi(s, \tilde{r}, b))^{\frac{\delta}{\delta-1}} (\psi(s, \tilde{\mu}, b))^{-\frac{1}{\delta-1}} - \int_0^s (C_2(t, x))^\delta \psi'(t, \tilde{\mu}, b) dt \right]$$

and

$$D_0(x, s) = rs - \psi(s, \tilde{\mu}, b) \log \frac{\psi(s, \tilde{r}, b)}{\psi(s, \tilde{\mu}, b)} - (1 - \psi(s, \tilde{\mu}, b)) \log \left( \frac{l}{x} \right)^L + \int_0^s Kt\psi'(t, \tilde{\mu}, b) dt.$$

*Proof.* This is an immediate consequence of (4.3), (4.4), and (4.5). □

## B. Negative Exponential Utility.

Not every utility function implies a linear relationship between the optimal investment process  $\pi$  and the initial wealth  $v$ . Negative exponential utility provides a counterexample.

Suppose an investor has the utility function of  $U_\theta(c) = -e^{-\theta c}$  with  $\theta > 0$ ,  $c > 0$ . Then we have  $U'_\theta(c) = \theta e^{-\theta c}$  and  $I(u) = (1/\theta) [\log(\theta/u)]^+$ . We define the intervals  $N_1 = (0, \theta (l/x)^{-L})$ ,  $N_2 = (0, \theta e^{-Kt} (l/x)^{-L})$  and  $N_3 = (0, \theta \psi(s, \tilde{\mu}, b)/\psi(s, \tilde{r}, b))$ . It is straightforward to show

that

$$\begin{aligned}
h(0) &= \frac{e^{rT}}{(1-F)\psi(T, \tilde{r}, b)} \left[ v - \frac{e^{-rs}}{\theta} \cdot 1_{N_1}(y) \cdot \log \left( y^{-1} \theta \left( \frac{l}{x} \right)^{-L} \right) \right] \\
h'(t) &= \frac{e^{rT}}{(1-F)\psi(T, \tilde{r}, b)} \cdot \psi(t, \tilde{r}, b) \cdot \frac{e^{-rs}}{\theta} K \cdot 1_{N_2}(y)
\end{aligned}$$

According to Lemma 4.4 we have an implicit representation for  $y$  in the form of

$$v = \frac{e^{-rs}}{\theta} \left\{ 1_{N_3}(y) \cdot \psi(s, \tilde{r}, b) \log \left( \frac{\theta \psi(s, \tilde{\mu}, b)}{y \psi(s, \tilde{r}, b)} \right) - \int_0^s 1_{N_2}(y) \cdot \psi'(t, \tilde{r}, b) \log \left( \frac{\theta e^{-Kt}}{y} \left( \frac{l}{x} \right)^{-L} \right) dt \right\}.$$

Unfortunately, the dependence of the right-hand side on  $y$  is complicated hence there is no simple way to extract  $y$  in order to write down a closed form expression. Lemma 4.2 guarantees that a unique  $y > 0$  satisfying the above identity exists. However, the optimal portfolio process  $\pi$  cannot be linear in the initial wealth  $v$ , unlike in the case of the power and logarithmic utility functions.

## 6 Numerical analysis for the logarithmic utility function

The logarithmic utility function is an important case worth further discussion, because maximizing the expected logarithm of the terminal wealth of an investor amounts to maximizing the expected return rate of her or his portfolio. In addition, the optimal investment process, given in (5.4) and (5.5) has a particularly simple form; it is linear in the initial wealth  $v$ , and does not depend on the terminal time  $s$ .

Identity (5.4) clearly implicates that an optimally behaving investor equipped with the log-

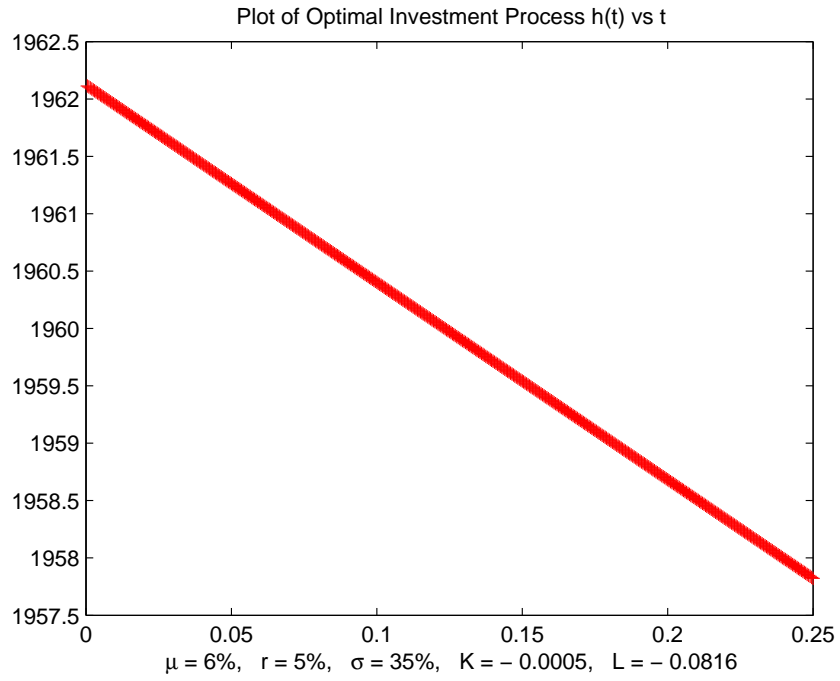


Figure 1: Case 1:  $\mu > r$  and  $\sigma^2 > \mu + r$ ,  $l/x = 0.5$ ,  $K$  and  $L$  are close to 0.

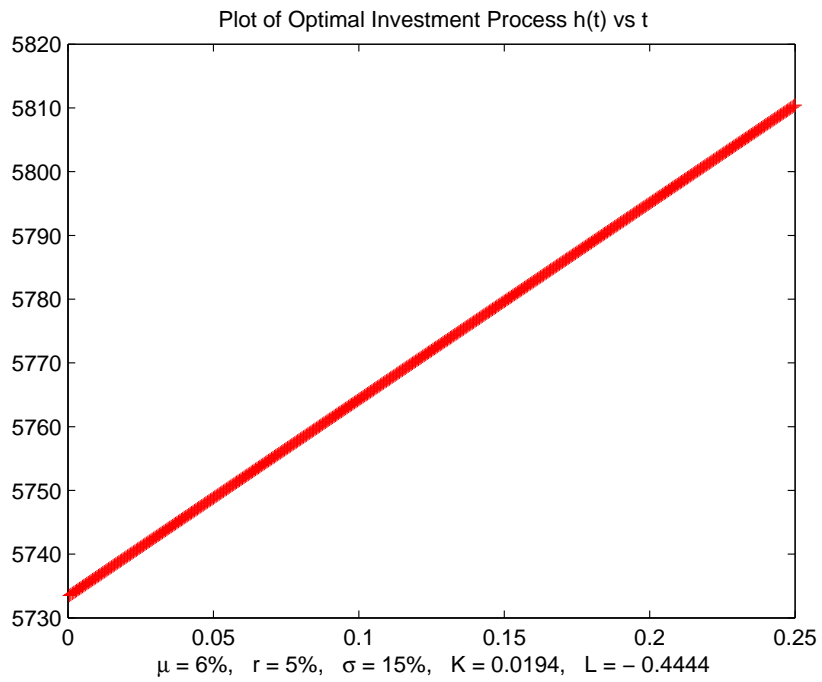


Figure 2: Case 2:  $\mu > r$  and  $\sigma^2 < \mu + r$ ,  $l/x = 0.5$ ,  $K$  and  $L$  are close to 0.

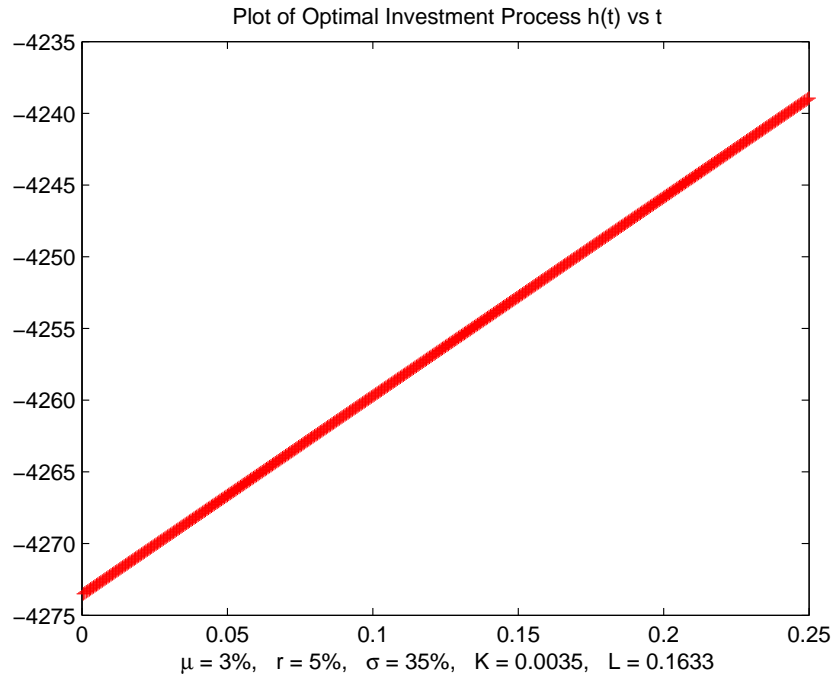


Figure 3: Case 3:  $\mu < r$  and  $\sigma^2 > \mu + r$ ,  $l/x = 0.5$ ,  $K$  and  $L$  are close to 0.

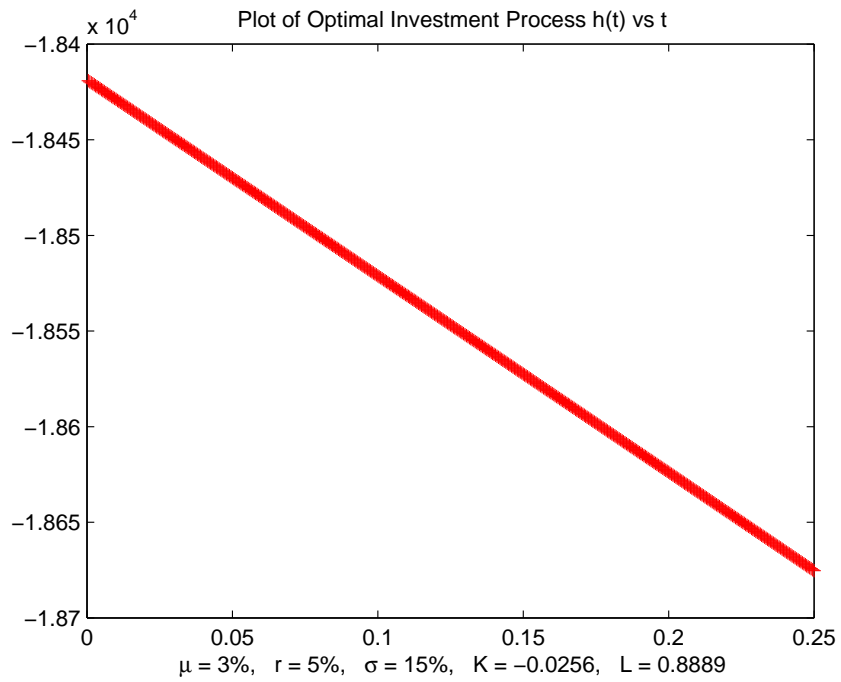


Figure 4: Case 4:  $\mu < r$  and  $\sigma^2 < \mu + r$ ,  $l/x = 0.5$ ,  $K$  and  $L$  are close to 0.

arithmetic utility function takes a long position in the bond at time zero if  $\mu > r$ , and a short position otherwise. The sign of the increment depends only on the sign of the constant  $K$  (see (5.5)). In Figure 1 through Figure 4 we show a plot of  $h_0(\cdot)$  in a typical example for each of the following four cases: case 1,  $\mu > r$  and  $\sigma^2 > \mu + r$ ; case 2,  $\mu > r$  and  $\sigma^2 < \mu + r$ ; case 3,  $\mu < r$  and  $\sigma^2 > \mu + r$ ; case 4,  $\mu < r$  and  $\sigma^2 < \mu + r$ . For the purpose of illustration, we set  $v = \$10,000$ ,  $T = 5$ ,  $F = 0.4$ ,  $l/x = .5$ . Various values of  $\mu$ ,  $r$  and  $\sigma$  are listed below the corresponding graphs. We plotted the graph of  $h(\cdot)$  on the time horizon  $[0, .25]$  since there are usually quarterly announcements of the firm's asset values.

We are going to interpret case 1 and 4 (Figure 1 and Figure 4 respectively). In case 1, the investor decides to buy the bond at time zero, because the firm runs well in the sense that it provides high expected asset return ( $\mu > r$ ). However, the firm's performance is volatile in the sense that  $\sigma$  is relatively high ( $\sigma^2 > \mu + r$ ), so a risk-averse investor reduces her or his position over time. In case 4 the investor shorts the bond at time zero since the firm is running poorly ( $\mu < r$ ); in addition, the firm's performance is relatively stable ( $\sigma^2 < \mu + r$ ) so the firm's asset process is not driven much by the diffusion component in (2.1). In other words, the firm tends to underperform consistently, which implies a high probability of default, so the investor continuously shorts more bonds.

As shown in the illustrations, in various scenarios the graph of the function  $h(\cdot)$  is almost linear. In the four cases illustrated in Figure 1 through 4 the absolute value of  $K$  is small, hence the almost linearity of the graphs. In Figure 5, the absolute value of  $K$  is relatively large ( $K = 9.99$ ), but  $L = -72$  is negative with a large absolute value which makes the graph

almost horizontal. There is some curvature in the case illustrated in Figure 8, where  $L$  is a large positive number and  $|K|$  is large. In this case our optimal policy requires shorting the bond at time zero at an astronomical level, which is quite intuitive considering that the drift  $\mu = -.15$  drives the asset value down fast. However, shorting that many bonds is clearly not realistic, and one can conclude only that the investor ought to short as many bonds as possible (presuming that there is a buyer for such bond). Finally,  $h(\cdot)$  has a strong curvature if we select the debt ratio very large ( $l/x = .95$ ), as illustrated in Figure 6 and 7.

In addition to the case depicted in Figure 5, even in cases illustrated in Figure 1 through 4 the rate of change of the portfolio is very small. Taking Figure 2 as an example, the investor purchases only about 76 additional bonds (after an initial purchase of 5,735 bonds) within a quarter of a year, buying roughly 1 bond a day. Over the period of a quarter of a year the investor increases her or his holdings by only a bit above 1% of the initial holding. In many realistic scenarios keeping the investment process at the constant level of  $h_0(0)$  is almost optimal, and has the advantage that transaction cost can be reduced significantly.

Another question that we consider is whether an investor would switch from holding long positions to holding short, or the other way around. We shall show that such switches are rare due to practical constraints. It can be seen from (5.4) and (5.5) that such switch occurs only if  $\sigma^2 > \mu + r$ . We first consider the case of  $\mu > r$ , which implies the investor starts with a long position ( $h_0(0) > 0$ ) and reduces her or his position over time ( $K < 0$  and  $h'_0(t) < 0$ ).

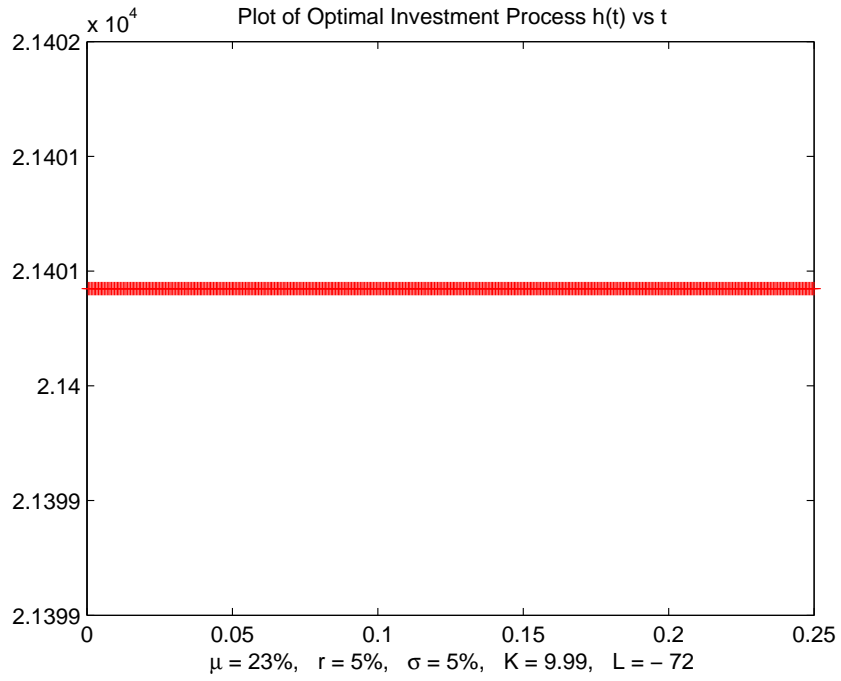


Figure 5:  $\mu > r$  and  $l/x = 0.5$ ,  $K$  and  $L$  are far from 0.

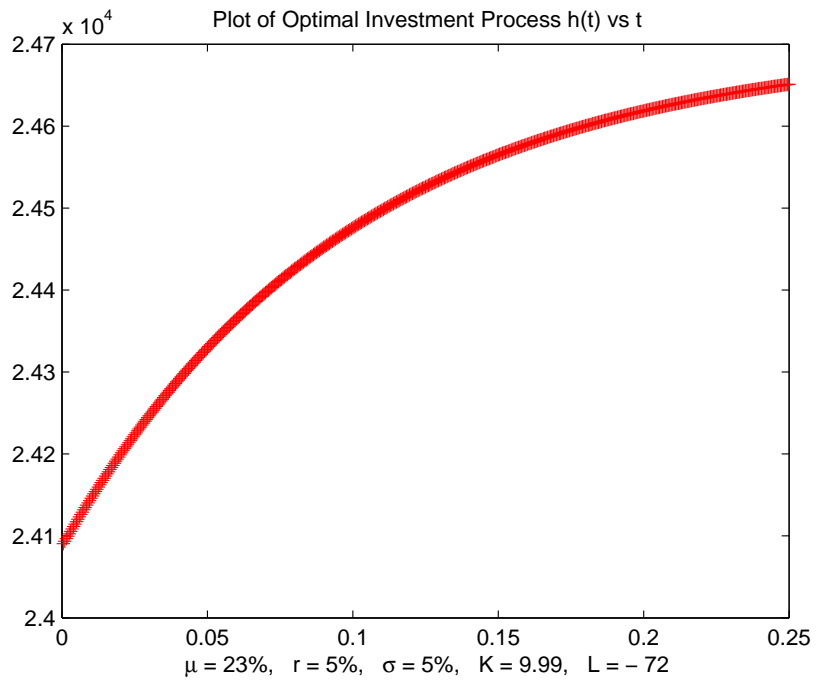


Figure 6:  $\mu > r$  and  $l/x = 0.95$ ,  $K$  and  $L$  are far from 0.

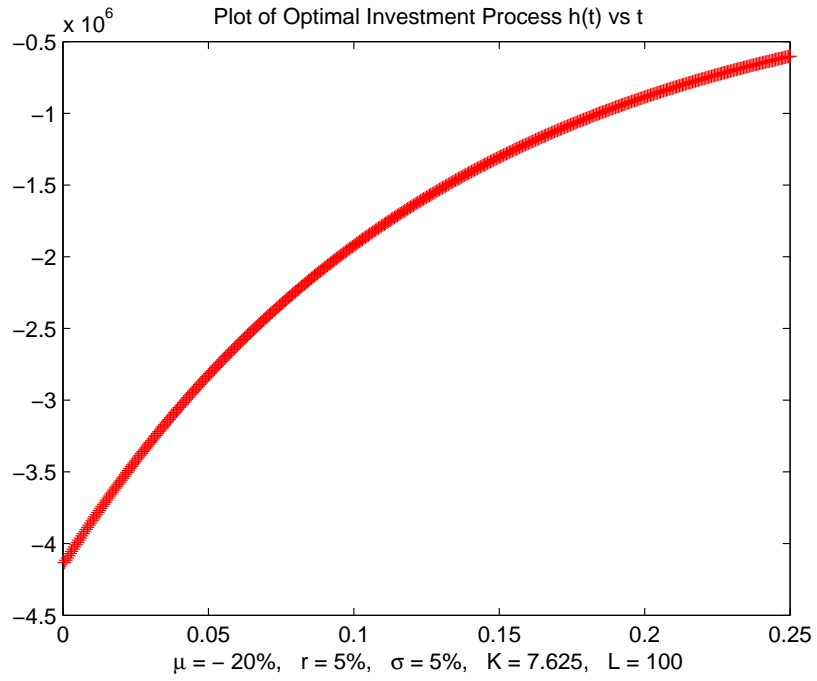


Figure 7:  $\mu < r$  and  $l/x = 0.95$ ,  $K$  and  $L$  are far from 0.

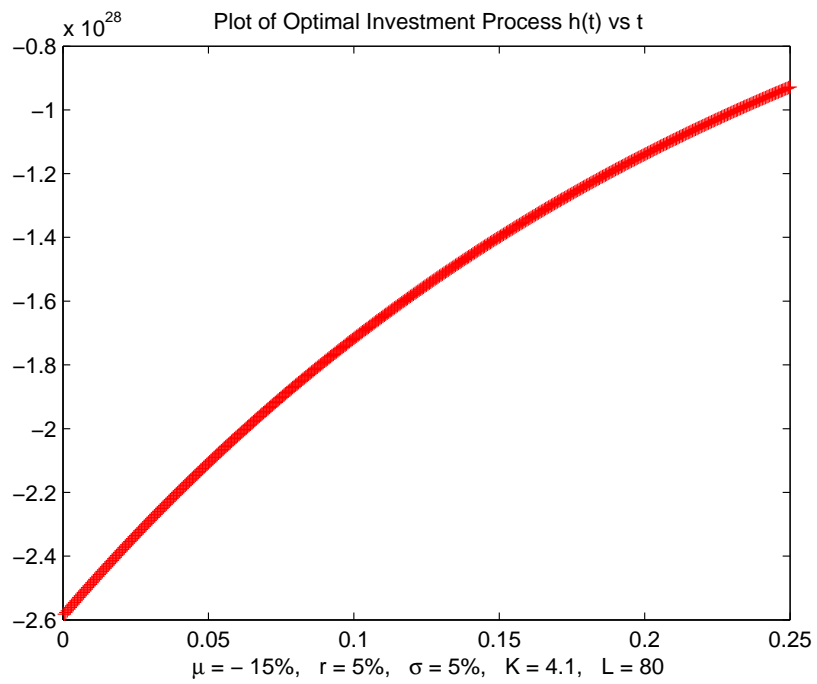


Figure 8:  $\mu < r$  and  $l/x = 0.5$ ,  $K$  and  $L$  are far from 0.

Clearly, the necessary condition of observing  $h_0(t) < 0$  for some  $t \leq .25$  is

$$1 - \left(\frac{l}{x}\right)^{-L} + K \left(\frac{l}{x}\right)^{-L} \int_0^t e^{-Ku} \psi(u, \tilde{r}, b) du < 0.$$

Since  $0 \leq \psi(t, \tilde{r}, b) \leq 1$  and  $t \leq .25$ , the above condition implies

$$1 - \left(\frac{l}{x}\right)^{-L} e^{-.25 \times K} < 0,$$

which gives

$$\frac{l}{x} > \exp \left\{ -\frac{1}{8} [\sigma^2 - (\mu + r)] \right\}.$$

In the case of  $\mu < r$ , the exactly same necessary condition is concluded.

The right-hand side of the above expression decreases in  $\sigma$ , so the higher the asset volatility is, the more likely we observe a position switch. However, even for a very large volatility the debt ratio  $l/x$  must be very large in order to observe a position switch within a quarter of a year. For example, suppose that  $\mu + r \geq 0$  and the volatility is as large as  $\sigma = .5$ . Our necessary condition for a position switch becomes  $l/x > \exp\{-\frac{1}{8} \times .5^2\} = .97$ . However, in this case the bond (presumably having a very low rating at this point) is a nuance away from default, and very likely will default before such position switch would take place.

## 7 Solution for the case of large time-horizon

In this section we allow the terminal time  $s$  to be an arbitrary time point between zero and  $T$ .

### A. Logarithmic utility

Here we consider the case of  $U(v) = \log v$ . We shall proceed by backwards induction, and in order to do so we have to consider initial times other than zero. Suppose that the present time is  $t_i < s$  for some index  $i$ , the asset value of the corporation has just been announced to be  $X(t_i) = x$ , and the current wealth of the investor is  $V(t_i) = v$ . Also assume that default did not happen yet, i.e.,  $\tau > t_i$ . Let  $P^{t_i, x}$  be the conditional probability measure given that  $X(t_i) = x$  and  $\tau > t_i$ , and let  $\mathbb{E}^{t_i, x}$  be the corresponding expectation. We shall now adapt the notations in the previous sections to this situation. The asset value of the corporation  $X_t$  now satisfies diffusion (2.1) with initial data  $(t_i, x)$ , and  $\tau = \tau_i$  since we assumed  $\tau > t_i$  ( $\tau_i$  was defined in (2.9)). The wealth  $V_u^\pi(t_i, v)$  at time  $u$  corresponding to an investment process  $\pi$  and to wealth  $v$  at time  $t_i$  is determined by the equation

$$e^{-ru} V_u^\pi(t_i, v) = e^{-rt_i} v + \int_{(t_i, u]} \pi_t dp_t^* .$$

Let  $\mathcal{A}^{(i)}(v, x)$  be the class of investment processes such that  $V_u^\pi(t_i, v) \geq 0$ , a.s., for all  $u \in [t_i, s]$  and  $\mathbb{E}^{t_i, x} [(\log V_s^\pi(t_i, v))^-] < \infty$ . The value function at time  $t_i$  becomes

$$W_0^{(i)}(v, x) = \sup \{ \mathbb{E}^{t_i, x} [\log V_s^\pi(t_i, v)]; \pi \in \mathcal{A}^{(i)}(v, x) \}$$

Suppose now that the present time is “just before”  $t_i$  for some  $t_i \leq s$  (in the case of  $s = T$  we consider only  $t_i < s$ ), so our information is represented by  $\mathcal{G}(t_i-)$ . In this case the information concerning the asset value of the corporation can be summarized as  $X(t_{i-1}) = x$ ,  $\tau > t_i$ . To reduce the burden of notation, let  $P^{t_i-, x}$  be the conditional probability measure given this information,  $\mathbb{E}^{t_i-, x}$  be the corresponding expectation, and suppose that  $V(t_i-) = v$ . It is important to note that  $x$  in this case corresponds to the asset value at  $t_{i-1}$  due to the discrete

nature of corporate news. The wealth  $V_u^\pi(t_i-, v)$  at time  $u$  corresponding to an investment process  $\pi$  and to wealth  $v$  at time  $t_i-$  is determined by

$$e^{-ru}V_u^\pi(t_i-, v) = e^{-rt_i}v + \int_{[t_i, u]} \pi_t dp_t^* .$$

Notice that in the above calculation the discounted gain, represented by the integral on the right-hand side, is calculated on the closed interval  $[t_i, u]$ , so gains or losses derived from the jump of the default bond at time  $t_i$  is included in the wealth at time  $u$ .

Let  $\mathcal{A}^{(i-)}(v, x)$  be the class of investment processes such that  $V_u^\pi(t_i-, v) \geq 0$ , a.s., for all  $u \in [t_i, s]$  and  $\mathbb{E}^{t_i-, x} [(\log V_s^\pi(t_i-, v))^-] < \infty$ . The value function at time  $t_i-$  now becomes

$$W_0^{(i-)}(v, x) = \sup \{ \mathbb{E}^{t_i-, x} [\log V_s^\pi(t_i-, v)]; \pi \in \mathcal{A}^{(i-)}(v, x) \} .$$

Let  $j$  be the index for which  $t_j < s \leq t_{j+1}$ .

**THEOREM 7.1.** *For all  $0 \leq i \leq j$  there exist functions  $c_i(x)$  and  $\tilde{c}_i(x)$  such that*

$$(7.1) \quad W_0^{(i)}(v, x) = \log v + c_i(x)$$

$$(7.2) \quad W_0^{(i-)}(v, x) = \log v + \tilde{c}_i(x)$$

where the functions  $c_i(x)$  and  $\tilde{c}_i(x)$  do not depend on  $v$ , but both may depend on  $s - t_i$ ,  $t_{i+1} - t_i$ ,  $t_{i+2} - t_i$ ,  $\dots$ ,  $t_j - t_i$ ,  $T - t_i$ , and  $\tilde{c}(x)$  may in addition depend on  $t_i - t_{i-1}$  as well.  $W_0^{(i-)}(v, x)$  has the form given in (7.2) even if  $s = t_{j+1} < T$  and  $i = j + 1$ .

*Proof.* In this proof we shall assume that  $s \neq t_{j+1}$  (the complementary case can be covered with a minimal modification of this proof and will be omitted). We are going to proceed by

backward induction. In step 1 we show that (7.1) holds for  $i = j$ , in step 2 we show that (7.2) holds assuming that (7.1) holds, and finally in step 3 we shall show that (7.1) holds with  $i$  replaced by  $i - 1$  provided that (7.2) holds.

Step 1 follows from (5.7) since

$$W_0^{(j)}(v, x) = W_0(v, x, s - t_j),$$

which by (5.7) indeed gives the form required by (7.1).

We continue with step 2, i.e., assume (7.1) and show (7.2). By the dynamic programming principle

$$W_0^{(i-)}(v, x) = \sup \left\{ \mathbb{E}^{t_i-, x} \left[ W_0^{(i)}(V_{t_i}^\pi(t_i-, v), X(t_i)) \right] \right\}$$

where the supremum is constrained to investment processes such that  $V_{t_i}^\pi(t_i-, v) \geq 0$ . Notice that this is a non-stochastic one-dimensional maximization problem, since we are maximizing with respect to  $\pi(t_i)$  which, by the predictability of  $\pi$  can not depend on  $X(t_i)$ , so it is non-random under  $P^{t_i-, x}$ . By our assumption (7.1) the above identity can be written as

$$W_0^{(i-)}(v, x) = \sup \left\{ \mathbb{E}^{t_i-, x} \left[ \log V_{t_i}^\pi(t_i-, v) \right] \right\} + \mathbb{E}^{t_i-, x} [c_i(X(t_i))].$$

Let  $\pi(t_i) = \lambda$ . By (2.11) the above can be written as

$$\begin{aligned} W_0^{(i-)}(v, x) = \sup_{\lambda} \left\{ \mathbb{E}^{t_i-, x} \left[ \log \left\{ v + \lambda e^{-r(T-t_i)} \left[ \gamma(0, T - t_i, X(t_i)) - \gamma(t_i - t_{i-1}, T - t_{i-1}, x) \right] \right\} \right] \right\} \\ + \mathbb{E}^{t_i-, x} [c_i(X(t_i))] \end{aligned}$$

with the constraint that

$$v + \lambda e^{-r(T-t_i)} \left[ \gamma(0, T - t_i, X(t_i)) - \gamma(t_i - t_{i-1}, T - t_{i-1}, x) \right] \geq 0, \quad P^{t_i-, x} - \text{a.s.}$$

By  $\tau > t_i$  the support of  $X(t_i)$  is  $(l, \infty)$ , so by Lemma A.2 in the Appendix our constraints can be written as

$$(7.3) \quad v + \lambda e^{-r(T-t_i)} [1 - \gamma(t_i - t_{i-1}, T - t_{i-1}, x)] \geq 0$$

$$(7.4) \quad v + \lambda e^{-r(T-t_i)} [F - \gamma(t_i - t_{i-1}, T - t_{i-1}, x)] \geq 0$$

(we must consider both constraints because  $\lambda$  may be negative when the investor shorts the bond). These become

$$(7.5) \quad \lambda \geq \frac{-v}{1 - \gamma(t_i - t_{i-1}, T - t_{i-1}, x)} e^{r(T-t_i)}$$

$$(7.6) \quad \lambda \leq \frac{v}{\gamma(t_i - t_{i-1}, T - t_{i-1}, x) - F} e^{r(T-t_i)}$$

We conclude that  $\pi(t_i) = \lambda$  is the solution of the (deterministic) maximization problem

$$(7.7) \quad \sup_{\lambda} \left\{ \mathbb{E}^{t_i^-, x} \left[ \log \left\{ v + \lambda e^{-r(T-t_i)} [\gamma(0, T - t_i, X(t_i)) - \gamma(t_i - t_{i-1}, T - t_{i-1}, x)] \right\} \right] \right\},$$

subject to constraints (7.5) and (7.6). The expression that we are maximizing is a strictly concave function of  $\lambda$ , hence it achieves its maximum over the compact interval determined by the constraints at a unique point. Due to the linearity of the constraints with respect to  $v$  and to the algebraic properties of the logarithm function the point where the maximum is achieved is a linear function of  $v$ . Let this maximum point be  $v\hat{\lambda}(x) = v\hat{\lambda}(x, t_i - t_{i-1}, \dots, T - t_{i-1})$ .

It follows that

$$\begin{aligned} W_0^{(i-)}(v, x) = & \log v + \mathbb{E}^{t_i^-, x} \left[ \log \left\{ 1 + \hat{\lambda}(x) e^{-r(T-t_i)} [\gamma(0, T - t_i, X(t_i)) - \gamma(t_i - t_{i-1}, T - t_{i-1}, x)] \right\} \right] \\ & + \mathbb{E}^{t_i^-, x} [c_i(X(t_i))] \end{aligned}$$

which is indeed the form required by (7.2).

In step 3 we are showing that  $W_0^{(i-1)}(v, x)$  satisfies (7.1) with  $i$  replaced by  $i - 1$  provided that (7.2) holds. By the dynamic programming principle we have

$$W_0^{(i-1)}(v, x) = \sup \left\{ E^{t_{i-1}, x} \left[ W_0^{(i-)}(V_{t_{i-}}^\pi(t_{i-1}, v), x) \right] \right\}$$

where the supremum is taken for admissible  $\pi$ 's. However, by (7.2) and (5.7) this can be cast into

$$\begin{aligned} W_0^{(i-1)}(v, x) &= \sup \left\{ E^{t_{i-1}, x} \log V_{t_{i-}}^\pi(t_{i-1}, v) \right\} + \tilde{c}_i(x) \\ &= \log v + D_0(x, t_i - t_{i-1}) + \tilde{c}_i(x) \end{aligned}$$

which is indeed the form required by (7.1). □

For the purposes of writing down the optimal investment process recall the function  $h_0(t, T, v, x)$  from (5.4) and (5.5). Also recall that the function  $\hat{\lambda}(x)$  is defined so that (7.7) achieves its unique maximum at  $\lambda = v\hat{\lambda}(x)$ , subject to constraints (7.5) and (7.6). In other words,  $\hat{\lambda}(x)$  is the point where (7.7) achieves the supremum under constraints (7.5) and (7.6) with  $v = 1$ .

**THEOREM 7.2.** *At a time  $t \leq s$  such that  $t \in (t_i, t_{i+1})$  for some  $i \leq j$  the optimal investment is*

$$\hat{\pi}_t 1_{\{\tau > t\}} = h_0(t - t_i, T - t_i, V(t_i), X(t_i)) 1_{\{\tau > t\}}.$$

*This formula is correct for  $t = 0$  as well with the selection of  $t_i = 0$ . Additionally, this gives the value of the optimal investment process at time  $t = T$  if  $s = T$  in which case  $i = n - 1$ .*

*For all  $i \leq j$  the optimal investment process at time  $t_i$  is*

$$(7.8) \quad \hat{\pi}(t_i) 1_{\{\tau > t_i\}} = V(t_i-) \hat{\lambda}(X(t_{i-1})) 1_{\{\tau > t_i\}}.$$

This is correct for  $i = j + 1$  as well if  $s = t_{j+1} < T$ .

*Proof.* This follows from the theorem 7.1 and its proof. In the case of  $s = T$  we have to consider the fact that the bond price is left-continuous at the maturity time  $T$ .  $\square$

**REMARK 7.3.** *An interesting feature of the logarithmic utility function is that the optimal investment process does not depend on the terminal time  $s$ .*

For the actual computation of  $\hat{\lambda}(x)$  we need the conditional distribution of  $X(t_i)$  given that  $X(t_{i-1}) = x$  and  $\tau > t_i$ . This is given in Lemma A.3 in the Appendix.

## B. Power Utility

In this subsection, the utility function is assumed to be the power function with parameter  $\delta < 1, \delta \neq 0$  given in (5.1). Since the results are essentially analogous to those in section A, we shall only state the main results and outline the proofs without providing details.

Define the value function for the power utility similarly to section A, i.e.,

$$W_\delta^{(i)}(v, x) = \sup \left\{ \mathbb{E}^{t_i, x} \left[ \frac{1}{\delta} (V_s^\pi(t_i, v))^\delta \right]; \pi \in \mathcal{A}^{(i)}(v, x) \right\},$$

and

$$W_\delta^{(i-)}(v, x) = \sup \left\{ \mathbb{E}^{t_i-, x} \left[ \frac{1}{\delta} (V_s^\pi(t_i-, v))^\delta \right]; \pi \in \mathcal{A}^{(i-)}(v, x) \right\}.$$

Recall that  $j$  is the index for which  $s \in (t_j, t_{j+1}]$ .

**THEOREM 7.4.** For all  $i \leq j$  there exist functions  $c_{\delta,i}(x)$  and  $\tilde{c}_{\delta,i}(x)$  such that

$$(7.9) \quad W_{\delta}^{(i)}(v, x) = \frac{1}{\delta} v^{\delta} \cdot c_{\delta,i}(x)$$

$$(7.10) \quad W_{\delta}^{(i-)}(v, x) = \frac{1}{\delta} v^{\delta} \cdot \tilde{c}_{\delta,i}(x)$$

$W_{\delta}^{(i-)}(v, x)$  has the form given in (7.10) even if  $s = t_{j+1} < T$  and  $i = j + 1$ .

*Proof Outline.* Step 1. We assume that  $s \neq t_{j+1}$ . From (5.6) follows that

$$W_{\delta}^{(j)}(v, x) = W_{\delta}(v, x, s - t_j) = D_{\delta}(x, s - t_j) \cdot \frac{1}{\delta} v^{\delta}.$$

Therefore,

$$(7.11) \quad c_{\delta,j}(x) = D_{\delta}(x, s - t_j).$$

Step 2: Assume that (7.9) holds. By the dynamic programming principle

$$(7.12) \quad \begin{aligned} W_{\delta}^{(i-)}(v, x) &= \sup \left\{ \mathbb{E}^{t_i-, x} \left[ W_{\delta}^{(i)}(V_{t_i}^{\pi}(t_i-, v), X(t_i)) \right] \right\} \\ &= \sup \left\{ \mathbb{E}^{t_i-, x} \left[ \frac{1}{\delta} \left( v + \lambda e^{-r(T-t_i)} [\gamma(0, T - t_i, X(t_i)) - \gamma(t_i - t_{i-1}, T - t_{i-1}, x)] \right)^{\delta} c_{\delta,i}(X(t_i)) \right] \right\} \end{aligned}$$

where  $\lambda = \pi(t_i)$ . Here the supremum is taken over all  $\lambda$ 's satisfying the admissibility constraints of (7.5) and (7.6).

The expression after the sup on the right-hand side of (7.12) is a strictly concave (deterministic) function of  $\lambda$ , therefore the supremum over a compact interval is achieved at a

unique point. In addition, the point where the supremum is achieved is a linear function of  $v$ . Denote by  $\hat{\lambda}(x)$  the function such that the supremum is achieved at  $\hat{\lambda}(x)v$ . Now we have

$$W_\delta^{(i-)}(v, x) = \frac{1}{\delta} v^\delta \mathbb{E}^{t_i-, x} \left[ \left( 1 + \hat{\lambda}(x) e^{-r(T-t_i)} [\gamma(0, T-t_i, X(t_i)) - \gamma(t_i - t_{i-1}, T-t_{i-1}, x)] \right)^\delta c_{\delta, i}(X(t_i)) \right]$$

which is indeed of the form required by (7.10), with

(7.13)

$$\tilde{c}_{\delta, i}(x) = \mathbb{E}^{t_i-, x} \left[ \left( 1 + \hat{\lambda}(x) e^{-r(T-t_i)} [\gamma(0, T-t_i, X(t_i)) - \gamma(t_i - t_{i-1}, T-t_{i-1}, x)] \right)^\delta c_{\delta, i}(X(t_i)) \right].$$

Step 3: Assume that (7.10) holds. Using again the dynamic programming principle we can show that

$$W_\delta^{(i-1)}(v, x) = \frac{1}{\delta} v^\delta \cdot D_\delta(x, t_i - t_{i-1}) \cdot \tilde{c}_{\delta, i}(x)$$

where  $D_\delta(\cdot, \cdot)$  is given in (5.8). It is clear that

$$(7.14) \quad c_{\delta, i-1}(x) = D_\delta(x, t_i - t_{i-1}) \cdot \tilde{c}_{\delta, i}(x).$$

□

In order to compute the optimal investment process, we have to calculate the functions  $\tilde{c}_{\delta, i}$  for all  $i = 1, 2, \dots, n-1$ . This makes the power utility much more computation extensive than the logarithmic utility. The computation can be done by backward induction using (7.11), (7.13), and (7.14).

The optimal investment process for  $t \in (t_i, t_{i+1})$  such that  $t \leq s$  is given by

$$\hat{\pi}_t 1_{\{\tau > t\}} = h_\delta(t - t_i, s - t_i, T - t_i, V(t_i), X(t_i)) 1_{\{\tau > t\}},$$

where the  $h_\delta$  function is determined by (5.2) and (5.3). This formula is correct for  $t = 0$  and  $t = T$  as well (the latter is relevant only if  $s = T$ ). In the case of  $t = 0$  we select  $t_i = 0$ , and in the case of  $t = T$  we select  $i = n - 1$  (for the  $t = T$  case keep in mind that the bond price is left-continuous at  $T$ ). For all  $1 \leq i \leq j$  the optimal investment process at time  $t_i$  is given by (7.8), where now  $\hat{\lambda}(x)$  is the unique point where the supremum on the right-hand side of (7.12) is achieved, under the constraints (7.5) and (7.6), and with  $v = 1$ . This is correct for  $i = j + 1$  as well if  $s = t_{j+1} < T$ . In order to compute  $\hat{\lambda}(x)$  we need the conditional distribution of  $X(t_i)$  given that  $X(t_{i-1}) = x$  and  $\tau > t_i$ , which is calculated in Lemma A.3 of the Appendix.

## 8 Conclusion

The major contribution of this paper is deriving the optimal investment in a defaultable zero coupon bond assuming a discrete information flow concerning the firm's asset value at specified time points. The methodology we present here can be applied to general utility functions. For the logarithmic and power utility functions we presented closed-form expressions for the investment strategy, which can be easily implemented. As an important example, we numerically analyzed the strategy for an investor who wants to maximize her or his expected return rate. We observed that under various reasonable parameter settings the investor changes her or his bond position very slowly between successive news releases, and most of the changes in the optimal portfolio happen “just before” and “just after” each news release.

## Appendix

**LEMMA A.1.** *For all  $\kappa < 0$ , we have the following alternative formula for  $\psi$ :*

$$(A.1) \quad \psi(t, \lambda, \kappa) = \Phi\left(-\frac{\kappa}{\sqrt{t}} + \lambda\sqrt{t}\right) - e^{2\lambda\kappa}\Phi\left(\frac{\kappa}{\sqrt{t}} + \lambda\sqrt{t}\right)$$

where  $\Phi$  is the standard normal distribution function.

*Proof.* The right-hand side converges to 1 as  $t \rightarrow 0$ , and it is straightforward to show that its derivative with respect to  $t$  agrees with that of the right-hand side of (2.7).  $\square$

**LEMMA A.2.** *For all  $u > 0$  we have  $\sup_{z>l} \gamma(0, u, z) = 1$  and  $\inf_{z>l} \gamma(0, u, z) = F$ .*

*Proof.* From the definition of  $\gamma(0, u, z)$  in (2.7) immediately follows that  $F \leq \gamma(0, u, z) \leq 1$  and  $\gamma(0, u, l) = F$ . Next we show that  $\lim_{z \rightarrow \infty} \gamma(0, u, z) = 1$ . By (2.7) it suffices to show  $\psi(t, \tilde{r}, b(z)) \rightarrow 1$ , as  $z \rightarrow \infty$ . However,

$$\psi(t, \tilde{r}, b(z)) = Q[\tau > t \mid X_0 = z] = Q\left(\inf_{0 \leq u \leq t} \widetilde{W}_u + \tilde{r}u > b(z)\right)$$

which indeed converges to 1 since  $\lim_{z \rightarrow \infty} b(z) = -\infty$ .  $\square$

**LEMMA A.3.** *For  $i = 1, 2, \dots, n - 1$  the conditional distribution of  $X(t_i)$  given that  $X(t_{i-1}) = x$  and  $\tau > t_i$  is given by*

$$P[X(t_i) > z \mid X(t_{i-1}) = x, \tau > t_i] = \begin{cases} \varphi(z, t_i - t_{i-1}, x), & \text{if } z > l; \\ 1, & \text{if } z \leq l. \end{cases}$$

where

$$\varphi(z, t, x) = (\psi(t, \tilde{\mu}, b))^{-1} \left[ \Phi \left( \frac{1}{\sigma\sqrt{t}} \log \frac{x}{z} + \tilde{\mu}\sqrt{t} \right) - e^{2b\tilde{\mu}} \Phi \left( \frac{1}{\sigma\sqrt{t}} \log \frac{l^2}{xz} + \tilde{\mu}\sqrt{t} \right) \right].$$

Here  $b = b(x)$  given in (2.6) and  $\Phi$  is the standard normal distribution function.

*Proof.* The case of  $z \leq l$  is straightforward so we only consider the case of  $z > l$ . By the time homogeneity of  $X_t$ , we have

$$P[X(t_i) > z \mid X(t_{i-1}) = x, \tau > t_i] = P[X(t_i - t_{i-1}) > z \mid X_0 = x, \tau > t_i - t_{i-1}].$$

For brevity, we use  $t = t_i - t_{i-1}$ , so the above probability is equal to

$$(A.2) \quad \frac{P[X_t > z, \tau > t \mid X_0 = x]}{P[\tau > t \mid X_0 = x]}.$$

The denominator is  $\psi(t, \tilde{\mu}, b(x))$  with  $b(x)$  given in (2.6); we need to compute the numerator.

Let  $Y_t = W_t + \tilde{\mu}t$ , then  $\log X_t = \log x + \sigma Y_t$  and  $\tau = \inf\{u \geq 0 : Y_u \leq b\}$ . Let  $Q_t$  be a probability measure, equivalent to  $P$ , given by

$$\frac{dQ_t}{dP} = \exp \left\{ -\tilde{\mu}W_t - \frac{\tilde{\mu}^2}{2}t \right\} = \exp \left\{ -\tilde{\mu}Y_t + \frac{\tilde{\mu}^2}{2}t \right\},$$

and  $E_t$  be the corresponding expectation. From Girsanov's theorem (Karatzas & Shreve, 1998) follows that  $\{Y_u; u \leq t\}$  is a Brownian Motion under  $Q_t$ . The numerator of (A.2) can be written as

$$P \left[ Y_t > \frac{1}{\sigma} \log \frac{z}{x}, \tau > t \right] = \mathbb{E}_t \left[ 1_{\{\tau > t, Y_t > \frac{1}{\sigma} \log \frac{z}{x}\}} \exp \left\{ \tilde{\mu}Y_t - \frac{\tilde{\mu}^2}{2}t \right\} \right].$$

Let  $b' = \frac{1}{\sigma} \log \frac{z}{x}$ . We assumed  $z > l$  so we have  $b' > b$ . The above expression is equal to

$$\mathbb{E}_t \left[ 1_{\{Y_t > b'\}} \exp \left\{ \tilde{\mu}Y_t - \frac{\tilde{\mu}^2}{2}t \right\} \right] - \mathbb{E}_t \left[ 1_{\{\tau \leq t, Y_t > b'\}} \exp \left\{ \tilde{\mu}Y_t - \frac{\tilde{\mu}^2}{2}t \right\} \right]$$

By straightforward calculation the first expectation in this expression is

$$\Phi\left(\frac{1}{\sigma\sqrt{t}}\log\frac{x}{z} + \tilde{\mu}\sqrt{t}\right).$$

In order to evaluate the second expectation, we apply the reflection principle. We reflect  $Y$  on the level  $b < 0$  after  $\tau$ , and denote the reflected process by  $\tilde{Y}$  which is also a Brownian motion under  $Q_t$ . Then  $Y_t = 2b - \tilde{Y}_t$  on the event  $\{\tau > t\}$ , hence the second expectation becomes

$$\mathbb{E}_t\left[1_{\{\tau \leq t, \tilde{Y}_t < 2b - b'\}} \exp\left\{\tilde{\mu}(2b - \tilde{Y}_t) - \frac{\tilde{\mu}^2}{2}t\right\}\right]$$

Since  $2b - b' < b$ , this is

$$\mathbb{E}_t\left[1_{\{\tilde{Y}_t < 2b - b'\}} \exp\left\{\tilde{\mu}(2b - \tilde{Y}_t) - \frac{\tilde{\mu}^2}{2}t\right\}\right],$$

which can be easily seen to be equal to

$$e^{2b\tilde{\mu}}\Phi\left(\frac{1}{\sigma\sqrt{t}}\log\frac{l^2}{xz} + \tilde{\mu}\sqrt{t}\right).$$

□

**REMARK.** Expression (A.1) for  $\psi$  verifies that the conditional distribution function computed in the above lemma is continuous at  $z = l$  as expected.

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