The Impact of Competition on Prices with Numerous Firms

Xavier Gabaix  David Laibson  Deyuan Li
Hongyi Li  Sidney Resnick  Casper G. de Vries

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Abstract

Markets with highly homogeneous goods sometimes exhibit robustly high mark-ups. This paper describes a behavioral mechanism that sustains high mark-ups, even when goods are homogeneous and there is a large number of competing firms. We show that random demand models with consumer errors driven by standard noise distributions predict robustly high equilibrium markups. With Gaussian noise, we show that markups are asymptotically proportional to $1/\sqrt{\ln n}$, where $n$ is the number of competing firms. This implies that an increase in the number of competing firms from 10 to 1000 firms results in only a halving of the equilibrium markup. In contrast, with uniform noise (or noiseless Cournot competition), such an increase would result in the markup becoming 100 times smaller. We show that insensitive mark-ups are the norm rather than the exception. The elasticity of the markup with respect to the number of firms asymptotically equals the extreme value theory tail exponent of the noise distribution. Only noise distributions with extremely thin tails or extremely heavy tails have markups with asymptotic elasticities different from zero. For noise distributions in the extremely heavy-tailed class, mark-ups increase as the number of competing firms increases.

*Gabaix: NYU Stern, CEPR and NBER, xgabaix@stern.nyu.edu; Laibson: Harvard University and NBER, dlaibson@harvard.edu; D. Li: Fudan University, deyuanli@fudan.edu.cn; H. Li: University of New South Wales, hongyi@unsw.edu.au; Resnick: Cornell, sir1@cornell.edu; de Vries: Erasmus University Rotterdam, Tinbergen Institute and Chapman University, cdevries@ese.eur.nl. We thank Timothy Armstrong, Jeremy Bulow, Thomas Chaney, Victor Chernozhukov, Robert Hall, Rustam Ibragimov, Paul Klemperer, Johan Walden and Glen Weyl for helpful conversations and seminar participants at the AEA, Chicago, MIT, NYU and SITE for useful comments. This research was partially supported by the NSF and the Swiss National Science foundation. D. Li’s research was partially supported by NNSFC Grant 11171074.
1 Introduction

The effect of competition on prices is one of the central questions in economics. Classical equilibrium models – for instance, Bertrand and Cournot competition – imply that competition quickly lowers prices. For example, when homogeneous consumers face a Cournot market, equilibrium markups are proportional to $1/n$, where $n$ is the number of competing firms. Hence, doubling the number of firms, halves the equilibrium mark-up.

In practice, however, markets with highly homogeneous goods and many competitors sometimes exhibit robustly high mark-ups. For example, Hortacsu and Syverson (2004) document high mark-ups in the mutual fund market, even in asset classes with hundreds of competing funds. Ausubel (1991) and Stango (2000) show that interest rates on credit cards have been much greater than the cost of funds, despite the presence of hundreds of competing card-issuing banks.

In this paper, we characterize a behavioral mechanism that sustains high mark-ups. We show that random demand models with standard (thin-tailed) ‘noise’ distributions predict robustly high markups, even when goods are homogeneous and there is a large number of competing firms. We show that competition only weakly drives down equilibrium mark-ups that arise from such noise.

Random demand models assume that consumer choice is influenced by noise, in addition to standard deterministic factors like observable product attributes (Luce 1959, McFadden 1981, see the comprehensive study by Anderson et al. 1992). The noise may arise because of variation in tastes across households and/or because of consumer errors. In this paper we emphasize the noise that arises from consumer errors. By focusing on this case, our analysis isolates and characterizes a behavioral mechanism that weakens classical competitive forces. Our analysis also contributes to the large and influential literature that uses random demand models to characterize imperfect competition (see Anderson et al., 1992).

Using extreme value theory (EVT), we develop tools that provide explicit expressions for equilibrium prices in symmetric random-utility models. Explicit expressions for equilibrium markups have previously been derived only for some specific distributions of noise. In these previously studied special cases, equilibrium markups turn out to be either completely un-
responsive or extremely responsive to competition. For instance, consider the Perloff-Salop (1985) model of competition, and assume that consumer noise has an exponential density or a logit (i.e., Gumbel) density. In this case, markups converge to a strictly positive value as \( n \), the number of competing firms, goes to infinity. Hence, asymptotic markups have zero elasticity with respect to \( n \) (Perloff and Salop 1985, Anderson et al. 1992). By contrast, when noise is uniformly distributed, markups are proportional to \( 1/n \), so markups have a unit elasticity and hence a strong negative relationship with \( n \) (Perloff and Salop 1985).

These special cases — exponential, logit, and uniform — are appealing for their analytic tractability rather than their realism. In comparison to the Gaussian distribution, the exponential and logit cases have relatively fat tails while the uniform case has no tails. It is important to understand how prices respond to competition when consumer noise follows more general distributions, particularly distributions that are considered to be empirically realistic.

The current paper solves this tractability problem for general noise distributions and derives a simple, useful formula for equilibrium markups. We show that markups are asymptotically proportional to \( 1/(nF''[F^{-1}(1-1/n)]) \), where \( F \) is the cumulative distribution function (CDF) of the noise. Moreover, we show that this markup turns out to be almost equivalent to the markup obtained under limit pricing. In other words, the markup is asymptotically proportional (and often equal) to the expected gap between the highest draw and second highest draw in a sample of \( n \) random draws of noise. This implies that for large \( n \) prices are pinned down by the tail properties of the distribution of taste shocks and that prices are independent of other institutional details.

We pay particular attention to the (thin-tailed) Gaussian case, because it is a leading approximation of many natural phenomena. Previously, equilibrium mark-ups associated with this distribution did not have a closed form solution. With Gaussian noise, we show that markups are asymptotically proportional to \( 1/\sqrt{\ln n} \). This implies that an increase in the number of competing firms from 10 to 1000 firms results in only a halving of the equilibrium markup. In contrast, with Cournot competition, such an increase would result in the markup becoming 100 times smaller. This example shows that competition with a realistic distribution of noise may only exert weak pressure on prices (even in the extreme case of homogeneous goods).

We show that insensitive prices are the norm rather than the exception. Specifically, we find that the elasticity of the markup with respect to the number of firms asymptotically equals the EVT tail exponent of the noise distribution, a magnitude that is easy to calculate.
Using this result, we show that markups have a zero asymptotic elasticity for empirically realistic noise distributions. Only noise distributions with extremely thin or no tails (like the uniform distribution) and very heavy tails (like the Pareto), have markups with asymptotic elasticities different from zero.

Moreover, our analysis implies that for distributions in the very heavy-tailed class (including subexponential distributions like the log-normal and power-law distributions like the Pareto distribution), mark-ups increase as the number of competing firms increase. This phenomenon, whereby prices rise with more intense competition, has recently attracted some attention; see, for example, Chen and Riordan (2008). More generally, and closely related to our work, Weyl and Fabinger (2014) show how comparative statics of pricing behavior hinge crucially on log-concavity of the demand function; relating this insight to our results, they point out that competition increases (decreases) prices if the distribution of consumer valuations is log-convex (log-concave); see also Quint (2015) for a related analysis.

Our findings exhibit “detail-independence”; they hold for all of the monopolistic competition models that we consider: Perloff-Salop (1995), Sattinger (1984), Hart (1985). Such detail-independence is useful for empirical analyses. It permits a more robust analysis than would be possible if results depended on the specifics of the noise distribution or the details of the demand specification. The Hart, Perloff-Salop, and Sattinger models differ in a host of important ways. Yet, these three models lead asymptotically to the same value of the markup up to a scaling constant for a wide range of different noise distributions.

One natural interpretation of our model is that noise in consumer evaluation of products arises from consumers’ confusion about true product quality. This relates to a growing literature that studies the effect of consumer confusion on market outcomes. In particular, a number of papers, including Spiegler (2006), Gabaix and Laibson (2006), Ellison and Ellison (2009), Armstrong and Vickers (2012) and Heidhues, Koszegi and Murooka (2014a, 2014b), study how sellers may take advantage of naive (boundedly rational) consumers by deliberately shrouding product characteristics; we discuss in Appendix A how our model may be

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3 For more background, see Bulow, Geanakoplos and Klemperer (1985).

4 A canonical example of “detail-independence” or “universality” is the central limit theorem: for large $n$, when forming the mean, we obtain a distribution (the Gaussian) that is independent of the details of the system.

5 For instance, in the Perloff-Salop model consumers need to buy one unit of the good. In the Sattinger model, they allocate a fixed dollar amount to the good. The Hart model does not impose either constraint.
augmented to consider deliberate shrouding by sellers. Relatedly, other papers (e.g. Bordalo, Gennaioli, Shleifer 2015) emphasize the impact of endogenous salience on market equilibrium.

Moving away from the specifics of random utility models, the tools that we develop allow us to calculate the asymptotic behavior of integrals for a class of functions $h(x)$, of the form

$$\int h(x) f^k(x) F(x)^n dx,$$

where $k \geq 1$, which appear in a very large class of economic situations, some of which we will review later. For instance, this integral can be used to calculate the expected value of a function of the maximum of $n$ random variables, or the gap between the maximum and the second largest value of those random variables. Using EVT we are able to derive robust approximations of this integral for a large $n$. We also review some other applications of this general result.

Extreme outcomes and EVT techniques are important in many parts of economics. For example, the Gumbel extreme value distribution is the foundation of the logit specification for discrete choice with random demand (Luce 1959, McFadden’s 1981). This specification has been used widely in the analysis of product differentiation, regional economics, geography and trade, see e.g. Anderson et al. (1992), Dagsvik (1994), Dagsvik and Karlstrom (2005), Ibragimov and Walden (2010), and Armstrong (2014). In international trade, Eaton and Kortum (2002) and Bernard et al. (2003), used the aggregation properties of the Fréchet extreme value distribution to analyze international trade at the producer level (see also Chaney 2008, 2014). Gabaix (2011) shows that the tails of the firm size distribution play a role in macroeconomic fluctuations; extremes in the size distribution of firms can emerge from random growth (Gabaix 1999, Luttmer 2007) or the network structure of the buyer-supplier relations amongst firms (Acemoglu et al. 2012, Chaney 2014). Jones (2005) models the distribution of innovative ideas and analyzes the impact of this distribution on the bias of technical change. In applied finance, much use has been made of EVT in risk management and systemic risk analysis; see e.g. Jansen and De Vries (1991) for an early contribution and Ibragimov, Jaffee and Walden (2009, 2011) for recent examples. EVT is also used extensively in the theory of auctions (e.g., Hong and Shum 2004). We anticipate that the techniques developed in the current paper will be useful in the setups discussed above, in addition to the basic price-theoretic setup we discuss here (e.g. Bulow and Klemperer 2002, 2012, Weyl and Fabinger 2014 for recent developments in that area). As an example, Gabaix and Landier (2008) use
some of the current paper’s results to analyze the upper tail of the distribution of CEO talents.

The paper proceeds as follows. Section 2 presents the main economic result using the random-utility model of Perloff and Salop (1985). Section 3 elaborates by presenting the main mathematical result: an asymptotic approximation of a key integral that is needed to characterize economic environments in which extremes matter. We show that the tail of the noise distribution – captured by the tail exponent – is the crucial determinant of prices. As many common noise distributions have a tail index of zero, our results imply that in a wide range of market contexts additional competition has little effect on prices, once the market goes beyond a small number of firms. Section 4 considers alternative models of monopolistic competition (Sattinger, 1984; and Hart, 1985), and shows that the details of the demand-side modeling matter little, or not at all, to markups. Section 5 concludes.

We prove our main results (Theorems 1 and 2) in the Appendix, and our other results in an online Appendix.

2 How Much Does Competition Affect Prices?

In this section, we introduce the model of oligopolistic competition from Perloff and Salop (1985). Skipping over the methodological details (which we explore in greater detail in Section 3), we produce our key result: an asymptotic expression for price markups under oligopolistic competition. We then discuss a number of economic implications.

2.1 The Perloff-Salop Model

Our analysis is based on the Perloff-Salop (1985) model of monopolistic competition. There is a single representative consumer and an exogenously specified number of firms, $n$. The consumer must purchase exactly one unit of the differentiated good from one firm. He perceives that he will receive net utility $U_i = X_i - p_i$ by purchasing the good of firm $i$, where $X_i$ is a noise term representing a random taste shock, i.i.d. across firms and consumers, and $p_i$ is the price charged by firm $i$. Thus the consumer chooses to purchase the good that maximizes $X_i - p_i$. The timing is as follows:

1. Firms simultaneously set prices;
2. Random taste shocks are realized;
3. Consumers make purchase decisions;

4. Profits are realized.

The key economic object of interest is the price markup in a symmetric equilibrium, which we derive by solving the first-order condition for each firm’s profit maximization problem. Firm $i$’s profit function is given by

$$\pi_i = (p_i - c)D(p_1, ..., p_n; i)$$

where $D(p_1, ..., p_n; i)$ is the demand function for firm $i$ given the price vector $(p_1, ..., p_n)$ of the $n$ goods, and where $c$ is the marginal cost of production. The first order condition for profit maximization implies the following equilibrium markup in a symmetric equilibrium

$$p - c = -\frac{D(p, p; n)}{D_1(p, p; n)}.$$  \hspace{1cm} (3)

Here $p$ is the symmetric equilibrium price, $D(p, p'; n)$ denotes the demand function for a firm that sets price $p$ when there are $n$ goods and all other firms set price $p'$, and $D_1(p, p'; n) \equiv \partial D(p, p'; n) / \partial p$. Denote the markup $p - c$ in a symmetric equilibrium with $n$ firms as $\mu_n$.

In a symmetric-price equilibrium, the demand function of firm $i$ is the probability that the consumer’s surplus at firm $i$, $X_i - p_i$, exceeds the consumer’s surplus at all other firms,

$$D(p_1, ..., p_n; i) = \mathbb{P}(X_i - p_i \geq \max \{X_j - p_j\}) = \mathbb{P}(X_i \geq \max _{j \neq i} \{X_j\}).$$ \hspace{1cm} (4)

Let $M_n$ denote $\max \{X_1, ..., X_n\}$, which has density $n f(x) F^{n-1}(x)$.

Evaluation of (3) gives the following markup expression for the symmetric equilibrium of the Perloff-Salop model:

$$\mu_n = \frac{1}{n \mathbb{E}[f(M_{n-1})]} = \frac{1}{n (n - 1) \int f^2(x) F^{n-2}(x) dx}.$$ \hspace{1cm} (5)

Here $F$ is the distribution function and $f$ is the corresponding density of $X_i$.

Before proceeding to our analysis of the markup expression (5), let us briefly discuss our modeling approach. We use a stripped-down version of monopolistic competition for our analysis. In the model, the consumer’s payoff function takes a simple additive form. We show in Section 4 that our results do not rely on this specification. There, we analyze three

\footnote{Indeed, $\mathbb{P}(M_n \leq x) = \mathbb{P}(X_i \leq x \text{ for } i = 1...n) = \mathbb{P}(X_i \leq x)^n = F(x)^n$.}
other models of monopolistic competition. These models have (as in Perloff-Salop 1985) a representative consumer who has random i.i.d. taste shocks over producers, but differ in the form of consumer preferences. Our results from the present section are preserved in these alternative models, suggesting that the impact of competition of markups is independent of many institutional details of market competition.

A second feature of our model is that firms are completely symmetric ex ante, and thus each firm gets an equal $1/n$ market share in equilibrium. This assumption is strong, but enables tractable analysis; we conjecture that our main findings will be preserved when we extend the model to incorporate firm heterogeneity.

### 2.2 Extreme Value Theory: Some Basics

Now, we’ll very briefly introduce some necessary machinery; we put off the methodological details to Section 3. Recall from Section 2.1 that we define $M_n \equiv \max_{i=1,...,n} X_i$, to be the maximum of $n$ independent random variables $X_i$ with distribution $F$. Also, define the countercumulative distribution function $F^-(x) \equiv 1 - F(x)$.\(^7\) We are particularly interested in the connection between $M_n$ and $F^{-1}(1/n)$; informally (in analogy with the empirical distribution function), one may think of $F^{-1}(1/n)$ as the “typical” value of $M_n$. In fact, the key to our analysis is to formalize this relationship between $F^{-1}(1/n)$ and $M_n$ for large $n$.

Our analysis is restricted to what we call well-behaved distributions:

**Definition 1** Let $F$ be a distribution function with support on $(w_l,w_u)$. We say $F$ is well-behaved iff $f = F'$ is differentiable in a neighborhood of $w_u$, $\lim_{x \to w_u} F/f = a$ exists with $a \in [0, \infty]$, and

\[
\gamma = \lim_{x \to w_u} \frac{d}{dx} \left( \frac{F(x)}{f(x)} \right) \tag{6}
\]

exists and is finite. We call $\gamma$ the tail index of $F$.

Being well-behaved imposes a restriction on the right tail of $F$. The case $\gamma < 0$ consists of thin-tailed distributions with right-bounded support such as the uniform distribution. The case $\gamma = 0$ consists of distributions with tails of intermediate thickness. A wide range of economically interesting distributions fall within this domain, ranging from the relatively

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\(^7\)Strictly speaking, we abuse notation in cases where $F$ is not strictly increasing by using $F^{-1}(t)$ to denote $F^-(t) = F^-(1-t)$, where $F^-(t) = \inf \{x \in (w_l,w_u) : F(x) \geq t \}$ is the generalized inverse of $F$ (Embrechts et al. 1997, p.130). This is for expositional convenience; our results hold with the generalized inverse as well.
thin-tailed Gaussian distribution to the relatively thick-tailed lognormal distribution, as well as other distributions in between, such as the exponential distribution. The case $\gamma > 0$ consists of fat-tailed distributions such as Pareto’s power-law and the Fréchet distributions.

Being well-behaved in the sense of Definition 1 is not a particularly strong restriction. It is satisfied by most distributions of interest, and is easy to verify. In Section 2.3, Table 1 lists a number of popular densities and the corresponding tail index $\gamma$. Note that distributions with an exponential-like upper tail all have $\gamma = 0$.

### 2.3 How do markups change with competition?

The next theorem is our key result: it characterizes, asymptotically, the equilibrium markup as a function of the noise distribution and the number of competing firms. For this result, we make a few quite mild assumptions. We assume that $F$ is well-behaved, and that $f^2 (x)$ is $[w_1, w_\infty)$-integrable, and that the tail index $-1.45 \leq \gamma \leq 0.64$. Our main economic result is the following.

**Theorem 1** The symmetric equilibrium markups in the Perloff-Salop model is, asymptotically (for $n \to \infty$),

$$
\mu_n \sim \frac{1}{nf \left( \frac{1}{n} \right) \Gamma (\gamma + 2)}. \tag{7}
$$

with $F(x) \equiv 1 - F(x)$.

Theorem 1 has a number of striking economic implications. Most directly, it allows us to characterize the equilibrium markup for various noise distributions. See Table 1.

The distributions in Table 1 are generally presented in increasing order of fatness of the tails. For the uniform distribution, which has the thinnest tails, the markup is proportional to $1/n$. This is the same equilibrium markup generated by the Cournot model. However the uniform cum Cournot case is unrepresentative of the general picture. Table 1 implies that markups scale with $n^\gamma$. For the distributions reported in Table 1, $\gamma$ is bounded below by $-1$, so the uniform distribution is an extreme case.

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8 Condition (6) is well-known in the EVT literature as a second-order von Mises condition.

9 This is the range over which the second order condition holds (see the online appendix for details); the first order condition holds whenever $\gamma > -2$. Note that this assumption on $\gamma$ is not restrictive: it permits thin-tailed distributions such as the Weibull, and all (fat-tailed) Pareto distributions with finite variance.

10 The proof relies on Theorem 2, proven later, but to help the reader we start with the main economic result.
Table 1: Asymptotic Expressions for Markups

This table lists asymptotic markups (under symmetric equilibrium) for the Perloff-Salop model for various noise distributions as a function of the number of firms $n$. $f$ specifies the density function, and $\gamma$ specifies the distribution’s tail index. Distributions are listed in order of increasing tail fatness. Asymptotic approximations are calculated using Theorem 1 except where the markup can be exactly evaluated.

<table>
<thead>
<tr>
<th>Distribution</th>
<th>$f$</th>
<th>$\gamma$</th>
<th>$\mu_n$</th>
<th>$\lim_{n \to \infty} \mu_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniform</td>
<td>$1, \ x \in [-1, 0]$</td>
<td>-1</td>
<td>$1/n$</td>
<td>0</td>
</tr>
<tr>
<td>Bounded Power Law</td>
<td>$\alpha (-x)^{\alpha - 1}$, $\alpha &gt; 0, x \in [-1, 0]$</td>
<td>$-1/\alpha$</td>
<td>$\Gamma(1-1/\alpha+n)\alpha(2-1/\alpha)^{(1+n)} \sim \frac{n^{-1/\alpha}}{\alpha(2-1/\alpha)}$</td>
<td>0</td>
</tr>
<tr>
<td>Weibull</td>
<td>$\alpha (-x)^{\alpha - 1} e^{-(x)^{\alpha}}$, $\alpha \geq 1, x &lt; 0$</td>
<td>$-1/\alpha$</td>
<td>$\frac{1}{\alpha(2-1/\alpha)} \frac{n^{1-1/\alpha}}{n-1} \sim \frac{n^{-1/\alpha}}{\alpha(2-1/\alpha)}$</td>
<td>0</td>
</tr>
<tr>
<td>Gaussian</td>
<td>$(2\pi)^{-1/2} e^{-x^2/2}$</td>
<td>0</td>
<td>$\sim (2 \ln n)^{-1/2}$</td>
<td>0</td>
</tr>
<tr>
<td>Rootzen class, $\phi &gt; 1$</td>
<td>$\kappa \lambda \phi x^{\alpha} + \phi - 1 e^{-x^\phi}$</td>
<td>0</td>
<td>$\sim \frac{1}{\phi^{1/\phi}} (\ln n)^{1/\phi-1}$</td>
<td>0</td>
</tr>
<tr>
<td>Gumbel</td>
<td>$\exp(-e^{-x} - x)$</td>
<td>0</td>
<td>$\frac{n}{n-1}$</td>
<td>1</td>
</tr>
<tr>
<td>Exponential</td>
<td>$e^{-x}$, $x &gt; 0$</td>
<td>0</td>
<td>$\frac{1}{n}$</td>
<td>1</td>
</tr>
<tr>
<td>Rootzen Gamma ($\tau &lt; 1$)</td>
<td>$\tau x^{\tau - 1} e^{-x^{\tau}}$, $x &gt; 0, \tau &lt; 1$</td>
<td>0</td>
<td>$\sim \frac{1}{\tau} (\ln n)^{1/\tau-1}$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>Log-normal</td>
<td>$\frac{\exp(-2^{-1} \log^2 x)}{x \sqrt{2\pi}}$, $x &gt; 0$</td>
<td>0</td>
<td>$\sim \frac{1}{\sqrt{2\ln n}} e^{\sqrt{2\ln n}}$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>Power law</td>
<td>$\alpha x^{-\alpha - 1}$, $\alpha &gt; 1, x \geq 1$</td>
<td>$1/\alpha$</td>
<td>$\Gamma(1+1/\alpha+n)\alpha(2+1/\alpha)^{(1+n)} \sim \frac{n^{1/\alpha}}{\alpha(2+1/\alpha)}$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>Fréchet</td>
<td>$\alpha x^{-\alpha - 1} e^{-x^{-\alpha}}$, $\alpha &gt; 1, x \geq 0$</td>
<td>$1/\alpha$</td>
<td>$\frac{1}{\alpha(2+1/\alpha)} \frac{n^{1+1/\alpha}}{n-1} \sim \frac{n^{1/\alpha}}{\alpha(2+1/\alpha)}$</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>
For the distributions with the fattest tails, markups paradoxically rise as the number of competitors increases.\textsuperscript{11} Intuitively, for fat-tailed noise, as \( n \) increases, the difference between the best draw and the second-best draw, which is proportional to \( nf\left( F^{-1}(1/n) \right) \), increases with \( n \) (see Section 2.5 below).

**Markup sensitivity** Thin-tailed distributions (e.g. uniform) and fat-tailed distributions (e.g. power-laws) are the extreme cases in Table 1. Most of the distributional cases imply that competition typically has remarkably little impact on markups. For instance with Gaussian noise, the markup \( \mu_n \) is proportional to \( 1/\sqrt{\ln n} \), and the elasticity of the markup with respect to \( n \) is asymptotically zero. So \( \mu_n \) converges to zero, but this convergence proceeds at a glacial pace. Indeed, the elasticity of the markup with respect to \( n \) converges to zero.

To illustrate the slow convergence, we calculate \( \mu_n \) when noise is Gaussian for a series of values of \( n \). Table 2 shows that in the models we study and with Gaussian noise, a highly competitive industry with \( n = 1,000,000 \) firms will retain a third of the markup of a highly concentrated industry with only \( n = 10 \) competitors. We also compare markups in our monopolistic competition models to those in the Cournot model, which features markups proportional to \( 1/n \) and a markup elasticity w.r.t. \( n \) of \( -1 \) (note that this is equal to markups in the Perloff-Salop model with uniformly distributed noise.)

More generally, in cases with moderate fatness, such as the Gumbel (i.e. logit), exponential, and log-normal densities, the markup again shows little (or no) response to changes in \( n \). Nevertheless, the markups become unbounded for the lognormal distribution. Finally, the case of Bounded Power Law noise shows that an infinite support is not necessary for our results. In this case the markup is proportional to \( n^{-1/\alpha} \) and markup decay remains slow for large \( \alpha \).

In practical terms, these results imply that in markets with demand noise we should not necessarily expect increased competition to dramatically reduce markups. The mutual fund industry exemplifies such stickiness. Currently, 10,000 mutual funds are available in the U.S. and many of these funds offer nearly identical portfolios. Even in a narrow class of homogenous products, such as medium capitalization value stocks or high-yield bonds, it is normal to find 100 or more competing funds. Despite the large and rising number of competitors in such sub-markets, Hortacsu and Syverson (2004) report that mutual funds still charge high annual fees:

\textsuperscript{11}However, even though markups rise with \( n \), profits per firm go to zero (keeping market size constant) since firm prices scale with \( n^\gamma \) but sales volume per firm is proportional to \( 1/n \) in the Perloff-Salop case and \( 1/n^{1+\gamma} \) in the Sattinger case.
Table 2: Markups with Gaussian Noise and Uniform Noise
Markups are calculated for (i) the symmetric equilibrium of the Perloff-Salop model for Gaussian noise and (ii) under Cournot Competition, for various values of the number of firms \( n \). Note that markups in the Sattinger model and the Hart model are asymptotically equal, up to a constant factor \( c \), to markups in the Perloff-Salop model. \( n \) is the number of firms in the market. Markups are normalized to equal one when \( n = 10 \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>Markup with Gaussian noise</th>
<th>Markup under Cournot Competition</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>100</td>
<td>0.61</td>
<td>0.1</td>
</tr>
<tr>
<td>1,000</td>
<td>0.47</td>
<td>0.01</td>
</tr>
<tr>
<td>10,000</td>
<td>0.40</td>
<td>0.001</td>
</tr>
<tr>
<td>100,000</td>
<td>0.35</td>
<td>0.0001</td>
</tr>
<tr>
<td>1,000,000</td>
<td>0.32</td>
<td>0.00001</td>
</tr>
</tbody>
</table>

for most industry sectors, more than 1% of assets under management. These fees have changed very little as the number of homogeneous competing funds has increased by a factor of 10 over the past several decades. We note that the issue is more general: for instance, Henderson and Pearson (2011) find that structured equity produces also have robustly high mark-ups, and they hypothesize that this is related to investor confusion. Bergresser, Chalmers and Tufano (2009) find that mutual funds sold by brokers have anomalously high fees and low net-of-fee returns. Another complementary explanation may be that investors like the psychological comfort given by specific mutual fund brokers (Gennaioli, Shleifer, Vishny 2015), which is a way of microfounding the random noise in our model.

The discussion so far suggests that the markup function depends on the nature of the distribution \( f \). In fact, we can state this point more precisely. The following proposition shows that the tail parameter \( \gamma \) in (6) has a concrete economic implication: it is the asymptotic elasticity of the markup with respect to the number of firms. In other words, interpreting \( n \) as a continuous variable, the markup behaves locally as \( \mu \sim kn^\gamma \). We assume that the conditions in Theorem 1 hold, and further that \( \log F(x)f^2(x) \) is \([w_l, w_u]-\)integrable.

**Proposition 1** The asymptotic elasticity of the Perloff-Salop markup with respect to the num-
ber of firms $n$ is:
\[
\lim_{n \to \infty} \frac{n \, d\mu_n}{\mu_n \, dn} = \gamma.
\]

The proof of Proposition 1 is provided in the online Appendix. For taste shocks with distributions fatter than the uniform ($\gamma > -1$), Proposition 1 shows that the mark-up falls more slowly than $1/n$. In particular, $\gamma = 0$ corresponds to the case of intermediate tail thickness; it includes distributions ranging from the Gaussian to the lognormal, recall Table 1 with a list of popular densities and limit elasticities. Within this range, Proposition 1 tells us that markup elasticity goes to zero for large $n$; this bolsters our earlier point that competition has remarkably little effect on markups for such distributions.

### 2.4 Consumer Surplus

The random utility framework is sometimes criticized for generating an unrealistically high value for consumer surplus. Indeed, if the distribution is unbounded, the total surplus tends to $\infty$ as the number of firms increases. Our analytical results allow us to examine this criticism. For brevity, we restrict ourselves to the case with unbounded distributions and $\gamma \geq 0$. In our setup, expected gross surplus is $E[M_n]$, where $M_n$ is the highest of $n$ draws. We may show that $E[M_n] \sim \Gamma(2 - \gamma) F^{-1}(1/n)$ for $\gamma \geq 0$.

For all the distributions that we study except the unbounded power law case, $F^{-1}(1/n)$ rises only slowly with $n$. Hence, even for unbounded distributions, and large numbers of producers, consumer surplus can be quite small. For example, for the case of Gaussian noise when consumer preferences have a standard deviation of $1$, $F^{-1}(1/n) \sim \sqrt{2 \ln n}$. So, with a million toothpaste producers, consumer surplus averages only $5.25 per tube. Hence, in many instances, the framework — even with unbounded distributions — does not generate counterfactual predictions about consumer surplus or counterfactual predictions about the prices that cartels would set.

### 2.5 Limit Pricing: An Alternative Interpretation

We now discuss an alternative model of oligopolistic competition, which is sometimes called “limit pricing” and has proven to be very useful in trade and macroeconomics (e.g. Bernard et al. 2003, see also Auer and Chaney 2009). The price-setting mechanism in the limit pricing model is remarkably simple, yet it produces (asymptotically) the same markups as the

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12This result is an immediate application of Theorem 2, which we present in Section 3.
Perloff-Salop (1985) model. This equivalence result implies that a similar logic underlies the equilibrium markups for these models, and generates an simple but useful interpretation of our economic results.

In the limit pricing model, each firm $i$ draws a quality shock $X_i$. Each firm simultaneously sets prices $p_i$ after observing every other firms quality shocks. (This is in contrast with the models of Section 2.1, where prices are set before taste shocks are observed.) The representative consumer needs to consume one unit of the good, and picks the firm which maximizes $X_i - p_i$. As before, call $M_n = \max_{i=1\ldots n} X_i$ the largest quality draw from the $n$ firms, and $S_n$ the second-largest draw. In the competitive equilibrium, the firm with the highest quality, $M_n$, gets all the market share, and sets a price $p = c + M_n - S_n$. (This is just enough to take all the market away from the firm with the second-highest quality.) So, the equilibrium markup is $\mu^L = M_n - S_n$.

The next Proposition analyzes the equilibrium markup under Limit Pricing. We assume that $F$ is well-behaved with tail index $\gamma < 1$, and that $xf(x)$ is $[w_1, w_\infty)$-integrable.

**Proposition 2** Call $M_n$ and $S_n$, respectively, the largest and second largest realizations of $n$ i.i.d. random variables with CDF $F$. Then limit pricing markup is $\mu^L = M_n - S_n$, and

$$E[\mu^L] \sim_{n \to \infty} \frac{\Gamma(1-\gamma)}{nf \left( F^{-1} \left( \frac{1}{n} \right) \right)}.$$  

Notice that this markup is asymptotic to the markup from Theorem 1. This suggests the following intuition for Theorem 1: to set its optimum price, a firm conditions on its getting the largest draw, then evaluates the likely draw of the second highest firm, and engages in limit pricing, where it charges a markup equal to the difference between its draw and the next highest draw: $E[\mu^L] \approx M_n - S_n$. This is analogous to the analysis of a first price sealed bid auction. In fact, this reasoning gets us approximately the correct answer: observe that $E[\mathcal{F}(M_n)] \approx \frac{1}{n+1}$ and $E[\mathcal{F}(S_n)] \approx \frac{2}{n+1}$, which suggests that $M_n$ (the highest draw) will be
close to $F^{-1}\left(\frac{1}{n}\right)$ and that $S_n$ (the second-highest draw) will be close to $F^{-1}\left(\frac{2}{n}\right)$. So,

$$
\mathbb{E}\left[\mu_n^{LP}\right] \approx M_n - S_n \approx F^{-1}\left(1/n\right) - F^{-1}\left(1/n + 1/n\right)
\approx -\frac{1}{n} \cdot \frac{dF^{-1}(x)}{dx}\bigg|_{x=\frac{1}{n}} \text{ by Taylor expansion}
= \frac{1}{nf\left(F^{-1}(1/n)\right)}.
$$

In fact, revisiting Theorem 1, we see that this heuristic argument generates the right approximation for the Perloff-Salop markups when $\gamma = 0$ (e.g. Gaussian, logit (Gumbel), exponential, and lognormal distributions), and that the approximation remains accurate up to a corrective constant in the other distributions.

### 3 Methodological Results

In this section we state our main mathematical results. Solving for the symmetric equilibrium outcome in the models discussed above, for distribution function $F$, requires the evaluation of integrals of the form

$$
\int x^j e^{\psi x} f^k(x) F(x)^{n-l} dx
$$

where $k, l \geq 1$ and $j, \psi \geq 0$. For large $n$, such integrals mainly depend on the tail of the distribution $F$, which suggests that techniques from Extreme Value Theory (EVT) may be applied. (See Resnick (1987), and Embrechts et al. (1997) for an introduction to EVT.) We introduce concepts and notation in Section 3.1. We state our main results in Section 3.2, and briefly discuss the application to our markup results. Proofs are provided in the Appendix.

#### 3.1 Concepts and Notation from EVT

This section develops the mathematical tools that we will use to asymptotically evaluate (9).

Definition 1 ensures that the right tail of $F$ behaves appropriately. To ensure that the integral (9) does not diverge, we also impose some restriction on the rest of $F$. The following notation will simplify the exposition of our results.
Definition 2 Let \( j : \mathbb{R} \to \mathbb{R} \) have support on \((w_l, w_u)\). The function \( j(x) \) is \([w_l, w_u]\)-integrable iff
\[
\int_{w_l}^{w} |j(x)| \, dx < \infty
\]
for all \( w \in (w_l, w_u) \).

For example, in Theorem 1 we require that \( f^2 \) be \([w_l, w_u]\)-integrable. Verification of this condition is typically straightforward; it is useful to note, for example, that \( f^2(x) \) is \([w_l, w_u]\)-integrable if \( f = F' \) is uniformly bounded.

Finally, the following definition of regular variation will be useful.

Definition 3 A function \( h : \mathbb{R}^+ \to \mathbb{R} \) is regularly varying at \( \infty \) with index \( \rho \) if \( h \) is strictly positive in a neighborhood of \( \infty \), and
\[
\forall \lambda > 0, \lim_{x \to \infty} \frac{h(\lambda x)}{h(x)} = \lambda^\rho.
\]
(10)

We indicate this by writing \( h \in RV^\infty_\rho \).

Analogously, we say that \( h : \mathbb{R}^+ \to \mathbb{R} \) is regularly varying at zero with index \( \rho \) if, \( \forall \lambda > 0, \lim_{x \to 0} h(\lambda x)/h(x) = \lambda^\rho \), and denote this by \( h \in RV^0_\rho \). Intuitively, a regularly varying function \( h(x) \) with index \( \rho \) behaves like \( x^\rho \) as \( x \) goes to the appropriate limit, perhaps up to logarithmic corrections. For instance, \( x^\rho \) and \( x^\rho |\ln x| \) are regularly varying (with index \( \rho \)) at both 0 and \( \infty \). Much of our analysis will require the concept of regular variation; specifically, we will require that certain transformations of the noise distribution \( F \) be regularly varying. In the case \( \rho = 0 \), we say that \( h \) is slowly varying (for example \( \ln x \) varies slowly at infinity and zero).

Finally, following the notation of Definition 1, define
\[
w_l = \inf\{x : F(x) > 0\} \quad \text{and} \quad w_u = \sup\{x : F(x) < 1\}
\]
(11)
to be the lower and upper bounds of the support of \( F \), respectively.

3.2 Core Mathematical Result, and Applications

Our core mathematical result documents an asymptotic relationship between \( M_n \) and \( F^{-1}(1/n) \).
Theorem 2 Let $F$ be a differentiable CDF with support on $(w_l, w_u)$, $\bar{F} = 1 - F$, $f = F'$, and assume that $F$ is strictly increasing in a left neighborhood of $w_u$. Let $G : (w_l, w_u) \rightarrow \mathbb{R}$ be a strictly positive function in a left neighborhood of $w_u$. Suppose that $\hat{G}(t) \equiv G \left( F^{-1}(t) \right) \in RV^\circ_\rho$ with $\rho > -1$, and that $|\hat{G}(t)|$ is integrable on $t \in (\overline{t}, 1)$ for all $\overline{t} \in (0, 1)$ (or, equivalently, $G(x)f(x)$ is $[w_l, w_u]$-integrable in the sense of definition 2). Then, for $n \rightarrow \infty$

$$\mathbb{E}[G(M_n)] = \int_{w_l}^{w_u} nG(x)f(x)F(x)^{n-1}dx \sim G \left( F^{-1} \left( \frac{1}{n} \right) \right) \Gamma(\rho + 1)$$

(12)

where $M_n$ is the largest realization of $n$ i.i.d. random variables with CDF $F$.

The intuition for equation (12) is as follows. By definition of $M_n$, if $X$ is distributed as $F$ and if $M_n$ and $X$ are independent, then $\mathbb{P}[X > M_n] = 1/(n+1)$; that is, $\mathbb{E}[\bar{F}(M_n)] = 1/ (n+1) \approx 1/n$. Consequently, we might conjecture (via heroic commutation of the expectations operator) that

$$\mathbb{E}[M_n] \approx \bar{F}^{-1} \left( \frac{1}{n} \right)$$

(13)

and that $\mathbb{E}[G(M_n)] \approx G(\mathbb{E}[M_n]) \approx G \left( \bar{F}^{-1}(1/n) \right)$. It turns out that this heuristic argument gives us the correct approximation, up to a correction factor $\Gamma(\rho + 1)$\textsuperscript{13}

We next present an intermediate result that is technically undemanding but will allow us to apply Theorem 2 to expressions of the form (9).

Lemma 1 Let $F$ be well-behaved with tail index $\gamma$. Then

1. $f \left( \bar{F}^{-1}(t) \right) \in RV^\circ_{\gamma+1}$.

2. If $w_u = \infty$, then $\bar{F}^{-1}(t) \in RV^0_{-\gamma}$. If $w_u < \infty$, then $w_u - \bar{F}^{-1}(t) \in RV^0_{-\gamma}$.

3. If $\alpha$ is finite, then $e^{\bar{F}^{-1}(t)} \in RV^0_{-\alpha}$.

\textsuperscript{13}To understand the correction factor, start with the linear case $G(x) = x$, in which case the theorem gives $E[M_n] \sim \bar{F}^{-1}(1/n) \Gamma(-\gamma + 1)$. Then the correction factor arises because the distribution of the maximum is $F^n(x)$, not $F(x)$. For distributions with an exponential type tail, $\gamma = 0$ and no correction is required. For distributions with a power type tail and finite mean, $\gamma \in (0, 1)$, an upward correction is needed. To provide some intuition for this, consider the log $(-\log \mathbb{P} \{ M_n \leq t \})$, and where the distribution $F$ is either Gumbel or Fréchet, see Table 3. In case of the Gumbel one finds $\log n - t$, while the Fréchet gives $n - \alpha \log t$. Take $n$ and $t$ large. In the Gumbel case $n$ plays a minor role, while in the case of the distribution $n$ and $t$ are of similar order of magnitude, so that $n$ affects the distribution and its moments. More generally, if $G(x)$ is not linear, the tail behavior of $G(x)$ interacts with the tail behavior of $F(x)$. Both functions then determine $\rho$ in the correction factor as indicated in the theorem. For example, take $G(x) = x^m$ and $F(x) = 1 - x^{-\alpha}$, $m < \alpha$, then $\mathbb{E}[(M_n)^m] \approx n^{m/\alpha} \Gamma(1 - m/\alpha)$. 17
Lemma 1 ensures that when $F$ is well-behaved, (9) satisfies the conditions imposed in Theorem 2 for a wide range of parameter values. The following proposition is then an immediate implication of Theorem 2 and Lemma 1.

**Proposition 3** Let $F$ be well behaved with tail index $\gamma$. Let $j, \psi \geq 0$, $k \geq 1$ and let $x^j e^{\psi x} f^k(x)$ be $[w_l, w_u]$-integrable. If $j > 0$, assume that $w_u > 0$. If $\psi = 0$, we can treat $\psi a = 0$ in the following expressions. If $(k-j-1) \gamma - \psi a + k > 0$, then as $n \to \infty$,

\[
\int_{w_l}^{w_u} x^j e^{\psi x} f^k(x) F(x)^{n-l} dx \\
\sim \begin{cases} 
\frac{1}{n^{l-1}} \left( \frac{1}{F^{-1}(1/n)} \right)^j e^{\psi F^{-1}(1/n)} (F^{-1}(1/n))^\Gamma ((k-j-1) \gamma - \psi a + k) : w_u = \infty \\
\frac{1}{n^{l-1}} w_u^j e^{\psi w_u} f^{k-1}(F^{-1}(1/n)) \Gamma ((k-1) \gamma + k) : w_u < \infty 
\end{cases}
\]

Proposition 3 allows us to approximate (9) for well-behaved distributions\(^{14}\). The parameter restriction $((k-j-1) \gamma - \psi a + k > 0$ is necessary to ensure that (9) does not diverge. For our purposes, this restriction is rather mild. One notable exception is that when $\psi > 0$, we cannot analyze heavy-tailed distributions (which have fatter-than-exponential tails) such as the lognormal distribution; for these distributions, $a = \infty$.\(^{15}\)

In fact, Theorem 1 is now an immediate corollary of Proposition 3. More generally, these results are relatively easy to apply. For example, the key mathematical objects in Theorem 1, $\gamma$ and $nf \left( F^{-1}(1/n) \right)$, are easy to calculate for most distributions of interest, and are listed for commonly used distributions in Table 3. The following fact, which is easily verified using Lemma 3 part 6, may often be useful to simplify calculations further for the case $\gamma \neq 0$: as $n \to \infty$,

\[
\frac{1}{nf \left( F^{-1}(1/n) \right)} \sim \begin{cases} 
\frac{\gamma F^{-1}(1/n)}{a}, & \gamma > 0 \\
-\gamma (w_u - F^{-1}(1/n)), & \gamma < 0 
\end{cases}
\]

\(^{14}\)For an antecedent to this result, see Maller and Resnick (1984).

\(^{15}\)Here we define a distribution to be heavy-tailed if $e^{\lambda x} F(x) \to \infty$ as $x \to \infty$ for all $\lambda > 0$. To see why $a = \infty$ in this case, note that $\lim_{x \to -\infty} F(x) / f(x) = \infty$ implies $-\frac{d}{d x} \log F(x) = o(1)$ as $x \to \infty$, so $-\log F(x) = o(x)$ and $e^{-\lambda x} = o(F(x))$ for all $\lambda$. 

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Table 3: Properties of Common Densities

Densities are listed in order of increasing tail fatness whenever possible.

<table>
<thead>
<tr>
<th>Density</th>
<th>$f$</th>
<th>$\gamma$</th>
<th>$nf\left(F^{-1}(1/n)\right)$</th>
<th>$F^{-1}(1/n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniform</td>
<td>$1, \ x \in [-1,0]$</td>
<td>$-1$</td>
<td>$n$</td>
<td>$-\frac{1}{n}$</td>
</tr>
<tr>
<td>Bounded Power Law</td>
<td>$\alpha (-x)^{\alpha-1}, \alpha \geq 1, \ x \in [-1,0]$</td>
<td>$-1/\alpha$</td>
<td>$\alpha n^{1/\alpha}$</td>
<td>$-n^{-1/\alpha}$</td>
</tr>
<tr>
<td>Weibull</td>
<td>$\alpha(-x)^{\alpha-1}e^{-(x)^{\alpha}}, \alpha \geq 1, \ x &lt; 0$</td>
<td>$-1/\alpha$</td>
<td>$\alpha n^{1/\alpha}$</td>
<td>$-n^{-1/\alpha}$</td>
</tr>
<tr>
<td>Gaussian</td>
<td>$(2\pi)^{-1/2}e^{-x^2/2}$</td>
<td>$0$</td>
<td>$\sim \sqrt{2\ln n}$</td>
<td>$\sim \sqrt{2\ln n}$</td>
</tr>
<tr>
<td>Rootzen Class</td>
<td>$\kappa_\lambda x^{\alpha+\phi-1}e^{-x^\phi}, \ x &gt; 0, \phi &gt; 1$</td>
<td>$0$</td>
<td>$\sim \phi \lambda^{1/\phi} (\ln n)^{1-1/\phi}$</td>
<td>$\sim (\ln n)^{1/\phi}$</td>
</tr>
<tr>
<td>Gumbel</td>
<td>$\exp(-e^{-x} - x)$</td>
<td>$0$</td>
<td>$\sim 1$</td>
<td>$\sim \ln n$</td>
</tr>
<tr>
<td>Exponential</td>
<td>$e^{-x}, \ x &gt; 0$</td>
<td>$0$</td>
<td>$1$</td>
<td>$\ln n$</td>
</tr>
<tr>
<td>Log-normal</td>
<td>$(2\pi)^{-1/2}x^{-1/2}e^{-(\log x)^2/2}, \ x &gt; 0$</td>
<td>$0$</td>
<td>$\sim \frac{\sqrt{2\ln n}}{F^{-1}(1/\alpha)}$</td>
<td>$\sim e^{\sqrt{2\ln n}}$</td>
</tr>
<tr>
<td>Power law</td>
<td>$\alpha x^{-\alpha-1}, \alpha &gt; 0, \ x \geq 1$</td>
<td>$1/\alpha$</td>
<td>$\alpha n^{-1/\alpha}$</td>
<td>$n^{1/\alpha}$</td>
</tr>
<tr>
<td>Fréchet</td>
<td>$\alpha x^{-\alpha-1}e^{-x^-\alpha}, \alpha &gt; 0, \ x \geq 0$</td>
<td>$1/\alpha$</td>
<td>$\alpha n^{-1/\alpha}$</td>
<td>$n^{1/\alpha}$</td>
</tr>
</tbody>
</table>

4 Applications

This section discusses a few applications that demonstrate the broader applicability of our methodological and economic results. In Section 4.1, we consider three alternative models of monopolistic competition with random demand (mentioned in Section 2.1), and show in Section 4.2 that they also obey the asymptotic markup rule of Theorem 1. In Section 4.3, we show that our methods apply to second-price auctions with random demand as well, and that a version of the “limit-pricing” logic from Section 2.5 holds in this setting.

4.1 Alternative Models of Monopolistic Competition

In this section, we describe two existing models of monopolistic competition with random demand, and introduce a new model that is a modification of Perloff-Salop (1985). These models share the same timing as Perloff-Salop (1985); so firms set prices simultaneously, before they learn their own (or others’) quality shocks. However, they differ from Perloff-Salop (1985) in the specification of consumer preferences. Some common notation and assumptions: in each of the following models, there is a single representative consumer and $n$ firms, indexed as $i = \{1, \ldots, n\}$, in a monopolistically competitive market. Also, $p_i$ is the price of good $i$, and the random shocks $X_i$ associated with each good $i$ are i.i.d. randomly distributed with
distribution function $F$.

**Sattinger (1984): Consumers demand a fixed dollar amount in the aggregate**

Sattinger (1984) analyzes the case of multiplicative random demand. There are two types of goods. Besides the monopolistically competitive market, there is separately a composite good purchased from an industry with homogenous output. Our focus is on markups in the monopolistically competitive market. The consumer has utility function

$$U = Z^{1-\theta} \sum_{i=1}^{n} A_i Q_i^\theta,$$  

(14)

where $Z$ is the quantity of the composite good, $A_i = \exp(X_i)$ is the random taste shock, and $Q_i$ is the quantity consumed of good $i$. The consumer faces the budget constraint $y = qZ + \sum_i p_i Q_i$ where $y$ is the consumer’s endowment and $q$ is the price of the composite good. We can show that firm $i$’s demand function is

$$D(p_1, \ldots, p_n; i) = \frac{\theta y}{p_i} \left( \frac{e^{X_i}}{p_i} \geq \max_{j \neq i} \frac{e^{X_j}}{p_j} \right).$$  

(15)

Evaluation of (3), which we relegate to the online appendix, gives the following symmetric equilibrium markup expression:

$$\frac{\mu_n^{Sat}}{c} = \frac{1}{n (n-1) \int f^2(x) F_{n-2}(x) dx} = \frac{1}{n \mathbb{E}[f(M_{n-1})]}.$$  

(16)

**Hart (1985): Consumers’s demand is flexible in quantity and value**

Hart (1985) analyzes a setup where both the quantity and the dollar amount spent depend on the prospective utility of the good purchased. In comparison, in the Perloff-Salop model, the quantity demanded is fixed; while in the Sattinger model, dollar expenditure is fixed. The Hart model thus allows us to study the impact of competition in a slightly richer economic
The consumer’s utility function is:

\[ U^{Hart} = \frac{\psi + 1}{\psi} \left( \sum_{i=1}^{n} A_i Q_i \right)^{\psi/(\psi+1)} - \sum_{i=1}^{n} p_i Q_i. \]  

(17)

where \( A_i = e^{X_i} \) is the associated random taste shock for good \( i \) and \( Q_i \) is the consumed quantity. Hart shows that the demand function for firm \( i \) is

\[ D^{Hart}(p_1, \ldots, p_n; i) = \mathbb{E} \left[ \frac{e^{\psi X_i}}{p_i + \psi} \left\{ e^{X_i/p_i} \mathbb{1}_{\{X_i \geq \max_{j \neq i} X_j \}} e^{X_j/p_j} \right\} \right]. \]  

(18)

We may use this to evaluate (3), which gives the following symmetric equilibrium markup expression:

\[ \frac{\mu_n^{Hart}}{c} = \psi + (n-1) \frac{1}{\mathbb{E}[e^{\psi f^2(x)F_n(x)dx}]} = \psi + (n-1) \frac{1}{\mathbb{E}[e^{\psi f^2(M_{n-1})}]} \mathbb{E}[e^{M_{n-1} f(M_{n-1})}]. \]  

(19)

Note that in the special case \( \psi = 0 \), by comparing (15) with (18) and (16) with (19), we see that \( D^{Hart}(p_1, \ldots, p_n; i) = D^{Sat}(p_1, \ldots, p_n; i) \) and \( \mu_n^{Hart} = \mu_n^{Sat} \); that is, the Hart model generates the same demand functions and markups as the Sattinger model.

**Enriched Linear Random Utility**

For our final alternative model, we add two features drawn from recent random demand models (see, for example, Berry, Levinsohn and Pakes 1995) to the Perloff-Salop model: an outside option good, and stochastic consumer price sensitivity. We call this extension the Enriched Linear Random Utility (ELRU) model. The consumer chooses either to purchase exactly one unit of the good from one firm in the monopolistically competitive market, or to take his outside option. The consumer’s utility from consuming firm \( i \)’s good is

\[ U_i^{ELRU} = -\beta p_i + X_i, \quad U_0^{ELRU} = \epsilon_0, \]

\[ U^{ELRU} = \sum_{i=1}^{n} U_i^{ELRU} = \sum_{i=1}^{n} (-\beta p_i + X_i) + \epsilon_0. \]
where $p_i$ is the price of good $i \in \{1, 2, 3, \ldots\}$ (set by firm $i$), $\beta \geq 0$ is a "taste for money", $\epsilon_0 \geq 0$ is the value of the consumer’s outside option, and $X_i$ is the random taste shock associated with good $i$. Each of $X_1, \ldots, X_n$ are (in addition to being i.i.d.) independent of and $(\beta, \epsilon_0)$, but $\epsilon_0$ may not be independent of $\beta$.

The demand function for good $i$ at price $p$ is the probability that the consumer’s payoff for good $i$ exceeds his payoff to all other goods, as well as the outside option:

$$D(p_1, \ldots, p_n; i) = \mathbb{P} \left( -\beta p_i + X_i \geq \max \left\{ \max_{j \neq i} \{ -\beta p_j + X_j \}, \epsilon_0 \right\} \right).$$

Evaluation of (3) gives the following symmetric equilibrium markup expression:

$$\mu_n^{ELRU} = \frac{\mathbb{E} \left[ 1 - F \left( \max \left\{ M_{n-1}, \beta p + \epsilon_0 \right\} \right) \right]}{\mathbb{E} \left[ f \left( \max \left\{ M_{n-1}, \beta p + \epsilon_0 \right\} \right) \right]}.$$ (20)

Note that if we set $\epsilon_0 = -\infty$ and $\beta$ to be a constant, this simplifies to the Perloff-Salop model.

### 4.2 Equilibrium Markups: Detail Independence

We now characterize equilibrium markups for the Sattinger (1984) and Hart (1985) models.

As in Theorem 1, we assume that $F$ is well-behaved, and that $f^2(x)$ is $[w_l, w_u]$-integrable. For the Sattinger model, assume that $-1.45 \leq \gamma \leq 0.64$.\(^{17}\) For the Hart model with parameter $\psi$, assume that $-1 < \gamma \leq 0$; if $\gamma = 0$, we further require that $1 - \psi a > 0$.\(^{18}\) Denote the Perloff-Salop, Sattinger and Hart markups as, respectively, $\mu_n$, $\mu_n^{Sat}$, and $\mu_n^{Hart}$. The following theorem states that (up to a factor $c$) that all three markups are asymptotically equal; in fact, the Sattinger markup is exactly equal to the Perloff-Salop markup.

**Theorem 3** The symmetric equilibrium markups in the Perloff-Salop, Sattinger and Hart models are asymptotically

$$\mu_n = \frac{\mu_n^{Sat}}{c} \sim \frac{\mu_n^{Hart}}{c} \sim \frac{1}{nf \left( \mathcal{F}^{-1} \left( \frac{1}{n} \right) \right) \Gamma(\gamma + 2)}.$$ (21)

with $\mathcal{F}(x) \equiv 1 - F(x)$.

\(^{17}\) As with the Perloff-Salop (1985) model, this is the range over which the second order condition holds (see the online appendix for details).

\(^{18}\) For distributions violating this condition, no symmetric price equilibrium can be calculated in the Hart model because each firm would face infinite demand.
Next, we turn our attention to the ELRU model. We limit the calculation to the case where the densities \( f(x) \) of the Rootzen type

\[
f(x) \sim_{x \to \infty} \kappa \lambda \phi x^{\phi + \nu - 1} \exp \left( -\lambda x^\phi \right), \quad \kappa > 0, \ \lambda > 0, \ \phi \geq 1, \ \nu \in \mathbb{R}
\]

such distributions have tail index \( \gamma = 0 \) and \( a < \infty \). This restriction allows us to retain common distributions such as the Gumbel, Gaussian, and Exponential.\(^{19}\) For example, the Gaussian distribution is a special case of (22), with \( \phi = 2, \ \lambda = 1/2, \ \kappa = 1/\sqrt{2\pi} \) and \( a = -1 \).

**Theorem 4** Assume that the density of the taste shock \( f \) takes the form (22). Assume also that the random variables \( \beta \) and \( \epsilon_0 \) have bounded support, and that \( \mathbb{E} [\beta]^2 > \text{Var}[\beta] \). Then the symmetric equilibrium markup in the ELRU model is asymptotically

\[
\mu_n^{ELRU} \sim \frac{1/\mathbb{E}[\beta]}{\phi \lambda^{1/\phi} (\ln n)^{1-1/\phi}}.
\]

Table 4 computes and compares the Perloff-Salop, Sattinger and Hart asymptotic markups using Theorem 3. It is easy to check that for Rootzen-class distributions, the ELRU markup expression from (23) is asymptotically equal (up to the factor \( \mathbb{E} [\beta] \)) to the Perloff-Salop / Sattinger / Hart markups.

Theorems 3 and 4 deliver the perhaps unexpected result that the Perloff-Salop (1985), Sattinger (1984), Hart (1985) and ELRU models generate asymptotically equal (up to a multiplicative constant) markups. Hence, they exhibit a sort of detail-independence: equilibrium markups do not depend on the details of the model of competition. The key ingredient in the modeling is the specification of the noise distribution, rather than the details of the particular oligopoly model. In particular, it suggests that the limit-pricing logic of Section 2.5 has broad applicability to random-demand models of monopolistic competition.

### 4.3 Auctions

Now, we stray from the setting of monopolistic competition and consider an application to auctions. Consider a second-price auction with a single good and \( n \) bidders where each bidder \( i \) privately values the good at \( X_i \), which is i.i.d. with CDF \( F \). It is well-known that if

\(^{19}\)These distributions produce equilibrium markups that are weakly decreasing with the degree \( n \) of competition. We may easily check that for this class, \( \gamma = 0 \) and \( a = 0 \) whenever \( \phi > 1 \).
Table 4: Asymptotic Expressions for Sattinger and Hart Markups
This table reproduces Table 1 and adds asymptotic markups for the Sattinger and Hart models. \( \mu_n, \mu_n^{Satt} \) and \( \mu_n^{Hart} \) are respectively the asymptotic markup expressions for the Perlo-Salop, Sattinger, and Hart models. Asymptotic approximations are calculated using Theorem 1 except where the markup can be exactly evaluated. The Hart markup is undefined for distributions fatter than the exponential.

<table>
<thead>
<tr>
<th>Distribution</th>
<th>( f )</th>
<th>( \mu_n = \mu_n^{Satt} / c )</th>
<th>( \mu_n^{Hart} / c )</th>
<th>( \lim_{n \to \infty} \mu_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniform</td>
<td>( 1, \ x \in [-1,0] )</td>
<td>( \frac{1}{n} )</td>
<td>( \sim \frac{n^{-1/\alpha}}{\alpha(2-1/\alpha)} )</td>
<td>0</td>
</tr>
<tr>
<td>Bounded Power Law</td>
<td>( \alpha(-x)^{\alpha-1} ) ( \alpha &gt; 0, x \in [-1,0] )</td>
<td>( \frac{\Gamma(1-1/\alpha+n)}{\alpha(2-1/\alpha)\Gamma(1+n)} \sim \frac{n^{1-1/\alpha}}{\alpha(2-1/\alpha)} )</td>
<td>( \sim \frac{n^{-1/\alpha}}{\alpha(2-1/\alpha)} )</td>
<td>0</td>
</tr>
<tr>
<td>Weibull</td>
<td>( \alpha(-x)^{\alpha-1}e^{-(x)^\alpha} ) ( \alpha \geq 1, x &lt; 0 )</td>
<td>( \frac{1}{\alpha(2-1/\alpha)} \frac{n^{1-1/\alpha}}{n-1} \sim \frac{n^{-1/\alpha}}{\alpha(2-1/\alpha)} )</td>
<td>( \sim \frac{n^{-1/\alpha}}{\alpha(2-1/\alpha)} )</td>
<td>0</td>
</tr>
<tr>
<td>Gaussian</td>
<td>( (2\pi)^{-1/2} e^{-x^2/2} )</td>
<td>( \sim (2 \ln n)^{-1/2} )</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>Rootzen class, ( \phi &gt; 1 )</td>
<td>( \kappa \lambda x^{\alpha+\phi-1}e^{-x^\phi} )</td>
<td>( \sim \frac{1}{\phi \lambda^{1/\phi}} (\ln n)^{1/\phi-1} )</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>Gumbel</td>
<td>( \exp(-e^{-x} - x) )</td>
<td>( \frac{n}{n-1} )</td>
<td>( \sim 1 )</td>
<td>1</td>
</tr>
<tr>
<td>Exponential</td>
<td>( e^{-x}, \ x &gt; 0 )</td>
<td>( 1 )</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>Rootzen Gamma</td>
<td>( \tau x^{\tau-1}e^{-x^\tau} ) ( \tau &gt; 0, \tau &lt; 1 )</td>
<td>( \sim \frac{1}{\tau} (\ln n)^{1/\tau-1} )</td>
<td></td>
<td>( \infty )</td>
</tr>
<tr>
<td>Log-normal</td>
<td>( \frac{\exp(-2^{-1} \log^2 x)}{x^{\sqrt{2}n}} ) ( x &gt; 0, \tau &lt; 1 )</td>
<td>( \sim \frac{1}{\sqrt{2 \ln n}} e^{\sqrt{2 \ln n}} )</td>
<td></td>
<td>( \infty )</td>
</tr>
<tr>
<td>Power law</td>
<td>( \alpha x^{-\alpha-1} ) ( \alpha &gt; 0, x \geq 1 )</td>
<td>( \frac{\Gamma(1+1/\alpha+n)}{\alpha(2+1/\alpha)\Gamma(1+n)} \sim \frac{n^{1/\alpha}}{\alpha(2+1/\alpha)} )</td>
<td></td>
<td>( \infty )</td>
</tr>
<tr>
<td>FrÃ¶lchert</td>
<td>( \alpha x^{-\alpha-1}e^{-x^{-\alpha}} ) ( \alpha &gt; 0, x \geq 0 )</td>
<td>( \frac{1}{\alpha(2+1/\alpha)} \frac{n^{1+1/\alpha}}{n-1} \sim \frac{n^{1/\alpha}}{\alpha(2+1/\alpha)} )</td>
<td></td>
<td>( \infty )</td>
</tr>
</tbody>
</table>
$F$ is strictly increasing on $(w_l, w_u)$, then the equilibrium outcome of this auction is that each bidder makes a bid equal to his private valuation; the bidder with the highest valuation ($M_n$) wins and pays the second-highest valuation ($S_n$). Thus the expected revenue in a second price auction equals $\mathbb{E}[S_n]$, and the expected surplus for the winner in a second price auction equals $\mathbb{E}[M_n - S_n]$. We can apply Theorem 2 and Lemma 1 to obtain asymptotic approximations for both of these expressions.\textsuperscript{20} Since this is closely related to the results for the limit pricing model, we state the following without proof.

**Proposition 4** Let $F$ be well-behaved with tail index $\gamma < 1$, and assume that $xf(x)$ is $[w_l, w_u)$-integrable. Then in a second-price auction where valuations are i.i.d. as $F$, the expected revenue to the seller, $\mathbb{E}[S_n]$, is

$$
\mathbb{E}[S_n] \sim_{n \to \infty} F^{-1}(1/n) \Gamma(2 - \gamma) \text{ if } w_u = \infty,
$$

$$
\mathbb{E}[S_n] = w_u - \left( w_u - F^{-1}(1/n) \right) \Gamma(2 - \gamma) + o\left( w_u - F^{-1}(1/n) \right) \text{ if } w_u < \infty,
$$

and the expected surplus for the winner of the auction is:

$$
\mathbb{E}[M_n - S_n] \sim_{n \to \infty} \frac{\Gamma(1 - \gamma)}{n f\left( F^{-1}(1/n) \right)}.
$$

(24)

## 5 Conclusion

The choice of noise distributions in random demand models is often influenced by tractability concerns. It is important to understand the consequences of these modeling choices and, when possible, to greatly expand the set of tractable models. With this challenge in mind, our paper makes three sets of contributions.

First, we derive equilibrium markups for general noise distributions in various types of monopolistically competitive markets. We show that markups are asymptotically determined by the tail behavior of the distribution of taste shocks.

Second, our results reveal a surprising degree of “detail-independence.” Specifically, the behavior of price markups are asymptotically identical (up to a constant factor) for all models that we study. Moreover, for the wide class of distributions with a zero extreme value tail

\textsuperscript{20}Result (24) appeared in Caserta (2002, Prop. 4.1) in the case $\gamma \neq 0$. Caserta does not have the key argument of the proof, with the integration by parts.
exponent — including the canonical case of Gaussian noise — we show that the elasticity of markups to the number of firms is asymptotically zero. Hence, we have shown that in many types of large markets, markups are relatively insensitive to the degree of competition.

Third, we show how to approximate a key integral that is useful for studying a wide range of economic environments in which extreme outcomes determine the equilibrium allocation.

The results in this paper can be used for many other applications in economics. In the appendix to this paper, we discuss applications to macroeconomics (with endogenous markups) and to endogenous product differentiation.

In conclusion, this paper uses extreme value theory to clarify the quantitative impact of competition on prices in the symmetric-firms case. We anticipate that many of our results will extend with quantitatively modest variations to the non-symmetric case. Extending our analysis to such cases poses an important technical problem for future research. Those minor economic variations will bring substantial additional mathematical challenges.
A Appendix: Further Extensions and Applications

This section discusses two extensions of our basic models and an application to trade/macroeconomics. Section A.1 endogenizes the degree of product differentiation between firms. Section A.2 develops a simple macroeconomic framework to demonstrate how the random demand specification may be used in place of the common Dixit-Stiglitz specification. Section A.3 demonstrates a connection between our results and the functional form of the Dixit-Stiglitz (1977) demand function.

A.1 Endogenous Product Differentiation or Noise

Until now, we have assumed that the standard deviation of the noise term is exogenous. We now relax this assumption and allow firms to choose the degree of product differentiation (in the traditional economic interpretation), or the degree of “confusion” (in a complementary behavioral interpretation). In the course of this analysis, we also show that the Hart model with (i) Gumbel-distributed noise and (ii) endogenous product differentiation, produces the familiar Dixit-Stiglitz (1977) demand function.

Assume that firms can choose the degree to which their own product is differentiated from the rest of the market; specifically, assume that each firm $i$ can choose $\sigma_i$ at a cost $c(\sigma_i)$ so that the firm’s taste shock is $X_i = \sigma_iX$, where $X$ has CDF $F$. The game then has the following timing:

1. firms simultaneously choose $(p_i, \sigma_i)$
2. random taste shocks are realized
3. consumers make purchase decisions
4. profits are realized

Firm $i$’s profit function is given by

$$\pi ((p_i, \sigma_i), (p, \sigma); n) = (p_i - c(\sigma_i)) D ((p_i, \sigma_i), (p, \sigma); n)$$

in step 1, where $D ((p_i, \sigma_i), (p, \sigma); n)$ is the demand for good $i$ when the firm chooses $(p_i, \sigma_i)$ and the remaining $n - 1$ firms choose $(p, \sigma)$. Each firm $i$ then chooses $(p_i, \sigma_i)$ to maximize
\[ \pi ((p_i, \sigma_i), (p, \sigma); n). \] The symmetric equilibrium is characterized by

\[ (p, \sigma) = \arg \max_{(p', \sigma')} \pi ((p', \sigma'), (p, \sigma); n). \]

Our techniques allow us to analyze the symmetric equilibrium for each of the Perloff-Salop, Sattinger and Hart models.

**Proposition 5** Consider the Perloff-Salop, Sattinger and Hart models where firms simultaneously choose \( p \) and \( \sigma \), under the same assumptions as Theorem 1. Assume that \( w_u > 0 \). Further, in the Perloff-Salop and Sattinger cases, assume that \( xf^2(x)dx \) is \( [w_l, w_u] \)-integrable, and that \( c' > 0, c'' > 0, \lim_{t \to -\infty} c'(t) = \infty \). In the Hart case, assume that \( c' > 0, (\ln c)' > 0, \lim_{t \to -\infty} (\ln c(t))' = \infty \).

Then the equilibrium outcome with \( n \) firms is asymptotically, as \( n \to \infty \)

\[
\mu_n^{PS} (\sigma_n) = \frac{\mu_n^{Sat} (\sigma_n)}{c_n^{Sat} (\sigma_n)} \sim \frac{\mu_n^{Hart} (\sigma_n)}{c_n^{Hart} (\sigma_n)} \sim n f \left( F^{-1} \left( 1 - \frac{1}{n} \right) \right) \Gamma (\gamma + 2),
\]

\[
c_n^{PS'} (\sigma_n) = \frac{c_n^{Sat'} (\sigma_n)}{c_n^{Sat} (\sigma_n)} \sim \frac{c_n^{Hart'} (\sigma_n)}{c_n^{Hart} (\sigma_n)} \sim \begin{cases} F^{-1} (1/n) : w_u < \infty \\ \frac{F^{-1}(1/n)}{\Gamma(\gamma+2)} : w_u = \infty. \end{cases}
\]

In other words, at the symmetric equilibrium, the normalized marginal cost of \( \sigma \) – that is \( c' (\sigma_n) \) in the Perloff-Salop case and \( c' (\sigma_n) / c (\sigma_n) \) in the Sattinger and Hart cases – asymptotically equals \( F^{-1} \left( \frac{1}{n} \right) \), up to a scaling constant. In particular, the normalized marginal cost of \( \sigma \) goes closer to the upper bound of the distribution as the number of firms increases. Hence, Proposition 5 quantitatively characterizes the monotonic relationship between the number of firms and the degree of endogenous product differentiation (in the traditional economic interpretation), and/or the relationship between the number of firms and the degree of endogenous confusion (in the behavioral interpretation). We note that this effect of competition on the supply of confusion or noise is potentially important (see e.g., Gabaix and Laibson 2006, Spiegler 2006, Carlin 2009, and Ellison and Ellison 2009).

We can use the limit pricing heuristic from Section 2.5 to obtain an intuition for this result. Consider the Perloff-Salop case. Since the firm engages in limit pricing, it can charge
a markup of $\sigma M_n - \sigma^* S_n$ where $\sigma$ is the firm’s product differentiation choice and $\sigma^*$ is the choice of all other firms, which we take as given. The marginal value of an additional unit of noise $\sigma$ is thus $M_n \simeq F^{-1}(\frac{1}{n})$.

A.2 A Trade / Macroeconomic Style Framework with Endogenous Markups

To model pricing power in trade and macroeconomic models, economists typically utilize the monopolistically competitive differentiated goods specification of Dixit and Stiglitz (1977) with a large number of goods. Shocks to the demand side are often modeled by shocking the coefficient of substitution in the Dixit Stiglitz specification; see e.g. Woodford (2003, ch. 6) and Smets and Wouters (2003). However, those shocks are arguably not structural. To meet this criticism, we investigate the implications of a random demand specification. As we show below, demand shocks in the random demand approach are taken into account by the firms when setting prices, rather than treating these exogenously. Here we take an extreme view of demand shocks and model these as a taste of the entire population for a specific item from the set of differentiated goods (one year everybody desires a BlackBerry, the next year the iPhone).

To be able to demonstrate the implications of random demand for macro, we develop two trade / macro style models. One is based on the traditional Dixit Stiglitz specification, the other is based on the random demand specification. (See the online appendix for more details.) The two models only differ with respect to utility function. The familiar Dixit-Stiglitz specification with endogenous labor supply is

$$U = Z^{1-\theta} \left[ \frac{1}{n} \sum_{i=1}^{n} Q_i^{1/(1+\tau)} \right]^{\theta(1+\tau)} - \frac{1}{1+\eta} L^{1+\eta},$$

where $Z$ is the composite good, the $Q_i$ are the differentiated goods and $L$ is labor. The substitution coefficient $\tau$ is constrained to $\tau \in (0, \infty)$, which implies concavity; $\theta \in (0,1)$.

The random demand model is based on Sattinger’s (1984) type utility function (14)

---

22See Atkeson and Burstein (2008), Melitz and Ottaviano (2008) and Auer and Chaney (2009) for other ways to generate varying markups in trade.
amended with the same disutility of labor as in the Dixit-Stiglitz specification

\[ U = Z^{1-\theta} \left[ \sum_{i=1}^{n} \exp(X_i) Q_i \right]^\theta - \frac{1}{1+\eta} L^{1+\eta}. \]  

(26)

In this setup the demand shock affects all consumers equally, i.e. the demand shocks \( X_i \) are identical across consumers.

The supply side technologies are linear:

\[ Z = BN \text{ and } Q_i = AN_i, \]  

(27)

where \( A \) and \( B \) are the labor productivity coefficients while \( N \) and \( N_i \) are the respective labor demands. Note that \( A \) and \( B \) also capture the supply side productivity shocks. Perfect competition in the composite goods market implies that prices equal the per unit labor costs. The differentiated goods producer exploits his direct pricing power, but ignores his pricing effect on the price index of the differentiated goods and the consumer income. For the random demand case, the markup is \( \mu_n \) from (16), in the Dixit-Stiglitz specification the markup is \( \tau \). Note that the markup factors \( \tau \) and \( \mu_n \) can take on similar values, cf. Table 1. The models’ solutions from the first order conditions is given in the online Appendix.

The respective labor productivities for the competitive good are as follows:

Lemma 2 The Dixit-Stiglitz and random demand labor productivities are

\[ Q_j/L = \frac{\theta A}{1 + (1-\theta) \tau} \text{ and } Q_i/L = \frac{\theta A}{1 + (1-\theta) \mu_n}. \]  

(28)

There is an immediate implication:

Corollary 1 The labor productivity in the Dixit-Stiglitz specification is a mixture of demand and supply shocks, while in the random demand specification there are only supply shocks.

The main difference between the two demand specifications stems from the way in which demand shocks impinge on the macro variables. Woodford (2003, ch. 6), and Smets and Wouters (2003) generate demand shocks by shocking \( \tau \). This is different from the demand shock that arises from the random utility concept. Letting \( \tau \) be random produces a time varying markup factor. The markup factor \( \mu_n \) in the case of random utility, though, is not
random, see (16). Only the amount demanded is random as \( X_i \) is part of the demand function. The deterministic markup in case of the random demand model can be explained by the fact that the uncertainty is anticipated on the supply side and ‘disappears as an expectation’.

In the random demand expression from (28) the number of competitors \( n \) plays a role through the markup \( \mu_n \). In the case of the Dixit-Stiglitz specification, however, \( n \) does not enter as \( \tau \) is exogenous. Consider the implications for the goods ratios \( Q/Z \):

**Proposition 6** In the Dixit-Stiglitz specification, the goods ratio \( Q_j/Z \) does not depend on \( n \). In the random demand case if the distribution of the fashion shock is bounded or has exponential like tails, then \( Q_i/Z \) (approximately) equals the ratio of the expenditure shares \( \theta/(1-\theta) \) times the ratio of the productivity shocks \( A/B \). But in the case that the preference shocks have fat tails, the goods ratio \( Q_i/Z \to 0 \).

**Proof of Proposition 6** Combining (55) and (56), and (52) and (53) yields for respectively the Dixit-Stiglitz and random demand specifications

\[
\frac{Q_j}{Z} = \frac{\theta}{(1-\theta)(1+\tau)} \frac{A}{B} \quad \text{and} \quad \frac{Q_i}{Z} = \frac{\theta}{(1-\theta)(1+\mu_n)} \frac{A}{B}.
\]

(29)

Then use Table 1 to plug in the details for \( \mu_n \) depending on the type of distribution. With \( \gamma \geq 0 \) and \( a \) unbounded, \( \lim_{n \to \infty} (Q_i/Z) = \lim_{n \to \infty} (1/\mu_n) = 0 \).

Thus in the case that the preference shocks have fat tails and with numerous competitors, the differentiated good becomes unimportant relative to the competitive good.

### A.3 A connection with Dixit-Stiglitz demand functions

Returning to Section 4.1, the Hart model with Gumbel-distributed noise generates the familiar demand function from Dixit-Stiglitz (1977), as the following proposition shows.\(^{23}\)

**Proposition 7** Let \( X_i \) be Gumbel distributed with parameter \( \phi \): \( F(x) = \exp(-e^{-x/\phi}) \). Then in the Hart model, demand for good \( i \) equals

\[
D(p_1, \ldots, p_n; i) = \Gamma (1 - \phi \psi \sigma) \frac{p_i^{-(1+\phi \psi \sigma)}}{(\sum_{i=1}^{n} p_j^{-(1+\phi \psi \sigma)})^{1-\phi \psi \sigma}}.
\]

\(^{23}\)Anderson et al (1992, pp. 85-90) derive this result for the case \( \psi = 0 \).
This result is of independent interest. For example, it implies that our framework generates a Dixit-Stiglitz demand functions with elasticity equal to \( \phi \sigma \). When \( \phi \sigma \) is time-varying or endogenous, then the Dixit-Stiglitz demand elasticity will be time-varying or endogenous as well.

**B Appendix: Proofs**

First, to clarify notation: denote \( f_n \sim g_n \) if \( f_n/g_n \to 1 \), \( f_n = o(g_n) \) if \( f_n/g_n \to 0 \) and \( f_n = O(g_n) \) if there exists \( M > 0 \) and \( n' \geq 1 \) such that for all \( n \geq n' \), \( |f_n| \leq M |g_n| \). We start by collecting some useful facts about regular variation, see Resnick (1987) or Bingham et al. (1989).

**Lemma 3**

1. If \( g(t) \in RV_a^0 \), then the limit \( \lim_{t \to 0} g(xt)/g(t) = x^\alpha \) holds locally uniformly with respect to \( x \) on \((0, \infty)\).

2. If \( \lim_{x \to 0} h(x)/s(x) = 1 \), \( \lim_{x \to 0} s(x) = 0 \) and \( g(x) \in RV^0_\rho \), then \( g(h(x)) \sim g(s(x)) \).

3. If \( g(t) \in RV^0_a \) and \( h(t) \in RV^0_b \), then \( g(t)h(t) \in RV^0_{a+b} \).

4. If \( g(t) \in RV^0_a \), \( h(t) \in RV^0_b \) and \( \lim_{t \to 0} h(t) = 0 \), then \( g \circ h(t) \in RV^0_{ab} \).

5. If \( g(t) \in RV^0_a \) and non-decreasing, then \( g^{-1}(t) \in RV^0_{a^{-1}} \) if \( \lim_{t \to 0} g(t) = 0 \).

6. Let \( U \in RV^0_\rho \). If \( \rho > -1 \) (or \( \rho = -1 \) and \( \int_0^x U(t) \, dt < \infty \)), then \( \int_0^x U(t) \, dt \in RV^0_{\rho+1} \) and

\[
\lim_{x \to 0} \frac{xU(x)}{\int_0^x U(t) \, dt} = \rho + 1.
\]

If \( \rho \leq -1 \), then for \( \overline{\rho} > 0 \), \( \int_{x}^\infty U(t) \, dt \in RV^0_{\rho+1} \) and

\[
\lim_{x \to 0} \frac{xU(x)}{\int_x^\infty U(t) \, dt} = -\rho - 1.
\]

7. If \( \lim_{t \to \infty} tj'(t)/j(t) = \rho \), then \( j \in RV^\infty_\rho \). Similarly, if \( \lim_{t \to 0} tj'(t)/j(t) = \rho \), then \( j \in RV^0_\rho \).

8. If \( g \in RV^\infty_\rho \) and \( \varepsilon > 0 \), then \( g(t) = o(t^{\rho+\varepsilon}) \) and \( t^{\rho-\varepsilon} = o(g(t)) \) as \( t \to \infty \); and if \( g \in RV^0_\rho \) and \( \varepsilon > 0 \), then \( g(t) = o(t^{\rho-\varepsilon}) \) and \( t^{\rho+\varepsilon} = o(g(t)) \) as \( t \to 0 \).
Our proof of Theorem 2 depends critically on the following result.

**Theorem 5** *(Karamata’s Tauberian Theorem)* Assume \( U : (0, \infty) \to [0, \infty) \) is weakly increasing, \( U(x) = 0 \) for \( x < 0 \), and assume \( \int_0^\infty e^{-sx}dU(x) < \infty \) for all sufficiently large \( s \). With \( \alpha \geq 0 \), \( U(x) \in RV_\alpha^0 \) if and only if

\[
\int_0^\infty e^{-sx}dU(x) \sim_{s \to \infty} U(1/s) \Gamma(\alpha + 1).
\]

For a proof, see Bingham et al. (1987, pp.38, Th. 1.7.1’) or Feller (1971, XIII.5, Th. 1) for another version of Karamata’s Tauberian theorem.

**Proof of Theorem 2** Assume for now that \( G(x) \geq 0 \) for all \( x \in (w_l, w_u) \), and show later that this assumption can be relaxed. Differentiation of \( P(M_n \leq x) = F^n(x) \) gives the density of \( M_n: f_n(x) = nf(x)F^{n-1}(x) \). Using the change of variable \( x = F^{-1}(t) \) and observing that \( dF^{-1}(t)/dt = -1/f(F^{-1}(t)) \)

\[
\mathbb{E}[G(M_n)] = \int_{w_l}^{w_u} G(x)nf(x)F^{n-1}(x)dx = n \int_{w_l}^{w_u} G(x)F^{n-1}(x)(f(x)dx) = n \int_0^1 G(F(t))[F(F^{-1}(t))]^{n-1}dt = n \int_0^1 \hat{G}(t)(1-t)^{n-1}dt.
\]

We next use the change in variables \( x = -\ln(1-t) \), so \( t = 1 - e^{-x} \), \( dt = e^{-x}dx \), and so

\[
\mathbb{E}[G(M_n)] = n \int_0^\infty \hat{G}(1-e^{-x})e^{-x}e^{-n't}dx
\]

where \( n' = n - 1 \).

Next, define \( h(x) = \hat{G}(1-e^{-x})e^{-x} \), and \( \mu(x) = \int_0^x h(y)dy \). Since \( \hat{G} \) is regularly varying at zero with index \( \rho > -1 \), Lemma 3(8) implies that \( \int_0^s |\hat{G}(t)|dt < \infty \) for sufficiently small \( s \). This, with the assumptions \( G(t) \geq 0 \) and \( \int_s^1 |\hat{G}(t)|dt < \infty \) for all \( s \in (0,1) \), ensure that \( \mu(x) = \int_0^{1-e^{-x}} \hat{G}(t)dt \) is finite and non-decreasing on \([0,\infty)\). By Lemma 3(2), \( h(x) \sim_{x \to 0} \)
\( \tilde{G}(x) \). So \( h \in RV_\rho^0 \), and by Lemma 3(6)

\[
\mu(x) = \int_0^x h(y) \, dy
\]

\[
\sim_{x \to 0} \frac{1}{1 + \rho} h(x) x
\]

\[
\sim_{x \to 0} \frac{1}{1 + \rho} \tilde{G}(x) x.
\]

Therefore \( \mu(x) \in RV_{\rho+1}^0 \).

Noting our assumption that \( \rho + 1 > 0 \), we can now apply Karamata’s Theorem 5 in combination with the last expression to obtain

\[
\int_0^\infty e^{-n'x} d\mu(x) \sim_{n' \to \infty} \mu (1/n') \Gamma (2 + \rho)
\]

\[
\sim_{n' \to \infty} \frac{1}{1 + \rho} \tilde{G} (1/n') (n')^{-1} \Gamma (2 + \rho)
\]

\[
\sim_{n \to \infty} \tilde{G} (1/n) n^{-1} \Gamma (1 + \rho).
\]

Thus

\[
\mathbb{E} \left[ G(M_n) \right] = n \int_0^\infty e^{-n'x} d\mu(x)
\]

\[
\sim n \tilde{G} (1/n) n^{-1} \Gamma (1 + \rho) = G(F^{-1} (1/n)) \Gamma (1 + \rho)
\]

holds when \( G(x) \geq 0 \) for all \( x \in (w_l, w_u) \).

Now relax the assumption that \( G(x) \geq 0 \) for all \( x \in (w_l, w_u) \). Choose \( \bar{t} \in (0, 1) \) such that \( G(t) > 0 \) for \( t \in [0, \bar{t}] \). The assumption that \( G(\cdot) \) is strictly positive in a left neighborhood of \( w_u \) ensures that such \( \bar{t} \) exists. Thus we can write

\[
\mathbb{E} \left[ G(M_n) \right] = n \int_0^\bar{t} \tilde{G} (t) \left( 1 - t \right)^{n-1} dt + n \int_{\bar{t}}^1 \tilde{G} (t) \left( 1 - t \right)^{n-1} dt
\]

Consider \( \tilde{G} : (0, 1) \to \mathbb{R} \) defined by

\[
\tilde{G}(t) \equiv \begin{cases} 
\tilde{G}(t) : t \leq \bar{t} \\
0 : t > \bar{t}
\end{cases}
\]
It is easy to check that \( \tilde{G} \) satisfies the conditions of the theorem and additionally is weakly positive everywhere on \((w_1, w_\omega)\). The argument above shows that as \(1/n \to 0\)

\[
n \int_0^\tau \tilde{G}(t) (1-t)^{n-1} dt = n \int_0^1 \tilde{G}(t) (1-t)^{n-1} dt \sim \tilde{G}(1/n) \Gamma(1+\rho) \sim \tilde{G}(1/n) \Gamma(1+\rho).
\]

(30)

To complete the proof we demonstrate that as \(n \to \infty\)

\[
\left| \int_\tau^1 \tilde{G}(t) (1-t)^{n-1} dt \right| = o \left( \int_0^\tau \tilde{G}(t) (1-t)^{n-1} dt \right).
\]

First, by (30) for \(n \to \infty\)

\[
\int_0^\tau \tilde{G}(t) (1-t)^{n-1} dt \sim n^{-1} \tilde{G}(1/n) \Gamma(1+\rho) \in RV_{-\rho-1}.
\]

Lemma 3(8) implies that \(\int_0^\tau \tilde{G}(t) (1-t)^{n-1} dt > n^{-\rho-1-\varepsilon}\) for sufficiently large \(n\) and given some \(\varepsilon > 0\). Also,

\[
\left| \int_\tau^1 \tilde{G}(t) (1-t)^{n-1} dt \right| \leq \int_\tau^1 \left| \tilde{G}(t) \right| (1-t)^{n-1} dt \\
\leq (1-\tau)^{n-1} \int_\tau^1 \left| \tilde{G}(t) \right| dt \\
\leq (1-\tau)^{n-1} \int_0^1 \left| \tilde{G}(t) \right| dt.
\]

By assumption \(\int_s^1 \left| \tilde{G}(t) \right| dt < \infty\) for all \(s \in (0,1)\), therefore

\[
\frac{\left| \int_\tau^1 \tilde{G}(t) (1-t)^{n-1} dt \right|}{\int_0^\tau \tilde{G}(t) (1-t)^{n-1} dt} \leq \frac{(1-\tau)^{n-1} \int_0^1 \left| \tilde{G}(t) \right| dt}{n^{-\rho-1-\varepsilon}} = o(1) \text{ as } n \to \infty.
\]

This completes the proof. \(\blacksquare\)

**Proof of Lemma 1** See the online appendix.

**Proof of Theorem 1** The Perloff-Salop and Sattinger cases follow immediately from Proposition 3; we will omit those calculations and focus on the Hart case. Applying Proposition 3
to (19), we immediately infer that

\[
\mu_n^{Hart} \frac{1}{c} \sim \frac{1}{\psi + n f \left( F^{-1} \left( 1/n \right) \right) \Gamma(\gamma+2-\psi a) \Gamma(1-\psi a)}
\]

under the conditions of the theorem. We will use the fact that \( a n f(U_a) \sim 1 \), which holds because

\[
\lim_{n \to \infty} \frac{1}{n f \left( F^{-1} \left( 1/n \right) \right)} = \lim_{x \to u_a} \frac{F(x)}{f(x)} = a.
\]

Consider first the case where \( a = 0 \). Then \( n f \left( F^{-1} \left( 1/n \right) \right) \to \infty \), and the expression simplifies to

\[
\mu_n^{Hart} \frac{1}{c} \sim \frac{1}{n f \left( F^{-1} \left( 1/n \right) \right)} \left[ \frac{\psi}{n f \left( F^{-1} \left( 1/n \right) \right)} + \frac{\Gamma(\gamma+2-\psi a)}{\Gamma(1-\psi a)} \right] \sim \frac{1}{n f \left( F^{-1} \left( 1/n \right) \right) \Gamma(\gamma+2)}.
\]

Next, consider the case \( 0 < a < \infty \), which implies \( \gamma = 0 \). We have

\[
\mu_n^{Hart} \frac{1}{c} \sim \frac{1}{\psi + n f \left( F^{-1} \left( 1/n \right) \right) \Gamma(2-\psi a) \Gamma(1-\psi a)} = \frac{1}{\psi + n f \left( F^{-1} \left( 1/n \right) \right) (1-\psi a)}
\]

\[
= \frac{1}{\psi \left( 1 - an f \left( F^{-1} \left( 1/n \right) \right) \right) + n f \left( F^{-1} \left( 1/n \right) \right)}
\]

\[
\sim \frac{1}{n f \left( F^{-1} \left( 1/n \right) \right)} = \frac{1}{n f \left( F^{-1} \left( 1/n \right) \right) \Gamma(2+\gamma)}
\]

when \( \gamma = 0 \).
References


