Online Appendix for “The Impact of Competition on Prices with Numerous Firms”
April 1, 2015

This online appendix discusses some additional issues. Section C contains details of the Perloff-Salop, Sattinger, and Hart models. Section D contains proofs that are omitted from the papers. Section E derives second-order conditions for equilibrium, for the Perloff-Salop, Sattinger and Hart models. Section G analyses the macroeconomic framework from Section A.2 in greater detail.

C Details of Monopolistic Competition Models

This section provides details for the derivation of the markup expressions for the four monopolistic competition models.

Perloff-Salop

Recall from (4) that in the Perloff-Salop model, the demand function for good $i$ is the probability that difference between the demand shock and the price is maximized by good $i$:

$$D(p_1, ..., p_n; i) = \mathbb{P} \left( X_i - p_i \geq \max_{j \neq i} X_j - p_j \right)$$

$$= \mathbb{E}_{X_i} \left[ \prod_{j \neq i} \mathbb{P}(x - p_i \geq X_j - p_j \mid X_i = x) \right]$$

$$= \mathbb{E}_{X_i} \left[ \prod_{j \neq i} F(x - p_i + p_j) \right]$$

$$= \int_{w_i}^{u_i} f(x) \prod_{j \neq i} F(x - p_i + p_j) \, dx.$$
Using $D(p_i, p; n)$ to denote the demand for good $i$ at price $p_i$ when all other firms set price $p$ and using $D_1(p_i, p; n)$ to denote $\partial D(p_i, p; n) / \partial p_i$, we may calculate

$$D(p_i, p; n) = \int_{u_i}^{w_i} f(x) F^{n-1}(x - p_i + p) \, dx$$

$$D_1(p_i, p; n) = -(n - 1) \int_{u_i}^{w_i} f(x) f(x - p_i + p) F^{n-2}(x - p_i + p) \, dx.$$  

Note that in a symmetric equilibrium

$$D(p, p; n) = \int_{u_i}^{w_i} f(x) F^{n-1}(x) \, dx = 1/n,$$

$$D_1(p, p; n) = -(n - 1) \int_{u_i}^{w_i} f^2(x) F^{n-2}(x) \, dx.$$  

It follows that

$$\mu_n^{PS} = -\frac{D(p, p; n)}{D_1(p, p; n)} = \frac{1}{n(n - 1) \int_{u_i}^{w_i} f^2(x) F^{n-2}(x) \, dx}.$$  

To interpret the Perloff-Salop markup equation, use the notation $M_{n-1}$ (the largest of the $n - 1$ noise realizations: $M_{n-1} \equiv \max_{j \in \{1, \ldots, n\}, j \neq i} X_j$). Then, $D(p, p; n) = \mathbb{P}(X_i > M_{n-1})$, so

$$D(p, p; n) = \mathbb{E}[\mathcal{F}(M_{n-1})].$$  

This formulation emphasizes that the demand for good $i$ is driven by the properties of the right-hand tail of the cumulative distribution function $\mathcal{F}$, as $M_{n-1}$ is likely to be large.

**Sattinger (1984)**

Under the utility specification (14), goods from the monopolistically competitive (MC) market are perfect substitutes. The consumer optimizes by buying only one monopolistically competitive good; the good $i$ which maximizes $e^{X_i}/p_i$. The consumer’s utility function is thus Cobb-Douglas in the composite good and the chosen MC good; it is then easy to show that the consumer spends fraction $\theta$ of his income on the chosen MC good. Without loss of generality, normalize the consumer’s endowment $y$ to equal $1/\theta$, so that the consumer always spends 1 unit of income on the MC good.

The demand function of firm $i$ is the probability that the good $i$ has a higher attraction-price ratio than all other goods, multiplied by the purchased quantity $1/p_i$ of the chosen good.
We may rewrite this expression as
\[ D (p_1, \ldots, p_n; i) = \frac{1}{p_i} \mathbb{P} \left( \frac{e^{X_i}}{p_i} = \max_{j=1, \ldots, n} \frac{e^{X_j}}{p_j} \right) = \frac{1}{p_i} \mathbb{P} \left( X_i - \ln p_i = \max_{j=1, \ldots, n} X_j - \ln p_j \right). \] (32)

Proceeding as in the case of the Perloff-Salop model, we get
\[
D (p_1, \ldots, p_n; i) = \frac{1}{p_i} \int f(x) \prod_{j \neq i} F(x - \ln p_j) \, dx.
\]

In a symmetric equilibrium
\[
D(p, p; n) = \int_{w_i}^{w_u} f(x) F^{n-1} (x - \ln p_i + \ln p) \, dx = \frac{1}{pn},
\]
\[
D_1(p, p; n) = -\frac{1}{p^2} \int_{w_i}^{w_u} f(x) F^{n-1} (x - \ln p_i + \ln p) \, dx
\]
\[
- \frac{(n-1)}{p^2} \int_{w_i}^{w_u} f(x) f(x - \ln p_i + \ln p) F^{n-2} (x - \ln p_i + \ln p) \, dx
\]

After some simple manipulations, it follows that the Sattinger markup in symmetric equilibrium is
\[
\mu_{p}^{\text{Sat}} = \frac{D(p, p; n)}{D_1(p, p; n)} = \frac{c}{n (n-1) \int_{w_i}^{w_u} f^2(x) F^{n-2}(x) \, dx}.
\]

### C.1 Hart (1985)

Recall that the consumer’s objective is to choose quantities to maximize:
\[
\max_{i=1, \ldots, n} \max_{Q_i \geq 0} U = \frac{\psi + 1}{\psi} \left( \sum_{i=1}^{n} e^{X_i} Q_i \right)^{\psi/(\psi+1)} - \sum_{i=1}^{n} p_i Q_i.
\] (33)
As in the Sattinger case, it is clear that because goods are perfect substitutes, the consumer will purchase only from one firm, which we denote by \( i \). The first-order condition of the consumer’s problem is then

\[
0 = \frac{d}{dQ_i} \left[ \frac{\psi + 1}{\psi} \left( e^{X_i Q_i} \right)^{\psi/(\psi+1)} - p_i Q_i \right] = e^{X_i \psi/(\psi+1)} Q_i^{-1/(\psi+1)} - p_i
\]

which gives us the optimal quantity for the chosen good \( i \): \( Q_i = e^{X_i \psi} / p_i^{1+\psi} \), and the total net utility is:

\[
V_i = \frac{\psi + 1}{\psi} \left( e^{X_i Q_i} \right)^{\psi/(\psi+1)} - p_i Q_i = \left( \frac{\psi + 1}{\psi} - 1 \right) p_i Q_i = \frac{1}{\psi} p_i e^{X_i \psi} / p_i^{1+\psi} = \frac{1}{\psi} \left( \frac{e^{X_i}}{p_i} \right)^\psi
\]

The consumer chooses the good that maximizes his net utility, i.e. \( \arg \max_i \left( e^{X_i} / p_i \right) \). We may then calculate the demand function for good \( i \) as

\[
D(p_1, \ldots, p_n; i) = \mathbb{E} \left[ \frac{e^{X_i}}{p_i^{1+\psi}} I \left\{ e^{X_i/p_i} = \max_{j=1, \ldots, n} e^{X_j/p_j} \right\} \right]
\]

(34)

\[
= \mathbb{E} \left[ \frac{e^{X_i}}{p_i^{1+\psi}} I \left( X_i - \ln p_i = \max_{j=1, \ldots, n} X_j - \ln p_j \right) \right]
\]

(35)

where \( I \{ \cdot \} \) is the indicator function. Writing out the expectation and differentiating gives

\[
D(p_i, p; n) = \frac{1}{p_i^{1+\psi}} \int_{u_i}^{u} e^{\psi x} f(x) F^{n-1}(x - \ln p_i + \ln p) \, dx,
\]

\[
D_1(p_i, p; n) = -\frac{1 + \psi}{p_i^{2+\psi}} \int_{u_i}^{u} e^{\psi x} f(x) F^{n-1}(x - \ln p_i + \ln p) \, dx
\]

\[
- \frac{n - 1}{p_i^{2+\psi}} \int_{u_i}^{u} e^{\psi x} f(x) f(x - \ln p_i + \ln p) F^{n-2}(x - \ln p_i + \ln p) \, dx.
\]

In a symmetric equilibrium

\[
D(p, p; n) = \frac{1}{p^{1+\psi}} \int_{u_i}^{u} e^{\psi x} f(x) F^{n-1}(x) \, dx
\]

\[
D_1(p, p; n) = -\frac{1 + \psi}{p^{2+\psi}} \left( (1 + \psi) \int_{u_i}^{u} e^{\psi x} f(x) F^{n-1}(x) \, dx + (n - 1) \int_{u_i}^{u} e^{\psi x} f^2(x) F^{n-2}(x) \, dx \right).
\]
With some simple calculations, it follows that the Hart markup in symmetric equilibrium is

$$\mu^{Hart}_p = -\frac{D(p, p; n)}{D_1(p, p; n)} = c \left( \psi + (n - 1) \int e^{\psi x} f^2(x) F^{n-2}(x) \right)^{-1}.$$ 

D Proofs Omitted from the Paper

Proof of Lemma 3

1. Follows upon inversion from Resnick (1987, Prop. 0.5).

2. This fact follows from the observation that for $$\delta(x) = \frac{s(x)}{h(x)} = \frac{g(s(x))}{g(h(x))} \sim \left( \frac{s(x)}{h(x)} \right)^p \to_0 1$$ where we can take the limit as $$x \to 0$$ because of Lemma 3(1). Going into more detail, choose $$\delta(x)$$ such that $$\lim_{t \to 0} \delta(t) = 0$$ and $$|s(t')/h(t') - 1| < \delta(t)$$ for $$t' < t$$. Such $$\delta(x)$$ exists by our assumptions on $$s$$ and $$h$$. Choose $$\varepsilon(\cdot, \cdot)$$ such that $$\lim_{t \to 0} \varepsilon(t, \delta) = \lim_{\delta \to 0} \varepsilon(t, \delta) = 0$$ and $$|g(xt')/g(t') - x^p| < \varepsilon(t, \delta)$$ for $$x \in (1-\delta, 1+\delta)$$ and $$t' < t$$. Lemma 3(1) ensures that such $$\varepsilon(\cdot, \cdot)$$ exists. Then

$$|g(s(t'))/g(h(t')) - 1| = \left| g\left( \frac{s(t')}{h(t')} h(t') \right) / g(h(t')) - 1 \right| < \varepsilon(h(t'), \delta(t)) + \rho \varepsilon(\delta(t))$$

for $$t' < t$$. Since the RHS goes to zero as $$t \to 0$$, the result follows.

3. Since $$\lim_{t \to 0} g(xt)/g(t) = x^a$$ and $$\lim_{t \to 0} h(xt)/h(t) = x^b$$, we have $$\lim_{t \to 0} \frac{g(xt)h(xt)}{g(t)h(t)} = x^{a+b}$$.

4. Follows upon inversion from Resnick (1987, Prop. 0.8, iv).

5. Follows upon inversion from Resnick (1987, Prop. 0.8, v).

6. Both parts follow upon inversion from Resnick (1987, Th. 0.6, a).

7. Follows from Resnick (1987, Prop. 0.7) and by inversion.

8. Directly by Resnick (1987, Prop. 0.8, ii) and upon inversion. ■

Proof of Lemma 1
1. Note that $F\left( F^{-1}(t) \right) = 1-t$ implies $f\left( F^{-1}(t) \right) \left( F^{-1}(t) \right)' = -1$. Let $x = F^{-1}(t)$, $j(t) = f\left( F^{-1}(t) \right)$. Then $tj'(t) / j(t) = -tf'\left( F^{-1}(t) \right) / f^2 \left( F^{-1}(t) \right) = -F(x)f'(x) / f^2 (x) = \left( F/f \right)'(x) + 1$, so $\lim_{t \to 0} tj'(t) / j(t) = \lim_{x \to F^{-1}(1)} \left( F/f \right)'(x) + 1 = \gamma + 1$ by Definition 1. Lemma 3(7) then implies the desired result.

2. Note that $-\frac{d}{dt} F^{-1}(t) = 1/f \left( F^{-1}(t) \right) \in RV^0_{-\gamma-1}$. So if $w_u < \infty$ (which implies $\gamma \leq 0$; see Embrechts et al., 1997) then Lemma 3(6) implies

$$F^{-1}(0) - F^{-1}(t) = \int_0^t 1/f \left( F^{-1}(s) \right) ds \in RV^0_{-\gamma}.$$ 

If $w_u = \infty$ (which implies $\gamma \geq 0$) then Lemma 3(6) implies, for any choice of $\bar{t} > 0$, that

$$F^{-1}(t) \sim F^{-1}(t) - F^{-1}(\bar{t}) = \int_t^{\bar{t}} 1/f \left( F^{-1}(s) \right) ds \in RV^0_{-\gamma};$$

see also Embrechts et al. (1997, pp. 160).

3. We have

$$\frac{t \frac{d}{dt} F^{-1}(t)}{e^{F^{-1}(t)}} = \frac{-t}{f \left( F^{-1}(t) \right)} = \frac{-F(x)}{f(x)} \text{ for } x = F^{-1}(t).$$

Lemma 3(7) then implies the desired result. ■

**Proof of Proposition 1**
Treating $n$ as continuous, we have

$$\frac{n}{\mu_n^{PS}} \frac{d\mu_n^{PS}}{dn} = -\left( \frac{2n-1}{n-1} + n \frac{\int f^2(x) F^{n-2}(x) \log F(x) dx}{\int f^2(x) F^{n-2}(x) dx} \right).$$

Noting that $-\log (1-x) \sim x \in RV^0_1$, applying Theorem 2 to $G(x) \equiv \frac{f(x)}{F(x)} \log F(x)$, using Lemma 3(3), we obtain

$$\int f^2(x) F^{n-2}(x) \log F(x) dx \sim -n^{-2} f(U_n) \Gamma(3 + \gamma).$$

Together with Theorem 1, it follows that

$$\frac{n}{\mu_n} \frac{d\mu_n}{dn} = -\left( 2 - \frac{n^{-2} f \left( F^{-1}(1/n) \right) \Gamma(3 + \gamma)}{n^{-2} f \left( F^{-1}(1/n) \right) \Gamma(2 + \gamma)} + o(1) \right) = \gamma + o(1).$$
Proof of Proposition 2 We first show a lemma that links differences between the two top order statistics to the behavior of the top tail statistics, and hence allows us to apply our general results.

Lemma 4 Call $M_n$ and $S_n$, respectively, the largest and second largest realizations of $n$ i.i.d. random variables with CDF $F$ and density $f = F'$, and $G$ a function such that $\int G(x) f(x) \, dx < \infty$, $\lim_{x \to F^{-1}(0)} G(x) F(x) = \lim_{x \to F^{-1}(1)} G(x) F'(x) = 0$. Then:

$$\mathbb{E}[G(M_n) - G(S_n)] = \mathbb{E} \left[ \frac{G'(M_n) F'(M_n)}{f(M_n)} \right]$$

(36)

Proof: Recall that the density of $M_n$ is $n f(x) F^{n-1}(x)$, and the density of $S_n$ is

$$n (n-1) f(x) F(x) F^{n-2}(x).$$

So

$$\mathbb{E}[G(S_n)] = \int n (n-1) G(x) f(x) F(x) F^{n-2}(x) \, dx$$

$$= n \left[ G(x) F(x) F^{n-1}(x) \right]^{F^{-1}(1)}_{F^{-1}(0)} - \int n G(x) F'(x) F^{n-1}(x) \, dx$$

$$= 0 + \int n G(x) f(x) F^{n-1}(x) \, dx - \int n \frac{G'(x) F(x)}{f(x)} f(x) F^{n-1}(x) \, dx$$

$$= \mathbb{E}[G(M_n)] - \mathbb{E} \left[ \frac{G'(M_n) F(M_n)}{f(M_n)} \right]$$

From this lemma, the proof follows for $G(x) = x$. As $f(F^{-1}(t)) \in RV_{1+\gamma}^0$, $t/f(F^{-1}(t)) \in RV_{-\gamma}^0$, and we may apply Theorem 2 to obtain the desired result.

Results for ELRU Model The next few results relate to the ELRU model, culminating in the proof of Theorem 4. We first present some results that we will use in the proof of Theorem 4, followed by the proof of the Theorem itself.

Lemma 5 Under the assumptions of Theorem 4,

$$D(p, p) \sim_{n \to \infty} \frac{1}{n}.$$
Proof of Lemma 5

Some notation: we use $x \vee y$ to denote $\max \{x, y\}$. Note that $M_{n-1}$ is independent of $Q = w + \beta p$ and therefore

$$D(p, p) = \mathbb{P} \left\{ X_i \geq \bigvee_{j \neq i} X_j \vee Q \right\}$$

$$= \mathbb{E}_Q \left[ \int_q^\infty F_{n-1}(s)f(s)ds \right]$$

$$= \frac{1}{n} - \frac{1}{n} \mathbb{E}_Q [F^n(q)]$$

$$= \frac{1}{n} + o\left(\frac{1}{n}\right). \blacksquare \quad (37)$$

To obtain the partial derivative $\partial D(p_i, p_j) / \partial p_i$ note that we can alternatively express demand in a symmetric equilibrium in terms of the distribution of $X_i$

$$D(p, p) = \mathbb{P} \left\{ X_i \geq \bigvee_{j \neq i} X_j \vee Q \right\}$$

$$= 1 - \mathbb{P} \{ X_i \leq M_{n-1} \vee Q \}$$

$$= 1 - \mathbb{E}_Q \mathbb{E}_{M_{n-1}} [F(m_{n-1} \vee q)].$$

This facilitates the differentiation with respect $p_i$ at a symmetric equilibrium

$$\frac{\partial D(p_i, p_j)}{\partial p_i} \bigg|_{p_i=p_j=p} = \frac{\partial \left\{ 1 - \mathbb{E}_{\beta,w} \mathbb{E}_{M_{n-1}} [F(m_{n-1} \vee w + \beta p_i)] \right\}}{\partial p_i}$$

$$= -\mathbb{E}_{\beta,w} \mathbb{E}_{M_{n-1}} [f(m_{n-1} \vee q) \times \beta]$$

$$= -\mathbb{E}_{\beta,w} \left[ \beta \int_q^{\infty} (n-1) f(s)F^{n-2}(s)f(s)ds \right]$$

We seek to apply Theorem 2. To this end we first need the asymptotic inverse of the Rootzen distributions

$$\mathcal{F}(x) = 1 - F(x) \sim \kappa x^{\nu} \exp \left( -\lambda x^\phi \right), \kappa > 0, \lambda > 0, \phi \geq 1, \nu \in \mathbb{R} \quad (38)$$
The asymptotic inverse of the upper tail of (38) is

\[
\bar{F}^{-1}(y) \sim \left(\frac{1}{\lambda}\right)^{1/\phi} \left[ \ln \left( \frac{\kappa \lambda^{-\nu/\phi}}{y} \times \left[ \ln \left( \frac{\kappa \lambda^{-\nu/\phi}}{y} \right) \right]^{\nu/\phi} \right) \right]^{1/\phi}
\]

for \( y \) close to zero, see Li (2008).

**Lemma 6** For the distribution with upper tail (38) it holds that \( f(\bar{F}^{-1}(y)) \in RV^{0}_{\rho} \) with \( \rho > -1 \).

**Proof of Lemma 6**

Write shorthand \( A = \kappa \lambda^{-\nu/\phi} \). Then, for \( y \) close to zero,

\[
f(\bar{F}^{-1}(y)) \sim \kappa \lambda^{\phi} \left[ \left(\frac{1}{\lambda}\right)^{1/\phi} \left[ \ln \left( \frac{A}{y} \times \left[ \ln \left( \frac{A}{y} \right) \right]^{\nu/\phi} \right) \right]^{1/\phi} \right]^{\phi+\nu-1} \times \\
\exp \left( -\lambda \left[ \left(\frac{1}{\lambda}\right)^{1/\phi} \left[ \ln \left( \frac{A}{y} \times \left[ \ln \left( \frac{A}{y} \right) \right]^{\nu/\phi} \right) \right]^{1/\phi} \right]^{\phi} \right) \\
= y \lambda^{1/\phi} \left[ \ln \left( \frac{A}{y} \right)^{-\nu/\phi} \left[ \ln \left( \frac{A}{y} \times \left[ \ln \left( \frac{A}{y} \right) \right]^{\nu/\phi} \right) \right]^{(\nu-1)/\phi} \phi \ln \left( \frac{A}{y} \times \left[ \ln \left( \frac{A}{y} \right) \right]^{\nu/\phi} \right) \right].
\]

Taking ratios shows

\[
\lim_{y \to 0} \frac{f(\bar{F}^{-1}(xy))}{f(\bar{F}^{-1}(y))} = \lim_{t \to \infty} \frac{f(\bar{F}^{-1}(x/t))}{f(\bar{F}^{-1}(1/t))} = x.
\]

Hence, \( f(\bar{F}^{-1}(y)) \in RV^{0}_{1} \), so that \( \rho > -1 \).

Define the function \( G(s) \) from the Theorem 2 as the density \( f(s) \) from (22)

\[
G(s) = \kappa \lambda^{\phi} x^{\phi+\nu-1} \exp \left( -\lambda x^\phi \right).
\]

Thus \( G(x) \) is positive, moreover \( \hat{G}(s) \) is integrable on \((0, 1)\). Lastly, recall that \( q \) is bounded.

**Lemma 7** Asymptotically (as \( n \to \infty \))

\[
\frac{\partial D(p, p)}{\partial p_{i}} \sim -\phi \lambda^{1/\phi} \frac{1}{n} (\ln n)^{1-1/\phi} \mathbb{E}[\beta].
\]

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Proof of Lemma 7

By the main Theorem 2 at given values of $\beta$ and $w$:

$$\frac{\partial D(p, p)}{\partial p_t} \bigg|_{w+\beta_p=q} = -(n-1) \beta \int_q^\infty f(s)F^{n-2}(s)f(s)ds$$

$$= - (n-1) \beta \int_q^\infty G(s)F^{n-2}(s)f(s)ds$$

$$\sim -\beta G \left( F^{-1}\left(1 - \frac{1}{n-1}\right) \right) \Gamma(2)$$

$$\sim -\beta \phi \lambda^{1/\phi} \frac{1}{n} (\ln n)^{1-1/\phi}. \quad (39)$$

By the boundedness assumption there exist $\underline{w} = \inf(w), \overline{w} = \sup(w)$ and $\underline{\beta} = \inf(\beta)$ and $\overline{\beta} = \sup(\beta)$. Let $m = n - 1$. Now

$$\mathbb{E}_{\beta,w} \left[ \beta \int_{\underline{w}+\beta_p}^\infty mF^{m-1}(s)f^2(s)ds \right] \geq 0$$

$$\mathbb{E}_{\beta,w} \left[ \beta \int_{\overline{w}+\beta_p}^\infty mF^{m-1}(s)f^2(s)ds \right] \geq 0$$

$$\mathbb{E}_{\beta,w} \left[ \beta \int_{\underline{w}+\beta_p}^\infty mF^{m-1}(s)f^2(s)ds \right] = \mathbb{E}_{\beta,w} \left[ \beta \int_{\overline{w}+\beta_p}^\infty mF^{m-1}(s)f^2(s)ds \right].$$

By (39) the inner integrals on the left and right are asymptotic to the same expression in $n$

$$\int_{\underline{w}+\beta_p}^\infty mF^{m-1}(s)f^2(s)ds \sim \int_{\overline{w}+\beta_p}^\infty mF^{m-1}(s)f^2(s)ds \sim \phi \lambda^{1/\phi} \frac{1}{n} (\ln n)^{1-1/\phi}.$$  

Hence, the middle term is asymptotic to

$$\mathbb{E}_{\beta,w} \left[ \beta \int_{\overline{w}+\beta_p}^\infty mF^{m-1}(s)f^2(s)ds \right] \sim \phi \lambda^{1/\phi} \frac{1}{n} (\ln n)^{1-1/\phi} \mathbb{E}_{\beta,w} [\beta]. \quad (40)$$

Using this middle term on the left and proceeding analogously gives

$$\mathbb{E}_{\beta,w} \left[ \beta \int_{\overline{w}+\beta_p}^\infty mF^{m-1}(s)f^2(s)ds \right] \leq \mathbb{E}_{\beta,w} \left[ \beta \int_{w+\beta_p}^\infty mF^{m-1}(s)f^2(s)ds \right]$$

$$\leq \mathbb{E}_{\beta,w} \left[ \beta \int_{0}^\infty mF^{m-1}(s)f^2(s)ds \right].$$

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By (39) and (40)

\[
\int_{0}^{\infty} m F^{m-1}(s)f^{2}(s)ds \sim \int_{\tau + \beta}^{\infty} m F^{m-1}(s)f^{2}(s)ds \sim \phi \lambda^{1/\phi} \frac{1}{n} (\ln n)^{1-1/\phi}.
\]

So that the middle term is asymptotic to

\[
\mathbb{E}_{\beta, w} \left[ \beta \int_{\tau + \beta}^{\infty} m F^{m-1}(s)f^{2}(s)ds \right] \sim \phi \lambda^{1/\phi} \frac{1}{n} (\ln n)^{1-1/\phi} \mathbb{E}_{\beta, w} [\beta].
\]

Thus, unconditionally

\[
\frac{\partial D(p, p)}{\partial p} = -\mathbb{E}_{\beta, w} \left[ \beta \int_{q}^{\infty} (n-1) f(s) F^{n-2}(s)f(s)ds \right] \\
\sim -\phi \lambda^{1/\phi} \frac{1}{n} (\ln n)^{1-1/\phi} \mathbb{E}[\beta].
\]

**Proof of Theorem 4.**

First, note that both \(D(p, p; n)\) and \(-D_1(p, p; n)\) are both decreasing in \(p_1\). Let \(M\) be large and independent of \(n\) and \(p \in [c, M]\). Since \(D(c, c; n)\) and \(D(M, M; n)\) are asymptotic to the same function of \(n\), see (37), it follows that \(D(p, p; n)\) converges uniformly to the same function of \(n\) on the interval \([c, M]\). We can make an identical argument to show that \(D_1(p, p; n)\) converges uniformly on the interval \([c, M]\). It then follows that \(-D(p, p; n)/D_1(p, p; n)\) converges uniformly to the asymptotic markup expression (which is a function of \(n\)) on \([c, M]\)

\[
\lim_{n \to \infty} -D(p, p)/D_1(p, p) \sim \phi \lambda^{1/\phi} \frac{1}{n} (\ln n)^{1-1/\phi} \mathbb{E}[\beta]
\]

uniformly on \(p \in [c, M]\).

Let \(q(p, n) = p - c + D(p, p; n)/D_1(p, p; n)\). Our equilibrium markup over \(p(n)\) is characterized by \(q(p(n), n) = 0\). Clearly, \(q(c, n) < 0\) for all \(n\), while for \(M\) sufficiently large \(q(M, n) > 0\) and sufficiently large \(n\). It follows that a solution \(p(n)\) exists to \(q(p, n) = 0\) in \([c, M]\) for sufficiently large \(n\); so a solution \(p(n)\) exists. This solution satisfies

\[
\lim_{n \to \infty} \frac{D(p(n), p(n))}{D_1(p(n), p(n))} \sim \phi \lambda^{1/\phi} \frac{1}{n} (\ln n)^{1-1/\phi} \mathbb{E}[\beta],
\]

as desired. ■
Proof of Proposition 5 First, some notation: \( \pi((p, \sigma), (p^*, \sigma^*); n) \) denotes the profit function of a firm that chooses \((p, \sigma)\) when the remaining \(n - 1\) firms choose \((p^*, \sigma^*)\). Also, \( \pi(p, \sigma; n) \) denotes the profit function of a firm when all \(n\) firms choose \((p, \sigma)\).

**Perloff-Salop Case**

Call \( \sigma^* \) and \( p^* \) the equilibrium choices of the other firms:

\[
\pi((p, \sigma), (p^*, \sigma^*); n) = (p - c(\sigma)) \mathbb{P} \left( \sigma X_1 - p \geq \max_{j \neq 1} \sigma^* X_j - p^* \right) \\
= (p - c(\sigma)) \mathbb{P} \left( \frac{\sigma}{\sigma^*} X_i + \frac{p^* - p}{\sigma^*} \geq \max_j X_j \right) \\
= (p - c(\sigma)) \int f(x) F^{n-1} \left( \frac{\sigma}{\sigma^*} x + \frac{p^* - p}{\sigma^*} \right) dx.
\]

The first-order conditions for profit maximization are as follows. Differentiating with respect to \( p \) yields

\[
p - c(\sigma) = \frac{\int f(x) F^{n-1} (x) dx}{\frac{1}{\sigma}(n - 1) \int f^2(x) F^{n-2} (x) dx}
\]

and differentiating with respect to \( \sigma \) gives

\[
c'(\sigma) \int f(x) F^{n-1} (x) dx = (n - 1) (p - c(\sigma)) \int x f^2(x) F^{n-2} (x) dx \frac{1}{\sigma}.
\]

Some manipulation reveals

\[
c'(\sigma) = \frac{\int x f^2(x) F^{n-2} (x) dx}{\int f^2(x) F^{n-2} (x) dx}.
\]

Now we consider two cases: \( w_u < \infty \) and \( w_u = \infty \). If \( w_u < \infty \), then

\[
\frac{\int x f^2(x) F^{n-2} (x) dx}{\int f^2(x) F^{n-2} (x) dx} = \frac{n^{-1} w_u f \left( \mathbb{F}^{-1} (1/n) \right) \Gamma (\gamma + 2)}{n^{-1} f \left( \mathbb{F}^{-1} (1/n) \right) \Gamma (\gamma + 2)} + o(1) = w_u + o(1).
\]

If \( w_u = \infty \) then

\[
\frac{\int x f^2(x) F^{n-2} (x) dx}{\int f^2(x) F^{n-2} (x) dx} \sim \frac{n^{-1} \mathbb{F}^{-1} (1/n) f \left( \mathbb{F}^{-1} (1/n) \right) \Gamma (2)}{n^{-1} f \left( \mathbb{F}^{-1} (1/n) \right) \Gamma (\gamma + 2)} \sim \frac{\mathbb{F}^{-1} (1/n)}{\Gamma (\gamma + 2)}.
\]

**Sattinger Case**

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We have

\[
\pi((p, \sigma, (p^*, \sigma^*) ; n) = \frac{p - c(\sigma)}{p} \mathbb{P} \left( \frac{e^{\sigma X_i}}{p} \geq \max_{j \neq i} \frac{e^{\sigma X_j}}{p^*} \right) = \frac{p - c(\sigma)}{p} \int f(x) F^{n-1} \left( \frac{\sigma}{\sigma^*} x + \frac{\log p^* - \log p}{\sigma^*} \right) dx
\]

so the first-order conditions for profit maximization become

\[
0 = \pi_2(p, \sigma ; n) = -\frac{c'(\sigma)}{p} \int f(x) F^{n-1}(x) dx + \frac{p - c(\sigma)}{\sigma p} (n - 1) \int x f^2(x) F^{n-2}(x) dx
\]

and

\[
0 = \pi_1(p, \sigma ; n) = \frac{c(\sigma)}{p^2} \int f(x) F^{n-1}(x) dx - \frac{p - c(\sigma)}{\sigma p^2} (n - 1) \int f^2(x) F^{n-2}(x) dx
\]

Rearranging, we get

\[
\frac{p - c(\sigma)}{c(\sigma)} = \frac{\sigma}{n(n-1) \int f^2(x) F^{n-2}(x) dx}
\]

and

\[
c'(\sigma) = \frac{\frac{p - c(\sigma)}{\sigma p} (n - 1) \int x f^2(x) F^{n-2}(x) dx}{\int f(x) F^{n-1}(x) dx},
\]

so

\[
\frac{c'(\sigma)}{c(\sigma)} = \frac{\int x f^2(x) F^{n-2}(x) dx}{\int f^2(x) F^{n-2}(x) dx}
\]

\[
= \begin{cases} 
F^{-1}(1/n) + o\left(F^{-1}(1/n)\right) = w_u + o(1) : w_u < \infty \\
\frac{F^{-1}(1/n) + o\left(F^{-1}(1/n)\right)}{F(\gamma+2)} : w_u = \infty
\end{cases}
\]

as calculated in the Perloff-Salop case.

**Hart Case**
We have

\[
\pi((p, \sigma), (p^*, \sigma^*) ; n) = (p - c(\sigma)) \mathbb{E} \left[ \frac{e^{\psi x_i}}{p^{1+\psi}} I \left\{ \frac{e^{\sigma x_i}}{p} \geq \max_{j \neq i} \frac{e^{\sigma^* x_j}}{p} \right\} \right]
\]

\[
= (p - c(\sigma)) \mathbb{E} \left[ \frac{e^{\psi x_i}}{p^{1+\psi}} I \left\{ \frac{\sigma x_i + \log p^* - \log p}{\sigma} = \max_{j \neq i} x_j \right\} \right]
\]

\[
= (p - c(\sigma)) \int \frac{e^{\psi x}}{p^{1+\psi}} f(x) F^{n-1} \left( \frac{\sigma}{\sigma^*} x + \frac{\log p^* - \log p}{\sigma^*} \right) dx
\]

so the first-order conditions for profit maximization become

\[
0 = \pi_2(p, \sigma; n) = -c'(\sigma) \int \frac{e^{\psi x}}{p^{1+\psi}} f(x) F^{n-1}(x) dx + (p - c(\sigma)) \left\{ \int \psi x \frac{e^{\psi x}}{p^{1+\psi}} f(x) F^{n-1}(x) dx \right\}
\]

\[
+ \frac{n-1}{\psi} \int x \frac{e^{\psi x}}{p^{1+\psi}} f^2(x) F^{n-2}(x) dx
\]

and

\[
0 = \pi_1(p, \sigma; n) = \int \frac{e^{\psi x}}{p^{1+\psi}} f(x) F^{n-1}(x) dx - (p - c(\sigma)) \left\{ (1 + \psi) \int \frac{e^{\psi x}}{p^{1+\psi}} f(x) F^{n-1}(x) dx \right\}
\]

\[
+ \frac{n-1}{\psi} \int \frac{e^{\psi x}}{p^{1+\psi}} f^2(x) F^{n-2}(x) dx
\]

so

\[
\frac{p - c(\sigma)}{c(\sigma)} = \frac{\int e^{\psi x} f(x) F^{n-1}(x) dx}{\psi \int e^{\psi x} f(x) F^{n-1}(x) dx + \frac{n-1}{\psi} \int e^{\psi x} f^2(x) F^{n-2}(x) dx}
\]

and

\[
\frac{c'(\sigma)}{c(\sigma)} = \frac{p - c(\sigma) \int \psi x e^{\psi x} f(x) F^{n-1}(x) dx + \frac{n-1}{\psi} \int x e^{\psi x} f^2(x) F^{n-2}(x) dx}{\int e^{\psi x} f(x) F^{n-1}(x) dx}
\]

\[
= \frac{\psi \int x e^{\psi x} f(x) F^{n-1}(x) dx + \frac{n-1}{\psi} \int x e^{\psi x} f^2(x) F^{n-2}(x) dx}{\psi \int e^{\psi x} f(x) F^{n-1}(x) dx + \frac{n-1}{\psi} \int e^{\psi x} f^2(x) F^{n-2}(x) dx}
\]

Now we consider two cases: \( w_u < \infty \) and \( w_u = \infty \). If \( w_u < \infty \), then (noting that \( a = 0 \) in this case)
\[
\frac{c'(\sigma)}{c(\sigma)} = \frac{\psi \int xe^{\psi \sigma x} f(x) F^{n-1}(x) \, dx + \frac{n-1}{\sigma} \int xe^{\psi \sigma x} f^2(x) F^{n-2}(x) \, dx}{\psi \int e^{\psi \sigma x} f(x) F^{n-1}(x) \, dx + \frac{n-1}{\sigma} \int e^{\psi \sigma x} f^2(x) F^{n-2}(x) \, dx} = \frac{\psi n^{-1} w_u \sigma x \psi \psi \omega_n \Gamma(1) + \frac{1}{\sigma} w_u \sigma x \psi \psi \omega_n \Gamma(1) (1 - \psi a) + \frac{1}{\sigma} \psi \psi \omega_n \Gamma(2 - \psi a) + o(1)}{\psi n^{-1} \psi \psi \omega_n \Gamma(1) (1 - \psi a) + \frac{1}{\sigma} \psi \psi \omega_n \Gamma(2 + \gamma - \psi a) + o(1)} = w_u + o(1).
\]

If \( w_u = \infty \), then noting that \( \gamma = 0 \),
\[
\frac{c'(\sigma)}{c(\sigma)} = \frac{\psi \int xe^{\psi \sigma x} f(x) F^{n-1}(x) \, dx + \frac{n-1}{\sigma} \int xe^{\psi \sigma x} f^2(x) F^{n-2}(x) \, dx}{\psi \int e^{\psi \sigma x} f(x) F^{n-1}(x) \, dx + \frac{n-1}{\sigma} \int e^{\psi \sigma x} f^2(x) F^{n-2}(x) \, dx} \sim \frac{\psi n^{-1} F^{-1}(1/\sigma) \epsilon \psi \psi \omega_n \Gamma(1) (1 - \psi a) + \frac{1}{\sigma} \psi \psi \omega_n \Gamma(2 - \psi a) + o(1)}{\psi n^{-1} \epsilon \psi \psi \omega_n \Gamma(1) (1 - \psi a) + \frac{1}{\sigma} \psi \psi \omega_n \Gamma(2 + \gamma - \psi a) + o(1)} = F^{-1}(1/\sigma).
\]

**Proof of Proposition 7** First, note that the demand function under the Hart specification is
\[
\mathbb{E} \left[ \frac{\epsilon^{\psi x_i}}{p_i^1+\psi} I \left\{ \epsilon^{\sigma x_i} \geq \max_{j \neq i} \epsilon^{\sigma x_j} \right\} \right] = \frac{1}{p_i^1+\psi} \int_{u_i}^{w_u} e^{\psi \sigma x} f(x) \prod_{j \neq i} F \left( x + \frac{\ln p_i - \ln p_j}{\sigma^*} \right) \, dx.
\]

We can then make the substitutions \( F(y) = \exp\left(-e^{-y/\phi}\right) \) and \( f(x) = \frac{1}{\phi} \exp\left(-\frac{x}{\phi} - e^{-x/\phi}\right) \) to calculate
D \left( p_1, \ldots, p_n; \sigma^* \right) = \frac{1}{p_i^{1+\psi}} \int_{w_i}^{u_i} e^{\psi \sigma^* x} f(x) \prod_{j \neq i} F \left( x + \frac{\ln p_i - \ln p_j}{\sigma^*} \right) dx
\begin{align*}
&= \frac{1}{p_i^{1+\psi}} \int_{w_i}^{u_i} \frac{1}{\phi} e^{\psi \sigma^* x} \exp \left( -\frac{x}{\psi} - e^{-x/\phi} \right) \exp \left( -\prod_{j \neq i} e^{\left( x + \frac{\ln p_i - \ln p_j}{\sigma^*} \right) / \phi} \right) dx \\
&= \frac{1}{\phi p_i^{1+\psi}} \int_{-\infty}^{\infty} \exp \left( x \left( \sigma^* \psi - \frac{1}{\phi} \right) - \sum_{j=1}^{n} \left( \frac{p_i}{p_j} \right)^{1/(\phi \sigma^*)} e^{-x/\phi} \right) dx \\
&= \Gamma \left( 1 - \phi \psi / \sigma^* \right) \frac{p_i^{-1/(\phi \sigma^*)}}{\left( \sum_{j=1}^{n} p_j^{-1/(\phi \sigma^*)} \right)^{1-\phi \psi / \sigma^*}}
\end{align*}

where, for the last equality, we use the identity
\[
\int_{-\infty}^{\infty} \exp \left( ax + be^{-x} \right) = b^a \Gamma \left( -a \right).
\]

E Second-Order Conditions for Profit Maximization

Recall that the profit function \( \pi \left( p_i, p \right) \) for firm \( i \) when it sets price \( p_i \) and all other firms set price \( p \) is
\[
\pi \left( p_i, p \right) = (p_i - c)D \left( p_i, p \right) - K. \tag{41}
\]
So far, we have analyzed the first-order condition for profit maximization, \( \pi_1 \left( p, p; n \right) = 0 \), which is necessary but not sufficient to ensure equilibrium. Anderson et al. (1992) show (Prop. 6.5, p.171 and Prop. 6.9, p.184) that symmetric price equilibria exist in the Perloff-Salop, Sattinger and Hart models when \( f \) is log-concave. Thus in these cases (41) defines the unique symmetric price equilibrium. However, their results do not cover distributions where \( f \) is not log-concave. We are unable to derive global conditions for existence of equilibrium in these cases. Instead, we verify in this appendix that the markups we study satisfy the second-order conditions for profit-maximization.

E.1 Perloff-Salop, Sattinger and Hart Models

The following three propositions show that the symmetric equilibrium markup expression (3) which we use in our calculations also satisfies the second-order condition for profit maximiza-
tion, \( \pi_{11} (p, p; n) < 0 \). It is useful to note that, via simple calculations, the second order condition is
\[
\pi_{11} (p, p; n) = 2D_1 (p, p; n) - \frac{D (p, p; n)}{D_1 (p, p; n)} D_{11} (p, p; n) < 0.
\] (42)

**Proposition 8** Assume that \( F \) satisfies the conditions for Theorem 1, that \( f^3 (x) \) is \([w_l, w_u]\)-integrable, and that
\[
-4 \Gamma (\gamma + 2)^2 + \Gamma (2\gamma + 3) < 0,
\]
which holds for \(-1.45 < \gamma < 0.64\). Then the second-order condition for profit maximization is satisfied in the symmetric equilibrium of the Perloff-Salop model.

Note that this covers all distributions with thin \((-1 \leq \gamma \leq 0)\) and medium fat tails \((\gamma = 0)\), and all the heavy tailed distributions with a finite variance, i.e. \( \gamma \in (0, 1/2] \).

**Proposition 9** Assume that \( F \) satisfies the conditions for Theorem 1, that \( f^3 (x) \) is \([w_l, w_u]\)-integrable, and that either \( \gamma > 0 \) or
\[
-4 \Gamma (\gamma + 2)^2 + \Gamma (2\gamma + 3) < 0,
\]
which holds for \(-1.45 < \gamma \leq 0\). Then the second-order condition for profit maximization is satisfied in the symmetric equilibrium of the Sattinger model.

**Proposition 10** Assume that the conditions for Theorem 1 are satisfied, and that \( e^{\psi x} f^3 (x) \) is \([w_l, w_u]\)-integrable. Then the second-order condition for profit maximization is satisfied in the symmetric equilibrium of the Hart model.

**Proof of Proposition 8**

We use \( U_n = F^{-1} (1/n) \) as a shortcut notation in several of the proofs below. Note, from Section C in this Appendix, that
\[
D (p_i, p) = \int f (x) F^{n-1} (x + p - p_i) \, dx \text{ and } D_1 (p_i, p) = - (n - 1) \int f (x) f (x + p - p_i) F^{n-2} (x + p - p_i) \, dx,
\]
from which we may calculate
\[
D_{11} (p, p) = \frac{(n - 1)(n - 2)}{2} \int f^3 (x) F^{n-3} (x) \, dx + \frac{n - 1}{2} f^2 (x) F^{n-2} (x) \bigg|_{-\infty}^{\infty}
\]
where the last term on the RHS vanishes. So, applying Proposition 3,

\[
\pi_{11}(p, p; n) = 2D_1(p, p; n) - \frac{D(p, p; n)}{D_1(p, p; n)} D_{11}(p, p; n)
\]

\[
= -2(n - 1) \int f^2(x) F^{n-2}(x) \, dx + \frac{(n-1)(n-2)}{2n(n-1)} \int f^3(x) F^{n-3}(x) \, dx
\]

\[
= -2(n - 1) \int f^2(x) F^{n-2}(x) \, dx + \frac{(n-2)}{2n} \int f^3(x) F^{n-3}(x) \, dx
\]

\[
\sim -2f(U_n) \Gamma(\gamma + 2) + \frac{f(U_n) \Gamma(2\gamma + 3)}{2\Gamma(\gamma + 2)}
\]

\[
= \frac{f(U_n)}{2\Gamma(\gamma + 2)} (-4\Gamma(\gamma + 2)^2 + \Gamma (2\gamma + 3)).
\]

since we can easily verify numerically that \(-4\Gamma(\gamma + 2)^2 + \Gamma (2\gamma + 3) < 0\) for \(-1.45 < \gamma \leq 0\), it follows that

\[
\pi_{11}(p, p; n) < 0 \quad \text{for} \quad \gamma \in [-1.45, 0.64].
\]

**Proof of Proposition 9**

Without loss of generality, let \(\theta y = 1\). Then, from Section C in this Appendix,

\[
D(p_i, p) = \frac{1}{p_i} \int f(x) F^{n-1}(x + \ln p - \ln p_i) \, dx \quad \text{and}
\]

\[
D_1(p_i, p) = -\frac{1}{p_i^2} \int f(x) F^{n-1}(x + \ln p - \ln p_i) \, dx
\]

\[- \frac{n-1}{p_i^3} \int f(x) f(x + \ln p - \ln p_i) F^{n-2}(x + \ln p - \ln p_i) \, dx,
\]

from which we may calculate

\[
D_{11}(p, p) = \frac{2}{p^3} \int f(x) F^{n-1}(x) \, dx + 3 \frac{n-1}{p^3} \int f^2(x) F^{n-2}(x) \, dx
\]

\[+ \frac{(n-1)(n-2)}{2p^3} \int f^3(x) F^{n-3}(x) \, dx + \frac{n-1}{2p^3} [f^2(x) F^{n-2}(x)]_\infty^{-\infty}
\]

where the last term on the RHS vanishes. We may then substitute our expressions for \(D(p, p; n), D_1(p, p; n), D_{11}(p, p; n)\) into (42) and apply Proposition 3. The asymptotic ex-
pression simplifies to

\[
\pi_{11}(p, p; n) = 2D_1(p, p; n) - \frac{D(p, p; n)}{D_1(p, p; n)}D_{11}(p, p; n)
\]

\[
= -\frac{2}{p^n} \left( \int f(x) F^{n-1}(x) \, dx + (n - 1) \int f^2(x) F^{n-2}(x) \, dx \right)
\]

\[
+ \frac{2 \int f(x) F^{n-1}(x) \, dx + 3(n - 1) \int f^2(x) F^{n-2}(x) \, dx}{p^n \left( \int f(x) F^{n-1}(x) \, dx + (n - 1) \int f^2(x) F^{n-2}(x) \, dx \right)}
\]

\[
= \frac{p^{-2}}{n} \left( -2 \left( 1 + nf(U_n) \Gamma(\gamma + 2) \right) + o(nf(U_n)) \right)
\]

In the case \( nf(U_n) = o(1) \), which implies \( \gamma \geq 0 \) and \( f(w_u) = 0 \), we get

\[
\pi_{11}(p, p; n) = \frac{p^{-2}}{n} \left( -2 \left( 1 + nf(U_n) \Gamma(\gamma + 2) \right) + o(nf(U_n)) \right)
\]

\[
= \frac{p^{-2}}{n} \left( -2 \left( 1 + nf(U_n) \Gamma(\gamma + 2) \right) + \frac{2 + 3nf(U_n)\Gamma(\gamma + 2) + 1}{1 + nf(U_n)\Gamma(\gamma + 2)} (nf(U_n))^2 + o(nf(U_n)^2) \right)
\]

\[
= \frac{p^{-2}}{n} \left( -2 \left( 1 + nf(U_n) \Gamma(\gamma + 2) \right) + \frac{2 + 3nf(U_n)\Gamma(\gamma + 2) + 1}{1 + nf(U_n)\Gamma(\gamma + 2)} (nf(U_n))^2 \right)
\]

\[
< 0.
\]
In the case $\lim_{n \to \infty} n f(U_n) \in (0, \infty)$, which implies $\gamma = 0$, we get

$$
\pi_{11}(p, p; n) = \frac{p^{-2}}{n} \left( \begin{array}{c} -2 (1 + n f(U_n) \Gamma(\gamma + 2)) + o(n f(U_n)) \\ 2 + 3n f(U_n)(\Gamma(\gamma + 2)) + \frac{1}{2} (n f(U_n))^2 + \frac{1}{2} (n f(U_n))^2 + (n f(U_n)) + o(n f(U_n))^2 \\ (1 + n f(U_n) \Gamma(\gamma + 2)) + o(n f(U_n)) \end{array} \right) 
$$

$$
= \frac{p^{-2}}{n} \left( -2 (1 + n f(U_n)) + \frac{2 + 3n f(U_n) + (n f(U_n))^2}{1 + n f(U_n)} + o(n f(U_n)) \right) 
$$

$$
= \frac{p^{-2}}{n} (-n f(U_n) + o(n f(U_n))) < 0.
$$

In the case $\lim_{n \to \infty} n f(U_n) = \infty$, which implies $\gamma \leq 0$, we get

$$
\pi_{11}(p, p; n) = \frac{p^{-2}}{n} \left( \begin{array}{c} -2 (1 + n f(U_n) \Gamma(\gamma + 2)) + o(n f(U_n)) \\ + \frac{2 + 3n f(U_n)(\Gamma(\gamma + 2)) + \frac{1}{2} (n f(U_n))^2 + \frac{1}{2} (n f(U_n))^2 + o(n f(U_n)) + o(n f(U_n))^2}{(1 + n f(U_n) \Gamma(\gamma + 2)) + o(n f(U_n))} \end{array} \right) 
$$

$$
= \frac{p^{-2}}{n} \left( -2 n f(U_n) \Gamma(\gamma + 2) + o(n f(U_n)) \right) 
$$

$$
= \frac{p^{-2}}{n} \left( \frac{1}{2} \Gamma(2 \gamma + 3) + \frac{1}{2} \Gamma(2 \gamma + 3) + o(n f(U_n)) \right) 
$$

since we can easily verify numerically that $-2 \Gamma(\gamma + 2) + \frac{1}{2} \Gamma(2 \gamma + 3) + \frac{1}{2} \Gamma(2 \gamma + 3) + \frac{1}{2} \Gamma(2 \gamma + 3) + o(n f(U_n)) < 0$ for $-1.45 < \gamma \leq 0$, it follows that

$$
\pi_{11}(p, p; n) < 0 \text{ for } \gamma \in [-1.45, 0]. \blacksquare
$$

**Proof of Proposition 10**
Note that in the Hart case, we are restricted to $\gamma \in [-1, 0]$. We have, from Section C in this Appendix,

$$D(p_i, p) = \frac{1}{p_i^{1+\psi}} \int e^{\psi x} f(x) F^{n-1}(x + \ln p - \ln p_i) \, dx$$

and

$$D_1(p_i, p) = -\frac{1}{p_i^{2+\psi}} \left\{ (1 + \psi) \int e^{\psi x} f(x) F^{n-1}(x + \ln p - \ln p_i) \, dx + (n - 1) \int e^{\psi x} f(x) f(x + \ln p - \ln p_i) F^{n-2}(x + \ln p - \ln p_i) \, dx \right\},$$

from which we may calculate

$$D_{11}(p, p) = \frac{1}{p^{3+\psi}} \left\{ (1 + \psi)(2 + \psi) \int e^{\psi x} f(x) F^{n-1}(x) \, dx + 3 \left(1 + \frac{\psi}{2}\right)(n - 1) \int e^{\psi x} f^2(x) F^{n-2}(x) \, dx + \frac{1}{2}(n - 1)(n - 2) \int e^{\psi x} f^3(x) F^{n-3}(x) \, dx \right\}.$$

We may then substitute our expressions for $D(p, p; n), D_1(p, p; n), D_{11}(p, p; n)$ into (42) and apply Proposition 3. This gives us

$$2D_1(p, p; n) - \frac{D(p, p; n)}{D_1(p, p; n)} D_{11}(p, p; n) = \frac{e^{\psi U_n}}{p_{i}^{2+\psi}} (A + B),$$

where

$$A \sim -2(1 + \psi) \Gamma(1 - \alpha \psi) - 2n f(U_n) \Gamma(\gamma + 2 - \alpha \psi)$$

and

$$B \sim \Gamma(1 - \alpha \psi) \left( \frac{(1 + \psi)(2 + \psi) \Gamma(1 - \alpha \psi)}{1 + \psi} + 3 \left(1 + \frac{\psi}{2}\right) n f(U_n) \Gamma(\gamma + 2 - \alpha \psi) + \frac{1}{2} \left(n f(U_n)\right)^2 \Gamma(2\gamma + 3 - \alpha \psi) \right) \Gamma(1 - \alpha \psi) + n f(U_n) \Gamma(\gamma + 2 - \alpha \psi).$$
After some tedious but straightforward calculations: if \( a = 0 \), then \( n f(U_n) \to_{n \to \infty} \infty \), and the asymptotic expression simplifies to

\[
\pi_{11}(p, p; n) \sim \frac{e^{\psi U_n}}{p_i^{2+\psi}} n f(U_n) \left( -2 \Gamma (\gamma + 2) + \frac{\Gamma (2 \gamma + 3)}{2 \Gamma (\gamma + 2)} \right)
\]

\(< 0 \) for \( \gamma \in [-1, 0] \)

Since we can verify that \(-2 \Gamma (\gamma + 2) + \frac{\Gamma (2 \gamma + 3)}{2 \Gamma (\gamma + 2)} < 0 \) for \( \gamma \in [-1, 0] \), our claim holds in the case \( a = 0 \).

If \( 0 < a < \infty \), then \( \gamma = 0 \), \( nU_n \to 1/a \) and the asymptotic expression simplifies to

\[
\pi_{11}(p, p; n) \sim \frac{e^{\psi U_n}}{p_i^{2+\psi}} \Gamma (1 - a \psi) \left( -2 (1 + 1/a) + \frac{(1 + \psi) (2 + \psi) + 3 (1 + \psi/2) (1/a - \psi)}{1 + 1/a} \right)
\]

\[= -\frac{e^{\psi U_n} \Gamma (1 - a \psi)}{p_i^{2+\psi} a} < 0.\]

**E.2 ELRU Model**

Finally, we check the second order condition for the ELRU model in the case that the density \( f(x) \) is of the Rootzen type (22).

**Proposition 11** Assume that the conditions for Theorem 4 are satisfied. Suppose, moreover, that the distribution for the “taste for money” is such that the variance is smaller than the square of the mean, i.e. \( V[\beta] < E[\beta]^2 \). Then the second-order condition for profit maximization is satisfied in the symmetric equilibrium of the ELRU model.
Proof of Proposition 11 First condition on $\beta$, $w$ and hence $q = \beta p + w$ having a fixed value.

Differentiation gives

$$\frac{\partial^2 D(p, p)}{\partial p_i^2} = -\beta^2 \int_0^{\infty} \frac{\partial f(s)}{\partial s} \left[ (n - 1) F^{n-2}(s)f(s) \right] ds$$

$$\sim -\beta^2 \int_0^{\infty} \left[ -\kappa \lambda^2 \phi^2 x^{2\phi + \nu - 2} \exp(-\lambda x^{\phi}) \right] \left[ (n - 1) F^{n-2}(s)f(s) \right] ds.$$ 

Moreover, for $s$ close to zero

$$f' \left( \overline{F}^{-1}(s) \right) \sim -\phi^2 \lambda^{2/\phi} s \left[ \ln \frac{A}{s} \right]^{-n/\phi} \left[ \ln \left( \frac{A}{s} \left[ \ln \frac{A}{s} \right]^{a/\phi} \right) \right]^{2+(\nu-2)/\phi}.$$ 

From this the regular variation at zero of $f' \left( \overline{F}^{-1}(y) \right)$ follows:

$$\lim_{y \to 0} \frac{f' \left( \overline{F}^{-1}(xy) \right)}{f' \left( \overline{F}^{-1}(y) \right)} = \lim_{t \to \infty} \frac{f' \left( \overline{F}^{-1}(x/t) \right)}{f' \left( \overline{F}^{-1}(1/t) \right)}$$

$$= x \lim_{t \to \infty} \frac{\left[ \ln \frac{At}{2} \right]^{-\nu/\phi} \left[ \ln \left( \frac{At}{2} \left[ \ln \frac{At}{2} \right]^{a/\phi} \right) \right]^{2+(\nu-2)/\phi}}{\left[ \ln At \right]^{-\nu/\phi} \left[ \ln \left( At \left[ \ln At \right]^{a/\phi} \right) \right]^{2+(\nu-2)/\phi}}$$

$$= x.$$ 

Hence, $f' \left( \overline{F}^{-1}(y) \right) \in RV_1^0$. So that by the main theorem at given values of $\beta$ and $w$

$$\frac{\partial^2 D(p, p)}{\partial p_i^2} \sim -\beta^2 f' \left( \overline{F}^{-1} \left( \frac{1}{n} \right) \right)$$

$$\sim \beta^2 \phi^2 \lambda^{2/\phi} \frac{1}{n} \left[ \ln A + \ln n \right]^{-\nu/\phi} \left\{ \ln A + \ln n + \frac{\nu}{\phi} \ln (\ln A + \ln n) \right\}^{2+(\nu-2)/\phi}$$

$$\sim \beta^2 \phi^2 \lambda^{2/\phi} \frac{1}{n} (\ln n)^{2-2/\phi}.$$ 

Subsequently apply the same sandwich arguments as were used in the proof to Lemma 7 and integrate out over the random taste for money and outside option. This gives

$$\frac{\partial^2 D(p, p)}{\partial p_i^2} \sim \phi^2 \lambda^{2/\phi} \frac{1}{n} (\ln n)^{2-2/\phi} \mathbb{E}[\beta^2]$$
The second order condition becomes

\[
2D_1 - \frac{D}{D_1}D_{11} \sim -2\phi \lambda^{1/\phi} \frac{1}{n} (\ln n)^{1-1/\phi} \mathbb{E}[\beta] \\
+ \frac{1/n}{\phi \lambda^{1/\phi} \frac{1}{n} (\ln n)^{1-1/\phi} \mathbb{E}[\beta]} \phi^2 \lambda^{2/\phi} \frac{1}{n} (\ln n)^{2-2/\phi} \mathbb{E}[\beta^2] \\
= -2\phi \lambda^{1/\phi} \frac{1}{n} (\ln n)^{1-1/\phi} \mathbb{E}[\beta] + \phi \lambda^{1/\phi} \frac{1}{n} (\ln n)^{1-1/\phi} \frac{\mathbb{E}[\beta^2]}{\mathbb{E}[\beta]} \\
= -\phi \lambda^{1/\phi} \frac{1}{n} (\ln n)^{1-1/\phi} \frac{1}{\mathbb{E}[\beta]} \left\{ 2\mathbb{E}[\beta]^2 - \mathbb{E}[\beta^2] \right\} \\
= -\phi \lambda^{1/\phi} \frac{1}{n} (\ln n)^{1-1/\phi} \frac{\mathbb{E}[\beta]^2 - \text{var}[\beta]}{\mathbb{E}[\beta]}.
\]

The SOC is satisfied if the variance of the taste for money is smaller than the square of the mean.