Boundedly Rational Dynamic Programming

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July 2015

Abstract

This paper proposes a way to model boundedly rational dynamic programming in a parsimonious and tractable way. The framework is quite general, and has at its core a behavioral version of the Bellman equation, in which the agent uses a simplified model of the world and the consequences of his actions. It is then applied to some of the canonical models in macroeconomics and finance.

In the consumption-savings model, the consumer can pay limited attention to the variables such as the interest rate and his income – this way using a simplified, “sparse” model of the world. Endogenously, the consumer pays little or no attention to the interest rate but pays more attention to his income.

The model yields a behavioral version of the New Keynesian model. In particular, it helps solve the “forward guidance puzzle”, the fact that in that model, shocks to very distant rates have a very powerful impact on today’s consumption and inflation: because the agent is de facto myopic, this effect is muted.

Ricardian equivalence partially fails, because the consumer is only partially attentive to future taxes. In a Merton-style portfolio choice problem, the agent endogenously pay limited or no attention to the varying equity premium and hedging demand terms. Finally, the paper gives a behavioral version of the canonical neoclassical growth model. Fluctuations are more persistent when agents are more boundedly rational.

*xgabaix@stern.nyu.edu. I thank Jerome Williams for very good research assistance. For useful comments I thank Nick Barberis, Robert Barro, John Campbell, Harrison Hong, David Laibson, Jennifer La’O, Ali Lazrak, Bentley MacLeod, Michael Rockinger, Thomas Sargent, Alp Simsek and seminar participants at various seminars and conferences. I am grateful to the Dauphine-CAM foundation, the Institute for New Economic Thinking, and the NSF (SES-1325181) for financial support.
1 Introduction

This paper proposes a way to model dynamic programming with boundedly rational agents. It lays out a fairly general procedure to formulate dynamic programming in situations of economic interest. Then, it shows how the framework applies to some of the canonical models in macro-finance and economics more generally: consumption saving problems, the basic New Keynesian model, the baseline neoclassical growth model (the Ramsey-Cass-Koopmans model), general linear-quadratic problems, and investment in risky assets (Merton’s problem). The upshot is that we have a portable, fairly general structure that applies to the some basic machines of macroeconomics, and allows to see where bounded rationality (BR) is important in those situations.

One of the criticisms of traditional economic models is the potential unrealism of the infinitely forward-looking agent who computes the whole equilibrium in her own head. This lack of realism has long been suspected to be the cause of some empirical misfits that we will review below. Behavioral economics aims to provide an alternative. However, the greatest successes of behavioral economics change the agents’ tastes (e.g. prospect theory or hyperbolic discounting) or their beliefs (e.g. overconfidence), but typically keep the assumption of rationality. When tackling the rationality assumption, there is much less agreement and the modelling of bounded rationality is much more piecemeal, different from one paper to the next.

This paper proposes a compromise that keeps much of the generality of the rational approach and injects some of the wisdom of the behavioral approach, mostly inattention and simplification. It does so by proposing a way to insert some bounded rationality into a large class of problems, the “recursive” contexts, i.e. with dynamic programming in some stochastic steady state.

To illustrate these ideas, let us consider a canonical consumption-savings problem. The agent maximizes utility from consumption, subject to a budget constraint, with a stochastic interest rate and stochastic income. In the rational model, the agent solves a complex DP problem with three state variables (wealth, income and the interest rate). This is a complex problem that requires a computer to solve it.

How will a boundedly rational agent behave? I assume that the agent starts with a much simpler model, where the interest rate and income are constant – this is the agent’s “default” model. Only one state variable remains, his wealth. He knows what to do then, but what will he do in a more complex environment, with stochastic interest rate and stochastic income? In the sparse version, he considers parsimonious enrichments to the value function, as in a Taylor expansion. He asks, for each component, whether it will matter enough for his decision. If a given feature (say, the interest rate) is small enough compared to some threshold (taken to be a fraction of standard deviation of consumption), then he drops the feature, or partially attenuates it. The result is a consumption policy that pays partial attention to income, and possibly no attention at all to the interest rate. This does seem realistic.

The result is a sparse version of the traditional permanent-income model. We see that it
is often simpler than the traditional model. Indeed, the agent typically ends up using a rule which is simpler (e.g., not paying attention to the interest rate).

Let us now turn to macro consequences of this approach. One is a behavioral version of the traditional New Keynesian model. In that model, “forward guidance” works very powerful, probably too powerful, emphasized by McKay, Nakamura and Steinsson (2015). The reason is that the traditional consumer always respects Euler equation, so that a movement of interest far in the future will have a strong impact today. However, in the behavioral model I put forth, this impact is muted by the agent’s myopia. That makes forward guidance less powerful. The model, in reduced form, takes the form of a “discounted Euler equation”, where the agent reacts in a discounted manner to future consumption growth. That apparently small modification, due to bounded rationality, has also powerful implications for the behavior of an economy at the zero lower bound. Depressions due to the ZLB are moderate, and bounded, even though they are unboundedly large with rational model (Werning 2012).

I also present a behavioral version of a large class of models, and work out in detail a BR version of most canonical of them, the neoclassical growth model of Ramsey-Cass-Koopmans. In this version, agents pay no attention to their own variables, less to aggregate variables. One upshot is that with BR, macroeconomic fluctuations are larger and more persistent. I illustrate this proposition, and qualify it, as it appears to hold for most reasonable values of the parameters, but can be overturned for extreme values. To see the idea, which is fundamentally quite simple, imagine first an economy with only one state variable, capital. It starts with a steady state amount of capital. Then, there is a positive shock to the endowment of capital. In a rational economy, agents would consume a certain fraction of it, say 6%, every period. That will lead the capital stock to revert quickly to its mean. However, in an economy with sparse agents, investors will not pay full attention to the additional capital. They will consume less of it than a rational agent would. Hence, capital will be depleted more slowly and will mean-revert more slowly. The shock has more persistent effects.

Given that shocks are more persistent, past shocks accumulate more. Mechanically, this leads to larger average deviations of capital from its trend. As a consequence, the interest rate and GDP also have larger, and more persistent, deviations from trend.

The model allows us to express those ideas in simple, quantitative ways. It allows us to explore them in richer environments, e.g. with shocks to both productivity and the capital stock.

Another application is a Merton-style dynamic portfolio choice problem, i.e. allocating one’s wealth between stocks and bonds when the expected returns are stochastic and correlated with past returns. This is a notoriously complicated problem for a rational agent. I study how a sparse agent would handle it. The sparse agent first anchors his action by imagining he’s facing a simpler problem – a world with a constant equity premium. Then, he can sparsely enrich his model to take into account the more complex features (the stochasticity of the equity premium, its correlation with past returns, which creates a hedging demand). Hence the agent will take these complex features into account only partially, or not at all. This may be a more satisfying description than the hyper-rational model of how
people behave in a complex environment. At the very least, it is important to have a concrete alternative to that hyper-rational model.


One key difference here is that the model is here much more systematic. It explicitly applies, in a unified manner, to a wide class of models. It relies on an earlier paper (Gabaix 2014) that proposes a “sparse max”, a behavioral, less than fully rational and attentive of the traditional max operator. That paper was concerned with static cases, here we explore dynamic ones. Similarly, that paper allows to give a behavioral formulation of some basic chapters of the microeconomic textbooks (consumer theory, equilibrium theory, Arrow-Debreu). The present paper allows to give a behavioral version of some chapters of the macroeconomics textbooks (mostly, consumption-savings problem and the basic neoclassical growth model). Hence, we have a more unified view of bounded rationality in micro and macro, whether those other papers are more piecemeal.

The other approaches have not (yet) yielded a systematic approach of dynamic programming, or of those basic building blocks. One partial exception is Maćkowiak and Wiederholt (forthcoming). They work out a New Keynesian model with an entropy-based penalty for precision à la Sims (2003), and show that it is quantitatively successful. The present paper is more analytical, develops tools that apply to those of other situations. It is also not quantitative, but more systematic in terms of theory.

The rest of the paper is as follows. Section 2 presents the general procedure. Then, we apply to a variety of canonical examples. Section 3 presents basic partial-equilibrium building blocks: the basic consumption-savings problem, including variants such as failure of Ricardian equivalents and the Merton portfolio problem. Section 5 works out the neoclassical growth model, e.g. a general equilibrium situation. Section 6 develops other models: the Merton dynamic consumption-investment model, linear-quadratic models, and the Becker-Murphy rational addiction model. Section 7 concludes. The Appendix and Online Appendix contain further extensions and proofs.

## 2 General Framework

### 2.1 A review of the static sparse max operator

In Gabaix (2014), I defined a sparse max or \( \text{smax} \) operator, which is a behavioral, partially inattentive version of the max operator. The agent faces a maximization problem which is, in its rational version, \( \max_a u(a, x) \). I state here the sparse max, using slightly different notations.
There is an attention vector \( m \), and an attention-dependent extension of the utility function, \( u(a,x,m) \). For instance, defining the Hadamard (component-wise) product as:

\[
m \odot x := (m_i x_i)_{i=1\ldots n}
\]

then

\[
u(a,x,m) = u(a,m \odot x)
\]

is the perceived utility function when the consumer is partially inattentive to \( x_i \). When \( m_i = 1 \), the agent fully perceives dimension \( i \), when \( m_i = 0 \), the agent is fully inattentive to it. There is a default attention vector \( m_d \), taken to be 0 in most applications, and a default action \( a^d := \arg\max_a u(a,x,m^d) \). We call \( a_{m_i} = \frac{\partial u}{\partial m_i} \), evaluated at \((a,m) = (a^d,m^d)\).

Hence, \( a_{m_i} = -u_{aa}^{-1} u_{am_i} \). When (2) holds, \( a_{m_i} = a_{x_i} x_i \).

There is a nonnegative parameter \( \kappa \), which is a taste for sparsity. When \( \kappa = 0 \), the agent is the traditional agent (unless the matrix \( \Lambda \) of Definition 1 is singular – in that case, the iterated sparse max below helps). The \( x_i \) are viewed by the agent as being drawn with a standard deviation \( \sigma_i \), and covariance \( \sigma_{ij} \).

**Definition 1 (Sparse max operator, Gabaix 2014)** The sparse max,

\[
\max_{a,m,m^d} u(a,x,m)
\]

is defined by the following procedure.

**Step 1:** Choose the attention vector \( m^* \):

\[
m^* = \arg\min_{m \in [0,1]^n} \frac{1}{2} \sum_{i,j=1\ldots n} (1-m_i) \Lambda_{ij} (1-m_j) + \kappa \sum_{i=1\ldots n} g(m_i - m^d_i)
\]

with the cost-of-inattention factors \( \Lambda_{ij} := -E[a_m u_{aa} a_m] \).

**Step 2:** Choose the action

\[
a^* = \arg\max_a u(a,x,m^*)
\]

and set the resulting utility to be \( u^* = u(a^*,x) \). In the expressions above, derivatives are evaluated at \( m = m^d \) and \( a^d = \arg\max_a u(a,x,m^d) \).

In other terms, the agent solves for the optimal \( m^* \) that trades off a proxy for the utility losses (the first term in the right-hand side of equation (3)) and a psychological penalty for deviations from a sparse model (the second term on the left-hand side of equation (3)). Then, the agent maximizes over the action \( a \), as if \( m^* \) were the true model.

This leads to define the attention function:

\[
A_g(v) := \sup \left[ \arg\min_{m \in [0,1]} \frac{1}{2} (m - 1)^2 |v| + g(m) \right].
\]

This represents the optimal attention to a variable with variance \(|v|\) an impact of 1 on the decision, with the cost of thinking \( \kappa \) is 1.

The following Lemma derives the main case.
Lemma 1 (Gabaix 2014) When variables are perceived to be uncorrelated, the $s_{\text{max}}$ operator yields:

$$a^* = \arg \max_a u(a, m_1^* x_1, \ldots, m_n^* x_n)$$

with

$$m_i^* = A_g \left( \sigma_i^2 a_{x_i} u_{aa} a_{x_i} / \kappa \right)$$

and $a_{x_i} = \frac{\partial a}{\partial x_i} = -u_{aa}^{-1} \cdot u_{a,x_i}$. In the expressions above, derivatives are evaluated at $x = 0$ and $a^d = \arg \max_a u(a, 0)$.

The intuition is that the $x_i$’s are truncated. If $\left| \frac{\partial a}{\partial x_i} \right|$ is small enough, so that $x_i$ shouldn’t matter much any way, then $m_i^* = 0$, and the agent doesn’t pay attention to $x_i$ (if $m_i^d = 0$).

This leads to the defining the truncation function:

$$\tau_g(b, k) := b A_g \left( \frac{b^2}{k^2} \right)$$

It is the coefficient $b$, times the attention to the coefficient, divided by the scaled cognition cost $k$.

The following lemma gives a more explicit version of the action.

Lemma 2 If the rational action is:

$$a^*(x) = a^d + \sum_i b_i x_i + O(\|x\|^2)$$

then the sparse action is

$$a^*(x) = a^d + \sum_i \tau \left( b_i \frac{\kappa_a}{\sigma_i} \right) x_i + O(\|x\|^2)$$

with $\kappa_a := (\kappa / |u_{aa}|)^{1/2}$.

When attention is chosen after seeing $x$ (“ex post”), we use the same expressions, with $\sigma_i := |x_i|$. For instance, the ex-post action becomes:

$$a^*(x) = a^d + \sum_i \tau \left( b_i x_i, \kappa_a \right) + O(\|x\|^2)$$

We see the contrast. In the first procedure, the slope is chosen before seeing $x_i$. Hence, the policy is still linear in $x_i$. In the second policy, the truncation is chosen after seeing the $x_i$. The policy is now non-linear in $x_i$. The linearity of policies make the first procedure useful for macro.
Attention and Truncation Functions  Here are some good truncation functions. In Gabaix (2014), I study attention functions \( A_\alpha (\sigma^2) \) corresponding to \( g (m) = m^\alpha 1_{m>0} \). For instance, for the values \( \alpha = 0, 1, 2 \), we have (Gabaix 2014):

\[
A_0 (\sigma^2) = 1_{\sigma^2 \geq 2}, \quad A_1 (\sigma^2) = \max \left( 1 - \frac{1}{\sigma^2}, 0 \right), \quad A_2 (\sigma^2) = \frac{\sigma^2}{2 + \sigma^2}
\]

Hence the truncation functions \( \tau_\alpha (b, k) \):

\[
\tau_0 (b, k) = b \cdot 1_{b^2 \geq 2k^2}, \quad \tau_1 (b, k) = b \max \left( 1 - \frac{k^2}{b^2}, 0 \right), \quad \tau_2 (b, k) = \frac{b^3}{b^2 + k^2}
\]

Figure 1 plots the attention functions, and Figure 2 the corresponding truncation functions.

Another useful cost function is \( g_{L_1} (m) = -\kappa \ln (1 - m) \), which generates \( A_{L_1} (\sigma^2) = \max \left( 1 - \frac{1}{|\sigma|}, 0 \right) \), and \( \tau_{L_1} (b, \kappa) = \text{sign} (b) \max (|b| - |\kappa|, 0) \). The subscript \( L_1 \) denotes that it often arises when doing an \( L_1 \) regularization, as in the sparsity literature in statistics (Candès and Tao 2006).

Equipped with this piece of machinery, we turn to dynamic problems.

2.2 Sparse Dynamic Programming: Definition

We express the notions when there is a finite horizon \( T \). The infinite-horizon case is similar with \( T = \infty \). The agent’s rational problem is:

\[
\max_{(a_t)} \sum_{t=0}^{T-1} \beta^t u (a_t, z_t, t) \quad \text{s.t.} \quad z_{t+1} = F (z_t, a_t, \varepsilon_{t+1}, t+1)
\]
and a terminal condition $z_T \in \mathcal{F}^T$ for a given set $\mathcal{F}^T$. Here state variable $z_t$ and action $a_t$ are vectors, while $\varepsilon_{t+1}$ is a mean-zero innovation.

The rational version of the dynamic programming (DP) problem is a series of value functions $V^{r,t}$ satisfying the Bellman equation:

$$V^{r,t}(z) = \max_a u(a, z, t) + \beta \mathbb{E}V^{r,t+1}_T(F^z(z, a, \varepsilon_{t+1}, t + 1))$$

(12)

for $t = 0, ..., T - 1$, and with $V^{r,T}(z) = 0$. A policy is then a function $a(z, t)$.

In the smax version, we are given attention-dependent functions $u(a, z, t; m)$, $F^z_z(z, a, \varepsilon_{t+1}; m)$ and $V^{r,T+1}(z; m)$ — in a way that will be detailed later. To fix ideas, we could take $u(a, z, t; m) = u(a, m \odot z, t)$, and for the $k$-th component of vector $F^z_z$

$$F^z_k(z, a, \varepsilon; m) = F^z_k(m^k \odot z, a, \varepsilon)$$

where $m^k$ denotes the attention to factors; generally $(m^k)^d = 1$: when predicting the future values of variables $z_k$, full attention is paid to its initial value.

We can formulate the BR Bellman equation.

**Definition 2** (Sparse dynamic programming) The sparse value function $V^t(z)$ is the solution of:

$$V^t(z) = \text{smax}_{a,m,m^d}[u(a, z, t; m) + \beta \mathbb{E}V^{t+1}_T(F^z(z, a, \varepsilon_{t+1}; m); m)]$$

(13)

for $t = 0, ..., T - 1$, and with $V^T(z) = 0$. The smax operator for sparse maximization is defined in Definition 1.

This is the same formulation as in the rational version, but with a smax rather than a max operator. The definition gives a construction of the value function by backward induction: starting from $V^T = 0$, we successively calculate $V^{T-1}, ..., V^0$. 

Figure 2: Three truncation functions. Because it gives sparsity and continuity, the $\tau_1$ function is recommended.
In an infinite-horizon problem, the Bellman equation is the same, with \( V^t = V^{t+1} \).\(^1\)

The existence of a value function with finite horizon is automatic. With infinite horizon, existence seems plausible in many applications—we shall see a variety of examples below—but it is mathematically open at this stage.

The problem may look complicated, but in many cases it is actually simple. Before proceeding to the examples, we present some results that help calculate the smax solution. The reader is encouraged to first skip to the applications starting in Section 3.

### 2.3 Tools for Boundedly Rational Dynamic Programming

This subsection present tools to compute BR dynamic programming. The reader is invite to skim it, read the main examples shown later, and then come back to it with those examples in mind.

#### 2.3.1 Tools to Expand a Simple Model Into a More Complex one

We decompose the vector of state variables into: \( z = (w, x) \) where \( w \) is a vector of variables that are fully taken into account \( (m^d = 1) \), in the default mode, while \( x \) is a vector of variables not taken into account \( (m^d = 0) \).

Here I develop the method to derive the Taylor expansion of a richer model, when starting from a simpler one. Here the methods are entirely paper and pencil. They draw from the techniques surveyed by Judd (1998, Chapter 14), who has a more computer-based perspective.

The state variables evolve according to:

\[
\begin{align*}
  w' &= F_w (w, x, a), \\
  x' &= F_x (x)
\end{align*}
\]

where variable \( x \) (which again is a vector) is like a macro disturbance, such as the deviation of the interest from trend, which evolves independently of the actions and state variable of the agent \( (w, a) \).

Consider the fully rational model:

\[
V^r (w, x) = \max_a u (w, x, a) + \beta \mathbb{E} V^r (F_w (w, x, a), F_x (x))
\]

We start with a simpler model, where \( x \) is always 0: \( x \equiv 0 \), \( F_{xx'} = 0 \), i.e.

\[
V^d (w) = \max_a u (w, 0, a) + \beta \mathbb{E} V^d (F_{ww'} (w, 0, a))
\]

We use the notation

\[
D_w f = \partial_w f + (\partial_a f) \frac{da}{dw}
\]

which is the total derivative with respect to \( w \) (e.g. the full impact of a change in \( w \), including the impact it has on a change in the action \( a \)).

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\(^1\) The existence of a value function with finite horizon is automatic. The existence in infinite horizon seems unproblematic in many applications, but it is mathematically open at this stage.
Proposition 1 The impact of a change \( x \) on the value function is:

\[
V_{w,x}(w,0) = \frac{D_wu_x + \beta D_w \left[ F'_x(w,0,a) V'_{w'}(w') \right]}{1 - \beta F'_{x'} D_w w'}
\]  

(15)

The impact of a change \( x \) on the optimal action is:

\[
da = -\Psi^{-1}_a \Psi_x dx
\]

\[
\Psi(a,x) = u_a(w,a) + \beta V'_{w'}F'_a
\]

\[
\Psi_a = u_{aa} + \beta F_{a}^{w'} V'_{w'w'} F'_{a}^{w'} F'_{w'} + \beta V''_{a} F'_{w'}
\]

\[
\Psi_x = u_{ax} + \beta V''_{w'} F'_{w'} + \beta V'_{w'} F'_{ax}
\]

They depend only on the transition functions and the derivatives of the simpler baseline value function \( V_{0w} \).

Proof Differentiating the Bellman equation (first with respect to the new variable \( x \), then with respect to the default variable \( w \)), we obtain:

\[
V_x(w,x) = u_x + \beta V'_{w'} F'_x(w,x,a) + \beta V''_{w'} F'_{xx}
\]

\[
V_{w,x}(w,x) = D_w u_x + \beta D_w \left[ V'_{w'} F'_{x}(w,x,a) \right] + \beta F'_{x} V'_{w',x} D_w w'
\]

so

\[
V_{w,x}(w,0) = \frac{D_w u_x + \beta D_w \left[ F'_x(w,0,a) V'_{w'}(w',0) \right]}{1 - \beta F'_{x'} D_w w'}
\]

\[\square\]

The same procedure can be followed when \( x' = F'x(w,x,a) \), with more complex algebra. We next show a useful consequence.

Proposition 2 For small \( x \), we have:

\[
V(w,x) = V^r(w,x) + x' \phi(w,x) x
\]

where matrix \( \phi(w,x) \) is continuous in \( (w,x) \) and twice differentiable at \( x = 0 \), with \( \phi(w,0) \) negative semi-definite. In other words, the sparse value function and the rational value functions differ only by second order terms in \( x \).

This basically generalizes the envelope theorem. It implies that, at \( x = 0 \):

\[
V_w = V^r_w, \quad V'_{w,w} = V^r_{w,w}, \quad V_x = V^r_x, \quad V'_{w,x} = V^r_{w,x}
\]

(16)

However, in most situations we have \( V''_{xx} \neq V''_{xx} \).

This leads to a simple proposition to calculate the value function.
Proposition 3 (Calculation of the optimal sparse policy). Consider the first order expansion of the optimal policy for small $x$,

$$a^*(w, x) = a^d(w) + \sum_i b_i(w) x_i + O(x^2)$$

Then, the sparse policy is, with ex-ante attention allocation:

$$a^s(w, x) = a^d(w) + \sum_i \tau \left( b_i(w), \frac{\kappa_a}{\sigma_i} \right) x_i + O(x^2)$$  \hspace{1cm} (17)

and with ex-post attention allocation:

$$a^s(w, x) = a^d(w) + \sum_i \tau (b_i(w) x_i, \kappa_a) x_i + O(x^2)$$  \hspace{1cm} (18)

This proposition will be quite useful. To derive policies, first we can simply do a Taylor expansion of the rational policy around the default model, and then truncate term by term.

I conclude with a remark which will be useful later, drawing again on Gabaix (2014). As $\kappa$ has the units of utils. One can make it more endogenous with the primitive, unitless parameter $\kappa$, by setting:

$$\kappa = \bar{\kappa}^2 \text{var} \left( u \left( a^d(x), x, m^d \right) \right)^{1/2}$$  \hspace{1cm} (19)

2.3.2 Simplification of functions

We develop here a bit of simple machinery to reflect how the agent can “simplify” a function (in practice a value function), by forcing them to have a given functional form.

A motivating example. Suppose that the agent consumes $c_1 = \frac{w}{2} + y_1$ and $c_2 = \frac{w}{2} + y_2$, where $y = (y_1, y_2)$ can be viewed as small. His value function is:

$$v(y) = u \left( \frac{w}{2} + y_1 \right) + u \left( \frac{w}{2} + y_2 \right)$$

The agent may wish to use a simplified representation of this function. We observe that $v(y) = v^S(y) + O(||y||^2)$ with

$$v^S(y) := 2u \left( \frac{w + y_1 + y_2}{2} \right)$$

We shall take this function $V^S$ as a “simplified” representation of $v$. We can then form a more general function: $v(y, m^V) := (1 - m^V) v^S(y) + m^V v(y)$. If $m^V = 1$, the agent uses the rational value function. If $m^V = 0$, the agent uses the proxy value function $v^S$, which is in some sense simpler.

The following Definition generalizes that thought and codifies the creation of a “simplified” value function.
**Definition 3** (Simplifying function) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function such that $f_{x_i}(x)|_{x=0} \neq 0$ for all $i$, and $\phi : \{1, \ldots, p\} \to \{1, \ldots, n\}$. Call $\mathcal{E}^f := \{v \in C^1(\mathbb{R}^p, \mathbb{R}) \text{ such that } v(0) = f(0)\}$. We define the simplification function $S_{f, \phi} : \mathcal{E}^f \to \mathcal{E}^f$ by:

$$
(S_{f, \phi}(v))(y) := f(b \cdot y)
$$

(20)

where $b$ is the uniquely determined matrix $b \in \mathbb{R}^{n \times p}$ such $b_{ij} = 0$ unless $i = \phi(j)$ and

$$
v(y) = f(b \cdot y) + o(\|y\|)
$$

(21)

Furthermore, $b_{ij} = \frac{v_{y_j}(y)|_{y=0}}{f_{x_i}(x)|_{x=0}}$ if $i = \phi(j), b_{ij}(x) = 0$ otherwise.

This also defines an attention-augmented function

$$
v(y, m^V) := (1 - m^V)v^S(y) + m^Vv(y)
$$

where parameter $m^V$ captures the attention to the true value $v$.

**Proof** We prove the $b$ is indeed unique. We want: $v(y) = f(b \cdot y) + o(\|y\|)$. This is equivalent to:

$$
v_{y_j}(y)|_{y=0} = \sum_i f_i b_{ij} = f_{\phi(j)}b_{\phi(i)j}.
$$

Inspecting the Taylor expansions gives the result. □

Basically $f(b \cdot y)$ is like a non-linear Taylor expansion of $v(y)$. For instance, in our introductory example, $f(x) = 2u(\frac{x+y}{2})$, $y = (y_1, y_2)$, $n = 1$, $p = 2$, $\phi(j) = 1$, and $b = (1, 1)$.

Here are two other variants of the same idea. Suppose that we have a stochastic variable, and a variant of the Black-Scholes model, with say stochastic volatility. Then, we may approximate the value function in by tweaking the implied volatility: $V(x_t, S, K, r, t) = V^{\text{BlackScholes}}(\sigma(x_t) + o(x_t), S, K, r, t)$, where $V^{\text{BlackScholes}}$ is the regular Black-Scholes formula, and $\sigma(x_t)$ could be affine.

Suppose that the agent estimates a distribution, $h(y)$, where $y$ are parameters of the distribution. The agent may wish to replace this distribution by a distribution with a simpler functional form, say a Gaussian: then $f$ is a Gaussian distribution approximating the distribution $h$, perhaps by matching $h$’s mean and variance.

### 2.4 Iterated Sparse Max

In some cases, it is useful to have a generalization of the basic sparse max.

**Definition 4** (Iterated sparse max) The $K$–times iterated sparse max, $\text{smax}^K_{x, m, m^d}u(a, x)$, is defined by the following procedure. Define $m^d(1) = 1$ to be the initial default attention, $m^d$.

Start at round $k = 1$. At each round $k \leq K$, apply the regular $\text{smax}$, using the default $m^d(k)$: $\text{smax}_{x, m, m^d(k)}u(a, x, m)$, and call $m^*(k)$ and $a^*(k)$ the resulting attention. Define then $m^d(k + 1) = m^*(k)$.

Stop at the end of round $k = K$, and return $m^*(K)$ and $a^*(K)$, the optimal attention and action at the last iteration.
**Illustration.** Suppose that
\[ u(a, x) = -\frac{1}{2} (a - x_1 (1 + x_2))^2 \]
so that the rational policy is \( a^r (x_1, x_2) = x_1 (1 + x_2) \). If the agent doesn’t think of \( x_1 \) (so, replaces it by \( x_1 = 0 \)), he should not think about \( x_2 \).

We next apply the iterated smax outlined in Definition 4, iterating twice \( (K = 2) \). Initial default attention is \( m^d (1) = (0, 0) \). We start step \( k = 1 \). We observe that so \( a^r_{x_1} = 1 + x_2 \), \( a^r_{x_2} = x_1 \), which gives:
\[ m^*_1 (1) = A \left( \frac{\sigma^2_1}{\kappa} \right), \quad m^*_2 (1) = 0 \]
So, at the beginning of the second step, the default is \( m^d (2) = m^* (1) \). Applying again the plain smax but with that default \( m^d (2) \), we have:
\[ m^*_1 (2) = A \left( \frac{\sigma^2_1}{\kappa} \right), \quad m^*_2 (2) = A \left( \frac{m^*_1 (1)^2 \sigma^2_2}{\kappa} \right) \]
Hence, the action is \( a = a^r (m^* (2) \otimes x) = m^*_1 (2) x_1 (1 + m^*_2 (2) x_2) \). We also see that as \( \kappa \to 0 \), the action converges to the rational action.\(^2\)

We have developed tools to define and compute BR dynamic programming. Now we turn to concrete examples.

## 3 Intertemporal Consumption: Behavioral Version

### 3.1 A 3 period model

It is useful to consider a simple 3-period model to clarify a number of notions. Here I assume that the discount factor and the gross interest rates are 1, so that utility is
\[ \sum_{t=0}^{2} u(c_t). \]
Calling \( w_t \) the wealth at the beginning of period \( t \), the budget constraints at times \( t = 0, 1, 2 \) are:
\[ w_1 = w_0 - c_0, \quad w_2 = w_1 - c_1, \quad 0 = w_2 + x - c_2. \]
The agent starts with an endowment \( w_0 \), and receives \( x \) at time 2. For instance, \( x \) could represent a negative income shock, such a tax to pay, or a decrease in income as retirement.

The question is: will the agent pay attention to time-2 payment \( x \)?

\(^2\)This iterated smax suffices for the problems considered in this paper. For other purposes, one could imagine a variant where the default is at say \( m^d = (\varepsilon, ..., \varepsilon) \), for some \( \varepsilon > 0 \), so as to better “probe” the importance of all variables.
The rational solution is to smooth consumption: total resources are \( w_0 + x \) (initial wealth \( w_0 \) and time-2 payment \( x \)), and they should be consumed equally in all periods:

\[
c_t = \frac{w_0 + x}{3} \quad \text{for } t = 0, 1, 2
\]

The corresponding dynamic policy is:

\[
c_0 = \frac{w_0 + x}{3}, \quad c_1 = \frac{w_1 + x}{2}, \quad c_2 = w_2 + x.
\]

as we will verify soon.

The rest of this subsection derive the BR solution. We first state the result, then derive it.

**Proposition 4** Take the 3-period life-cycle problem. The BR policy is

\[
c_0 = \frac{w_0 + m_0 x}{3}, \quad c_1 = \frac{w_1 + m_1 x}{2}, \quad c_2 = w_2 + x.
\]

where \( m_t \) are the attention parameters given in (24) and (29). If \(|x|\) is not too large, they satisfy \( m_0 \leq m_1 \). In particular, this implies

\[
\frac{\partial c_0 (w_0, x)}{\partial x} \leq \frac{\partial c_1 (w_0, x)}{\partial x} \leq \frac{\partial c_2 (w_0, x)}{\partial x}
\]

with at least one strict inequality if \( \kappa \) large enough. If the agent was rational, we would have:

\[
\frac{\partial c_0 (w_0, x)}{\partial x} = \frac{\partial c_1 (w_0, x)}{\partial x} = \frac{\partial c_2 (w_0, x)}{\partial x}.
\]

**Proof of Proposition 4** We apply the smax procedure of Definition 2, using backward induction.

At time 2, the agent consumes all his disposable wealth:

\[
V^2 (w_2, x) = u (w_2 + x)
\]

At time 1, the agent’s problem is:

\[
\text{smax}_{c_1, m_1} v^2 (c_1, x, m_1) \quad \text{with } v^2 (c_1, x, m_1) := u (c_1) + V^2 (w_1 - c_1, m_1 x).
\]

The first order condition \( v^2_{c_1} = 0 \) reads:

\[
u' (c_1) = V_w^2 (w_1 - c_1, m_1 x) = u' (w_1 - c_1 + m_1 x)
\]

so \( c_1 = w_1 - c_1 + m_1 x \), and

\[
c_1 = \frac{w_1 + m_1 x}{2}
\]

Hence, the agent pays partial attention \( m_1 \) to the time-2 income \( x \).
To calculate attention $m_1$, we apply (5). Noting that $v_{cc}^2(c, x, m_1)_{|m_1=0} = 2u''(c^d)$, with $c^d = \frac{w_0}{2}$ is the optimal consumption with $m_1 = 0$, we have $m_1 = A \left( \frac{1}{\kappa} 2u''(c^d) \var(r \left( \frac{x}{x} \right)) \right)$, so

$$m_1 = A \left( \frac{1}{2\kappa} u'' \left( \frac{w_1}{2} \right) \sigma_x^2 \right). \quad (24)$$

Hence, the value function at time 1 is:

$$V^1(w_1, x) = u \left( \frac{w_1 + m_1x}{2} \right) + u \left( \frac{w_1 + (2 - m_1)x}{2} \right) \quad (25)$$

This is a little complicated. This is where the simplification operator $S$ (defined in Definition 3) intervenes. Applying it (with the same notations as in the motivating example before and after Definition 3), we obtain $V^{1,S} := S(V^1)$, i.e.

$$V^{1,S}(w, x) = 2u \left( \frac{w_1 + x}{2} \right) \quad (26)$$

The value is the same as $V^1$, up to $O(x^2)$ terms: $V^1(w_1, x) = V^{1,S}(w, x) + O(x^2)$. The attention-augmented value function at time 1 is:

$$V^1(w, x, m^V) = m^V V^1(w, x) + (1 - m^V) V^{1,S}(w, x) \quad \text{At time 0, the agent does } \text{smax}_{c_0, m^V} v^0(c_0, x, m_0), \text{ with } m_0 = (m_0^x, m^V) \text{ and}$$

$$v^0(c_0, w_0, x, m_0) := u(c_0) + V^1(w_0 - c_0, m_0^x, m^V) \quad (27)$$

The FOC is $v^0_{c_0} = 0$ with

$$v^0_{c_0} = u'(c_0) - V^1_w (w_0 - c_0, m_0^x, m^V).$$

We have $V^1_{w,m^V} = 0$ at the default $m_0^d = (0, 0)$, so $\frac{\partial v^0_{c_0}}{\partial m^V} |_{m^V=0} = 0$ and the optimal attention is $m^V = 0$: the agent uses the proxy value function, not the exactly rational one (we will see soon that attention $m^V$ can be non-zero using the 2-step smax, but it is still likely to be 0 if $\kappa$ is not too small). So, the FOC for consumption is:

$$u'(c_0) = V^1_w (w_0 - c_0, m_0^x, m^V) |_{m^V=0} = u' \left( \frac{w_0 - c_0 + m_0^x}{2} \right) \quad \text{i.e. } c_0 = \frac{w_0 - c_0 + x}{2}$$

i.e. $c_0 = \frac{w_0 - c_0 + x}{2}$ and

$$c_0 = \frac{w_0 + m_0^x}{3} \quad (28)$$

To determine attention $m_0 = m_0^x$, we again use (5); we calculate:

$$v^0_{cc} = u''(c^d) + V^{1}_{w,c|m=0} u''(c^d) + \frac{1}{2} u''(c^d) = \frac{3}{2} u''(c^d)$$
so that $m_0 = A \left( \frac{1}{\kappa} \frac{3}{2} u'' \left( c^d \right) \var \left( \frac{x}{3} \right) \right)$, i.e.

$$m_0 = A \left( \frac{1}{6\kappa} u'' \left( \frac{w_0}{3} \right) \sigma_x^2 \right). \quad (29)$$

The consumptions are:

$$c_0 (w_0, x) = \frac{w_0}{3} + \frac{m_0}{3} x, \quad c_1 (w_0, x) = \frac{w_0}{3} + \left( \frac{m_1}{2} - \frac{m_0}{6} \right) x, \quad c_2 (w_0, x) = \frac{w_0}{3} + \left( 1 - \frac{m_0 + m_1}{2} \right) x.$$ \quad (30)

Comparing (24) and (29), we see that

$$m_0 \geq m_1 \text{ iff } \frac{1}{6} \left| u'' \left( \frac{w_0}{3} \right) \right| \leq \frac{1}{2} u'' \left( \frac{w_1}{2} \right).$$

When $x = 0$, this is automatically verified, as $\frac{w_0}{3} = \frac{w_1}{2}$. Hence, we have $m_0 \leq m_1$ iff $x$ is not too large. \footnote{If $x$ was very large and positive, we could have the following effect: the agent realizes at time 1 that he's actually quite wealthy, so pays less attention. This effect needs a very large $x$, so is not operative in most situations.}

Given $0 \leq m_0 \leq m_1 \leq 1$, equation (30) implies (22). When $\kappa$ is very large, then $m_0 = m_1 \to 0$, so that $\frac{\partial c_0 (w_0, x)}{\partial x} \to 0$ and $\frac{\partial c_2 (w_0, x)}{\partial x} \to 1$, so that one inequality in (22) is strict.

We note that if $m^V > 0$, the FOC is more complex. The FOC is:

$$u' (c_0) = (1 - m^V) u' \left( \frac{w_1 + m_0 x}{2} \right) + m^V \frac{1}{2} \left[ u' \left( \frac{w_1 + m_1 x}{2} \right) + u' \left( \frac{w_1 + (2 - m_1) x}{2} \right) \right]$$

Still, to the first order, the decision is the same (as per Proposition 3). Making the problem simpler at every period, via the $m^V = 0$ device, makes the problem more tractable for both the agent and the economist examining him. Section 8.1 details this. \hfill \square

This example (Proposition 4) illustrates a few general features that are specific to a dynamic setting.

Sparse agents are locally myopic like in hyperbolic agents, but globally patient like rational agents. As a result, they differ from both the rational and sparse agents. Indeed, Agents here invest their wealth $w$ very patiently here, exactly like rational agents. At the same time, they tend to be myopic about the future small shocks (the time-2 shock $x$), as in models of hyperbolic discounting (Laibson 1997, O'Donoghue and Rabin 1999). In other terms, in the present model, agents are only partially myopic (e.g. don't react to a schedule increase in taxes).

Agents react more to “near” shocks than to “distant” shocks: That’s equation (22). The main reason is that, normatively, the shock $x$ should impact $c_0$ as $\frac{x}{2} \left( \frac{\partial c_0 (w_0, x)}{\partial x} = \frac{1}{3} \right)$, while it should impact $c_1$ as $\frac{x}{2} \left( \frac{\partial c_1 (w_0, x)}{\partial x} = \frac{1}{3} \right)$. Hence, attention is greater to the last period shock $x$ is lower at earlier dates ($t = 0$) than at late dates ($t = 1$).

The sparse agent exhibits partial sophistication in this understanding of its future actions. The rational value function (25) endows the agent with a perfectly sophisticated understanding of his future actions (in particular, the agent understand that he will not fully optimize at period 1. However, the simplified value function $V^1$ in (26) with $m^V = 0$ gives him a rougher understanding of his future actions: the agent is more like a naive agent. Hence, the
agent is sophisticated with \( m^V = 1 \) and more naive when \( m^V = 0 \). The agent optimizes on
the degree of sophistication.

The Euler equation fails. The Euler equation holds under the BR-perceived consumption,
but not under the actual consumption.

Now that this basic issues are in place, we move on to an infinite horizon problem.

3.2 Infinite-Horizon Problem

I now turn to the canonical consumption-investment problem, with an infinite horizon. The
agent has utility \( \mathbb{E} \sum_{t=0}^{\infty} \beta^t c_t^{1-\gamma} / (1 - \gamma) \). Wealth \( w_t \) evolves as:

\[
    w_{t+1} = (1 + r_t) (w_t - c_t) + y_t.
\]

(that is, wealth at \( t + 1 \) is savings at \( t \), \( w_t - c_t \), invested at rate \( r_t \), plus current income, \( y_t \)).

We decompose:

\[
    r_t = \overline{r} + \tilde{r}_t, \quad y_t = \overline{y} + \tilde{y}_t
\]

where \( \tilde{r}_t \) and \( \tilde{y}_t \) are deviations of the interest rate and income from their means, respectively,
and follow AR(1) processes:

\[
    \tilde{r}_{t+1} = \rho_r \tilde{r}_t + \varepsilon_{t+1}^r, \quad \tilde{y}_{t+1} = \rho_y \tilde{y}_t + \varepsilon_{t+1}^y
\]

\( \varepsilon_{t+1} \) are independent disturbances with mean zero. For simplicity, assume here that
\( R \equiv 1 + \overline{r} = \beta^{-1} \).

In the default model, the agent assumes that \( m^d_r = m^d_y = 0 \); this is, he assumes that
future interest rate and income will be constant. Then, the optimal consumption is
\( c^d (w_t) = (\overline{r} w_t + \overline{y}) / R \), and the value function is

\[
    V^d (w_t) = A (\overline{r} w_t + \overline{y})^{1-\gamma}
\]

for a constant \( A = \frac{1}{\overline{r}^{1-\gamma}} \).

To calculate the BR policy, we first use Proposition 1. This gives the following Lemma, shown in the appendix.

**Lemma 3** The rational policy is:

\[
    \ln c^r (w_t, \tilde{y}_t, \tilde{r}_t) = \ln c^d (w_t) + b_y \tilde{y}_t + b_r \tilde{r}_t + O \left( \| \tilde{y} \|^2 + \| \tilde{r} \|^2 \right)
\]

(31)

where

\[
    b_y = \frac{\overline{r}}{R (R - \rho_y)} c_t^d, \quad b_r = \overline{r} \left( \frac{w_t}{c_t^d} - 1 \right)^{-1 / \gamma}
\]

(32)

The agent does a sparse truncation of (31), as in Proposition 2. Hence, we obtain the
following.
Proposition 5 A sparse agent has the following consumption policy, up to second order terms:

\[ \ln c_t^s = \ln c^d (w_t) + b^s_y y_t + b^s_r r_t \]  

(33)

where (for \( x = y, r \)) \( b^s_x := \tau \left( b_x, \frac{\sigma_x}{\bar{c}_x} \right) \) and \( b_x \) are in (32).

Equation (33) shows a “feature-by-feature” truncation. It is useful because it embodies in a compact way the policy of a sparse agent in quite a complicated world. Note that the agent can solve this problem without solving the 3-dimensional (and potentially 21-dimensional, say, if there are 20 state variables besides wealth) problem. Only local expansions and truncations are necessary.

In this manner, we arrive at a quite simple way to do sparse dynamic programming. There is just one continuously-tunable parameter, \( \kappa \). When \( \kappa = 0 \), the agent is (to the leading order) the traditional rational agent. When \( \kappa \) is large enough, the agent is fully sparse, and does not react to any variable. Hence, we have a simple, smooth way to parametrize the agent, from very sparse to fully rational.

Numerical illustration To get a feel for the effects, consider a calibration with (using annual units): \( \gamma = 1, r = 5, \bar{w} = 2\bar{c}, \bar{c} = 1, \sigma_r = 0.8\%, \sigma_y = 0.2\bar{c}, \rho_y = 0.95, \sigma_{\ln c} = 5\% \), and \( \rho_r = 0.7 \): as income shocks are persistent, they are important to the consumer’s welfare. We use the \( \tau_1 \) truncation function.

Then, Figure 3 shows the impact of a change in the interest rate and income on consumption. Consider the left panel, \( b^s_r \). If the cost of rationality is \( \kappa = 0 \), then the agent reacts like the rational agent: if interest rates go up by 1%, then consumption falls by 2.8% (the agent saves more). However, for a sparsity parameter \( \kappa \approx 0.5 \), the agent essentially does not respond to interest rates. Psychologically, he thinks “the interest rate is too unimportant, so let me not take it into account.” Hence, the agent does not react much to the interest rate, but will react more to a change in income (right panel of Figure 3), which is more important: the sensitivity of consumption to income remains high even for a high cognitive friction \( \kappa \).

Note that this “feature-by-feature” selective attention could not be rationalized by just a fixed cost to consumption, which is not feature-dependent.

The same reasoning holds in every period. The above describes a practical way to do sparse dynamic programming. In some cases, this is simpler than the rational way (as the agent does not need to solve for the equilibrium), and it may also be more sensible.

Active decision: Consumption or Savings? Here we assume that the active decision was one of consumption. One could imagine that it would be in savings. Does this matter? First, for many variables, it does not matter: the impact of interest rates, future taxes, future income shocks etc. are the same whether a sparse agent uses the consumption frame or saving frame. However, the frame does matter for one variable: current income. Indeed,
take the permanent-income setup. 4

Which frame does the agent use? One might posit that the agent takes the frame that yields the higher expected utility. To analyze this, we note the following result.

Proposition 6 (Welfare under the consumption vs savings frame) The consumption frame yields greater utility than the savings frame if and only if \( \phi_y > r \), i.e. if income shocks mean-revert faster than the interest rate.

When \( \phi_y > r \) (which is probably the relevant case), the “consumption” frame is indeed better for the agent. The reason is that consumption should be smooth, while savings could be bumpy as they absorb transitory income shocks. When the agent chooses consumption in an inattentive manner, it makes consumption automatically rather smooth. However, if the agent chooses savings inattentively, he makes savings smooth, but consumption needs to absorb the shocks, hence is quite volatile. Hence, generally, to keep consumption smooth, choosing consumption inattentively is better than choosing savings inattentively. However, when income shocks are a random walk \( (\phi_y = 0) \), the savings frame is better. An inattentive

---

4Recall that \( \hat{c}_t^* = \frac{r}{r + \phi} \hat{y}_t \), so

\[
\hat{c}_t^* = \frac{r}{r + \phi} m \hat{y}_t \quad \text{under the consumption frame}
\]

However, if the consumer choose savings, \( S_t \), and then consumes \( c_t = w y_t - S_t \), the rational amount is \( \hat{S}_t = \hat{y}_t - \hat{c}_t^* \), i.e. \( \hat{S}_t = \frac{\phi}{r + \phi} \hat{y}_t \). Hence, the savings of a sparse agent is \( \hat{S}_t^* = \frac{\phi}{r + \phi} m \hat{y}_t \), and the deviation of consumption is: \( \hat{c}_t^* = \hat{y}_t - \hat{S}_t^* \), i.e.

\[
\hat{c}_t = \left( 1 - \frac{m \phi}{r + \phi} \right) \hat{y}_t \quad \text{under the savings frame}
\]

which is generally not the same as \( \hat{c}_t^* \) under the consumption frame.
agent will keep a constant savings, and let consumption react one for one to income shock, which is the normatively correct behavior when income shocks are completely persistent.

4 A Boundedly Rational New Keynesian model, and the Forward Guidance Puzzle

4.1 A Discounted Euler Equation

Here I explore the simplest specification of BR dynamic programming. I find that it leads to a natural modification of the New Keynesian framework, with important consequences for “forward guidance”.

The agent’s dynamic programming problem is:

\[ V(z) = \max_{c,m} u(c) + \beta V(F_z(m \circ z)) \]  

(34)

with \( m = (m_w, m_x) \) with \( m_w = 1 \) and \( m_x = m(1,..,1) \), i.e. \( m_x \) uniformly depends all quantities (this can be relaxed later, and gives similar results, though in a more complex way).

For instance, suppose that the true model can be linearized as:

\[
\begin{align*}
    w_{t+1} &= (R + \hat{r}_t) (w_t + y - c_t) + \hat{y}_{t+1} \\
    z_{t+1} &= Az_t + \varepsilon_{t+1}
\end{align*}
\]

where \( z_t \) is a state vector, and \( \hat{y}_{t+1} = k^y z_{t+1} \) and \( \hat{r}_t = k^r z_t \) for some vectors \( k^y, k^r \). Equation (34) means that the agent’s subjective model of the world is:

\[
\begin{align*}
    w_{t+1} &= (R + m \hat{r}_s) (w_t + y - c_t) + m \hat{y}_{t+1} \\
    z_{t+1} &= mAz_t + \varepsilon_{t+1}
\end{align*}
\]

i.e. future events have an extra discounting term in their transition function. For now, imagine \( \hat{r}_s = \hat{r}_t \), so that the interest rate is perceived correctly.

In that case, we then have the following Euler equation.

**Proposition 7** (Discounted Euler equation) *If the consumer follows the BR dynamic programming approach, then we have the (linearized) discounted Euler equation:*

\[
ME_t[\hat{c}_{t+1}] - \hat{c}_t = \tilde{\sigma} \hat{r}_t^s + [(M - 1) + M (\hat{r}_t - \hat{r}_t^s)] \frac{r}{R} w_t
\]  

(35)

with

\[
M := \frac{m}{R - rm}.
\]

(36)

\[
\tilde{\sigma} := M c_{t} \psi \frac{1}{R}
\]

---

\[5\] This is related to the “perceived law of motion” of the literature on learning. But the concept of a mental representation also holds in static contexts, and the model is not about learning.
Intuition and proof sketch The appendix contains a formal proof, here we provide the intuition for the major terms.

First, take the case where there are no disturbance to the interest rate, (35) gives: 
\[ ME_t [\hat{c}_{t+1}] - \hat{c}_t = 0, \] rather than the Euler equation 
\[ E_t [\hat{c}_{t+1}] - \hat{c}_t = 0. \] To gain some intuition for this, suppose that there is an extra lump-sum payment \( \hat{y}_T \) at time \( T \). At time 0, the agent perceives only a fraction \( m^T \) of it, so feels richer by the present value \( \frac{m^T}{R^T} \hat{y}_T \). Given the MPC is \( \frac{r}{R} \), he changes consumption by \( \hat{c}_0 = \frac{r}{R} m^T \hat{y}_T \). At time 1, the consumer’s financial wealth has changed by \( \hat{w}_1 = - (R + \hat{r}_0) \hat{c}_0 \simeq -R \hat{c}_0 \) to the leading order (the consumer had dissaved, so is poorer at time 1). In addition, he now perceives an income \( m^{T-1} \hat{y}_T \) in \( T-1 \) periods, whose present value is \( \frac{m^{T-1}}{R^{T-1}} \hat{y}_T \). So he consumes a fraction \( \frac{r}{R} \) of this perceived wealth:

\[
\hat{c}_1 = \frac{r}{R} \left( \hat{w}_1 + \frac{m^{T-1}}{R^{T-1}} \hat{y}_T \right) = \frac{r}{R} \left( -R \hat{c}_0 + \frac{m^{T-1} R^{T+1}}{R^{T-1}} \frac{1}{rm^T} \hat{c}_0 \right) = \left( -r + \frac{R}{m} \right) \hat{c}_0 = -\frac{rm + R}{m} \hat{c}_0 = \frac{1}{M} \hat{c}_0
\]

This gives \( M \hat{c}_1 = \hat{c}_0 \), a discounted Euler equation.

On the right-hand side of (35), consider the first term, \( \hat{\sigma} \hat{r}_t^s \). It is the perceived interest rate, not the actual one that matters for the consumption decision, hence for the term \( \hat{\sigma} \hat{r}_t^s \) in the Euler equation. 6 There are two more terms. One, \( (M-1) \frac{1}{R} \hat{y}_t \) is a mechanical compensation term: recall that the consumer consumes a fraction \( \frac{r}{R} \) of wealth; if there’s an innovation in wealth, \( c_t = \frac{r}{R} \hat{y}_t = c_{t+1} \), so that \( ME_t [\hat{c}_{t+1}] - \hat{c}_t = (M-1) \frac{1}{R} \hat{y}_t \). The last term, \( (\hat{r}_t - \hat{r}_t^s) \frac{1}{R} \hat{y}_t \), reflects “surprise income”: the interest rate income at time \( t+1 \) is higher by \( (r_t - r_t^s) \hat{w}_t \) at time \( t+1 \), and a fraction \( \frac{r}{R} \) of it is consumed at time \( t+1 \). That gives the extra adjustment term.

The distinction between \( m \) and \( M \) is not very important: they are same up to second order terms. In the limit of small time intervals, write \( m = 1 - \xi \) for a small \( \xi \). Then, \( M = \frac{1}{1+\xi} \) \( \simeq 1 - \xi + O (\xi r) \).

Model with discounted Euler equation for firms This subsection can be skipped at the first reading. Here I explore what happens if firms also don’t fully process the future. Suppose indeed the same setup for firms that for consumers, but now with a coefficient \( M_f \) for firms. That is, given their state variable \( z_t \), the firm’s dynamic programming problem is:

\[
V(z_t) = \operatorname{smax}_{n_t \in [m]} v(n_t, z_t) + \beta \mathbb{E} [V(F^z (m \odot z_t))]
\]

6To understand that expression, suppose that there is a disturbance \( \hat{r}_0 \), lasting for 1 period, and take the simplest case with no initial wealth. In the rational model, we should have \( \hat{c}_0 = -\frac{\psi \psi^{d \hat{r}_0}}{R} \) (To see this, observe that \( \hat{c}_0 \) creates \( \hat{w}_1 = -R \hat{c}_0 \), so \( \hat{c}_1 = \frac{r}{R} \hat{c}_0 = -R \hat{c}_0 \); the Euler equation imposes \( \psi \psi^{d \hat{r}_0} = \hat{c}_1 - \hat{c}_0 = -R \hat{c}_0 \), hence \( \hat{c}_1 = \frac{\psi \psi^{d \hat{r}_0}}{R} \)). The BR agent instead uses a perceive interest rate \( \hat{r}_0^s \), and discounts by \( m \) it’s perceived effects, so his consumption is \( \hat{c}_0 = -\frac{m \psi \psi^{d \hat{r}_0}}{R} \). As \( \hat{c}_1 = -R \hat{c}_0 \),

\[
M \hat{c}_1 - \hat{c}_0 = - (Mr + 1) \hat{c}_0 = - \frac{RM}{m} \hat{c}_0 = -M \frac{m \psi \psi^{d \hat{r}_0}}{R} \hat{r}_0^s = \hat{\sigma} \hat{r}_0^s.
\]

This gives the term \( \hat{\sigma} \hat{r}_0^s \) on the RHS of (35).
where \( v(N_t, z_t) \) is the profit function, \( N_t \) is the quantity of labor to employ, and with \( m = (m_w, m_x) \) with \( m_w = 1 \) and \( m_x = M^f (1, \ldots, 1) \), i.e. \( m_x \) uniformly depends all quantities (this can be relaxed later, and gives similar results, though in a more complex way).

We apply this to the NK model, as exposited in Gali (2015, Chapter 3). Then, calculations show that the firm \( i \)'s optimal price is:

\[
p_t^* = (1 - \beta \theta) \sum_{k=0}^{\infty} (\beta \theta)^k E^{BR}_t [\hat{p}_{t+k} - \Theta \hat{\pi}_{t+k}]
\]

where \( E^{BR} \) is the expectation with the beliefs induced by the partially inattentive model: \( z_{t+1} = F^s (m \circ z_{t+1}) \). This gives:

\[
p_t^* = (1 - \beta \theta) \sum_{k=0}^{\infty} (\beta \theta M^f)^k [\hat{p}_{t+k} - \Theta \hat{\pi}_{t+k}]
\]

By following the steps described in Gali, we have (with \( \kappa = (1 - \theta) (1 - \beta \theta M^f) / \Theta \))

\[
\pi_t = \beta M^f E_t [\pi_{t+1}] + \kappa x_t \quad (39)
\]

This is the NK Phillips curve, with partially attentive firms. The case \( M^f = 1 \), fully attention, is the traditional curve.

I should emphasize that those firms can be fully attentive to all idiosyncratic terms (something that will be easy to include in a future version of the paper), e.g. idiosyncratic part of productivity of demand. They simple have pay attention \( M^f \) to macro outcomes. If we include idiosyncratic terms, and firms are fully attentive to them, the aggregate NK curve doesn’t change.

### 4.2 Implication of the discounted Euler equation for the Behavioral NK model

In the traditional NK model, there is no capital or government spending, so GDP is consumption. Calling \( x_t = (c_t - c_t^*) / c^d \) the output gap, the traditional NK model is:

\[
x_t = E_t [x_{t+1}] - \sigma (i_t - E_t \pi_{t+1} - r_t^a)
\]

\[
\pi_t = \beta E_t [\pi_{t+1}] + \kappa x_t \quad (41)
\]

where \( r_t^a \) is the real risk-free rate in the frictionless economy. \(^7\) I now state its inattentive version.

---

\(^7\) See e.g. Gali (2015) for a textbook exposition. In the traditional proof, output is consumption (there’s no investment), and the Euler equation holds: \( 1 = E \left[ \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} \beta_t R_t \right] \), which leads to (with \( \beta_t = e^{-\rho_t} \))

\[
0 = -\gamma (Ec_{t+1} - c_t) + r_t - \rho_t, \quad \text{and} \quad c_t = E_t c_{t+1} - \psi (r_t - \rho_t) \quad (42)
\]

we set \( \sigma = \psi \), and \( r_t^a = \rho_t \) and \( r_t = i_t - E_t \pi_{t+1} \).
Proposition 8 The BNK (behavioral, New Keynesian) model is:
\[ x_t = ME_t [x_{t+1}] - \sigma (i_t - E_t \pi_{t+1} - r^n_t) \] (IS curve)  \hspace{1cm} (43)
\[ \pi_t = \beta M^f E_t [\pi_{t+1}] + \kappa x_t \] (Phillips curve) \hspace{1cm} (44)
where \( M, M^f \in [0, 1] \) are the attention of consumers and firms, respectively, to macroeconomic outcomes. In the traditional model, \( M = M^f = 1 \). In addition, \( \sigma = M \psi \frac{1}{\bar{\pi}} \)

Proof. The most challenging part (43). To derive it, we just apply Proposition 7, with \( w_t = 0 \): there is no wealth in the NK model. \( \square \)

The behavioral “New Keynesian IS curve”, (43). It is the more drastic modification of the framework. It implies, if \( \hat{r}^*_t = \hat{r}_t \),
\[ \hat{c}_t = \tilde{\sigma} \sum_{s \geq t} M^{s-t} E_t [i_t - E_t \pi_{t+1} - r^n_t] \]
i.e. it’s the discounted value of future interest rates that matters, rather than the undiscounted sum. This will be important soon when we study forward guidance.

In particular, in the NK model with the corrective tax, there’s no wealth, so that we plainly have
\[ ME_t [\hat{c}_{t+1}] - \hat{c}_t = \tilde{\sigma} \hat{r}^*_t \]
where \( \hat{r}^*_t \) is the interest rate perceived at \( t \). McKay, Nakamura and Steinsson (MNS, 2015) do argue that this equation fit better. They provide a microfoundation based on heterogeneous rational agents with limited risk sharing. In their model, wealthy, unconstrained agents with no unemployment risk would still satisfy the usual Euler equation. Piergallini (2006), Nistico (2012), Del Negro, Giannoni, Patterson (2015), other micro-foundations with heterogeneous mortality shocks, as in a perpetual-youth models (this severely limit how myopic agents can be, given that life expectancies are quite high). Here I keep the representative agent, but make him more boundedly rational; wealth, behavioral agent would still satisfy this discounted Euler equation.

The modified Philipps curve (44) Its change is less drastic, as it simply changes \( \beta \) into \( \beta M^f \). Let me reiterate that firms are still forward looking (with discount parameter \( \beta \)) in the deterministic steady state. It’s only there sensitivity to deviations around the deterministic steady state that they’re partially myopic.

We now study two impact of this modifications, to the forward guidance puzzle.

4.3 Application to the Forward Guidance Puzzle

4.3.1 Thought experiment: An announcement of a rate cut at time \( T \)
Here I follow McKay, Nakamura and Steinsson (2015). Suppose that the central bank announces a time \( 0 \) that it will cut the rate at time \( T \), following a policy \( \delta_t = 0 \) for \( t \neq T \), \( \delta_T < 0 \). What is the impact?
Figure 4: This Figure shows the response of current inflation to forward guidance about interest rate in $T$ periods, compared to an immediate rate change of the same magnitude. Units are yearly. The left panel is the traditional New Keynesian model, the right panel the behavioral model. Parameters are the same in both models, except that (annualized) attention is $M = e^{-\xi} = 0.7$ in the behavioral model, and $M = 1$ in the traditional model.

We have $x_t = M x_{t+1} - \sigma \delta_t$, so $x_t = -\sigma M^{T-t} \delta_T$ for $t \leq T$ and $x_t = 0$ for $t > T$. Next, inflation is

$$\pi_0 = \kappa \sum_{t \geq 0} \beta^t x_t = -\kappa \sigma T \sum_{t=0}^T \beta^t M^{T-t} \delta_T = -\kappa \sigma \frac{M^{T+1} - \beta^{T+1}}{M - \beta} \delta_T$$

In the traditional case, $M = 1$, so that a rate cut in a very long future has a big impact on inflation, $\pi_0 = \frac{-\kappa \sigma}{1-\beta}$. In contrast, when $M < 1$, the effect is just 0.

Figure 4 illustrate the effect.

4.3.2 Thought experiment: A persistently binding ZLB

In the behavioral NK model, write $M = e^{-\xi \Delta t}$, where $\xi$ is cognitive discounting parameter due to myopia. The model (43) becomes:

$$\dot{x}_t = \xi x_t + \sigma (i_t - r_t - \pi_t)$$

(45)

$$\dot{\pi}_t = \rho \pi_t - \kappa x_t$$

(46)

When $\xi = 0$, we recover Werning (2012)'s formulation, in which agents are all rational. 8

A ZLB scenario. A follow a thought experiment of Werning (2012), but with behavioral agents. I take $r_t = \underline{r}$ for $t \leq T$, and $r_t = \bar{r}$ for $t > T$, with $\underline{r} < 0 < \bar{r}$. I assume that

8Note that the way to read (45) is as a forward equation. Calling $\delta_t := i_t - r_t - \pi_t$ the excess interest rate. We take into account the fact that the model is stationary ($\lim_{t \to \infty} x_t = 0$). We have:

$$\dot{x}_t = -\sigma \int_t^\infty e^{-\xi (s-t)} \delta_s ds$$

Hence, the effect of future rate changes is dampened by myopia.
for $t > T$, the central bank implements, $x_t = \pi_t = 0$, by setting $\dot{i}_t = \bar{r}$. At time $t < T$, I suppose that the CB is at the ZLB, so that $\dot{i}_t = 0$.

**Proposition 9** In the traditional case ($\zeta = 0$), we obtain an unboundedly intense recession as the length of the ZLB is longer:

$$\lim_{t \to -\infty} x_t = -\infty.$$  

This also holds when myopia is mild, $\xi \in [0, \frac{\sigma\kappa}{\rho}]$.

However when cognitive myopia is strong enough ($\xi > \frac{\sigma\kappa}{\rho}$), we obtain a boundedly intense recession

$$\inf_{t \in (-\infty, T]} x_t = \lim_{t \to -\infty} x_t = \frac{\rho\sigma\bar{r}}{\rho\xi - \sigma\kappa} < 0.$$  

**Proof of Proposition 9** We have: $\dot{x}_t = \xi x_t - \sigma (\bar{r} + \pi_t)$. To solve for the system, we note:

$$\begin{align*}
\dot{x}_t &= \xi \dot{x}_t - \sigma \dot{x}_t = \xi \dot{x}_t - \sigma (\rho \pi_t - \kappa x_t) = \xi \dot{x}_t + \sigma \kappa x_t - \rho \sigma \pi_t \\
&= \xi \dot{x}_t + \sigma \kappa x_t + \rho (\dot{x}_t - \xi x_t + \sigma \bar{r}) = (\rho + \xi) \dot{x}_t + (\sigma \kappa - \rho \xi) x_t + \rho \sigma \bar{r}
\end{align*}$$

so that:

$$\dot{x}_t - (\rho + \xi) \dot{x}_t + (\rho \xi - \sigma \kappa) x_t = \rho \sigma \bar{r} \quad (48)$$

and the boundary conditions are: $x_T = \pi_T = 0$, hence (taking the left derivative):

$$x_T = 0, \dot{x}_T = -\sigma \bar{r}. \quad (49)$$

To analyze (48), we look for solutions of the type $x_t = e^{\lambda t}$. Call $\lambda \leq \lambda'$ the two roots of:

$$\lambda^2 - (\rho + \xi) \lambda + \rho \xi - \sigma \kappa = 0 \quad (50)$$

Then, with $D = \frac{\rho \sigma \bar{r}}{\rho \xi - \sigma \kappa}$, the solution is:

$$x_t = D + \frac{(D \lambda - \sigma \bar{r}) e^{\lambda(t-T)} - (D \lambda' - \sigma \bar{r}) e^{\lambda'(t-T)}}{\lambda' - \lambda}. \quad (51)$$

In the traditional case, $\xi = 0$, so that $\lambda < 0 < \lambda'$. As $D > 0$, this implies that, as $t \to -\infty$, $x_t \to -\infty$. We obtain an unbounded large recession. This is the logic that Werning analyzes.

However, take the case where cognitive myopia is strong enough, $\xi > \frac{\sigma\kappa}{\rho}$. Then, both roots of (50) are positive. Hence, we have a bounded recession. Indeed, as $D < 0$ in that case, $x_t$ is increasing in $t$. □

We see how impactful myopia can be. We see that myopia has to be stronger when the agent are highly sensitive to the interest rate ($\sigma$ high) and price flexibility is high (high
Figure 5: This Figure shows the output gap $x_t$. The economy is at the Zero Lower Bound during times $0$ to $T = 15$ years. The left panel is the traditional New Keynesian model, the right panel the behavioral model. Parameters are the same in both models, except that (annualized) attention is $M = e^{-\xi} = 0.7$ in the behavioral model, and $M = 1$ in the traditional model.

$\kappa$). High price flexibility makes the system very reactive, and a high myopia is useful to counterbalance that.

The dynamics are illustrated in Figure 5. The left panel shows the traditional model, the right one the behavioral model. The parameters are the same in both models, except that attention is lower (set an annualized rate of $M = e^{-\xi} = 0.7$) in the behavioral model (against its value $M = 1$ in the traditional model). \(^9\)

5 Neoclassical Growth Model: A Boundedly Rational Version

We can now study a BR version of the baseline neoclassical growth model, the Ramsey-Cass-Koopmans model.

5.1 Setup

The utility function is still $E \sum_t \beta^t C_t^{\frac{1}{\gamma}} / (1 - \gamma)$, and we again call $\psi = \frac{1}{\gamma}$ and $\beta = e^{-\rho}$. In the aggregate, the capital stock follows:

$$K_{t+1} = f(K_t, L) + (1 - \delta) K_t - C_t + \varepsilon_{t+1}$$  \hspace{1cm} (52)

where $\varepsilon_{t+1}$ are mean-zero shocks, whose distribution we’ll specify later. This way, there is just one state variable in the economy, the capital stock. In the most basic neoclassical model, $\varepsilon_{t+1}$ is always 0, and $L$ is fixed.

\(^9\)The other parameters are: $\rho = 3\%$, $\kappa = 0.1$, $\sigma = 0.2$, $\tau = -5\%$. 

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Figure 6: This Figure shows the traditional approach to the neoclassical growth model. Arguably, this is psychologically quite absurd. The present paper proposes a more behavioral approach.

This is a textbook example, which can be found e.g. in Acemoglu (2009, Chapter 8), Blanchard-Fischer (1989, Chapter 2), Romer (2012, Chapter 2); it introduces generations of students to macroeconomics. However, it looks somewhat odd (in my opinion), with these infinitely-rational forward looking agents that calculate the whole macroeconomic equilibrium in their heads. I present here an alternative to that model.

**Let us first review some mechanics of convergence.** If there were no shocks, the economy would be at the steady state, with capital stock $K^*$. I use the hat notation for deviations from the mean, e.g. $\hat{K}_t = K_t - K^*$. The law of motion for capital (52) is, in linearized form:

$$\hat{K}_{t+1} = (1 + r) \hat{K}_t - \hat{C}_t + \varepsilon_{t+1}$$

(53)

where $r$ is the steady state interest rate, $r = \beta^{-1} - 1$.

As there is one state variable, the linear policy function of the agent (rational or not) is:

$$\hat{C}_t = b\hat{K}_t$$

(54)

for some constant $b$ to be determined.

Plugging this into (53) we obtain: $\hat{K}_{t+1} = (1 + r - b) \hat{K}_t + \varepsilon_{t+1}$, i.e.

$$\hat{K}_{t+1} = (1 - \phi) \hat{K}_t + \varepsilon_{t+1}$$

(55)

where $\phi$ is the speed of mean-reversion:

$$\phi = b - r.$$  

(56)
When agents are more reactive to shocks (when \( b \) is higher), the economy mean-reverts faster to the steady state (\( \phi \) is higher).

Finally, squaring equation (55), we obtain: \( \text{var} \hat{K}_{t+1} = (1 - \phi)^2 \text{var} \hat{K}_t + \sigma^2 \). As in the steady state, \( \text{var} \hat{K}_{t+1} = \text{var} \hat{K}_t \), \( \text{var} \hat{K}_t = \frac{\sigma^2}{1 - (1 - \phi)^2} \), i.e., in the limit of small time intervals:

\[
\sigma_K = \frac{\sigma^2}{\sqrt{2\phi}} \tag{57}
\]

When shocks mean-revert more slowly (lower \( \phi \)), the average deviation of the stock price from trends is higher (shocks “pile up” more).

The rational agent has a value function \( V(K_t) \), which satisfies:

\[
V(K) = \max_c c + \beta \mathbb{E} [V(K')] \quad \text{with} \quad K' = F(K, L) - c + \varepsilon_t
\]

where \( F(K) := f(K, L) - \delta K \) is output net of depreciation.

The steady state is at \( K = K^* \), \( C_t = C^* \) with:

\[
F'(K^*) = \rho
\]

which determines \( K^* \) and consumption is determined by:

\[
C^* = F(K^*)
\]

We define:

\[
\xi = \frac{u'(C^*)}{u''(C^*)} F''(K^*) \geq 0 \tag{58}
\]

which plays an important role later.

### 5.2 Boundedly Rational Version

The agent has wealth \( k_t \), and we normalize the population to be 1, so that in equilibrium will be equal to \( k_t = K_t \). The agent’s wealth evolves as:

\[
k_{t+1} = (1 + r_t) (k_t + y_t - c_t)
\]

where \( y_t = F(K_t) - K_tF'(K_t) \) is labor income, and \( r_t = F'(K_t) \) is the interest rate. Taylor expansion yields, to the leading order:

\[
\hat{r}_t = F''(K^*) \hat{K}_t, \quad \hat{y}_t = -K^*F''(K^*) \hat{K}_t \tag{59}
\]

In the agent’s model, income and interest rate evolve as:

\[
\hat{r}_{t+1} = (1 - \phi^s) \hat{r}_t, \quad \hat{y}_{t+1} = (1 - \phi^s) \hat{y}_t
\]

where \( \phi^s \) is the perceived speed of mean-reversion. We again parametrize it as: \( \phi^s = (1 - m \phi) \phi^d + m \phi \phi^r \), where \( \phi \) is the equilibrium speed of mean-reversion, and \( \phi^d \) is a default value – perhaps coming from some empirical experience, saying that “business cycles” have a half-life of a few years.
**Rational agent**  Lemma 3 reads here:

\[ \hat{c}_t = r\hat{k}_t + \frac{r}{r + \phi}\hat{y}_t + \frac{rk^* - c^*\psi}{r + \phi}\hat{r}_t \]

and using (59) we obtain:

\[ \hat{c}_t = r\hat{k}_t + \frac{-rK^*F''(K^*) + (rk^* - c^*\psi)F''(K^*)}{r + \phi}\hat{K}_t \]

i.e.

\[ \hat{c}_t = r\hat{k}_t + \frac{\xi}{r + \phi}\hat{K}_t \]  \hspace{1cm} (60)

Hence, the aggregate consumption follows \( \hat{C}_t = b\hat{K}_t \) with:

\[ b = r + \frac{\xi}{r + \phi} \]

and using (56), the speed of mean reversion is:

\[ \phi = \frac{\xi}{r + \phi} \]

Solving for the fixed point in \( \phi \):

\[ \phi = \frac{-r + \sqrt{r^2 + 4\xi}}{2}. \]  \hspace{1cm} (61)

**Behavioral agent**  The behavioral agent partially does not think about those aggregate shocks \( \hat{K}_t \). Hence, instead of (60), the BR agent pays an attention \( m_K \) to the capital stock, and perceives that it mean-reverts at a speed

\[ \phi^s = (1 - m_\phi)\phi^d + m_\phi \phi. \]

where \( \phi^d \) is a default speed of mean-reversion (which in practice could be the empirical speed of mean-reversion of business cycles). This gives:

\[ \hat{c}_t = r\hat{k}_t + \frac{\xi}{r + \phi^s}m_K\hat{K}_t \]  \hspace{1cm} (62)

Endogenizing \( m_K \), we obtain:

\[ \hat{c}_t = r\hat{k}_t + \tau\left(\frac{\xi}{r + \phi^s} \frac{\kappa_c}{\sigma_K}\right)\hat{K}_t \]
Hence, \( b = r + \tau \left( \frac{\xi}{r + \phi^d}, \frac{\kappa}{\sigma_K} \right) \), and using (56) we obtain:

\[
\phi = \tau \left( \frac{\xi}{r + \phi^d}, \frac{\kappa}{\sigma_K} \right).
\]

Now, what is \( m_\phi \), the attention to the accurate speed of mean-reversion? Given (62), \( \frac{\partial \phi}{\partial m_{\phi} \mid m_K = 0} = 0 \). This means that using the sparse max, \( m_\phi = 0 \). (With the iterated sparse max, we can have \( m_\phi > 0 \) if \( \kappa \) is small enough). Hence, we have

\[
\phi = \tau \left( \phi_0, \frac{\kappa}{\sigma_K} \right), \quad \phi_0 := \frac{\xi}{r + \phi^d} \quad \text{(63)}
\]

When \( \kappa \) is exogenous, then recalling that \( \kappa_c = \sqrt{\frac{\pi}{u_{cc}}} \) and (57), we have:

\[
\phi = \tau \left( \phi_0, \frac{\kappa}{\sigma_K} \right) = \tau \left( \phi_0, \frac{\sqrt{\frac{\kappa}{u_{cc}}}}{\sigma_x \sqrt{2\phi}} \right) = \tau \left( \phi_0, \sqrt{B\phi} \right), \quad B := \frac{2\kappa}{|u_{cc}| \sigma_x^2}
\]

For simplicity, we use the \( \tau_1 \) truncation function here (see equation 10): \( \tau (b, k) = \max \left( 1 - \frac{k^2}{b^2}, 0 \right) \), so \( \phi = \phi_0 - \frac{B\phi}{\phi_0} \), and

\[
\phi = \frac{\phi_0}{1 + \frac{B\phi}{\phi_0}}. \quad \text{(64)}
\]

Another solution is to use the “scale-free” version of \( \kappa \), equation (19). This gives \( \frac{\kappa}{\sigma_K} = \frac{\pi\sigma_c}{\sigma_K} = \pi (r + \phi_0) \), and

\[
\phi = \tau \left( \phi_0, \kappa (r + \phi_0) \right) \quad \text{(65)}
\]

Note that \( \phi \) has the same comparative statics as the rational case: \( \frac{\partial \phi (\xi, r)}{\partial \xi} \geq 0 \), \( \frac{\partial \phi (\xi, r)}{\partial r} \leq 0 \). There is a new comparative statics: \( \frac{\partial \phi}{\partial \kappa} \leq 0 \).

**Proposition 10** The speed of mean-reversion of the economy is:

\[
\phi = \left( \frac{\xi}{r + \phi^d - \kappa r} \right)_+.
\]

If the default perception \( \phi^d \) is the actual \( \phi \) (as in rational expectations), \( \phi^d = \phi \), so that:

\[
\phi = \left( \frac{\xi}{r + \phi - \kappa r} \right)_+,
\]

whose solution is

\[
\phi = \frac{1}{2} \left( -r \left( 1 + \kappa \right) + \sqrt{r^2 (1 - \kappa)^2 + 4\xi} \right)_+.
\]

In particular, \( \phi \) is decreasing in \( \kappa \), \( \phi (\kappa = 0) = \phi^* \). We have \( \phi > 0 \) iff \( \kappa \leq \kappa^* := (1 + r^2 / \xi)^{-2} \).
The impact of fluctuations. Hence, (provided $\phi^d \geq \phi$) the variance of shocks will be larger in the sparse economy than in the rational economy.

**Proposition 11** Suppose that $\phi^d \geq \phi$. Then, $\phi \leq \phi^r$, so that in the sparse economy, the speed of mean-reversion is slower, and the variance of shocks is bigger, than in the rational economy.

### 6 Behavioral Version of a Few Other Models

To probe the validity of the framework, we study here a few other models.

#### 6.1 Partial Failure of Ricardian Equivalence

Intuitively, a sparse agent will violate Ricardian equivalence (Barro (1974)). I study the magnitude and dynamics of that violation.

For simplicity, we use continuous time. The interest rate is $r = -\ln \beta$. The government needs to collect a present value of $G/r$. This could be done by taxing the population (of size normalized to 1) by $H = Ge^{rT}$, starting at a period $T$.\(^{10}\) Hence, the path of taxes is: 0 for $t < T$, and $H$ for $t \geq T$.

What is a consumer’s response at time $t < T$? If the consumer is perfectly attentive, then he should start saving at time 0. However, a sparse agent might not pay attention to those future taxes increases, and start cutting on consumption only later, or indeed perhaps just when the tax cuts are enacted.

Let us analyze this more in detail. At $T$, the tax $H$ is enacted, so that for $t \geq T$, the agent is aware of it. This yields: $\hat{c}_t = r\hat{w}_t - H$.

Before the enactment of taxes ($t < T$), will the consumer think of the tax $H$? That tax lowers the present value of his income by $H e^{-r(T-t)}$, so the consumer’s response is:

$$\hat{c}_t = r\hat{w}_t - \kappa \left( H e^{-r(T-t)}, \kappa \right)$$

Hence, the consumer will not think about the tax increase $H$ when $H e^{-r(T-t)} \leq \kappa$. Call $s \in [0,T)$ the first moment when he thinks about them (if it exists, i.e. if $H > \kappa$), otherwise we set $s = T$.

The next Proposition details the dynamics.

**Proposition 12** (Myopic behavior and failure of Ricardian equivalence) Suppose that taxes will go up at time $T$. While a rational agent would cut consumption at time 0, a sparse agent cuts consumption later, at a time $s = \max \left( 0, \min \left( T, \frac{1}{r} \ln \frac{\kappa}{H e^{-rT}} \right) \right)$. His consumption path is:

$$\hat{c}_t = \begin{cases} 0 & \text{for } t < s \\ -H e^{-r(T-s)} + \kappa (1 - r(t-s)) & \text{for } s \leq t < T \\ r\hat{w}_T - H & \text{for } t \geq T \end{cases}$$

\(^{10}\)If taxes are collected later, then to guarantee the same present value, they need to be larger by a factor $e^{rT}$. 31
Figure 7: Reaction of consumption and wealth to an increase of future taxes, for different level of $\kappa$. Notes. At time 0, it is announced that taxes will be paid start at time $T = 10$. This Figure plots the change in consumption and wealth. The solid line is the prediction of the rational model (i.e. $\kappa = 0$), the other lines the reaction for different value of $\kappa$ ($\kappa = 0.01$ (blue, dotted), $\kappa = 0.025$ (red, dashed-dotted), $\kappa = 0.1$ (green, dashed)). The very BR agents does not react at first, but starts reacting when he is closer to $T$. He reacts even more when taxes are in effect. As he delayed his savings, he needs to cut more on consumption when taxes start. Units are percentage points of previous steady state consumption. The amount is $G = 2\%$ of permanent income.

Let us take an example illustrated in Figure 7, with $r = 5\%$, $G = 2\%$, $T = 10$ years. This Figure plots the change in consumption and wealth for the rational actor $\kappa = 0$ (black, solid), and progressively less rational agents: $\kappa = 0.01$ (blue, dotted), $\kappa = 0.025$ (red, dashed-dotted), $\kappa = 0.1$ (green, dashed). The traditional Ricardian consumer ($\kappa = 0$) immediately decreases his consumption by 2%, which leads to wealth accumulation at until time $T$. In contrast the very BR consumer ($\kappa = 0.1$) doesn’t react at all until $T = 10$ (hence he doesn’t accumulated any wealth), and then cuts a lot on consumption. The value $\kappa = 0.01$ and $\kappa = 0.025$ display an intermediary behavior. For $\kappa = 0.025$, the consumer initially doesn’t pay attention to the future tax. However, at a time $s = 4.5$ years, (i.e., when there are 3.6 years remaining until the taxes are effective), he starts paying attention, and starts savings for the future taxes. As the tax looms larger, the agent saves more. As the agent delayed his savings, he ends up cuttings down on consumption more drastically when taxes are in effect.

Smaller taxes generate a more delayed reaction. Controlling for the PV of taxes, consumers are better off with early rather than delayed taxes (as this allows them to smooth more).
6.2 The life-cycle model

By life-cycle model, I mean a model that features a finite life (unlike the previous infinite-horizon model), and emphasizes the need to save for retirement. I develop the BR version here.

The agent lives for $T$ periods, receives income $y_t = y$ for times $t \in [0, L]$, (where $L$ is the time where labor income shocks) then $y_t = y + \hat{y}$ (with $\hat{y} < 0$) for $t \in [L, T]$. We call $B = T - L$ the length of retirement. Utility is: $\sum_{t=0}^{T-1} u(c_t)$. The interest rate and the discount rate are both 0.

In the rational model, the optimal policy is to smooth consumption: consume

$$c_t = \frac{w_0 + Ly + B(y + \hat{y})}{T} = \frac{w_0}{T} + y + \frac{B}{T}\hat{y}$$

**Proposition 13** In the BR life-cycle model, the optimal consumption policy is, before retirement ($t < L$)

$$c_t = \frac{w_t}{T-t} + \tau \left( \frac{B\hat{y}}{T-t}, \kappa \right) + y.$$  

and after retirement $c_t = \frac{w_t}{T-t} + y + \hat{y}$ for $t \geq L$. Hence, when $\kappa > 0$ and $\hat{y} < 0$, consumption weakly falls over time, and discretely falls at retirement. After retirement, consumption is constant.

6.3 Dynamic Portfolio Choice

I now study a Merton problem with dynamic portfolio choice. The agent’s utility is:

$$E \left[ \frac{1}{1-\gamma} \int_0^\infty e^{-\rho s} c_s^{1-\gamma} ds \right],$$

and his wealth $w_t$ evolves according to:

$$dw_t = (-c_t + rw_t) dt + w_t \Sigma_t \frac{\sigma}{\sqrt{\tau}} dZ_t$$

where $\theta_t$ is the allocation to equities.

I start by describing the rational problem, then the behavioral solution. I call $\psi = \frac{1}{\gamma}$ the IES. Though for simplicity I use a CRRA utility function, I try to write the expressions in a way that involves both $\gamma$ and $\psi$, in a way that would generalize correctly to Epstein-Zin utility, where the two notions are disentangled.

6.3.1 Taylor expansions of the value function: rational case

We examine the problem in the rational case first, with a reminder of notions of portfolio choice. In a deterministic context with interest rate $r_t$, the SDF is simply $M_t = e^{-\int_0^t r_s ds}$. Next, suppose that there is a stochastic opportunity set: A set of assets with risk premium $\pi_t$, and covariance matrix $\Sigma_t$. In a static maximization, the optimal portfolio the certainty equivalent is a return: $R_t(\theta_t) = r_t + \theta_t \pi_t - \frac{1}{2} \theta_t \Sigma_t \theta_t$, so that the (static) optimal portfolio
choice is \( \theta_t = \arg \max_\theta R_t(\theta) \), i.e. \( \theta_t = \frac{1}{\gamma} \Sigma^{-1}_t \pi_t \), and the certainty equivalent is finally: 
\[
R_t = \max_{\theta_t} R_t(\theta_t)
\]
where
\[
\Lambda_t = \pi_t \Sigma^{-1}_t \pi_t
\]
the “squared Sharpe ratio” of the investment opportunity set. Suppose that the process is driven by a Brownian motion \( B_t \) (which may be multidimensional). If the price of risk is \( \lambda_t \) (so that \( \Lambda_t = \|\lambda_t\|^2 \)), the stochastic discount factor can be represented as:
\[
M_t = \exp \left[ - \int_0^t \left( r_s + \frac{\Lambda_t}{2} \right) ds - \lambda_s dB_s \right]
\]
The value function is as follows.

**Lemma 4** (Value function, traditional case) Suppose that the interest rate \( r_t \) and and the price of risk \( \lambda_t \) are deterministic, and that the agent is the traditional rational agent. The value function is
\[
V_{w_t}(w_t, x_t) = (\mu_t w_t)^{-\gamma}
\]
and the optimal policy is to consume \( c_t = \mu_t w_t \), where:
\[
\mu_t^{-1} = E_t \left[ \int_0^\infty e^{-\psi ps} \left( \frac{M_{t+s}}{M_t} \right)^{1-\psi} ds \right] = E_t \left[ \int_0^\infty e^{-f_t(\psi, \psi'u + (1-\psi)R_u)du} dt \right]
\]
where
\[
R_t = r_t + \frac{1}{2\gamma} \Lambda_t.
\]
is the certainty equivalent of expected portfolio returns (comprising stocks and bonds), with \( \Lambda_t = \|\lambda_t\|^2 \) is the square Sharpe ratio of the investment opportunity set.

When the opportunity set is constant, we have \( R_t = R_* \) and \( \mu_t = \mu_* \) with
\[
\mu_* = \psi \rho + (1 - \psi) R_*,
\]
When it is not constant, we have, up to second order terms:
\[
\mu_t = \psi \rho + (1 - \psi) \bar{R}_t
\]
where \( \bar{R}_t = \mu_* V_t^R \) is the average future portfolio returns, and \( V_t^R \) is the present value of future portfolio returns.
\[
V_t^R := E_t \left[ \int_t^\infty e^{-\mu_*(s-t)} R_s ds \right]
\]
Here $\hat{R}_t$ is the future average return of the portfolio (including stocks and bonds). Hence, the marginal propensity to consume is a weighted average (with weights $\psi$ and $1 - \psi$) of the pure rate of time preference $\rho$, and the average future return of the portfolio.

Lemma 4 summarizes and somewhat generalizes well-known notions, particularly from the work of Campbell and Viceira (2002). It indicates that what matters is the risk-adjusted rate of return of the portfolio, $R_t$: it is the safe short-term rate $r_t$, plus the square Sharpe ratio $\lambda$, divided by two times risk aversion. The future average return $\hat{R}_t$ is key to capture the (leading order of) the value function.

To structure the problem, suppose that the vector of asset returns $d\hat{r}_t$ (where $d\hat{r}_i$ is the return of asset $i$) follows:

$$d\hat{r} = (r + \pi_\star + \hat{\pi}_t) dt + \sigma dZ_t$$
$$\hat{\pi}_t = f'X_t$$

where $X_t$ is a vector of factors, following an AR(1):$^{11}$

$$dX_t = -\Phi X_t dt + \sigma^X dZ_t$$

and $f$ is a matrix of weights. We call

$$\Sigma^rX = \text{cov}(d\hat{r}, dX_t') / dt = \sigma\sigma^X.$$ 

the matrix of covariance, i.e. $\Sigma^r_{ij} = \text{cov}(d\hat{r}_i, dX_{jt}) / dt$. We define $\theta_\star = \frac{1}{\gamma} \Sigma^{-1}_\pi \pi_\star$ the portfolio choice in the model with constant variance and expected returns.

Then, the portfolio return is

$$R_t = \frac{1}{2\gamma} (\pi_\star + \hat{\pi}_t)' \Sigma^{-1}_t (\pi_\star + \hat{\pi}_t) = \frac{1}{2\gamma} \pi_\star \Sigma^{-1}_t \pi_\star + \theta_\star' \hat{\pi}_t + O(\|X_t\|^2)$$

$$= R_\star + \theta_\star' \hat{\pi}_t$$

$$= R_\star + \theta_\star' f'X_t = R_\star + b'X_t$$

i.e. the return is augmented by $\theta_\star' \hat{\pi}_t$, with

$$b := f\theta_\star.$$

Then, the present value of returns (72) is:

$$V_t^R = \frac{R_\star}{\mu_\star} + b'(\mu_\star I + \Phi)^{-1} X_t$$

(73)

where $I$ is the identity matrix of the $X$'s dimension.

For instance, if $X_t$ is a one-dimensional, so that $bX_t = \hat{R}_t := R_t - R_\star$, and $\hat{R}_t := R_\star + \frac{\mu_\star}{\mu_\star + \Phi_R} R_t$.  

$$\mu_t = \mu_\star + (1 - \psi) \frac{\mu_\star}{\mu_\star + \Phi} \hat{R}_t$$

(74)

Hence, we obtain a tractable representation of the value function, to the leading order.

$^{11}$ Or $X_t$ could be a linearity-generating twisted-AR(1), so that the derivations below can be exact (Gabaix 2009).
6.3.2 The hedging demand

We can calculate the hedging demand.

**Lemma 5** (Hedging demand, rational) The stock demand is

\[ \theta_t = \frac{1}{\gamma} \Sigma_t^{-1} (\pi_t + H_t) \] (75)

where \( H_t \) is the hedging demand premium, equal to (up to second order terms):

\[ H_{it} = (1 - \gamma) \text{cov} (d\tilde{r}_t, dV_t^R) \] (76)

de i.e. \( H_{it} \) is \( (1 - \gamma) \) times the covariance between asset i’s return \( (d\tilde{r}_i) \) and the present value of future returns \( V_t^R \) (equation 72).

In the AR(1) framework above,

\[ H_t = (1 - \gamma) \Sigma^{r,X} (\mu_s I + \Phi')^{-1} b. \] (77)

Suppose that returns mean-revert, i.e. \( \text{cov} (d\tilde{r}_{it}, d\frac{\tilde{R}_t}{\tilde{\mu}_t}) < 0 \). So, if \( \psi < 1 \), then investors load more on stocks because of the hedging demand.

We next state the modification of the value function.

**Lemma 6** (Value function with hedging demand, rational) In the hedging demand context, we have:

\[ \mu_t = \psi \rho + (1 - \psi) (\tilde{R}_t + \theta' H_t) \] (78)

where \( \tilde{R}_t = \mu_s V_t^R \) is the expected present value of returns, and \( H_t \) is the hedging demand term; they are explicit in (73) and (77).

The intuition for (75) is that \( H_{it} \) is a risk-adjusted risk premium of asset \( i \). This intuition carries over to (78). Compared to (71), the expression for \( \mu(X_t) \) offers one more term, the term \( (1 - \psi) \theta' H_t \).

A tractable case The equity premium \( \pi_t = \pi + \tilde{\pi}_t \) has a variable part \( \tilde{\pi}_t \), which follows

\[ d\tilde{\pi}_t = -\phi_R \tilde{\pi}_t dt - \chi_t \sigma dZ^1_t + \sigma' dZ^2_t \]

where the return is \( d\tilde{r}_t = (r_t + \pi_t) dt + \sigma dZ^1_t \). The parameter \( \chi_t \geq 0 \) indicates that equity returns mean-revert: good returns today lead to lower returns tomorrow. That will create a hedging demand term.

We call \( \theta_s := \frac{\pi}{\pi^2} \) the standard, myopic demand for stocks.
6.3.3 The sparse agent’s investment and consumption

We can calculate the sparse agent’s demand. Recall that $\psi = 1/\gamma$ is the IES.

**Proposition 14** (Behavioral dynamic portfolio choice) The fraction of wealth allocated to equities is, with $\theta_* := \frac{\pi_t}{\gamma \sigma^2}$

$$\theta_t^s = \theta_* + \tau \left( \frac{\hat{\pi}_t}{\gamma \sigma^2}, \kappa \right) + \tau \left( \frac{H_t}{\gamma \sigma^2}, \kappa \theta \right)$$

while consumption is: $c_t^s = \mu_t^s w_t$ with

$$\mu_t^s = \mu_* + \tau \left( 1 - \psi \right) \frac{\mu_*}{\mu_* + \Phi} \theta_* \hat{\pi}_t, \kappa c/w \right) + \tau \left( 1 - \psi \right) \theta_* H_t, \kappa c/w \right)$$

where $H_t$ is the hedging demand term (79)

$$H_t = (1 - \gamma) \text{cov} \left( dR_t, dV_t^R \right) = - (1 - \gamma) \theta_* \frac{1}{\mu_* + \Phi} \sigma^2 \chi_t$$

**Proof** We first calculate the rational values. In that case

$$\hat{R}_t = r_* + \frac{\Lambda_*}{2\gamma} + \theta_* \frac{\mu_*}{\mu_* + \Phi} \hat{\pi}_t$$

$$H_t = (1 - \gamma) \text{cov} \left( dR_t, d \left( \frac{\hat{R}_t}{\mu_*} \right) \right) = - (1 - \gamma) \theta_* \frac{1}{\mu_* + \Phi} \sigma^2 \chi_t$$

so that

$$\theta_t = \frac{\pi_* + \hat{\pi}_t + H_t}{\gamma \sigma^2}$$

In addition

$$\mu_t = \psi \rho + (1 - \psi) \left( \hat{R}_t + \theta'_s H_t \right) = \mu_* + (1 - \psi) \left( \theta_* \frac{\mu_*}{\mu_* + \Phi} \hat{\pi}_t + \theta'_s H_t \right)$$

As in Proposition 3, with ex-post attention, the BR agent just truncates those terms.

\\

Proposition 14 predicts the choice of a sparse agent. When $\kappa = 0$, it is the policy of a fully rational agent, e.g. as in Campbell and Viceira (2002). When $\kappa > 0$, it is the policy of a sparse agent. When $\kappa$ is larger, portfolio choice becomes insensitive to the change in the equity premium, $\hat{\pi}_t$, and the agent thinks less about the mean-reversion of asset, the $B \chi$ terms.

In addition, the agents’ consumption function pays little attention to the mean-reversion of assets. [Next iteration should have a calibration.]
6.4 Linear-Quadratic models

A lot of economic problems can be conveniently expressed as linear-quadratic (LQ) models (Ljungqvist and Sargent 2012). We show here how to systematically derive a BR version of those models.

We again write \( z = (w, x) \), where \( w \) is the set of variables known under the default model, and \( x \) is the set of variables that are not considered in the default model. Utility is:

\[
u(z, a) := \frac{1}{2} \begin{pmatrix} z \\ a \end{pmatrix}^T \begin{pmatrix} U_{zz} & U_{za} \\ U_{az} & U_{aa} \end{pmatrix} \begin{pmatrix} z \\ a \end{pmatrix}
\]

and the law of motion is:

\[
z' = F^z(z, a) := \Gamma^z z + \Gamma^a a
\]

where \( U \) and \( \Gamma \) are constant matrices. The rational value function is also LQ

\[
V(z) = -\frac{1}{2} z' V_{zz} z = -\frac{1}{2} (w' V_{ww} w + 2 w' V_{wx} x + x' V_{xx} x)
\]

Under the default model, \( V_{ww} \) is known, and

\[
a^d(w) = A_w w
\]

for \( A_w \) a constant. Our goal is to find \( V_{wx} \), which affects the value function. To do so, we apply from (259).

Lemma 7 In the linear-quadratic problem, the cross-partial of the value function is

\[
V_{wx} = V_{wx} = \left[ 1 - \beta (D_w w') \cdot \Gamma^w_x \right]^{-1} \left[ U_{wx} + U_{xa} A_w + \beta \Gamma^w_x V_{ww} (D_w w') \right].
\]

where \( D_w w' = \Gamma^w_w + \Gamma^w_a A_w \). The impact on the action is \( a = A_w w + A_x x \), where \( A_w \) is the default value, and

\[
A_x = -\Psi_a^{-1} \Psi_x,
\]

where

\[
\Psi_a = U_{aa} + \beta \Gamma^w_a V_{ww} \Gamma^w_a \\
\Psi_x = U_{xa} + \beta V_{wx} \Gamma^w_a
\]

This illustrates that the value function can be written:

\[
V(z) = -\frac{1}{2} z' V_{zz} z = -\frac{1}{2} w' V_{ww} w + w' V_{wx} x + O \left( \|x\|^2 \right)
\]

with matrix \( V_{wx} \) is expressed in closed form above.

Hence, the BR value function is simply:

\[
V^s(z, m) = -\frac{1}{2} z' V_{zz} z = -\frac{1}{2} w' V_{ww} w + w' V_{wx} M(m) x + O \left( \|x\|^2 \right)
\]

for the diagonal attention matrix \( M(m) = \text{diag} (m_i) \).
Proposition 15 (Behavioral version of linear-quadratic problems) In a linear-quadratic problem, the optimal attention is

\[ m_{x_i} = A \left( A_{x_i} \Psi A_{x_i} \sigma_i^2 / \kappa \right) \tag{81} \]

and the optimal sparse action is

\[ a = A_w w + A_x M x \]

where \( M = \text{diag}(m_{x_i}) \). Here we use the notations of Lemma 7.

6.5 The Becker-Murphy model of rational addiction

The Becker-Murphy (1988) model of rational addiction is a peak of the use of rationality in economics. We will give a behavioral version of it. We shall see that the qualitative evidence in favor of the model (the fact that future increase in prices lower consumption today) are also consistent with this BR version: it is shows that agent are at least partially rational (as in the present model), not that they are fully rational (as assumed by Becker-Murphy). This distinction is important: If people are BR enough, they’d be better off under a high tax, or a ban, of the addictive substance – while the optimal tax is 0 in the Becker-Murphy model. This analysis is in the spirit of Gruber and Koszegi (2001), who study a hyperbolic discounting addict, rather than a boundedly rational one in the sense of this paper.

We call \( c \) the consumption, and \( x \) the level of addition. Utility function is:

\[ u(c, x) = -\frac{1}{2} (c - x - A)^2 - Bx \]

Addition \( x_t \) evolves as:

\[ x_{t+1} = \rho x_t + h c_t. \]

The BR agent has in mind the model:

\[ x_{t+1} = \rho^* x_t + h^* c_t. \]

We posit that in the default model the agent does not perceive any addiction dynamics: he perceives addition as being constant.

\[ \rho^d = 1, \quad h^d = 0. \]

When the agent has partial attention \( m \) to inattention dynamics, we have:

\[ x_{t+1} = (1 - m) x_t + m (\rho x_t + h c_t) \]

so

\[ \rho^s = (1 - m) + m \rho, \quad h^s = m h \]

Let us now study the BR dynamics.
Warm-up: 2 period model  As before, it is helpful to study a 2-period model, with $t = 1, 2$.

Behavior at the last period, $t = 2$. The agent should and does consume his optimal consumption

$$c^d (x) = \arg \max_c u (c, x) = x + A$$

We define the resulting utility as $\bar{u} (x)$

$$\bar{u} (x) := \max_c u (c, x) = u (c^d (x), x) = -Bx.$$ To, the time-1 value function is

$$V^1 (x) = \bar{u} (x). \quad (82)$$

Behavior at period 1, $t = 1$. Given perceived dynamics, the problem is:

$$\max_{c, m} v (c, x, m)$$

$$v (c, x, m) := u (c, x) + \beta V (\rho^s (m) x + h^s (m) c)$$

which gives:

$$0 = u_c + \beta h^s V' (\rho^s x + h^s c)$$

$$= -c + x + A - \beta h^s B$$

$$c = x + A - \beta h^s B \quad (83)$$

An interesting case is to impose $c \geq 0$. Then, first period consumption is $> 0$ iff $A - h^s B > 0$. So, if $h^s B < A \leq hB$, then the rational agent consumes 0, while the very behavioral agent consumes a positive amount, and gets addicted.

The optimal attention is $m = A (\nu e c^2 m / \kappa) = A (-u e B^2 h^2 / \kappa)$.

Infinite horizon model  The value function satisfies:

$$V (x) = \max_{c, m} u (c, x) + \beta V (\rho^s (m) x + h^s (m) c)$$

The FOC is:

$$u_c (c, x) + \beta V' (\rho^s (m) x + h^s (m) c) h^s (m) = 0$$

i.e. the agent takes into account only part of the addiction costs, as $h^s (m) \leq h$. As a result, the agent is more addicted in the steady state. The greater the myopia, the greater the optimal tax.
Proposition 16  In the Becker-Murphy model with boundedly rational agents, the consumption \( c \) given the stock of addition \( x \) is:

\[
c(x) = x + A + \beta b(m) m^b h
\]

using \( m = (m^b, m^V) \); the value function is

\[
V(x, m) = a(m) + b(m) x
\]

where \( b(m) = -\frac{B}{1-\beta(1+m^V(\rho+h-1))} \) and \( a(m) \) is in the proof. When using the plain (as opposed to iterated) sparse max, \( m^V = 0 \) and attention to addition is \( m^h = A \left( \frac{1}{\alpha} \left( \frac{\beta B h}{1-\beta} \right)^2 \right) \).

7  Conclusion

I presented a practical way to do boundedly rational dynamic programming. It is portable and to the first order has just one free continuous parameter, \( \kappa \), the penalty for lack of sparsity, which can also be interpreted as a cost of complexity.

It allows us to revisit canonical models in economics, and give them a behavioral flavor. From the micro point of view, we obtain inattention and delayed response. Those are not necessarily very surprising features – however, it is useful to have clean model that generates those things and can be calibrated.

From the macro point of view, the model allows us to think about bounded rationality in general equilibrium. The upshot is that compared to the rational model, sparsity leads to larger and more persistent fluctuations. The reason is that rational actors tend to “dampen” fluctuations. For instance, they consume more when more capital is available. This channel is muted with sparse agents. Hence, fluctuations are more persistent, innovations have a longer-lasting effect, and the average fluctuations (deviations from the mean) are larger.

Given that it seems easy to use and sensible, we can hope that this model may be useful for other extent issues in macroeconomics and finance.


8 Appendix: Extensions

8.1 Extension of the basic 3-period example

We have so far used the plain sparse max. This led to $m^V = 0$, the exclusive reliance on the simplified value function. We now calculate what happens when using the twice-iterated sparse max of Definition 4.

To endogenize $m^V$, we use the twice-iterated smax: $\text{smax}^2_{c|m} v^0(c_0, w_0, x, m)$ with $m = (m_0^x, m^V)$. At the first round, $v^0_{c_0, m^V} = 0$, so $m^V_0 (1) = 0$, and as before $m^x_0 (1) = A \left( \frac{1}{6k} u'' \left( \frac{w_0}{3} \right) \sigma^2_x \right)$.

At the second round, now $m^d = (m_0^x (1), 0)$. The easy part is the attention to $x$, which is slightly different than at step 1:

$$m^x_0 (1) = A \left( \frac{1}{6k} u'' \left( \frac{w_0 + m_0^x (0) x}{3} \right) \sigma^2_x \right)$$

The more novel part is to calculate $m^V$. We have, with $w_1 = w_0 - c_0$, and calling $x^s := m^x_0 x$

$$v^0_{c, m^V} (c_0, w_0, x, m_0^x, m^V) = \partial_c \left[ V^1 (w_0 - c_0, x^s) - V^1 (w_0 - c_0, x^s, m^V = 0) \right]$$

$$= -\frac{1}{2} u' \left( \frac{w_1 + m_1 x^s}{2} \right) - \frac{1}{2} u' \left( \frac{w_1 + (2 - m_1) x^s}{2} \right) + u' \left( \frac{w_1 + x^s}{2} \right)$$

Doing a Taylor expansion of the consumptions $\frac{w_1 + m_1 x^s}{2}$ and $\frac{w_1 + (2 - m_1) x^s}{2}$ around their mean

$$c^d = \frac{w_1 + x^s}{2} = \frac{w_1 + m_0^x x}{2}$$

we obtain:

$$v^0_{c, m^V} = -\frac{1}{2} u' \left( c^d + (m_1 - 1) \frac{x^s}{2} \right) - \frac{1}{2} u' \left( c^d - (m_1 - 1) \frac{x^s}{2} \right) + u' \left( c^d \right)$$

$$= -\frac{1}{2} u'' \left( c^d \right) (m_1 - 1)^2 \left( \frac{x^s}{2} \right)^2 \times 2 + o \left( x^2 \right)$$

$$= -\frac{1}{4} u'' \left( c^d \right) (m_1 - 1)^2 \left( m_0^x x \right)^2 + o \left( x^2 \right)$$

Likewise, $v^0_{c_c | m = m_0^d (1)} = \frac{3}{2} u'' \left( c^d \right)$. So, the impact of $m^V$ is

$$\frac{\partial c_0}{\partial m^V} = -\frac{v^0_{c, m^V}}{v^0_{c_c}} = -\frac{1}{6} u'' \left( c^d \right) (m_1 - 1)^2 \left( m_0^x x \right)^2 + o \left( x^2 \right)$$

Hence, for a small $x$, the attention $m^V$ to the difference between the difference between
the true and proxy value functions (i.e., \(V^1(w_1, x, m^V)\) for \(m^V = 1\) vs \(m^V = 0\)) is:

\[
m^V_0 = A \left( \frac{1}{\kappa} E \left[ \left( \frac{\partial c_0}{\partial m^V} \right)^2 v^0_{cc} \right] \right)
= A \left( \frac{1}{\kappa} E \left[ \left( \frac{1}{6} u''(c^d) (m_1 - 1)^2 (m^x_0)^2 x^2 \right)^2 \right] \right) \frac{3}{2} u''(c^d)
= A \left( \frac{1}{24 \kappa} \left( \frac{u''(c^d)}{u''(c^d)} \right)^2 (m_1 - 1)^4 (m^x_0)^4 E [x^4] u''(c^d) \right)
\]

(84)

It is instructive to take the limit of small \(\kappa\), using a sparsity-inducing cost function \((g'(0) > 0)\). To have \(m^x_0 > 0\), we need \(\sigma^2_\kappa\) large enough, so \(\sigma_x \geq \kappa^{1/2}\). To have \(m^V_0 > 0\), we need \(\sigma^2_\kappa\) large enough, i.e. \(\sigma_x \geq \kappa^{1/4}\), which is a much higher hurdle \((\kappa^{1/4}/\kappa^{1/2} \rightarrow \infty)\) for small \(\kappa\). We formalizing this.

**Proposition 17** (Attention to a variable, vs attention to the fine properties value function depending on that variable) Suppose a succession of problems (indexed by \(\kappa\) going to 0) such that there are positive constants \(B, B', \varepsilon\) such that for \(\kappa\) small enough: \(B\kappa^{1/2-\varepsilon} \leq \sigma_x(\kappa) \leq B'\kappa^{1/4+\varepsilon}\). Then, the agent will have \(m^x_0 > 0\) and \(m^V_0 = 0\) when \(\kappa\) is small enough. This is, the agent pays attention to the disturbance \(x\), but not to the subtle difference between the true and proxy value functions (i.e., \(V^1(w_1, x, m^V)\) for \(m^V = 1\) vs \(m^V = 0\)).

In plain terms: because thinking about the nuances \(m^V\) in \(V(x, m^V)\), one needs to think about \(x\) at all. Hence, in many situations, we have \(m^V = 0\) and \(m^x > 0\). Indeed, we cannot have (with just one state variable \(m^x = 0\) and \(m^V > 0\)).

In particular, for our 3-period problem for \(\kappa\) small enough but not too small, \(m^V = 0\) and \(m^x_0 > 0\): the agent uses the simplified value function, as still pays attention to \(x\), like in the basic smax case. This is one reason it is useful to use the basic smax: it gets to the essence of the more complex patterns that can later be refined using the iterated smax.

### 8.2 Continuous time

Calculations are typically cleaner in continuous time, so we develop the continuous-time version of the machinery. We take for now problems without stochastic terms (those should be added later).

The laws of motion are:

\[
\dot{w}_t = F^w(w, x, a) \\
\dot{x}_t = F^x(w, x)
\]

and the Bellman equation is:

\[
\rho V(w, x) = u(w, x, a) + V_w(w, x) F^w(w, x, a) + V_x(w, x) F^x(w, x, a)
\]
In the more complex case $\dot{x}_t = F^w (w, x, a)$, we need to solve for a matrix Ricatti equation—but not here.

Call $D_w = \partial_w + a_w \partial_a$ the “total impact” of a change in $w$. Then:

$$\rho V_x = u_x + V_w F^w_x + V_x F^x_x + V_{xx} F^x_{xx}$$

(85)

Now, we differentiate and evaluated at $x = 0$:

$$\rho V_{wx} = D_w (u_x + V_w F^w_x) + V_{wx} F^x_x + V_x F^x_{wx}$$

so

$$V_x = (\rho - F^x_x)^{-1} [u_x + V_w F^w_x]$$

(86)

$$V_{wx} = (\rho - F^x_x)^{-1} [D_w (u_x + V_w F^w_x) + V_x F^x_{wx}]$$

(87)

As $a$ satisfies $\Psi = 0$ with

$$\Psi (a, w, x) = u_a + V_w F^w_a$$

Hence, the impact of $x$ on the optimal action is

$$a_x = -\Psi^{-1}_a \Psi_x$$

$$\Psi_a = u_{aa} + V_w F^w_{aa}$$

$$\Psi_x = u_{ax} + V_{wx} F^w_a + V_w F^w_{ax}$$

Calculation of $V_{xx}$. We now turn to the more difficult case of $V_{xx}$. Using $D_x = \partial_x + a_x \partial_a$ the “total impact” of a change in $x$, we have:

$$\rho V_x = D_x u + V_w D_x F^w_x + V_x F^x_x + V_{xx} F^x_{xx}$$

$$= a_x (u_a + V_w F^w_a) + u_x + V_w F^w_x + V_x F^x_x + V_{xx} F^x_{xx}$$

Next, differentiating at $x = 0$,

$$\rho V_{xx} = a_x D_x (u_a + V_w F^w_a) + D_x [u_x + V_w F^w_x + V_x F^x_x] + V_{xx} F^x_{xx}$$

$$= a_x [u_{ax} + u_{aa} a_x + V_{wx} F^w_a + V_w F^w_{ax} + V_w F^w_{aa} a_x]$$

$$+ u_{xx} + u_{xa} a_x + V_{xx} F^w_x + V_w D_x F^w_x + 2 V_{xx} F^x_x + V_{xx} F^x_{xx}$$

hence

$$(\rho - 2 F^x_x) V_{xx} = a_x [u_{ax} + u_{aa} a_x + V_{wx} F^w_a + V_w F^w_{ax} + V_w F^w_{aa} a_x]$$

$$+ u_{xx} + u_{xa} a_x + V_{xx} F^w_x + V_w D_x F^w_x + V_{xx} F^x_{xx}$$

This is a bit of a complicated expression. Let us note it can be written

$$(\rho - 2 F^x_x) (V_{xx}^s - V_{xx}^r) = a_x A + a_x B a_x + C$$

with $B = u_{aa} + V_w F^w_{aa}$.

We use the following elementary Lemma:
Lemma 8 Let \( f(a) = Aa + a'Ba + C \), for \( B \) symmetric negative definite. Let \( a^* = \arg\max_a f(a) \), so \( a^* = -\frac{1}{2}B^{-1}A \). Then, for any \( a \),

\[
f(a) - f(a^*) = (a - a^*)B(a - a^*).
\]

Let’s compare \( V_{xx} \) under the sparse vs rational model: the difference is just in the \( D_x \) vs \( D^s_x \) term. Indeed,

\[
D^s_x - D_x = (a^s - a^r)\partial_a
\]

so, using the previous Lemma,

\[
V^s_{xx} - V^r_{xx} = (\rho - 2F^x_x)^{-1}(a^s - a^r)(u_{aa} + V_w F^w_{aa})(a^s - a^r)
\]  

(88)

We gather the results.

Proposition 18 (What are the losses from a suboptimal policy?) Consider the value function \( V^r \) under the optimal policy and \( V^s \) under a potentially suboptimal policy, and \( V^\Delta (w, x) = V^s(w, x) - V^r(w, x) \). Then, evaluating at \( x = 0 \), we have:

\[
V^\Delta = 0, V^\Delta_w = 0, V^\Delta_{ww} = 0, V^\Delta_x = 0, V^\Delta_{wx} = 0
\]  

(89)

and

\[
V^\Delta_{xx} = (\rho - 2F^x_x)^{-1}(a^s - a^r)(u_{aa} + V_w F^w_{aa})(a^s - a^r)
\]  

(90)

Equation (90) has an intuitive interpretation. At a point in time, as a function of \( a \), present and continuation utility is \( v(a) = u(a, w_t) dt + (1 - \rho dt)V(w_t + F^w(w_t, a_t) dt) \). Hence (omitting for \( dt \) to remove the notational chatter), \( v'(a) = u_a + V_w F^w_{aa} \) and \( v''(a) = u_{aa} + V_w F^w_{aa} \). Hence, reacting imperfectly to a small \( x_t \) (with \( a^\delta_t = a^s_t - a^r_t \)) creates an instantaneous utility loss of \( \Delta_t = -\frac{1}{2}x_t a^\delta_{aa} a^\delta_t x_t \). The full utility loss is the present discounted value of that, i.e.

\[
2\Lambda = \int_0^\infty e^{-\rho t} 2\Lambda_t dt = -\int_0^\infty e^{-\rho t} x_t a^\delta_{aa} a^\delta_t x_t \text{ with } x_t = e^{-\phi t} x_0
\]

\[
= -\int_0^\infty e^{-\rho t} e^{-2\phi t} x_0 a^\delta_{aa} a^\delta_t x_0 = \frac{1}{2}\phi x_0 a^\delta_{aa} a^\delta_t x_0
\]

\[
= -x_0(\rho - 2F^x_x)^{-1} a^\delta_{aa} (u_{aa} + V_w F^w_{aa}) a^\delta_t x_0 \text{ as } F^x_x = -\phi
\]

It is enough to study the “static” utility losses to derive the dynamic utility losses. This proposition 18 is a dynamic application of the Proposition 26 in Gabaix (2014, online appendix) regarding losses from a suboptimal policy. For convenience, we restate this Proposition here. With static problem \( \max u(a, x) \) s.t. \( b(a, x) \geq 0 \), and a Lagrangian \( L(a, x) = u(a, x) + \lambda b(a, x) \), the losses from a suboptimal policy \( a^\delta = a - a^r \) (where \( a^r \) is the optimal policy) are to the leading order: \( \frac{1}{2}a^\delta t L_{aa} a^\delta \).
Here the Lagrangian is

\[ L = \int e^{-\rho t} [u(a_t, z_t) + \lambda_t (-z_t + F^z(a_t, z_t))] dt, \]

where \( z_t = (w_t, x_t) \) is the state vector. Hence, the loss \( \Lambda \) is expressed by (to the leading order)

\[
2\Lambda = a' L a = \int a^\delta_t L a t a^\delta_t = \int e^{-\rho t} a^\delta_t [u_{at} + \lambda_t F_{ata}] a^\delta_t dt
\]

Suppose that we can linearize, \( a^\delta_t = A x_t \), we have

\[
2\Lambda = \int e^{-\rho t} x_t' A^\delta_t [u_{at} + \lambda_t F_{ata}] A^\delta x_t dt
\]

Consider the ergodic limit, where \( x_t \) has a distribution independent of \( t \). Recall that

\[
\mathbb{E} x_t' B x_t = \mathbb{E} \sum_{i,j} x_t B_{ij} x_j = \sum_{i,j} B_{ij} \mathbb{E} [x_i x_j] = \text{Trace} (B \mathbb{E} [x x'])
\]

Hence,

\[
2\Lambda = \frac{1}{\rho} \text{Trace} (B \mathbb{E} [x x'])
\]

9 Appendix: Proofs

**Proof of Proposition 2** The rational reaction function satisfies:

\[
a^r(x) = a^d + \sum_i b_i x_i + \lambda(x)
\]

for a function \( \lambda(x) = O \left( \| x \|^2 \right) \).

So, \( \partial a / \partial x_i = b_i \) and:

\[
m^*_i = \tau \left( 1, \frac{\kappa_a}{\sigma_i \cdot \partial a / \partial x_i} \right) = \tau \left( 1, \frac{\kappa_a}{\sigma_i b_i} \right)
\]

We shall use the notation \( \overline{x}(x) := \lambda((m^*_i x_i)_{i=1...n}) \), which also satisfies \( \overline{x}(x) = O \left( \| x \|^2 \right) \).

The sparse reaction function is:

\[
a^s(x) = \arg \max_a u(a, m^*_1 x_1, ..., m^*_n x_n) = a^s(m^*_1 x_1, ..., m^*_n x_n)
\]

\[
= a^d + \sum_i b_i m^*_i x_i + \lambda((m^*_i x_i)_{i=1...n}) = a^d + \sum_i b_i \tau \left( 1, \frac{\kappa_a}{\sigma_i b_i} \right) x_i + \overline{x}(x)
\]

\[
= a^d + \sum_i \tau \left( b_i, \frac{\kappa_a}{\sigma_i} \right) x_i + \overline{x}(x) = a^d + \sum_i \tau \left( b_i, \frac{\kappa_a}{\sigma_i} \right) x_i + O \left( \| x \|^2 \right).
\]
Proof of Proposition 3  Let us consider two functions $U$ and $u^s$

\[
U^* (a, w, x) := u (a, w, x) + \beta \mathbb{E} V (F^w (w, x, a), F^x (w, x, a)) \\
U^{**} (a, w, x) := u (a, w, x) + \beta \mathbb{E} V^s (F^w (w, x, a), F^x (w, x, a))
\]

and define the associated optimal actions:

\[
a^* (w, x) := \arg \max_a U^* (a, w, x), \quad a^{**} (w, x) := \arg \max_a U^{**} (a, w, x)
\]

In $U^{**}$, there is no inattention: however, the continuation policy $V^s$ is used: the agent will be inattentive in the future.

First, we will prove:

Lemma 9 Suppose that $F^x_a = 0$. We have, at $x = 0$, \( \frac{\partial a^* (w, x)}{\partial x} = \frac{\partial a^{**} (w, x)}{\partial x} \)

Proof. The key fact comes from Proposition 2, and is:

\[
V_w (w, 0) = V^s_w (w, 0) \\
V_{ww} (w, 0) = V^s_{ww} (w, 0) \\
V_x (w, x)_{|x=0} = V^s_x (w, x)_{|x=0} \\
V_{wx} (w, x)_{|x=0} = V^s_{wx} (w, x)_{|x=0}
\]

and

\[
U^*_a = u_a (a, w, x) + \beta \mathbb{E} [V_w \cdot F^w_a (w, x, a) + V_x \cdot F^x_a (w, x, a)]
\]

\[
U^{*_a} = u^{*_a} + \beta \mathbb{E} [F^x_a \cdot V^s_{xx} F^s_a + V_x \cdot F^x_a]
\]

Likewise, for $U^{**}$,

\[
U^{**}_a = u^{**}_a + \beta \mathbb{E} [F^s_a \cdot V^s_{xx} F^s_a + V_x \cdot F^s_a]
\]

Hence, we have

\[
U^{**}_a = U^{*_a} \quad \text{at } x = 0
\]

Note that we used $F^x_a = 0$. This is necessary, because in general $V_{xx} \neq V^s_{xx}$.

Likewise,

\[
U^{*_a} = u^{*_a} (a, w, x) + \beta \mathbb{E} [F^w_a (w, x, a) \cdot V_{ww} \cdot F^w_a (w, x, a) + V_w \cdot F^w_a (w, x, a)] \\
+ \beta \mathbb{E} [F^s_a (w, x, a) \cdot V^s_{xx} F^s_a (w, x, a)]
\]

Hence, we have

\[
U^{**}_a = U^{*_a} \quad \text{at } x = 0
\]
and a similar expression for $U_{aa}^{**}$, which leads to:

$$U_{aa}^{**} = U_{aa}^* \text{ at } x = 0$$

Finally, we have:

$$\frac{\partial a^{**} (w, x)}{\partial x} \bigg|_{x=0} = -U_{aa}^{**-1} \cdot U_{ax}^{**} \bigg|_{x=0} = -U_{aa}^{*1} \cdot U_{ax}^* \bigg|_{x=0}$$

$$= \frac{\partial a^* (w, x)}{\partial x} \bigg|_{x=0}.$$

Given $a^r (w, x) = a^d (w) + \sum_i b_i (w) x_i + O (x^2)$, we have

$$\frac{\partial a^r (w, x)}{\partial x_i} = b_i (w)$$

Hence, the lemma gives:

$$\frac{\partial a^{**} (w, x)}{\partial x_i} = b_i (w)$$

so

$$a^{**} (w, x) = a^d (w) + \sum_i b_i (w) x_i + O (x^2)$$

Finally,

$$a^s (x) = a^{**} (m_i^* x_i)$$

$$= a^d (w) + \sum_i b_i (w) m_i^* x_i + O (x^2)$$

$$= a^d (w) + \sum_i b_i (w) \tau \left(1, \frac{\kappa_a}{b_i (w) \sigma_i} \right) x_i + O (x^2)$$

$$= a^d (w) + \sum_i \tau \left(b_i (w), \frac{\kappa_a}{\sigma_i} \right) x_i + O (x^2).$$

**Proof of Lemma 3** The Bellman equation is:

$$V (w, r) = \max_c u (c) + \beta V' ((R + r) (w - c) + y', r') \tag{91a}$$

I suppress the expectation operator, as the shocks are assumed to be small. We assume a law of motion:

$$r' = \rho r + \epsilon'$$

Call next-period wealth $w'$:

$$w' = (R + r) (w - c) + y'$$
We assume that the agent knows the simple model where the interest rate is always at its average, \( r \equiv 0 \). As is well-known, the optimal policy is \( c = rw + y \), and, with \( R = 1 + \tau \),

\[
V(w) = A\left( w + w^H \right)^{1-\gamma} / (1 - \gamma), \quad w^H = Y/\tau, \quad A = \left( \frac{\tau}{R} \right)^{-\gamma}
\]

First, we differentiate the Bellman equation with respect to the new variable:

\[
V_r(w, r) = \beta V_{w'}(w', r') \frac{\partial w'}{\partial r} + \beta V_{r'}(w', r') \frac{\partial r'}{\partial r}
\]

Evaluating at \( r = 0 \), this leads to:

\[
V_r(w, 0) = V^d_w(w) \frac{\beta(w-c)}{1 - \beta \rho}
\]

We now take the total derivative with respect to \( w \), \( D_w f = \partial_w f + \frac{dc}{dw} \partial_a f \), e.g. the full impact of a change in \( w \), including the impact it has on a change in the consumption \( c \). The baseline policy is \( c(w) = \tau w/R + \bar{y} \), so \( D_w c = \bar{r} \), and \( D_w w' = d \left( \bar{R} (w - c) \right) / dw = \bar{R} - \bar{R} \tau / \bar{R} = 1 \).

\[
D_w c = \bar{r} / \bar{R} \\
D_w w' = 1
\]

This means that one extra dollar of wealth received today translates into exactly one dollar of wealth next period: its interest income, \( r \), is entirely consumed.

So differentiate (using the total derivative) equation 92. We obtain:

\[
\beta^{-1} V_{wr}(w, r) = V_{w'w'}(w', r')(D_w w) \cdot (w - c) + V_{w'}(w', r') D_w(w - c) + V_{w'r'}(w', r') \rho D_w w'
\]

so, using

\[
V_{w'r'}(w', r') = -\gamma V_{w'} \cdot \frac{1}{w + w^H} = -\gamma V_{w'} \cdot \frac{\bar{r}}{R c}
\]

\[
V_{w,r} = \frac{\beta V_{w'} \bar{r} (1 - \gamma \bar{r} \left( \frac{w-c}{c} \right))}{1 - \rho \beta}
\]

Finally, let’s derive the impact of a change in \( r \) on \( c \) : We have

\[
V_w = \beta (\bar{R} + r) V_{w'} = u'(c)
\]

so

\[
\frac{dc}{dr} = \frac{V_{w r}}{u''(c)} = \frac{-1}{u''(c)} \frac{V_w}{\bar{R}} \frac{1 - \gamma \bar{r} \left( \frac{w-c}{c} \right)}{\bar{R} - \rho_r \beta}
\]

\[
= \frac{-1}{\gamma u''(c) c \frac{\bar{R}}{\bar{R} - \rho_r}} \frac{V_w}{\bar{R} - \rho_r}
\]

\[
\frac{dc}{c} = \frac{1}{\bar{R}} \frac{\bar{r} \left( \frac{w-c}{c} \right) - 1}{\gamma} dr
\]
We note that the result
\[ b_y = \frac{\tau}{R(R - \rho_y)} c^d_t, \quad b_r = \frac{\tau \left( \frac{w_t}{c^d_t} - 1 \right) - 1/\gamma}{R - \rho_r} \]
becomes, in continuous time:
\[ b_y = \frac{\tau}{\tau + \phi_y}, \quad b_r = \frac{\tau w_t - 1/\gamma}{\tau + \phi_r} \]  \hspace{1cm} (93)

**Proof of Proposition 7** Let’s first study the mechanics of accumulation. We have
\[ w_{t+1} = (R + \hat{r}_t)(w_t + y - c_t) + k^y z_{t+1} \]
\[ z_{t+1} = Az_t \]
and a policy:
\[ c_t = \frac{r}{R} w_t + y + B_z z_t \]  \hspace{1cm} (94)
as the income is \( k^y z_t \). This implies that next period consumption is:
\[ c_{t+1} = \frac{r}{R} ((R + \hat{r}_t)(w_t + R y - R c_t + k^y z_{t+1}) + y + B_z z_{t+1}) \]
\[ = \frac{r}{R} (R + \hat{r}_t)w_t + R y - R \left( \frac{r}{R} w_t + y + B z z_t + k^y z_{t+1} \right) + B_z z_{t+1} \]
\[ = \frac{r}{R} w_t + \frac{r}{R} \hat{r}_t w_t + \left( -r B_z + \left( \frac{r}{R} k^y + B_z A \right) z_t + y \right) \]
\[ c_{t+1} = \frac{r}{R} w_t + \left( - (R - A) B_z + \frac{r}{R} k^y + \frac{r}{R} w_t k^r \right) z_t + y \]  \hspace{1cm} (95)
as \( \hat{r}_t = k^r z_t \). Let us first study the rational case.

**The rational case.** The (rational) Euler equation imposes:
\[ c_{t+1} - c_t = \psi c^d_t \frac{\hat{r}_t}{R} \]
i.e., using (94) and (95),
\[ -(R - A) B_z + \frac{r}{R} k^y + \frac{r}{R} w_t k^r = \psi c^d_t \frac{k^r}{R} \]
This gives the normatively correct sensitivity:
\[ B_z = (R - A)^{-1} \left[ \left( r w_t - c^d_t \psi \right) k^r + r k^y A \right] \frac{1}{R} \]  \hspace{1cm} (96)

**The behavioral case.** The subjective model is:
\[ z_{t+1} = m A z_t + \varepsilon_{t+1} \]
\[ w_{t+1} = (R + m \hat{r}_t)(w_t + y - c_t) + m k^y z_{t+1} \]
\[ = (R + k^r m z_t) w_t - R c_t + k^y m z_{t+1} \]

\[ \text{as } \hat{r}_t = k^r z_t. \]
Here, $\hat{r}_t^s = k^r z_t$ is a subjectively perceived interest rate, that may not be the correct one (e.g. because of inattention, or money illusion). Note that I replaced $(R + \hat{r}_t) c_t$ by $R c_t$, by Taylor expansion.

The policy has still the form (94), but with a different policy vector $B_z$. With the DP approach, the policy is as (96), replacing $A, k^y$ and $k$ by $mA, mk^y, mk^r$. Hence (96) is replaced by:

$$B_z = (R - mA)^{-1} \left[ \left( \frac{r}{R} w_t - c^{d \psi}_t \right) \frac{mk^r}{R} + \frac{r}{R} k^r mA \right]$$

(97)

$$= \left( \frac{R}{m} - A \right)^{-1} \left[ \left( \frac{r}{R} w_t - c^{d \psi}_t \right) \frac{k^r}{R} + \frac{r}{R} k^y A \right]$$

(98)

Using the formal value of $c_{t+1}$ in (95), with $M$ to be determined, and with $\hat{r}_t = k^r z_t$, and set $\xi = \frac{1}{M} - 1 \geq 0$, we calculate:

$$c_{t+1} - \frac{1}{M} c_t = -\xi \frac{r}{R} w_t + \left( -\left( \frac{1}{M} + r - A \right) B_z + \frac{r}{R} k^y A + \frac{r}{R} w_t k^r \right) z_t$$

$$= -\xi \frac{r}{R} w_t + \left( -\left( \frac{1}{M} + r - A \right) B_z + \frac{r}{m} \left( -\frac{1}{M} + r - \frac{R}{m} \right) B_z + c^{d \psi}_t \frac{k^r}{R} \right) z_t$$

$$= -\xi + (k^r - k^r) z_t \frac{r}{R} w_t + \left( -\left( \frac{1}{M} + r - \frac{R}{m} \right) B_z + c^{d \psi}_t \frac{k^r}{R} \right) z_t$$

To get a clean Euler equation, we choose $M$ to satisfy: $\frac{1}{M} + r = \frac{R}{m}$, i.e.

$$M = \frac{m}{R - rm}$$

Then:

$$c_{t+1} - \frac{1}{M} c_t = \left( -\xi + (k^r - k^r) z_t \right) \frac{r}{R} w_t + c^{d \psi}_t \frac{\hat{r}^s}{R}$$

i.e., as $\xi = \frac{1}{M} - 1$

$$ME_t [c_{t+1}] - c_t = \left( -M \xi + M (k^r - k^r) z_t \right) \frac{r}{R} w_t + M c^{d \psi}_t \frac{\hat{r}^s}{R}$$

$$= \left( M - 1 + M (r_t - \hat{r}_t^s) \right) \frac{r}{R} w_t + M c^{d \psi}_t \frac{\hat{r}^s}{R}$$

$\square$

**Proof of Proposition 12** Taxes lower the present value of his income by $He^{-r(T-t)}$, so the consumer’s response is:

$$\hat{c}_t = r \hat{w}_t - r \left( He^{-r(T-t)} \right)$$

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so wealth accumulation is: \( \frac{d}{dt} \hat{b}_t = r \hat{w}_t - \hat{c}_t = \tau \left( He^{-(T-t)}, \kappa \right) \). The consumer starts thinking about it at a time \( s \) s.t. \( He^{-(T-s)} = \kappa \) (assuming that the solution is in \((0, T)\)), i.e.

\[
s = \max \left( 0, \min \left( T, \frac{1}{r} \ln \frac{\kappa}{He^{-(T-t)}} \right) \right)
\]

(99)

First, consider the case: \( s < T \).

Then, for \( t \in [s, T) \),

\[
\frac{d}{dt} \hat{w}_t = \tau \left( He^{-(T-t)}, \kappa \right) - \kappa
\]

\[
\hat{w}_t = \int_s^t \left( He^{-(T-t')} - \kappa \right) dt' = \frac{H}{r} e^{-rT} \left( e^{rt} - e^{rs} \right) - \kappa (t-s)
\]

\[
\hat{c}_t = r \hat{w}_t - \tau \left( He^{-(T-t)}, \kappa \right)
\]

\[
= r \left( \frac{H}{r} e^{-rT} \left( e^{rt} - e^{rs} \right) - \kappa (t-s) \right) - \left( He^{-(T-t)} - \kappa \right)
\]

\[
\hat{c}_t = -He^{-(T-s)} + \kappa (1-r(t-s))
\]

(100)

So at \( t = T \)

\[
\hat{w}_T = \frac{H}{r} \left( 1 - e^{-r(T-s)} \right) - \kappa (T-t)
\]

At \( T \), the tax \( H \) is enacted, so that for \( t \geq T \), the agent is aware of it. This yields:

\[
\hat{c}_t = r \hat{w}_t - H
\]

\[
\frac{d}{dt} \hat{w}_t = r \hat{w}_t - H - \hat{c}_t = \text{investment income} - \text{taxes} - \text{consumption change}
\]

\[
= 0
\]

hence for \( t > T \), \( \hat{w}_t = \hat{w}_T \), and \( \hat{c}_t = r \hat{w}_T - H \).

We conclude that consumption is:

\[
\hat{c}_t = \begin{cases} 
0 & \text{for } t < s \\
-He^{-(T-s)} + \kappa (1-r(t-s)) & \text{for } s \leq t < T \\
r \hat{w}_T - H & \text{for } t \geq T 
\end{cases}
\]

and wealth is

\[
\hat{w}_t = \begin{cases} 
0 & \text{for } t < s \\
\frac{H}{r} e^{-rT} \left( e^{rt} - e^{rs} \right) - \kappa (t-s) & \text{for } s \leq t \leq T \\
\frac{H}{r} \left( 1 - e^{-r(T-s)} \right) - \kappa (T-s) = \hat{w}_T & \text{for } t \geq T 
\end{cases}
\]
**Proof of Proposition 13**  The BR agent has value function, which satisfies

\[ V^t(w_t) = u(c_t) + V^{t+1}(w_t - c_t + y_t) \]

Note that the rational value function is, for \( t \)

\[ V^{t,r} = \frac{1}{T-t} u \left( \frac{w + \sum_{s=t}^{T-1} y_s}{T-t} \right) \]

Proposition 2 shows that for small \( \hat{y} \), this is also the Taylor expansion (up to \( O(\hat{y}^2) \) terms) of the value function under a BR policy. Hence, we will study an agent who uses the simplified value function

\[ V^{t,S}(w, m^V)_{m^V=0} = (T-t) u \left( \frac{w + \sum_{s=t}^{T-1} y_s}{T-t} \right) \]

Given this, consumption before retirement \( (t < L) \) is given by:

\[ \max_{c_t; m} u(c_t) + V^{t+1}(w_t - c_t, m^V)_{m^V=0} \]

i.e.

\[ \max_{c_t; m} u(c_t) + (T-t-1) u \left( \frac{w - c_t + \sum_{s=t+1}^{T-1} y_s}{T-t-1} \right) \]

The FOC is

\[ u'(c_t) = u' \left( \frac{w - c_t + \sum_{s=t+1}^{T-1} y_s}{T-t-1} \right) \]

so \( c_t = \frac{w - c_t + \sum_{s=t+1}^{T-1} y_s}{T-t-1} \), i.e.

\[ c_t = \frac{w + \sum_{s=t+1}^{T-1} y_s}{T-t} \]

so the agent consumes the perceived permanent income.

The choice of attention comes from (assuming that \( m^V = 0 \), e.g. because \( \kappa^V \) is large enough)

\[ \max_{c_t; m} v^t(c_t, m) := u(c_t) + (T-t-1) u \left( \frac{w_t - c_t + \sum_{s=t+1}^{T-1} (y + m\hat{y})}{T-t-1} \right) \]

so we obtain

\[ c_t = \frac{w + m_t B \hat{y} + y}{T-t} \]

We have \( v^t_{cc}(c_t, m)_{m=0} = (1 + \frac{1}{T-t-1}) u''(c) \), so \( v^t_{cc}(c_t, m)_{m=0} = u''(c) \) in the continuous time limit, so

\[ c_t = \frac{w_t}{T-t} + \tau \left( \frac{B \hat{y}}{T-t}, \kappa \right) + y. \]
Proof of Proposition 10  When $\phi > 0$, we saw that

$$
\phi = \left( \frac{\xi}{r + \phi} - \kappa \left( r + \frac{\xi}{r + \phi} \right)^2 \frac{r + \phi}{\xi} \right)
$$

Let $\psi := \frac{r + \phi}{\xi} \neq 0$. Then

$$
\phi = \psi^{-1} - \kappa(r + \psi^{-1})^2 \psi,
$$

which is equivalent to

$$
\psi(\xi \psi - r) = \psi \phi = 1 - \kappa[(r + \psi^{-1})^2]
= 1 - \kappa(r \psi + 1)^2
= 1 - \kappa(r^2 \psi^2 + 2r \psi + 1).
$$

Rearranging yields

$$
(\xi + \kappa r^2)\psi^2 + (2\kappa - 1)r \psi + (\kappa - 1) = 0.
$$

The quadratic formula then gives

$$
\psi = \frac{(1 - 2\kappa)r \pm \sqrt{\Delta}}{2(\xi + \kappa r^2)},
$$

where

$$
\Delta = [(2\kappa - 1)r]^2 - 4(\xi + \kappa r^2)(\kappa - 1)
= r^2 [(2\kappa - 1)^2 - 4\kappa(\kappa - 1)] + 4\xi(1 - \kappa)
= r^2 [(4\kappa^2 - 4\kappa + 1) - (4\kappa^2 - 4\kappa)] + 4\xi(1 - \kappa)
= r^2 + 4\xi(1 - \kappa).
$$

In the case $\kappa = 0$, the correct root is the higher one for $\psi$ (i.e., it’s the higher root of $\phi = \frac{\xi}{r + \phi}$, the one with the $+\sqrt{\Delta}$ sign). Hence, $\psi = \frac{(1 - 2\kappa)r + \sqrt{\Delta}}{2(\xi + \kappa r^2)}$

Finally,

$$
\phi = \xi \psi - r
= \frac{\xi \left[ (1 - 2\kappa)r + \sqrt{\Delta} \right] - 2(\xi + \kappa r^2)r}{2(\xi + \kappa r^2)}
= \frac{\left[ \xi(1 - 2\kappa) - 2(\xi + \kappa r^2) \right] r + \xi \sqrt{\Delta}}{2(\xi + \kappa r^2)}
= \frac{-[2\kappa r^2 + 2\xi \kappa + \xi] r + \xi \sqrt{\Delta}}{2(\xi + \kappa r^2)}
= \frac{-[2\kappa r^2 + 2\xi \kappa + \xi] r + \xi \sqrt{r^2 + 4\xi(1 - \kappa)}}{2(\xi + \kappa r^2)}
$$
Proof of Proposition 6. We use the content\(^{12}\) and notations of Proposition 18. We set \(x_t = y_{t}\). We have \(F_w(w, x, c) = rw + x_t - c_t\) and \(F_x(w, x) = -\phi x\).

Under the consumption frame, \(a_t = c_t\), and \(F_{ww} = 0\), so by Proposition 18, noting \([V_{xx}^\delta]C\) the value of \(V_{xx}^\delta(w, 0)\) under the consumption frame:

\[
[V_{xx}^\delta]C = \frac{u''(c)}{r + 2\phi_y} (c^e_y - c^r_y)^2
\]

(101)

and as \(c^e_y = mc^r_y\) with \(c^r_y = \frac{r}{r + \phi}\),

\[
[V_{xx}^\delta]C = \frac{u''(c)}{r + 2\phi_y} (1 - m)^2 \left( \frac{r}{r + \phi} \right)^2
\]

and the expected losses are (with \(\sigma^2_y = E[\tilde{y}^2]\)):

\[
L^C = -\frac{1}{2} [V_{xx}^\delta]C \sigma^2_y = -\frac{1}{2} \frac{u''(c)\sigma^2_y}{r + 2\phi_y} (1 - m)^2 \left( \frac{r}{r + \phi} \right)^2
\]

\[
= A (1 - m)^2 \phi^2
\]

Under the savings frame, \(a_t\) is savings, so \(F_w = a_t\), and \(c_t = rw_t + x_t - a_t\). Hence:

\[
[V_{xx}^\delta]S = \frac{u''(c)}{r + 2\phi} (S^e_y - S^r_y)^2
\]

and as \(S^e_y = mS^r_y\), with \(S^r_y = 1 - c^r_y = \frac{\phi}{r + \phi}\),

\[
[V_{xx}^\delta]S = \frac{u''(c)}{r + 2\phi} (1 - m)^2 \left( \frac{\phi}{r + \phi} \right)^2
\]

and expected losses are:

\[
L^S = -\frac{1}{2} [V_{xx}^\delta]S \sigma^2_y = A (1 - m)^2 \phi^2
\]

The consumption frame yields greater utility than the savings frame if and only if \(\phi_y > r\).

Losses from a general variable \(x\). Using the same reasoning, the losses from not paying attention to a variable \(x\) is:

\[
L^x = -\frac{u''(c)}{r + 2\phi} \sigma_x^2 \left( c^e_x - c^{rat}_x \right)^2 = -\frac{u''(c)}{r + 2\phi} \sigma_x^2 c^2_x (1 - m_x)^2
\]

We parametrize the losses by the “equivalent permanent tax” \(\lambda^x\) such that \(L^x = E \int_0^\infty e^{-\mu t} \left[ u(c_t) - u(c_t) \right] dt\). Hence, using a Taylor expansions, \(\lambda^x = \frac{L}{u''(c)/r}\). This gives:

\[
\lambda^x = \frac{1}{2} \frac{u''(c)}{r + 2\phi} \sigma_x^2 c^2_x (1 - m_x)^2
\]

\(^{12}\)We could also draw on the results in Cochrane (1989), with a variety of adjustments. Proposition 18 extend Cochrane’s results (derived for consumption) to general dynamic problems.
The losses from paying only attention \( m_x \) to variable \( x \), expressed in terms of an “equivalent proportional losses in consumption”, \( \lambda^x \) are:

\[
\lambda^x = \frac{1}{2} \left( \frac{r}{\phi} \right) \left[ \frac{c_x \sigma_x}{c} (1 - m_x) \right]^2
\]

Proposition 19  The losses from paying only attention \( m_x \) to variable \( x \), expressed in terms of an “equivalent proportional losses in consumption”, \( \lambda^x \) are:

\[
\lambda^x = \frac{1}{2} \left( \frac{r}{\phi} \right) \left[ \frac{c_x \sigma_x}{c} (1 - m_x) \right]^2
\]

where \( \sigma_x \) is the standard deviation of \( x \), and \( c_x = \frac{\partial c}{\partial x} \).

The calibration gives:

\[
\lambda^x = (1 - m_x)^2 \times 0.03\%, \quad \lambda^y = (1 - m_y)^2 \times 3.0\%
\]

It may be useful to see the effect in a simpler context. Take a 3 period model with \( \beta = R = 1 \), and an income shock with persistence \( \rho \): \( \hat{y}_t = \rho^{t-1} \varepsilon \) for \( t = 0, 1, 2 \), with \( \varepsilon \) a mean-0 shock. Normatively, that should induce the change \( \tilde{c} = (\tilde{c}_t)_{t=0,1,2} = (1, 1, 1) \left( 1 + \rho + \rho^2 \right) \varepsilon \) (indeed, the total value of income has increased by \( (1 + \rho + \rho^2) \varepsilon \)). Let us now consider a BR agent with \( m = 0 \). However, under the consumption frame, \( \tilde{c}^C = (0, 1, \frac{1}{2}, \frac{1}{2} + \rho + \rho^2) \varepsilon \) (as there is no reaction of \( c_0 \), so that time-1 wealth increases by \( \tilde{w}_1 = \varepsilon \), of which half is consumed at time 1, so \( \tilde{c}_1^C = \frac{\varepsilon}{2} \)). Under the savings frame, we get \( \tilde{c}^S = (1, \rho, \rho^2) \varepsilon \) (savings doesn’t change, consumption absorbs all the shocks). It is easy to verify that for \( \rho \) small, the utility is higher under the consumption frame, while the opposite for large \( \rho \). Indeed, when \( \rho = 0 \), \( \tilde{c}^C = (0, 1, \frac{1}{2}, \frac{1}{2} + \rho + \rho^2) \varepsilon \) and \( \tilde{c}^S = (1, 0, 0) \varepsilon \), so there is more smoothing under the consumption frame. Other the other hand, with \( \rho = 1 \), \( \tilde{c}^C = (0, 1, \frac{1}{2}, \frac{1}{2}) \varepsilon \) and \( \tilde{c}^S = (1, 1, 1) \varepsilon \), and there is more smoothing under the savings frame.

Proof of Lemma 4  Here we present a proof sketch, in part because those notions are well-known. We record the values with a time-discounting of \( \phi \). However, under the consumption frame, \( \tilde{c}^C = (0, 1, \frac{1}{2}, \frac{1}{2} + \rho + \rho^2) \varepsilon \) (savings doesn’t change, consumption absorbs all the shocks). It is easy to verify that for \( \rho \) small, the utility is higher under the consumption frame, while the opposite for large \( \rho \). Indeed, when \( \rho = 0 \), \( \tilde{c}^C = (0, 1, \frac{1}{2}, \frac{1}{2} + \rho + \rho^2) \varepsilon \) and \( \tilde{c}^S = (1, 0, 0) \varepsilon \), so there is more smoothing under the consumption frame. Other the other hand, with \( \rho = 1 \), \( \tilde{c}^C = (0, 1, \frac{1}{2}, \frac{1}{2}) \varepsilon \) and \( \tilde{c}^S = (1, 1, 1) \varepsilon \), and there is more smoothing under the savings frame.

\[
\mu_0^{-1} = E \left[ \int_0^\infty D_t^\psi M_t^{1-\psi} dt \right]
\]

To the leading order, \( \tilde{u} = \frac{1}{2} u''(c^d) E \sum \tilde{c}_t^2 \), so \( \tilde{u}^C = \frac{1}{2} u''(c^d) \sigma^2_t \left( \frac{1}{4} + (\frac{1}{2} + \rho + \rho^2)^2 \right) \) and \( \tilde{u}^S = \frac{1}{2} u''(c^d) \sigma^2_t (1 + \rho^2 + \rho^4) \). This yields \( \tilde{u}^C \geq \tilde{u}^S \) iff \( \rho < \rho^* \simeq 0.32 \).
When $M_t$ follows (68), routine calculations show that:

$$
\mu_0^{-1} = E \left[ \int_0^\infty D_t^\psi e^{-(1-\psi) \int_0^t R_u \, du} \, dt \right]
$$

We next proceed to a Taylor expansion:

$$
\mu_0^{-1} = E \left[ \int_0^\infty D_t^\psi e^{-(1-\psi) \int_0^t (R_t + \hat{R}_u) \, du} \, dt \right] = E \left[ \int_0^\infty D_t^\psi e^{-(1-\psi) R_t} \left( 1 - \frac{1}{(1-\psi) t} \right) \int_0^t \hat{R}_u \, du \right] \, dt
$$

With an infinite horizon, $D_t = e^{-\mu t}$ and

$$
\mu_0^{-1} = E \left[ \int_0^\infty e^{-\mu t} \left( 1 - (1-\psi) \int_0^t \hat{R}_u \, du \right) \, dt \right] = \frac{1}{\mu*} - (1-\psi) E \left[ \int_0^\infty e^{-\mu u} \int_0^t \hat{R}_u \, du \right] = \frac{1}{\mu*} - (1-\psi) \frac{1}{\mu*} E \left[ \int_0^t \mu* e^{-\mu u} \hat{R}_u \, du \right] = \frac{1}{\mu*} - (1-\psi) \frac{1}{\mu*} (\hat{R}_0 - \hat{R}_*) \text{ with } \hat{R}_0 - \hat{R}_* = E \left[ \int_0^t \mu* e^{-\mu u} \hat{R}_u \, du \right]
$$

so

$$
\mu_0 = \mu* + (1-\psi) (\hat{R}_0 - \hat{R}_*) = \psi \rho + (1-\psi) \hat{R}_* + (1-\psi) (\hat{R}_0 - \hat{R}_*) = \psi \rho + (1-\psi) \hat{R}_0
$$

When the consumer has a finite horizon and only cares about date $T$ consumption, then $D_t = \delta (t - T)$, and

$$
\mu_0^{-1} = E \left[ \int_0^\infty D_t^\psi e^{-(1-\psi) R_t} \left( 1 - (1-\psi) \int_0^t \hat{R}_u \, du \right) \, dt \right] = e^{-\mu* T} - e^{-\mu* T} (1-\psi) E \left[ \int_0^T \hat{R}_u \, du \right]
$$

so the MPC is 0 but we have

$$
\mu_t^{-1} = e^{-\mu* (T-t)} \left( 1 - (1-\psi) E \left[ \int_t^T \hat{R}_u \, du \right] \right)
$$

so again $\mu_t$ is related to the present value of future portfolio returns.

\[ \square \]
Proof of Lemma 5  In semi-discrete notation the asset demand at time $t$ comes from:

$$\max_\theta E_t[V(w(1 + r_t dt + \theta d\bar{r}_t), X_t + dX_t)]$$

where, with $\pi_t = \pi_* + f'X_t$,

$$E_t[dV_t] = E_t[V(w(1 + r_t dt + \theta d\bar{r}_t), X_t + dX_t) - V(w, X_t)]$$

$$= V_w(w(r_t + \theta'\pi_t) dt + V_{w,X}w(\theta'd\bar{r}_t, dX_t) + V_{w,w}w^2\theta'\Sigma_t\theta dt + \frac{1}{2}Tr(V_{XX}\Sigma^{X,X}dt$$

$$= V_ww\left[\theta'(\pi_t + H_t) - \frac{\gamma}{2}\theta'\Sigma_t\theta dt\right] + \frac{1}{2}Tr(V_{XX}\Sigma^{X,X})dt$$

where

$$\theta' H_t = \frac{V_{w,X}}{V_w}\langle \theta d\bar{r}_t, dX_t\rangle$$

is the hedging demand premium term. This implies

$$\theta = \frac{1}{\gamma}\Sigma_t^{-1}(\pi_t + H_t)$$

To calculate $H_t$ more fully, recall that $V_w = \mu(X_t)^{-\gamma}w^{-\gamma}$, so that $\ln V_w = -\gamma \mu(X_t) - \gamma \ln w$, and

$$\frac{V_{w,X}}{V_w} = -\gamma\frac{\mu_X}{\mu_*} = -\gamma (1 - \psi) \frac{\bar{R}_X}{\mu_*} = (1 - \gamma) \frac{\bar{R}_X}{\mu_*} = b'$$

with

$$\frac{\bar{R}_X}{\mu_*} = b' (\mu_* I + \Phi_R)^{-1}$$

Note that $R_t = r + \frac{1}{2}\pi_t\Sigma_t^{-1}\pi_t$ with $\pi_t = \pi_* + \hat{\pi}_t$, so, with $R_* = r + \frac{1}{2}\pi'_*\Sigma_\pi^{-1}\pi_*$

$$R_t = R_* + \frac{1}{2}\pi'_*\Sigma^{-1}\hat{\pi}_t = R_* + \theta'_*\hat{\pi}_t$$

$$R_t = R_* + \theta'_*f'X_t$$

hence

$$b = f\theta$$

Hence,

$$\theta' H_t = \frac{V_{w,X}}{V_w}\langle \theta d\bar{r}_t, dX_t\rangle = \sum_{i,j} \theta_i \langle d\bar{r}_{it}, B_{ij}dX_j \rangle = \theta_i \sum_{ij}^{r} B_{ij}$$

so that

$$H_t := \Sigma^{r,X}B = (1 - \gamma) \Sigma^{r,X} \frac{\bar{R}_X}{\mu_*} = (1 - \gamma) \text{cov} \left( d\bar{r}, d\bar{R}_t \right)$$

$$= (1 - \gamma) \Sigma^{r,X} (\mu_* I + \Phi_R')^{-1} b$$

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Proof of Lemma 6 Suppose
\[ d\tilde{r} = (r + \pi_\star + fX_t) dt + \sigma dZ_t \]
and that that agents have a constant MPC \( \mu_\star \):
\[
\begin{align*}
\frac{dw_t}{w_t} &= (r - \mu_\star) dt + \theta' d\tilde{r} = (r - \mu_\star + \theta\pi_\star) dt + \theta\sigma dZ_t \\
\frac{dw_t}{w_t} &= (g_\star + \theta' f'X_t) dt + \theta\sigma dZ_t \\
\end{align*}
\]
with \( b = f\theta \) and \( g_\star := r + \theta\pi_\star - \mu_\star \)

We want to calculate (assuming the policy \( c_t = \mu_\star w_t \), which leads only to second order losses)
\[
U = \mathbb{E} \left[ \frac{1}{1 - \gamma} \int_0^\infty e^{-\rho s} c_1^{1-\gamma} ds \right] = \frac{\mu_\star^{1-\gamma}}{1 - \gamma} \mathbb{E} \left[ \int_0^\infty e^{-\rho s} w_s^{1-\gamma} ds \right]
\]

Call \( m_s = e^{-\rho s} w_s^{1-\gamma} \)

We calculate:
\[
\begin{align*}
\frac{dm_t}{m_t} &= -\rho + (1 - \gamma) \left( g_\star + b'X_t - \frac{\gamma}{2} \|\theta\sigma\|^2 \right) + (1 - \gamma) \theta' \sigma dZ_t \\
&= (a + 1 - \gamma) b'X_t dt + (1 - \gamma) \theta' \sigma dZ_t \\
\end{align*}
\]

\[
\begin{align*}
a &= \rho - (1 - \gamma) \left( g_\star - \frac{\gamma}{2} \|\theta\sigma\|^2 \right) \\
&= \rho - (1 - \gamma) \left( r + \theta\pi_\star - \mu_\star - \frac{\gamma}{2} \|\theta\sigma\|^2 \right) = \rho - (1 - \gamma) (R_\star - \mu_\star) \\
\end{align*}
\]

We calculate LG moments. We assume \( dX_t = -\Phi X_t dt + \sigma X dZ_t + O (\|X_t\|^2) \):
\[
\begin{align*}
E \left[ \frac{dm_t}{m_t} \right] / dt &= -\mu_\star + (1 - \gamma) b'X_t \\
E \left[ d\left( \frac{m_t}{m_t} X_t \right) \right] / dt &= (\mu_\star + (1 - \gamma) b'X_t) X_t - \Phi X_t + (1 - \gamma) \langle \theta' d\tilde{r}_t, dX_t \rangle \\
&= (1 - \gamma) \theta' \langle d\tilde{r}_t, dX_t \rangle + (-\mu_\star - \Phi) X_t + O (\|X_t\|^2) \\
\end{align*}
\]
so the LG generator (Gabaix, 2009) is
\[
\omega = \begin{pmatrix}
\mu_* \\
(1 - \gamma) \Sigma^{X,r} \theta \\
\mu_* + \Phi
\end{pmatrix}
\]

Hence, the present value is \( V = (1, 0) \omega^{-1} \cdot (1, X_t) \)

We use the formula for the inversion of the block matrix:
\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1} BD^{-1} \\ * & * \end{pmatrix}
\]

where * are terms we will not use. We have:
\[
(1, 0) \omega^{-1} = (f, f (1 - \gamma) b' (\mu_* + \Phi)^{-1})
\]
\[
f := (\mu_* - (1 - \gamma)^2 b' (\mu_* + \Phi)^{-1} \Sigma^{X,r} \theta)^{-1}
\]

so
\[
V = f (1 + (1 - \gamma) b' (\mu_* + \Phi)^{-1} X_t)
\] (109)

The value function has the Taylor expansion: \( V(w_t, X_t) = v(X_t) \mu_*^{1-\gamma} w_t^{1-\gamma} \)

\[
v(X_t) = \frac{1 + (1 - \gamma) b' (\mu_* + \Phi)^{-1} X_t}{\mu_*}
\]
\[
\mu_{**} = \mu_* - (1 - \gamma)^2 b' (\mu_* I + \Phi)^{-1} \Sigma^{X,r} \theta
\]
\[
= \mu_* - \frac{(1 - \gamma)^2}{\left(1 - \frac{1}{\psi}\right)} \psi H_t' \theta \text{ using (77), } H_t = \left(1 - \frac{1}{\psi}\right) \Sigma^{r,X} (\mu_* I + \Phi')^{-1} b
\]
\[
= \mu_* - (1 - \gamma) H_t' \theta
\]

Rewrite
\[
V = v(X_t) \mu_*^{1-\gamma} = \frac{1 + K}{\mu_* + L} \mu_*^{1-\gamma} \text{ with } \\
K = (1 - \gamma) b' (\mu_* + \Phi)^{-1} X_t \\
L = -(1 - \gamma) H_t' \theta \\
V = (\mu_* + C)^{-\gamma} = \mu_*^{-\gamma} (1 - \gamma \mu_*^{-1} C) \\
= \mu_*^{-\gamma} (1 + K - \mu_*^{-1} L)
\]

hence,
\[
\mu_t - \mu_* = C = -\frac{\mu_*}{\gamma} K + \frac{1}{\gamma} L = (1 - \psi) b' \mu_* (\mu_* + \Phi)^{-1} X_t + \frac{1}{\gamma} (1 - \gamma) H_t' \theta
\]
\[
= (1 - \psi) b' \mu_* (\mu_* + \Phi)^{-1} X_t + (1 - \psi) H_t' \theta
\]
\[ \mu^I := \frac{1}{\gamma} L = \frac{1}{\gamma} (1 - \gamma) H'_t \theta = \left( 1 - \frac{1}{\psi} \right) H'_t \theta \]

Intuition: the extra present value of returns is
\[
\frac{\bar{R}_t - R_s}{\mu_s} = \frac{C}{(1 - \psi) \mu_s} = b' (\mu_s + \Phi)^{-1} X_t - \psi (1 - \psi) \frac{1}{\mu_s} b' (\mu_s I + \Phi)^{-1} \theta \Sigma_r X
\]
\[
= b' (\mu_s + \Phi)^{-1} \left( X_t + (1 - \gamma) \frac{1}{\mu_s} \theta \Sigma_r X \right)
\]

**Proof of Proposition 16** We’re looking for a solution of the form: \( V(x) = a + bx \), for \( a, b \) to be determined. The FOC is: \( u_c + \beta V_x h^s = 0 \), i.e. \( -(c - x - A) + \beta bh^s = 0 \) and
\[
c = x + A + \beta b^s h^s
\]
\[
u (c(x), x) = -\frac{1}{2} (\beta b^s h^s)^2 - Bx
\]

The self-consistency condition is:
\[
V(x) = u(c(x), x) + \beta V(\rho x + hc)
\]
i.e.
\[
a + bx = -\frac{1}{2} (\beta b^s h^s)^2 - Bx + \beta \left[ a + b (\rho x + h (x + A + \beta b^s h^s)) \right]
\]
This gives:
\[
b = \frac{-B}{1 - \beta \rho}
\]
\[
a = \beta \frac{hA + \beta b^s hh^s - \frac{1}{2} \beta (b^s h^s)^2}{1 - \beta}
\]

When the agent perceives \( \rho' = 1 - m^V + m^V \rho \) and \( h' = m^V h \) when forming the value function, we have the same expressions,
\[
b^s(m) = \frac{-B}{1 - \beta (\rho' + h')} = \frac{-B}{1 - \beta (1 + m^V (\rho + h - 1))}
\]
\[
a(m) = \beta \frac{hA + \beta b^s (m) hh^s - \frac{1}{2} \beta (b^s (m) h^s)^2}{1 - \beta}
\]

To determine optimal attention \( m \), observe that in the 1-step smax, at the beginning, \( m^V = 0 \), so the perceived value function is
\[
V(x, m^V = 0) = u(c(x), x) + \beta V(x, m^V = 0)
\]
so
\[
V(x, m^V = 0) = \frac{u(c(x), x)}{1 - \beta} = -\frac{1}{2} (\beta b^s h^s)^2 - Bx
\]
This implies: 
\[ b^* = -\frac{B}{1-\beta}, \]
and
\[
\begin{align*}
c &= x + A + \beta b^* \left( m^V = 0 \right) h m \\
&= x + A - \frac{\beta B}{1-\beta} h m
\end{align*}
\]
so that the impact of thinking more about \( h \), while keeping the future value function constant is:
\[
\frac{\partial c}{\partial m} = \beta b^* \left( m^V = 0 \right) h = -\frac{\beta Bh}{1-\beta}
\]
Hence, optimal attention is:
\[
m = A \left( \frac{1}{\kappa} \left( \frac{\partial c}{\partial m} \right)^2 u_{cc} \right) = A \left( \frac{1}{\kappa} \left( \frac{\beta Bh}{1-\beta} \right)^2 \right)
\]
References


