Boundedly Rational Dynamic Programming: 
A Sparse Approach

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Abstract

This paper proposes a way to model boundedly rational dynamic programming in a parsimonious and tractable way. The framework is quite general, and has at its core a behavioral version of the Bellman equation, in which the agent uses a simplified model of the world and the consequences of his actions. It is then applies to some of the canonical models in macroeconomics and finance. In the consumption-savings model, the consumer can pay attention to the variables such as the interest rate and his income, or replace them, in his mental model, by their average values – this way using a “sparse” model of the world. Endogenously, the consumer pays little attention to the interest rate but pays more attention to his income. This helps resolve some extant puzzles in consumption behavior, especially the tenuous link between interest rates and consumption growth. Ricardian equivalence partially fails, because the consumer is only partially attentive to future taxes. In a Merton-style portfolio choice problem, the agent endogenously pay limited or no attention to the varying equity premium and hedging demand terms. Finally, the paper gives a behavioral version of the canonical neoclassical growth model. Fluctuations are more persistent with when agents are more boundedly rational.

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1 Introduction

This paper proposes a way to model dynamic programming with boundedly rational agents. It lays out a fairly general procedure to formulate dynamic programming in situations of economic interest. Then, it shows how the framework applies to some of the canonical models in macro-finance and economics more generally: consumption saving problems, the baseline neoclassical growth model (the Ramsey-Cass-Koopmans model), investment in risky assets (Merton’s problem), general linear-quadratic problems, and models of rational addition (à la Becker-Murphy). The upshot is that we have a portable, fairly general structure that applies to the some basic machines of macroeconomics, and allows to see where bounded rationality (BR) is important in those situations.

One of the criticisms of traditional economic models is the potential unrealism of the infinitely forward-looking agent who computes the whole equilibrium in her own head. This lack of realism has long been suspected to be the cause of some empirical misfits that we will review below. Behavioral economics aims to provide an alternative. However, the greatest successes of behavioral economics change the agents’ tastes (e.g. prospect theory or hyperbolic discounting) or their beliefs (e.g. overconfidence), but typically keep the assumption of rationality. When tackling the rationality assumption, there is much less agreement and the modeling of bounded rationality is much more piecemeal, different from one paper to the next.

This paper proposes a compromise that keeps much of the generality of the rational approach and injects some of the wisdom of the behavioral approach, mostly inattention and simplification. It does so by proposing a way to insert some bounded rationality into a large class of problems, the “recursive” contexts, i.e. with dynamic programming in some stochastic steady state.

To illustrate these ideas, let us consider a canonical consumption-savings problem. The agent maximizes utility from consumption, subject to a budget constraint, with a stochastic interest rate and stochastic income. In the rational model, the agent solves a complex DP problem with three state variables (wealth, income and the interest rate). This is a complex problem that requires a computer to solve it.

How will a boundedly rational agent behave? I assume that the agent starts with a much simpler model, where the interest rate and income are constant – this is the agent’s “default” model. Only one state variable remains, his wealth. He knows what to do then, but what will he do in a more complex environment, with stochastic interest rate and stochastic income? In the sparse version, he considers parsimonious enrichments to the value function, as in a Taylor expansion. He asks, for each component, whether it will matter enough for his decision. If a given feature (say, the interest rate) is small enough compared to some threshold (taken to be a fraction of standard deviation of consumption), then he drops the feature, or partially attenuates it. The result is a consumption policy that pays partial attention to income, and possibly no attention at all to the interest rate. This does seem realistic.

The result is a sparse version of the traditional permanent-income model. We see that it
is often *simpler* than the traditional model. Indeed, the agent typically ends up using a rule which is simpler (e.g., not paying attention to the interest rate).

One application is a Merton-style dynamic portfolio choice problem, i.e. allocating one’s wealth between stocks and bonds when the expected returns are stochastic and correlated with past returns. This is a notoriously complicated problem for a rational agent. I study how a sparse agent would handle it. The sparse agent first anchors his action by imagining he’s facing a simpler problem – a world with a constant equity premium. Then, he can sparsely enrich his model to take into account the more complex features (the stochasticity of the equity premium, its correlation with past returns, which creates a hedging demand). Hence the agent will take these complex features into account only partially, or not at all. This may be a more satisfying description than the hyper-rational model of how people behave in a complex environment. At the very least, it is important to have a concrete alternative to that hyper-rational model.

Let us now turn to macro consequences of this approach. To do so, I present a behavioral version of a large class of model, and work out in detail a BR version of most canonical of them, the neoclassical growth model of Ramsey-Cass-Koopmans. In this version, agents pay of lot of attention to their own variables, less to aggregate variables. One upshot is that with BR, macroeconomic fluctuations are larger and more persistent. I illustrate this proposition, and qualify it, as it appears to hold for most reasonable values of the parameters, but can be overturned for extreme values.

To see the idea, which is fundamentally quite simple, imagine first an economy with only one state variable, capital. It starts with a steady state amount of capital. Then, there is a positive shock to the endowment of capital. In a rational economy, agents would consume a certain fraction of it, say 6%, every period. That will lead the capital stock to revert quickly to its mean. However, in a an economy with sparse agents, investors will not pay full attention to the additional capital. They will consume less of it than a rational agent would. Hence, capital will be depleted more slowly and will mean-revert more slowly. The shock has more persistent effects.

Given that shocks are more persistent, past shocks accumulate more. Mechanically, this leads to larger average deviations of capital from its trend. As a consequence, the interest rate and GDP also have larger, and more persistent, deviations from trend.

The model allows us to express those ideas in simple, quantitative ways. It allows us to explore them in richer environments, e.g. with shocks to both productivity and the capital stock.


One key difference here is that the model is here much more systematic. It explicitly applies, in a unified manner, to a wide class of models. It relies on an earlier paper
(Gabaix 2014) that proposes a “sparse max”, a behavioral, less than fully rational and attentive of the traditional max operator. That paper was concerned with static cases, here we explore dynamic ones. Similarly, that paper allows to give a behavioral formulation of some basic chapters of the microeconomic textbooks (consumer theory, equilibrium theory, Arrow-Debreu). The present paper allows to give a behavioral version of some chapters of the macroeconomics textbooks (mostly, consumption-savings problem and the basic neoclassical growth model). Hence, we have a more unified view of bounded rationality in micro and macro, whether those other papers are more piecemeal.

The other approaches have not (yet) yielded a systematic approach of dynamic programming, or of those basic building blocks. One partial exception is Maćkowiak and Wiederholt (forthcoming). They work out a New Keynesian model with an entropy-based penalty for precision à la Sims (2003), and show that it is quantitatively successful. The present paper is more analytical, develops tools that apply to those of other situations. It is also not quantitative, but more systematic in terms of theory.

The rest of the paper is as follows. Section 2 presents the general procedure. Then, we apply to a variety of canonical examples. Section 3 presents basic partial-equilibrium building blocks: the basic consumption-savings problem, including variants such as failure of Ricardian equivalents and the Merton portfolio problem. Section 4 works out the neoclassical growth model, e.g. a general equilibrium situation. Section 8 concludes. The Appendix contains the more technical material and derivations.

2 General Framework

2.1 A review of the static sparse max operator

In Gabaix (2014), I defined a sparse max or smax operator, which is a behavioral, partially inattentive version of the max operator. The agent faces a maximization problem which is, in its rational version, \( \max_a u(a, x) \). The \( x_i \) are viewed by the agent as being drawn with a standard deviation \( \sigma_i \), and covariance \( \sigma_{ij} \). There is a nonnegative parameter \( \kappa \), which is a taste for sparsity. When \( \kappa = 0 \), the agent is the traditional agent.

**Definition 1** (Sparse max operator, Gabaix 2014) The sparse max, \( \text{smax}_a u(a, x) \), is defined by the following procedure.

1. **Step 1**: Choose the attention vector \( m^* \):

   \[
   m^* = \arg \min_{m \in [0,1]^n} \frac{1}{2} \sum_{i,j=1 \ldots n} (1 - m_i) \Lambda_{ij} (1 - m_j) + \kappa \sum_{i=1 \ldots n} g(m_i) \tag{1}
   \]

   with the cost-of-inattention factors \( \Lambda_{ij} := -\sigma_{ij} a_x u_{aa} a_x \). Define \( x_i^s = m_i^* x_i \), the sparse representation of \( x \).

2. **Step 2**: Choose the action

   \[
   a^* = \arg \max_a u(a, x^s) \tag{2}
   \]
and set the resulting utility to be \( u^* = u(a^*, x) \). In the expressions above, derivatives are evaluated at \( x = 0 \) and \( a^d = \arg\max_a u(a, 0) \).

In other terms, the agent solves for the optimal \( m^* \) that trades off a proxy for the utility losses (the first term in the right-hand side of equation (1)) and a psychological penalty for deviations from a sparse model (the second term on the left-hand side of equation (1)). Then, the agent maximizes over the action \( a \), as if \( m^* \) were the true model.

This leads to define the attention function:

\[
A_g(v) := \sup \left[ \arg\min_{m \in [0, 1]} \frac{1}{2} (m - 1)^2 |v| + g(m) \right].
\]

This represents the optimal attention to a variable with variance \(|v|\) an impact of 1 on the decision, with the cost of thinking \( \kappa \) is 1.

The following proposition derives the main case.

**Proposition 1** (Gabaix 2014) When variables are perceived to be uncorrelated, the smax operator can be equivalently formulated as:

\[
a^* = \arg\max_a u(a, m_1^* x_1, ..., m_n^* x_n)
\]

with

\[
m_i^* = A_g \left( \sigma_i^2 a_{x_i} u_{aa} a_{x_i} / \kappa \right)
\]

and \( a_{x_i} = \frac{\partial a}{\partial x_i} = -u_{aa}^{-1} u_{a,x_i} \).

The intuition is that the \( x_i \)'s are truncated. If \( |\frac{\partial a}{\partial x_i}| \) is small enough, so that \( x_i \) shouldn’t matter much any way, then \( m_i^* = 0 \), and the agent doesn’t pay attention to \( x_i \) (if \( m_i^d = 0 \)).

This leads to defining the truncation function:

\[
\tau_g(b, k) := b A_g \left( \frac{b^2}{k^2} \right).
\]

The following proposition gives a more explicit version of the action.

**Proposition 2** If the rational action is:

\[
a^*(x) = a^d + \sum_i b_i x_i + O(\|x\|^2)
\]

then the sparse action is: with attention chosen ex ante:

\[
a^*(x) = a^d + \sum_i \tau \left( b_i \frac{\kappa a}{\sigma x_i} \right) x_i + O(\|x\|^2)
\]
while with attention chosen ex post:

$$a^s(x) = a^d + \sum_i \tau(b_i x_i; \kappa_a) + O(||x||^2)$$  \hspace{1cm} (5)

with

$$\kappa_a := (\kappa / |u_{aa}|)^{1/2}.$$

For a quadratic utility function $u = -\frac{1}{2}(a - \sum_i b_i x_i)^2$, the above expressions are exact (i.e. hold without the $O(||x||^2)$ terms).

We see the contrast. In the first procedure, the slope is chosen before seeing $x_i$. Hence, the policy is still linear in $x_i$. In the second policy, the truncation is chosen after seeing the $x_i$. The policy is now non-linear in $x_i$. The linearity of policies make the first procedure useful for macro. Equipped with this piece of machinery, we turn to dynamic problems.

### 2.2 Sparse Dynamic Programming

We express the notions in the finite-horizon case. The agent’s problem is:

$$\max_{(a_t), z_t} \sum_{t=0}^{T-1} \beta^t u(a_t, z_t, t) \quad \text{s.t.} \quad z_{t+1} = F^z(z_t, a_t, \varepsilon_{t+1}, t + 1)$$  \hspace{1cm} (6)

and a terminal condition $z_T \geq 0$. Here state variable $z_t$ and action $a_t$ are vectors, while $\varepsilon_{t+1}$ is a mean-zero innovation.

The rational version of the dynamic programming (DP) problem is a series of value functions $V^{r,t}$ satisfying the Bellman equation:

$$V^{r,t}(z) = \max_a u(a, z, t) + \beta \mathbb{E}V^{r,t+1}(F^z(z, a, \varepsilon_{t+1}, t + 1))$$  \hspace{1cm} (8)

for $t = 0, ..., T - 1$, and with $V^{r,T}(z) = 0$. A policy is then a function $a(z, t)$.

In the smax version, we are given “simplified” value functions $u(a, z, t; m), V^{r,t+1}(z; m), F^z(z, a, \varepsilon_{t+1}; m)$—in a way that will be detailed later. To fix ideas, we could take

$$u(a, z, t; m) = u(a, m \odot z, t)$$

where

$$m \odot z := (m_i z_i)_{i=1...n}.$$

We define the value function as follows.

**Definition 2** The sparse value function $V^t(z)$ is the solution of:

$$V^t(z) = \max_{a, m} u(a, z, t; m) + \beta \mathbb{E}V^{t+1}(F^z(z, a, \varepsilon_{t+1}; m); m)$$  \hspace{1cm} (9)

for $t = 0, ..., T - 1$, and with $V^{r,T}(z) = 0$. The smax operator for sparse maximization is defined in Definition 1.
This is the same formulation as in the rational version, but with a $\text{smax}$ rather than a $\text{max}$ operator. The definition is recursively for $t = T - 1, \ldots, 1$, with $V^T := 0$.

In an infinite-horizon problem, the Bellman equation is the same, except that $V^{r,t} = V^{r,t+1}$.  \footnote{The existence of a value function with finite horizon is automatic. The existence in infinite horizon seems unproblematic in many applications, but, strictly speaking, is mathematically open at this stage.}

The problem may look complicated, but in many cases it is actually simple. Before proceeding to the examples, we present some results that help calculate the $\text{smax}$ solution. The reader is encouraged to skip to the applications Section 3 in a first reading.

### 2.3 Tools for Boundedly Rational Dynamic Programming

This subsection presents tools to compute BR dynamic programming. The reader is invited to skim it, read the main examples shown later, and then come back to it with those examples in mind.

#### 2.3.1 Tools to Expand a Simple Model Into a More Complex one

We now decompose the vector of state variables into:

$$z = (w, x)$$

where $w$ is a vector of variables that are fully taken into account ($m^d_i = 1$), in the default mode, while $x$ is a vector of variables not taken into account ($m^d_i = 0$).

Here I develop the method to derive the Taylor expansion of a richer model, when starting from a simpler one. Here the methods are entirely paper and pencil. They draw from the techniques surveyed by Judd (1998, Chapter 14), who has a more computer-based perspective.

Consider the fully rational model:

$$V^r(w, x) = \max_a u(w, x, a) + \beta \mathbb{E} V^r(w', x')$$

The state variables are $w$ and $x$, and the decision variable is $a$. The state variables evolve according to:

$$w' = G^w(w, x, a)$$
$$x' = G^x(x)$$

where variable $x$ (which again is a vector) is like a macro disturbance, such as the deviation of the interest from trend, which evolves independently of the actions and state variable of the agent, $(w, a)$.

We start with a simpler model, where $x \equiv 0$, i.e.

$$V^d(w) = \max_a u(w, 0, a) + \beta \mathbb{E} V^d(w'(a))$$
where \( w' = G_w^r(w, 0, a) \).

Using the notation
\[
D_w f = \partial_w f + (\partial_a f) \frac{da}{dw}
\]  
which is the total derivative with respect to \( w \) (e.g. the full impact of a change in \( w \), including the impact it has on a change in the action \( a \)). Differentiating the Bellman equation (first with respect to the new variable \( x \), then with respect to the default variable \( w \)), we obtain:

\[
V_x (w, x) = u_x + \beta V'_w G_w^r (w, x, a) + \beta V_x G_x^r
\]

\[
V_w, x (w, x) = D_w u_x + \beta D_w \left[ V'_w G_w^r (w, x, a) \right] + \beta G_x^r V'_w, x D_w w'
\]

so

\[
V_{w, x} (w, 0) = \frac{D_w u_x + \beta D_w \left[ G_w^r (w, 0, a) V'_w (w', 0) \right]}{1 - \beta G_x^r D_w w'}
\]  

Proposition 3  The impact of a change \( x \) on the value function is:

\[
V_{w, x} (w, 0) = \frac{D_w u_x + \beta D_w \left[ G_w^r (w, 0, a) V'_w (w', 0) \right]}{1 - \beta G_x^r D_w w'}
\]  

The impact of a change \( x \) on the optimal action is:

\[
da = -\Psi_a^{-1} \Psi_x dx
\]

\[
\Psi (a, x) = u_a (w, a) + \beta V'_w G_a^w
\]

\[
\Psi_a = u_{aa} + \beta V'_w G_a^w G_a^w + \beta V'_w G_a^w
\]

\[
\Psi_x = u_{ax} + \beta V'_w G_a^w + \beta V'_w G_a^w
\]

They depend only on the transition functions and the derivatives of the simpler baseline value function \( V'_w (w') \).

The same procedure can be followed when \( x' = G_x^r (w, x, a) \), with more complex algebra. We next show a useful consequence.

Proposition 4  For small \( x \), we have:

\[
V^s (w, x) = V^r (w, x) + x' \phi (w, x) x
\]

where matrix \( \phi (w, x) \) is continuous in \((w, x)\) and twice differentiable at \( x = 0 \), with \( \phi (w, 0) \) negative semi-definite. In other words, the sparse value function and the rational value functions differ only by second order terms in \( x \).
This basically generalizes the envelope theorem. It implies that, at \( x = 0 \):
\[
V_w^s = V_w^r, \quad V_{ww}^s = V_{ww}^r, \quad V_x^s = V_x^r, \quad V_{xx}^s = V_{xx}^r
\]  
(13)
However, in most situations we have \( V_{xx}^s \neq V_{xx}^r \), and indeed \( V_{xx}^s < V_{xx}^r \); this is because lack of optimization leads to second order losses in \( x \).

This leads to a simple proposition to calculate the value function.

**Proposition 5** (Calculation of the optimal sparse policy). Suppose that \( F_{a|x=0}^x = 0 \). Consider the first order expansion of the optimal policy for small \( x \),
\[
a^r (w, x) = a^d (w) + \sum_i b_i (w) x_i + O (x^2)
\]
Then, the sparse policy is, with ex-ante attention allocation:
\[
a^s (w, x) = a^d (w) + \sum_i \tau \left( b_i (w), \frac{\kappa_a}{\sigma_{x_i}} \right) x_i + O (x^2)
\]
and with ex-post attention allocation:
\[
a^s (w, x) = a^d (w) + \sum_i \tau (b_i (w) x_i, \kappa_a) x_i + O (x^2)
\]
(14)
(15)
Condition \( F_{a|x=0}^x = 0 \) is often satisfied in models. This proposition will be quite useful. To derive policies, first we can simply do a Taylor expansion of the rational policy around the default model, and then truncate term by term.

I conclude with a remark which will be useful later, drawing again on Gabaix (2014). As \( \kappa \) has the units of utils, it cannot be a primitive parameter. One can make it more endogenous with the primitive, unitless parameter \( \pi \), by setting:
\[
\kappa = \pi^2 \text{var} \left( u \left( a^d (x), x, m^d \right) \right)^{1/2}
\]
(16)

### 2.3.2 Simplification of functions
We develop here a bit of simple machinery to reflect how the agent can “simplify” a function (in practice a value function), by forcing them to have a given functional form.

A motivating example. Suppose that the agent consumes \( c_1 = \frac{w}{2} + y_1 \) and \( c_2 = \frac{w}{2} + y_2 \), so that there is no smoothing if \( y_1 \neq y_2 \). His value function is:
\[
V (y) = u \left( \frac{w}{2} + y_1 \right) + u \left( \frac{w}{2} + y_2 \right)
\]
Up to second order terms, \( V (y) = V^S (y) + O (\|y\|^2) \), where
\[
V^S (y) = 2u \left( \frac{w + y_1 + y_2}{2} \right)
\]
which is the value function of a rational agent who would perfectly smooth consumption (assuming an interest rate and discounting of 0 for now). We can then form a more general function:

\[ V(y, m^V) = (1 - m^V) V^S(y) + m^V V(y) \]

If \( m^V = 1 \), the agent uses the rational value function. If \( m^V = 0 \), the agent uses the proxy value function \( V^S \), which is in some sense simpler.

We next generalize that thought.

We have \( x \in \mathbb{R}^n \), \( \hat{x} \in \mathbb{R}^p \). The following lemma is very elementary mathematically, but useful nonetheless because it records how the “\( b \)” term below can be made unique.

**Lemma 1** (Simplifying function) Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be a function such that \( f_{x_i}(x)_{x=0} \neq 0 \) for all \( i \), and \( \phi \) be a mapping mapping \( \phi : \{1, \ldots, p\} \rightarrow \{1, \ldots, n\} \). We are given a function \( g : \mathbb{R}^p \rightarrow \mathbb{R} \) such that \( g(0) = f(0) \). There is a uniquely determined matrix \( b \in \mathbb{R}^{n \times p} \) such that \( b_{ij} = 0 \) unless \( i = \phi(j) \) and

\[ g(\hat{x}) = f(b \cdot \hat{x}) + o(||\hat{x}||) \quad (17) \]

Furthermore, \( b_{ij} = \frac{g_{\hat{x}_j}(\hat{x})|_{\hat{x}=0}}{f_{x_i}(\hat{x})|_{\hat{x}=0}} \) if \( i = \phi(j) \), \( b_{ij} = 0 \) otherwise.

Call \( \mathcal{E}^f := \{ g \text{ such that } g(0) = f(0) \} \subset C^1(\mathbb{R}^p, \mathbb{R}) \). This define a simplification function: \( S : \)

\[ \mathcal{E}^f \rightarrow \mathcal{E}^f \]

\[ g \mapsto (Sg) \text{ such that } (Sg)(\hat{x}) = f(b \cdot \hat{x}) \]

Basically, this is like a non-linear Taylor expansion.

**Proof** We want: \( g(\hat{x}) = f(b \cdot \hat{x}) + o(||\hat{x}||) \). This is equivalent to:

\[ g_{\hat{x}_j}(\hat{x})|_{\hat{x}=0} = \sum_i f_i b_{ij} = f_{\phi(j)} b_{\phi(i)j}. \]

Inspecting the Taylor expansions gives the result. □

For instance, in our introductory example, \( f(x) = 2u\left(\frac{w+x}{2}\right), \hat{x} = (y_1, y_2), g(\hat{x}) = V(\hat{x}) \), \( n = 1, p = 2, \phi(j) = 1, \) and \( b = (1, 1) \).

Here are two other variants of the same idea. Suppose that we have a stochastic variable, and a variant of the Black-Scholes model, with say stochastic volatility. Then, we may approximate the value function by tweaking the implied volatility: \( V(x_t, S, K, r, t) = V_{\text{BlackScholes}}(\sigma(x_t) + o(x_t)), S, K, r, t) \), where \( V_{\text{BlackScholes}} \) is the regular Black-Scholes formula, and \( \sigma(x_t) \) could be affine.

Suppose that the agent estimates a distribution, \( g(\hat{x}) \), where \( \hat{x} \) are parameters of the distribution. The agent may wish to replace this distribution by a distribution with a simpler functional form, say a Gaussian: then \( f(x) \) is a Gaussian distribution, while \( g(\hat{x}) \) is a more general distribution. For instance, the procedure might match take the mean and variance of the Gaussian to be that of \( g \).
2.4 Iterated Sparse Max

In some cases, it is useful to have a generalization of the basic sparse max.

**Definition 3** (Iterated sparse max) The $K$-times iterated sparse max, $\text{smax}^K_{a,m,m^d} u(a,x)$, is defined by the following procedure. Define $m^d(k)$ to be the initial default attention, $m^d$. Start at round $k = 1$. At each round $k \leq K$, apply the regular smax, using the default $m^d(k)$:

$$\text{smax}^K_{a,m,m^d} u(a,x,m)$$

and call $m^*(k)$ and $a^*(k)$ the resulting attention. Define then $m^d(k+1) = m^*(k)$. Stop at the end of round $k = K$, and return $m^*(K)$ and $a^*(K)$, the optimal attention and action at the last iteration.

**Illustration.** Suppose that

$$u(a,x) = -\frac{1}{2} (a - x_1 (1 + x_2))^2$$

So that the agent should think about $x_2$ only if he already thinks about $x_1$. We now that $a^r(x_1, x_2) = x_1 (1 + x_2)$, so $a^r_{x_1} = 1 + x_2$, $a^r_{x_2} = x_1$, and call $\sigma_i^2 = \text{var}(x_i)$.

We next apply the iterated smax outlined in Definition 3, iterating twice ($K = 2$). Initial default attention is $m^d(1) = (0, 0)$. Then, at step $k = 1$

$$m^*_1(1) = \mathcal{A}\left(\frac{\sigma_1^2}{\kappa}\right), \quad m^*_2(1) = 0$$

So, at the beginning of the second step, the default is $m^d(2) = (m^*_1(1), m^*_2(1))$. Applying again the plain smax but with that default $m^d$, we have:

$$m^*_1(2) = \mathcal{A}\left(\frac{\sigma_1^2}{\kappa}\right), \quad m^*_2(2) = \mathcal{A}\left(\frac{m^*_1(1)^2 \sigma_2^2}{\kappa}\right)$$

Hence, the action is $a = a^r(m^*_1(2) \odot x) = m^*_1(2) x_1 (1 + m^*_2(2) x_2)$. We also see that as $\kappa \to 0$, the action converges to the rational action.$^2$

We have developed tools to define and compute BR dynamic programming. Now we turn to concrete examples.

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$^2$This iterated smax suffices for the problems considered in this paper. For other purposes, one could imagine a variant where the default is at say $m^d = (\varepsilon, \ldots, \varepsilon)$, for some $\varepsilon > 0$, so as to “probe” the importance of all variables.
3 Intertemporal Consumption: Behavioral Version

3.1 Warm-up: A 3 period model

To clarify a number of notions, let us consider a simple 3-period model. The discount factor and the gross interest rates are 1. Utility is

\[ \sum_{t=0}^{2} u(c_t) \]

Calling \( w_t \) the wealth at the beginning of period \( t \), the budget constraints at times \( t = 0, 1, 2 \) are:

\[ w_1 = w_0 - c_0, \quad w_2 = w_1 - c_1, \quad 0 = w_2 + x - c_2. \]

The agent starts with an endowment \( w_0 \), and receives \( x \) at time 2. For instance, \( x \) could represent a negative income shock, such a tax to pay, or a decrease in income as retirement.

The question is: will the agent pay attention to time-2 payment \( x \)?

The rational solution is to smooth consumption: total resources are \( w_0 + x \) (initial wealth \( w_0 \) and time-2 payment \( x \)), and they should be consumed equally in all periods:

\[ c_t = \frac{w_0 + x}{3} \quad \text{for } t = 0, 1, 2 \]

The corresponding dynamic policy is:

\[ c_0 = \frac{w_0 + x}{3}, \quad c_1 = \frac{w_1 + x}{2}, \quad c_2 = w_2 + x. \]

as we will verify soon.

Let us apply the smax procedure, by backward induction. At the terminal date, everything is consumed:

\[ V^2(w_2, x) = u(w_2 + x) \]

At time 1, the agent’s problem is:

\[ \text{smax}_{c,m} v^2(c, x, m) \]

with

\[ v^2(c, x, m) := u(c) + V^2(w - c, mx). \]

This yields:

\[ u'(c_1) = V^2_w (w_1 - c_1, m_1 x) = u'(w_1 - c_1 + m_1 x) \]

so \( c_1 = w_1 - c_1 + m_1 x \), and

\[ c_1 = \frac{w_1 + m_1 x}{2} \]

Hence, the agent pays partial attention \( m_1 \) to the time-2 income \( x \).
To calculate attention $m_1$, we apply (3). Noting that take $v^2(c, x, m) = u''(c^d)$, with $c^d = \frac{w_1}{x}$ is the optimal consumption with $m = 0$, we have $m_1 = A \left( 2u''(c^d) \text{var}(x) \right) / \kappa$, so

$$m_1 = A \left( \frac{1}{2\kappa} u'' \left( \frac{w_1}{2} \right) \sigma^2_x \right). \quad (19)$$

Hence, the value function at time 1 is:

$$V^1(w, x) = u \left( \frac{w + m_1 x}{2} \right) + u \left( \frac{w + (2 - m_1) x}{2} \right) \quad (20)$$

This is a little complicated. There’s where the simplification operator $SV^1$ (defined in Lemma 1) intervenes. Applying it, we obtain

$$V^1(w, x, m^V = 0) = 2u \left( \frac{w_1 + x}{2} \right) \quad (21)$$

The value is the same as $V^1$, up to $O(x^2)$ terms. As a partial coincidence, this is also the rational value. The general simplified value function at time 1 is:

$$V^1(w, x, m^V) = m^V V^1(w, x) + (1 - m^V)V^1(w, x, m^V = 0)$$

At time 0, the agent does $\text{smax}_{x_0, m} v^0(c_0, x, m)$ where $m = (m^x, m^V)$, with:

$$v^0(c_0, w_0, x, m) := u(c_0) + V^1(w_0 - c_0, m_0^x x, m^V) \quad (22)$$

The FOC is $v^0_{c_0} = 0$ with

$$v^0_{c_0}(c_0, w_0, x, m) = u'(c_0) - V^1_w \left( w_0 - c_0, m_0^x x, m^V \right).$$

We have $V^1_{wm} = 0$ at the default $m_0^x = 0$, so $\frac{\partial v^0}{\partial m} |_{m^x=0} = 0$ and $m^V = 0$: the agent uses the proxy value function, not the exactly rational one (we will see soon that attention $m^V$ can be non-zero using the 2 step smax, but it is still likely to be 0 if $\kappa$ is not too small). So, the FOC for consumption is:

$$u'(c_0) = u' \left( \frac{w_0 - c_0 + m_0^x x}{2} \right)$$

i.e. $c_0 = \frac{w_0 - c_0 + x}{2}$ and

$$c_0 = \frac{w_0 + m_0^x x}{3} \quad (23)$$

To determine attention $m_0 = m_0^x$, we again use (3); we calculate:

$$v^0_{oc} = u''(c^d) + (V^1_{ww})_{m=0} = u''(c^d) + \frac{1}{2} u''(c^d) = \frac{3}{2} u''(c^d)$$

13
so that $m_0 = A \left( \frac{\frac{\partial^2 u''(\omega)}{\partial \omega^2}}{\kappa} \right)$, i.e.

$$m_0 = A \left( \frac{1}{6\kappa} u'' \left( \frac{w_0}{3} \right) \sigma^2_x \right). \quad (24)$$

Note that this implies $m_0 \leq m_1$.

The resulting dynamics are:

$$c_0 = \frac{w_0}{3} + \frac{m_0}{3} x, \quad c_1 = \frac{w_0}{3} + \left( \frac{m_1}{2} - \frac{m_0}{6} \right) x, \quad c_2 = \frac{w_0}{3} + \left( 1 - \frac{m_0 + m_1}{2} \right) x. \quad (25)$$

**Proposition 6** Take the 3-period life-cycle problem with $m_0^V = 0$. The BR policy is

$$c_0 = \frac{w_0 + m_0 x}{3}, \quad c_1 = \frac{w_1 + m_1 x}{2}, \quad c_2 = w_2 + x.$$ 

where $m_t$ are the attention parameters given in (19) and (24). In particular, this implies

$$\frac{\partial c_0 (w_0, x)}{\partial x} \leq \frac{\partial c_1 (w_0, x)}{\partial x} \leq \frac{\partial c_2 (w_0, x)}{\partial x} \quad (26)$$

with at least one strict inequality if $\kappa$ large enough, while if the agent was rational, we would have: $\frac{\partial c_0 (w_0, x)}{\partial x} = \frac{\partial c_1 (w_0, x)}{\partial x} = \frac{\partial c_2 (w_0, x)}{\partial x}$. 

**Proof** Most was already derived. Given $0 \leq m_0 \leq m_1 \leq 1$, equation (25) implies (26). When $\kappa$ is very large, then $m_0 = m_1 \to 0$, so that $\frac{\partial c_0 (w_0, x)}{\partial x} \to 0$ and $\frac{\partial c_2 (w_0, x)}{\partial x} \to 1$, so that one inequality in (26) is strict.

\[\square\]

We note that if $m^V > 0$, the FOC is more complex. The FOC is:

$$u'(c_0) = (1 - m^V) u' \left( \frac{w_0 + m_0 x}{2} \right) + m^V \frac{1}{2} \left[ u' \left( \frac{w_1 + m_1 x}{2} \right) + u' \left( \frac{w_1 + (2 - m_1) x}{2} \right) \right]$$

Still, to the first order, the decision is the same (as per Proposition 5). Making the problem simpler at every period, via the $m^V = 0$ device, makes the problem more tractable for both the agent and the economist examining him.

This example illustrates a few general features that are specific to a dynamic setting.

*Those sparse agents are locally myopic like in hyperbolic agents, but globally patient like rational agents. As a results, they differ from both the rational and sparse agents. Agents here invest their wealth $w$ very patiently here, exactly like rational agents. At the same time, they tend to be myopic about the future small shocks (the time-2 shock $x$), as in models of hyperbolic discounting (Laibson 1997, O’Donoghue and Rabin 1999). In other terms, in the present model, agents are only partially myopic (e.g. don’t react to a schedule increase in taxes).*
Agents react more to “near” shocks than to “distant” shocks: That’s equation (26). The main reason is that, normatively, the shock \( x \) should impact \( \chi_0 \) as \( \frac{x}{3} \left( \frac{\partial c_0(w_0, x)}{\partial x} = \frac{1}{3} \right) \), while it should impact \( c_1 \) as \( \frac{x}{2} \left( \frac{\partial c_1(w_1, x)}{\partial x} = \frac{1}{2} \right) \). Hence, attention is greater to the last period shock \( x \) is lower at earlier dates \((t = 0)\) than at late dates \((t = 1)\).

The sparse agent exhibit partial sophistication and naiveté in understanding its future agents. The rational value function (20) endows the agent with a perfectly sophisticated understanding of his future actions (in particular, the agent understand that he will not fully optimize at period 1. However, the simplified value function \( V^1 \) in (21) with \( m^V = 0 \) gives him a rougher (hence, more naive) understanding of his future actions: the agent is more like a naive agent. Hence, the agent is sophisticated with \( m^V = 1 \) and more naive when \( m^V = 0 \). The agent optimizes on the degree of sophistication.

The Euler equation fails. The Euler equation holds under the BR-perceived consumption, but not under the actual consumption.

Now that this basic issues are in place, we move on to an infinite horizon problem.

**Attention to the proxy vs true value function** We have so far used the plain sparse max. This led to \( m^V = 0 \), the exclusive reliance on the simplified value function. We now calculate what happens when using the twice-iterated sparse max of Definition 3. We calculate the twice-iterated smax: \( \text{smax}_{c, m}^{2} v^0 (c_0, w_0, x, m) \) with \( m = (m^x_0, m^V) \).

At the first round, \( v^{0}_{c_0, m^V} = 0 \), so \( m^V_0 (1) = 0 \), and as before \( m^x_0 (1) = A \left( \frac{1}{6\kappa} u'' \left( \frac{w_0}{3} \right) x^2 \right) \).

At the second round, now \( m^d = (m^x_0 (1), 0) \). The easy part is the attention to \( x \), which is slightly different than at step 1:

\[
m^x_0 (1) = A \left( \frac{1}{6\kappa} u'' \left( \frac{w_0 + m^x_0 (0) x}{3} \right) x^2 \right)
\]

The more novel part is to calculate \( m^V \). We have, with \( w_1 = w_0 - c_0 \), and calling \( x^s := m^x_0 x \)

\[
v^{0}_{c, m^V} (c_0, w_0, x, m^x_0, m^V) = \partial_c [V^1 (w_0 - c_0, x^s) - V^1 (w_0 - c_0, x^s, m^V = 0)]
\]

\[
= -\frac{1}{2} u' \left( \frac{w_1 + m_1 x^s}{2} \right) - \frac{1}{2} u' \left( \frac{w_1 + (2 - m_1) x^s}{2} \right) + u' \left( \frac{w_1 + x^s}{2} \right)
\]

Doing a Taylor expansion of the consumptions \( \frac{w_1 + m_1 x^s}{2} \) and \( \frac{w_1 + (2 - m_1) x^s}{2} \) around their mean

\[
c^d = \frac{w_1 + x^s}{2} = \frac{w_1 + m^x_0 x}{2}
\]
and proxy value functions (i.e., essence of the more complex patterns that can later be re
in the basic smax case. This is one reason it is useful to use the basic smax: it gets to the
about \( \theta \)
\( \mu \)
 depending on that variable)
Proposition 7
depending on that variable)
Suppose a succession of problems (indexed by \( \kappa \) going to 0) such that there are positive constants \( B, B', \varepsilon \) such that for \( \kappa \) small enough: \( B\kappa^{1/2-\varepsilon} \leq \sigma_x(\kappa) \leq B'\kappa^{1/4+\varepsilon} \). Then, the agent will have \( m^o_0 > 0 \) and \( m^v_0 = 0 \) when \( \kappa \) is small enough. This is, the agent pays attention to the disturbance \( x \), but not to the subtle difference between the true and proxy value functions (i.e., \( V^1(w_1, x, m^V) \) for \( m^V = 1 \) vs \( m^v = 0 \)).

In plain terms: because thinking about the nuances \( m^{V} \) in \( V(x, m^{V}) \), one needs to think about \( x \) at all. Hence, in many situations, we have \( m^V = 0 \) and \( m^x > 0 \). Indeed, we cannot have (with just one state variable \( m^x = 0 \) and \( m^{V} > 0 \)).

In particular, for our 3-period problem for \( \kappa \) small enough but not too small, \( m^V = 0 \) and \( m^o_0 > 0 \): the agent uses the simplified value function, as still pays attention to \( x \), like in the basic smax case. This is one reason it is useful to use the basic smax: it gets to the essence of the more complex patterns that can later be refined using the iterated smax.

we obtain:
\[
v_{c,m^v}^o = -\frac{1}{2} u'(c^d + (m_1 - 1) \frac{x^s}{2}) - \frac{1}{2} u'(c^d - (m_1 - 1) \frac{x^s}{2}) + u'(c^d)
\]
\[
= -\frac{1}{2} u''(c^d) (m_1 - 1)^2 \left( \frac{x^s}{2} \right)^2 \times 2 + o(x^2)
\]
\[
= -\frac{1}{4} u''(c^d) (m_1 - 1)^2 (m^x_0 x)^2 + o(x^2)
\]
Likewise, \( v_{c|m=m^o_0(1)}^o = \frac{3}{2} u''(c^d) \). So, the impact of \( m^V \) is
\[
\frac{\partial c_0}{\partial m^V} = \frac{-v_{c,m^V}^o - \frac{1}{6} u''(c^d)(m_1 - 1)^2 (m^x_0 x)^2 + o(x^2)}{v_{c,c}^o}
\]
Hence, for a small \( x \), the attention \( m^V \) to the difference between the difference between the true and proxy value functions (i.e., \( V^1(w_1, x, m^V) \) for \( m^V = 1 \) vs \( m^v = 0 \)) is:
\[
m^V_0 = A \left( \frac{1}{\kappa} E \left( \left( \frac{\partial c_0}{\partial m^V} \right)^2 \right) \right)
\]
\[
= A \left( \frac{1}{\kappa} E \left( \left( \frac{1}{6} \frac{u''(c^d)}{u''(c^d)} \right)(m_1 - 1)^2 (m^x_0 x)^2 \right)^2 \right) \frac{3}{2} u''(c^d)
\]
\[
= A \left( \frac{1}{24\kappa} \frac{u''(c^d)}{u''(c^d)} \right)^2 (m_1 - 1)^4 (m^x_0)^4 E \left[ x^4 \right] u''(c^d)
\]
(27)

It is instructive to take the limit of small \( \kappa \), using a sparsity-inducing cost function (\( g'(0) > 0 \)). To have \( m^o_0 > 0 \), we need \( \frac{\alpha}{\kappa} \) large enough, so \( \sigma_x \geq \kappa^{1/2} \). To have \( m^V_0 > 0 \), we need \( \frac{\alpha}{\kappa} \) large enough, i.e. \( \sigma_x \geq \kappa^{1/4} \), which is a much higher hurdle (\( \kappa^{1/4} \rightarrow \infty \)) for small \( \kappa \). Formalizing this.

Proposition 7 (Attention to a variable, vs attention to the nuances of the value function depending on that variable) Suppose a succession of problems (indexed by \( \kappa \) going to 0) such that there are positive constants \( B, B', \varepsilon \) such that for \( \kappa \) small enough: \( B\kappa^{1/2-\varepsilon} \leq \sigma_x(\kappa) \leq B'\kappa^{1/4+\varepsilon} \). Then, the agent will have \( m^o_0 > 0 \) and \( m^v_0 = 0 \) when \( \kappa \) is small enough. This is, the agent pays attention to the disturbance \( x \), but not to the subtle difference between the true and proxy value functions (i.e., \( V^1(w_1, x, m^V) \) for \( m^V = 1 \) vs \( m^v = 0 \)).
3.2 Infinite-Horizon Problem

I now turn to the canonical consumption-investment problem, with an infinite horizon. The agent has utility $E \sum_{t=0}^{\infty} \beta^t c_t^{1-\gamma} / (1-\gamma)$. Wealth $w_t$ evolves as:

$$w_{t+1} = (1 + r_t) (w_t - c_t) + y_t.$$

(that is, wealth at $t+1$ is savings at $t$, $w_t - c_t$, invested at rate $r_t$, plus current income, $y_t$).

We have decompose:

$$r_t = \tau + \tilde{r}_t, \quad y_t = \bar{y} + \tilde{y}_t$$

where $\tilde{r}_t$ and $\tilde{y}_t$ are deviations of the interest rate and income from their means, respectively, and follow AR(1) processes:

$$\tilde{r}_{t+1} = \rho \tilde{r}_t + \varepsilon_{t+1}^r, \quad \tilde{y}_{t+1} = \rho \tilde{y}_t + \varepsilon_{t+1}^y$$

$\varepsilon_{t+1}$ are independent disturbances with mean zero. For simplicity, assume here that $R \equiv 1 + \tau = \beta^{-1}$.

In the default model, the agent assumes that $m^d_r = m^d_y = 0$; this is, he assumes that future interest rate and income will be constant. Then, the optimal consumption is $c^d (w_t) = (\tau w_t + \bar{y})/R$, and the value function is

$$V^d (w_t) = A (\tau w_t + \bar{y})^{1-\gamma}$$

for a constant $A = \frac{1}{\tau^{1-\gamma}}$.

To calculate the BR policy, we first use Proposition 3. This gives:

$$\ln c^r (w_t, \tilde{y}_t, \tilde{r}_t) = \ln c^d (w_t) + b_y \tilde{y}_t + b_r \tilde{r}_t + \text{2nd-order terms}. \quad (28)$$

where:

$$b_y = \frac{\tau}{R (R - \rho_y) c_t^{d-1}}, \quad b_r = \frac{\tau \left( \frac{w_t}{c_t^d} - 1 \right) - 1/\gamma}{R - \rho_r}. \quad (29)$$

as shown in calculations in the appendix.

The agent does a sparse truncation of (28), as in Proposition 2. Hence, we obtain the following.

**Proposition 8** A sparse agent has the following consumption policy, up to second order terms:

$$\ln c^s_t = \ln c^d (w_t) + b_x^s \tilde{y}_t + b_x^s \tilde{r}_t \quad (30)$$

where $(for \ x = y, r) b_x^s := \tau \left( b_x, \frac{\sigma_y}{\sigma_x} \right)$ and $b_x$ are in (29).
Equation (30) shows a “feature-by-feature” truncation. It is useful because it embodies in a compact way the policy of a sparse agent in quite a complicated world. Note that the agent can solve this problem without solving the 3-dimensional (and potentially 21-dimensional, say, if there are 20 state variables besides wealth) problem. Only local expansions and truncations are necessary.

In this manner, we arrive at a quite simple way to do sparse dynamic programming. There is just one continuously-tunable parameter, $\kappa$. When $\kappa = 0$, the agent is (to the leading order) the traditional rational agent. When $\kappa$ is large enough, the agent is fully sparse, and does not react to any variable. Hence, we have a simple, smooth way to parametrize the agent, from very sparse to fully rational.

**Numerical illustration** To get a feel for the effects, consider a calibration with (using annual units): $\gamma = 1$, $r = 5$, $\varpi = 2\pi$, $\tau = 1$, $\sigma_r = 0.8\%$, $\sigma_y = 0.2\pi$, $\rho_y = 0.95$, $\sigma_{inc} = 5\%$, and $\rho_r = 0.7$: as income shocks are persistent, they are important to the consumer’s welfare.

Then, Figure 1 shows the impact of a change in the interest rate and income on consumption. Consider the left panel, $b_r^\kappa$. If the cost of rationality is $\kappa = 0$, then the agent reacts like the rational agent: if interest rates go up by 1%, then consumption falls by 2.8% (the agent saves more). However, for a sparsity parameter $\kappa \approx 0.5$, the agent essentially does not respond to interest rates. Psychologically, he thinks “the interest rate is too unimportant, so let me not take it into account.” Hence, the agent does not react much to the interest rate, but will react more to a change in income (right panel of Figure 1), which is more important: the sensitivity of consumption to income remains high even for a high cognitive friction $\kappa$. Note that this “feature-by-feature” selective attention could not be rationalized by just a fixed cost to consumption, which is not feature-dependent.

The same reasoning holds in every period. The above describes a practical way to do sparse dynamic programming. In some cases, this is simpler than the rational way (as the agent does not need to solve for the equilibrium), and it may also be more sensible.
Figure 2: Measured intertemporal elasticity of substitution (IES), \( \hat{\psi} \), if the consumer is sparse with cost \( \kappa \), while the econometrician assumes he is fully rational. The true IES is \( \psi = 1 \).

Consequence. A behavioral solution to puzzles and controversies regarding the intertemporal elasticity of substitution

For many finance applications (e.g., Bansal and Yaron 2004, Barro 2009, Gabaix 2012), a high intertemporal elasticity of substitution (IES, denoted \( \psi = 1/\gamma \)) is important (\( \psi > 1 \)). However, micro studies point to an IES of less than 1 (e.g., Hall 1988). I show how this may be due to the way econometricians proceed, by fitting the Euler equation, which yields

\[ \ln c_{t+1} - \ln c_t = \hat{\psi} r_{t+1} + \text{constant}, \]

where \( \hat{\psi} \) is the measured IES. If the consumer “under-reacts to the interest rate,” the measured IES will be biased towards 0. Using the above model, we can more precisely calculate that if consumers are boundedly rational (in the sense laid out above), the estimated IES will be:

\[ \hat{\psi} = \tau \left( \frac{w_t}{c_t^d} - 1 \right) - b_r^s \bar{R} \left( \bar{R} - \rho_R \right). \]

This is a point prediction that goes beyond Chetty (forth.)’s prediction of an interval bound. Hence we obtain:

**Proposition 9** An econometrician fitting an Euler equation even though the agent is sparse will estimate a downwardly-biased IES (intertemporal elasticity of substitution):

\[ \hat{\psi} = \psi - \bar{R} \left( \bar{R} - \rho_R \right) (b_r^s - b_r) < \psi \]

where \( \hat{\psi} \) is the estimated IES, \( \psi \) the true IES and \( b_r^s - b_r \) is the difference between the sparse agent’s and the traditional agent’s interest-rate sensitivity of consumption.

The above calibration yields Figure 2, which plots the measured IES \( \hat{\psi} \) if the consumer is sparse with sparsity cost \( \kappa \). If \( \kappa = 0 \), the consumer is the traditional, frictionless rational agent. We see that as \( \kappa \) increases, the IES becomes more and more biased. Hence, inattention
may explain why while macro-finance studies require a high IES, microeconomic studies find a low IES.  

### 3.3 Failure of Ricardian Equivalence

Intuitively, a sparse agent will violate Ricardian equivalence (Barro (1974)). I study the magnitude and dynamics of that violation.

For simplicity, we use continuous time. The interest rate is $r = -\ln \beta$. The government needs to collect a present value of $G/r$. This could be done by taxing the population (of size normalized to 1) by $H = Ge^{rT}$, starting at a period $T$. Hence, the path of taxes is: 0 for $t < T$, and $H$ for $t \geq T$.

What is a consumer's response at time $t < T$? If the consumer is perfectly attentive, then he should start saving at time 0. However, a sparse agent might not pay attention to those future taxes increases, and start cutting on consumption only later, or indeed perhaps just when the tax cuts are enacted.

Let us analyze this more in detail. At $T$, the tax $H$ is enacted, so that for $t \geq T$, the agent is aware of it. This yields: $\tilde{c}_t = r\tilde{w}_t - H$.

Before the enactment of taxes ($t < T$), will the consumer think of the tax $H$? That tax lowers the present value of his income by $He^{-r(T-t)}$, so the consumer's response is:

$$\tilde{c}_t = r\tilde{w}_t - \tau \left(He^{-r(T-t)}, \kappa\right)$$

Hence, the consumer will not think about the tax increase $H$ when $He^{-r(T-t)} \leq \kappa$. Call $s \in [0, T)$ the first moment when he thinks about them (if it exists, i.e. if $H > \kappa$), otherwise we set $s = T$.

The next Proposition details the dynamics.

**Proposition 10** (Myopic behavior and failure of Ricardian equivalence) *Suppose that taxes will go up at time $T$. While a rational agent would cut consumption at time 0, a sparse agent cuts consumption later, at a time $s = \max \left(0, \min \left(T, \frac{1}{r} \ln \frac{\kappa}{He^{-rT}}\right)\right)$. His consumption path is:

$$\tilde{c}_t = \begin{cases} 0 & \text{for } t < s \\ -He^{-r(T-s)} + \kappa (1 - r (t - s)) & \text{for } s \leq t < T \\ r\tilde{w}_t - H & \text{for } t \geq T \end{cases}$$

with $\tilde{w}_T = \frac{H}{r} \left(1 - e^{-r(T-s)}\right) - \kappa (T - s)$.*
Figure 3: Reaction of consumption and wealth to an increase of future taxes, for different level of \( \kappa \). Notes. At time 0, it is announced that taxes will be paid start at time \( T = 10 \). This Figure plots the change in consumption and wealth. The solid line is the prediction of the rational model (i.e. \( \kappa = 0 \)), the other lines the reaction for different value of \( \kappa \) (\( \kappa = 0.01 \) (blue, dotted), \( \kappa = 0.025 \) (red, dashed-dotted), \( \kappa = .1 \) (green, dashed)). The very BR agents does not react at first, but starts reacting when he is closer to \( T \). He reacts even more when taxes are in effect. As he delayed his savings, he needs to cut more on consumption when taxes start. Units are percentage points of previous steady state consumption. The amount is \( G = 2\% \) of permanent income.

Let us take an example illustrated in Figure 3, with \( r = 5\% \), \( G = 2\% \), \( T = 10 \) years. This Figure plots the change in consumption and wealth for the rational actor \( \kappa = 0 \) (black, solid), and progressively less rational agents: \( \kappa = 0.01 \) (blue, dotted), \( \kappa = 0.025 \) (red, dashed-dotted), \( \kappa = .1 \) (green, dashed). The traditional Ricardian consumer (\( \kappa = 0 \)) immediately decreases his consumption by 2%, which leads to wealth accumulation at until time \( T \). In contrast the very BR consumer (\( \kappa = 0.1 \)) doesn’t react at all until \( T = 10 \) (hence he doesn’t accumulated any wealth), and then cuts a lot on consumption. The value \( \kappa = 0.01 \) and \( \kappa = 0.025 \) display an intermediary behavior. For \( \kappa = 0.025 \), the consumer initially doesn’t pay attention to the future tax. However, at a time \( s = 4.5 \) years, (i.e., when there are 3.6 years remaining until the taxes are effective), he starts paying attention, and starts savings for the future taxes. As the tax looms larger, the agent saves more. As the agent delayed his savings, he ends up cuttings down on consumption more drastically when taxes are in effect.

Smaller taxes generate a more delayed reaction. Controlling for the PV of taxes, consumers are better off with early rather than delayed taxes (as this allows them to smooth more).
3.4 The life-cycle model

The life-cycle model is that of Modigliani. It features a finite life (unlike the previous infinite-horizon model) and emphasizes the need to save for retirement. We develop the BR version here.

The agent lives for \( T \) periods, receives income \( y_t = y \) for times \( t \in [0, L] \), (where \( L \) is the time where labor income shocks) then \( y_t = y + \hat{y} \) (with \( \hat{y} < 0 \)) for \( t \in [L, T] \). We call \( B = T - L \) the length of retirement. Utility is

\[
\sum_{t=0}^{T-1} u(c_t)
\]

The interest rate and the discount rate are both 0.

In the rational model, the optimal policy is to smooth consumption: consume

\[
c_t = \frac{w_0 + L y + B (y + \hat{y})}{T} = \frac{w_0}{T} + y + \frac{B}{T} \hat{y}
\]

The BR agent has value function, which satisfies

\[
V^t (w_t) = u(c_t) + V^{t+1} (w_t - c_t + y + \hat{y}_t)
\]

Note that the rational value function is, for \( t \neq 0 \)

\[
V^{t,r} = \frac{1}{T-t} u \left( \frac{w + \sum_{s=t}^{T-1} y_s}{T-t} \right)
\]

Proposition 4 shows that for small \( \hat{y} \), this is also the Taylor expansion (up to \( O(\hat{y}^2) \) terms) of the value function under a BR policy. Hence, we will study an agent who uses the simplified value function

\[
V^t (w, m^V = 0) = (T-t) u \left( \frac{w + \sum_{s=t}^{T-1} y_s}{T-t} \right)
\]

Given this, consumption before retirement (\( t < L \))

\[
smax_{c_t;m} u(c_t) + V^{t+1} (w_t - c_t, m^V = 0)
\]

\[
smax_{c_t;m} u(c_t) + (T-t-1) u \left( \frac{w - c_t + \sum_{s=t+1}^{T-1} y_s}{T-t-1} \right)
\]

\[
u'(c_t) = u' \left( \frac{w - c_t + \sum_{s=t+1}^{T-1} y_s}{T-t-1} \right)
\]
so \( c_t = \frac{w - c_t + \sum_{s=t+1}^{T-1} y_s}{T-t-1} \) and
\[
c_t = \frac{w + \sum_{s=t+1}^{T-1} y_s}{T-t}
\]
so the agent consumes the perceived permanent income.

The choice of attention comes from (assuming that \( \mu^V = 0 \), e.g. because \( \kappa^V \) is large enough)
\[
\text{smax}_{c_t,m} v^t (c_t, m) := u (c_t) + (T - t - 1) u \left( \frac{w_t - c_t + \sum_{s=t+1}^{T-1} (y + m \hat{y})}{T-t-1} \right)
\]
so we get
\[
c_t = \frac{w_t + m_t B \hat{y}}{T-t} + y.
\]

We have \( v^t_t (c_t, m)_{|m=0} = (1 + \frac{1}{T-t-1}) u'' (c) \), so \( \text{v}^t_{cc} (c_t, m)_{|m=0} = u'' (c) \) in the continuous time limit, so
\[
c_t = \frac{w_t}{T-t} + \tau \left( \frac{B \hat{y}}{T-t}, \kappa \right) + y.
\]

**Proposition 11** In the BR life-cycle model, the optimal consumption policy is, before retirement \((t < L)\)
\[
c_t = \frac{w_t}{T-t} + \tau \left( \frac{B \hat{y}}{T-t}, \kappa \right) + y.
\]
and after retirement \( c_t = \frac{w_t}{T-t} + \hat{y} \) for \( t \geq L \). Hence, when \( \kappa > 0 \) and \( \hat{y} < 0 \), consumption weakly falls over time, and discretely falls at retirement (after which it is constant).

### 4 Neoclassical Growth Model: A Boundedly Rational Version

We study here the basic neoclassical growth model, i.e. the Ramsey-Cass-Koopmans model. We derive a BR version of it.

#### 4.1 Setup

The utility function is still \( E \sum_i \beta^i C_i^{1-\gamma} / (1 - \gamma) \), and we again note \( \psi = \frac{1}{\gamma} \). In the aggregate, the capital stock follows:
\[
K_{t+1} = F (K_t, L) + (1 - \delta) K_t - C_t + \varepsilon_{t+1}
\]
(31)
where \( \varepsilon_{t+1} \) are mean-zero shocks, whose distribution we’ll specify later. This way, there is just one state variable in the economy, the capital stock. In the most basic neoclassical model, \( \varepsilon_{t+1} \) is always 0, and \( L \) is fixed.
Figure 4: This Figure shows the traditional approach to the neoclassical growth model. Arguably, this is psychologically quite absurd. The present paper proposes a more behavioral approach.

This is a textbook example, which can be found e.g. in Acemoglu (2009, Chapter 8), Blanchard-Fischer (1989, Chapter 2), Romer (2012, Chapter 2); it introduces generations of students to macroeconomics. However, it looks somewhat odd (in my opinion), with these infinitely-rational forward looking agents that calculate the whole macroeconomic equilibrium in their heads. I present here an alternative to that model.

Let us first review some mechanics of convergence. If there were no shocks, the economy would be at the steady state, with capital stock $K^*$. I use the hat notation for deviations from the mean, e.g. $\hat{K}_t = K_t - K^*$. The law of motion for capital (31) is, in linearized form:

$$\hat{K}_{t+1} = (1 + \rho) \hat{K}_t - C_t + \varepsilon_{t+1}$$

where $\rho$ is the steady state interest rate, $\rho = \beta^{-1} - 1$.

Given there is one state variable, the policy function of the agent (rational or not) will take the form of a deviation of consumption from trend:

$$\hat{C}_t = b\hat{K}_t$$

for some positive $b$ to be determined later.

Plugging this into (32) we obtain: $\hat{K}_{t+1} = (1 + r - b) \hat{K}_t + \varepsilon_{t+1}$, i.e.

$$\hat{K}_{t+1} = (1 - \phi) \hat{K}_t + \varepsilon_{t+1}$$

with a speed of mean-reversion:

$$\phi = b - r.$$  

When agents are more reactive to shocks (when $b$ is higher), the economy mean-reverts faster to the steady state ($\phi$ is higher).
The rational agent has a value function $V(K_t)$, which satisfies:

$$V(K) = \max_c u(c) + \beta \mathbb{E}[V(K')]$$

$$K' = f(K, L) + (1 - \delta) K - c + \varepsilon_t$$

where $f(K, L)$ is gross output, and $F(K) := f(K, L) - \delta K$ is output net of depreciation.

The solution is that small deviations of the capital stock mean revert at a speed $\phi$ ($\hat{K}_t = e^{-\phi t} \hat{K}_0$) that we will characterize soon.

The steady state is at $K = K^*, C_t = C_*$ with:

$$F'(K^*) = \rho$$

which determined $K^*$ and

$$C^* = F(K^*)$$

which determined $C^*$. We define:

$$\xi = \frac{-u'(C^*)}{u''(C^*)} F''(K^*) = -\psi_c F''(K^*)$$

(36)

which plays an important role later.

### 4.2 Boundedly Rational Version

The agent has wealth $k_t$ (and we normalize the population to be 1, so that in equilibrium will be equal to $K_t$, the aggregate wealth). It evolves as:

$$k_{t+1} = (1 + r_t)(k_t + y_t - c_t)$$

where $y_t = F(K_t) - K_tF'(K_t)$ is labor income, and $r_t = F'(K_t)$ is the interest rate. We have, by Taylor expansion:

$$\hat{r}_t = F''(K^*) \hat{K}_t, \quad \hat{y}_t = -K^*F''(K^*) \hat{K}_t$$

In the agent’s model, income and interest rate evolve as

$$\hat{r}_{t+1} = (1 - \phi^s) \hat{r}_t, \quad \hat{y}_{t+1} = (1 - \phi^s) \hat{y}_t$$

where $\phi^s$ is the perceived speed of mean-reversion. As is now routine, we parametrize it as:

$$\phi^s = (1 - m_\phi) \phi^d + m_\phi \phi^r$$

where $\phi$ is the equilibrium speed of mean-reversion, and $\phi^d$ is a default value – perhaps coming from some empirical experience, saying that “business cycles” have a half-life of a few years.
Rational agent  This leads to the optimal policy (using the sensitivities derived earlier, see equation (29))

\[
\hat{c}_t = r\hat{k}_t + \frac{r}{r + \phi} \hat{y}_t + \frac{rK^* - c^*\psi}{r + \phi} \hat{r}_t \\
= r\hat{k}_t + \frac{(-rK^*F''(K^*) + (rK^* - c^*\psi)F''(K^*))}{r + \phi} \hat{K}_t
\]

i.e.

\[
\hat{c}_t = r\hat{k}_t + \frac{\xi}{r + \phi} \hat{K}_t \tag{37}
\]

Hence, from (33)

\[
b = r + \frac{\xi}{r + \phi}
\]

and using (35),

\[
\phi = \frac{\xi}{r + \phi}
\]

i.e., solving for the fixed point:

\[
\phi = \frac{-r + \sqrt{r^2 + 4\xi}}{2}. \tag{38}
\]

Behavioral agent  The behavioral agent partially does not think about those aggregate shocks \( \hat{K}_t \). Hence, instead of (37), the BR agent does:

\[
\hat{c}_t = r\hat{k}_t + \frac{\xi}{r + \phi^*} \frac{\kappa_c}{\sigma_K} \hat{K}_t \tag{39}
\]

\[
= r\hat{k}_t + \tau \left( \frac{\xi}{r + \phi^*} \frac{\kappa_c}{\sigma_K} \right) \hat{K}_t \tag{40}
\]

Hence, \( b = r + \tau \left( \frac{\xi}{r + \phi^*} \frac{\kappa_c}{\sigma_K} \right) \), and we have

\[
\phi = \tau \left( \frac{\xi}{r + \phi^*} \frac{\kappa_c}{\sigma_K} \right) \\
= \left( \frac{\xi}{r + \phi^*} - \frac{\kappa_c}{\sigma_K} \right)_+
\]

Now, what is \( m_\phi \), the attention to the accurate speed of mean-reversion? Given (39),

\[
\frac{\partial c}{\partial \phi|_{m_K=0}} = 0.
\]

This means that using the sparse max, \( m_\phi = 0 \). (With the iterated sparse max, we can have \( m_\phi > 0 \) if \( \pi \) is small enough). Hence, we have

\[
\phi = \left( \frac{\xi}{r + \phi^d} - \frac{\kappa_c}{\sigma_K} \right)_+
\]

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To solve for \( \frac{\kappa_c}{\sigma_K} \), we use the “scale-free” version of \( \kappa \), equation (16).

\[
\frac{\kappa_c}{\sigma_K} = \left( \frac{\kappa}{-u_{cc}} \right)^{1/2} \frac{1}{\sigma_K} \left( \frac{-\kappa^2 u_{cc} \var(r \hat{K}_t)}{-u_{cc}} \right)^{1/2} = \frac{r \sigma K}{\sigma_K} = \hat{r}
\]

Hence, we obtain:

\[
\hat{c}_t = r \hat{K}_t + \tau \left( \frac{\xi}{r + \phi^s}, \hat{r} \right) \hat{K}_t
\]

so that in equilibrium, \( \hat{c}_t = b^s \hat{K}_t \)

\[
b^s = r + \tau \left( \frac{\xi}{r + \phi^s}, \hat{r} \right)
\]

Hence, the equilibrium mean-reversion \( \phi \) is: \( \phi = b^s - r \):

\[
\phi = \tau \left( \frac{\xi}{r + \phi^s}, \hat{r} \right)
\]

(41)

When \( m_\phi = 0 \), we have \( \phi = \tau \left( \frac{\xi}{r + \phi^d}, \hat{r} \right) \)

\[
\phi = \left( \frac{\xi}{r + \phi^d - \hat{r}} \right)_+
\]

(42)

Note that \( \phi \) has the same comparative statics as the rational case: \( \frac{\partial \phi (\xi, r)}{\partial \xi} \geq 0 \), \( \frac{\partial \phi (\xi, r)}{\partial r} \leq 0 \).

A new comparative statics emerges: \( \frac{\partial \phi (\xi, r)}{\partial \xi} \leq 0 \).

**Proposition 12** The speed of mean-reversion of the economy is:

\[
\phi = \left( \frac{\xi}{r + \phi^d - \hat{r}} \right)_+
\]

In particular, \( \phi \) is decreasing in \( \kappa \), \( \phi (\kappa = 0) = \phi^r \). We have \( \phi > 0 \) iff \( \kappa \leq \kappa^* := (1 + r^2 / \xi)^{-2} \).

If the default \( \phi^d = \phi \) (as in a rational expectation fixed point), when the solution is

\[
\phi = \left( \frac{\xi}{r + \phi - \hat{r}} \right)_+
\]

whose solution is

\[
\phi = \frac{1}{2} \left( -r (1 + \kappa) + \sqrt{r^2 (1 - \kappa)^2 + 4 \xi} \right)_+
\]
The impact of fluctuations Finally, squaring equation (34), we obtain: \( \text{var}\, \hat{K}_{t+1} = (1 - \phi)^2 \text{var}\, \hat{K}_t + \sigma^2 \). As in the steady state, \( \text{var}\, \hat{K}_{t+1} = \text{var}\, \hat{K}_t \),

\[ \text{var}\, \hat{K}_t = \frac{\sigma^2}{1 - (1 - \phi)^2} \]

When shocks mean-revert more slowly (lower \( \phi \)), the average deviation of the stock price from trends is higher (shocks “pile up” more). Hence, (provided \( \phi^d \geq \phi \)) the variance of shocks will be larger in the sparse economy than in the rational economy.

**Proposition 13** Suppose that \( \phi^d \geq \phi \). Then, \( \phi \leq \phi^* \), so that in the sparse economy, the speed of mean-reversion is slower, and the variance of shocks is bigger, than in the rational economy.

5 Dynamic Portfolio Choice

I now study a Merton problem with dynamic portfolio choice. The agent’s utility is:

\[ E \left[ \frac{1}{1 - \gamma} \int_0^\infty e^{-\rho s} c_s^{1-\gamma} ds \right], \]

and his wealth \( w_t \) evolves according to:

\[ dw_t = (-c_t + rw_t) dt + \omega_t \pi_t (\pi_t dt + \sigma dZ_t) \]

where \( \omega_t \) is the allocation to equities.

I start by describing the rational problem, then the behavioral solution. I call \( \psi = \frac{1}{\gamma} \) the IES. Though for simplicity I use CRRA utility function, I try to write the expressions in a way that involves both \( \gamma \) and \( \psi \), in a way that would generalize correctly to Epstein-Zin utility, where the two notions are disentangled.

5.1 Taylor expansions of the value function: rational case

We examine the problem in the rational case first, with a reminder of notions of portfolio choice. In a deterministic context with interest rate \( r_t \), the SDF is simply \( M_t = e^{-\int_0^t r_s ds} \). Next, suppose that there is a stochastic opportunity set: A set of assets with risk premium \( \pi_t \), and covariance matrix \( \Sigma_t \). In a static maximization, the optimal portfolio the certainty equivalent is a return: \( R_t(\theta_t) = r_t + \theta_t \pi_t - \frac{1}{2} \theta_t \Sigma_t \theta_t \), so that the (static) optimal portfolio choice is \( \theta_t = \text{arg max}_\theta R_t(\theta) \), i.e. \( \theta_t = \frac{1}{\gamma} \Sigma_t^{-1} \pi_t \), and the certainty equivalent is finally:

\[ R_t = \max_{\theta_t} R_t(\theta_t) \]

\[ R_t = r_t + \frac{1}{2\gamma} \Lambda_t \]  

where

\[ \Lambda_t = \pi_t \Sigma_t^{-1} \pi_t \]
the “squared Sharpe ratio” of the investment opportunity set. Suppose that the process is
driven by a Brownian motion $B_t$ (which may be multidimensional). If the price of risk is $\lambda_t$
(so that $\Lambda_t = \|\lambda_t\|^2$), the stochastic discount factor can be represented as:

$$M_t = \exp \left[ - \int_0^t \left( r_s + \frac{\Lambda_t}{2} \right) ds - \lambda_s dB_s \right]$$  \hspace{1cm} (45)

The value function is as follows.

**Lemma 2** Suppose that $r_t$ and $\lambda_t$ are deterministic. The value function is

$$V_{w_t}(w_t, x_t) = (\mu_t w_t)^{-\gamma}$$

and the optimal policy is to consume $c_t = \mu_t w_t$, where $\mu_t^{-1}$ is given by:

$$\mu_t^{-1} = E_t \left[ \int_0^\infty e^{-\psi r_s} \left( \frac{M_{t+s}}{M_t} \right)^{1-\psi} ds \right] = E_t \left[ \int_0^\infty e^{-\int_t^s (\psi \rho_u + (1-\psi) R_u) du} dt \right]$$  \hspace{1cm} (46)

where

$$R_t = r_t + \frac{1}{2\gamma} \Lambda_t.$$

is the certainty equivalent of expected portfolio returns (comprising stocks and bonds), with
$\Lambda_t = \|\lambda_t\|^2$ is the square Sharpe ratio of the investment opportunity set.

When the opportunity set is constant, we have $R_t = R_*$ and $\mu_t = \mu_*$ with

$$\mu_* = \psi \rho + (1 - \psi) R_*.$$  \hspace{1cm} (47)

When it is not constant, we have, up to second order terms:

$$\mu_t = \psi \rho + (1 - \psi) \bar{R}_t$$  \hspace{1cm} (48)

where $\bar{R}_t = \mu_* V_t^R$ is the average future portfolio returns, and $V_t^R$ is the present value of
future portfolio returns.

$$V_t^R := E_t \left[ \int_t^\infty e^{-\mu_*(s-t) R_s} ds \right]$$  \hspace{1cm} (49)

Here is the future average return of the portfolio (including stocks and bonds). Hence,
the marginal propensity to consume is a weighted average (with weights $\psi$ and $1 - \psi$) of the
pure rate of time preference $\rho$, and the average future return of the portfolio. The discount
rate for “future return” is the subjective discount rate $\mu_*$. 

Lemma 2 summarizes and somewhat generalizes well-known notions, particularly from
the work of Campbell and Viceira (2002). It indicates that what matters is the risk-adjusted
rate of return of the portfolio, $\bar{R}_t$: it is the safe short-term rate $r_t$, plus the square Sharpe
ratio $\Lambda_t$, divided by two times risk aversion. The future average return $\bar{R}_t$ is key to capture
the (leading order of) the value function.
Suppose that the vector of asset returns $\tilde{d}r_t$ (where $\tilde{d}r_{it}$ is the return of asset $i$) follows:

$$d\tilde{r} = (r + \pi + \hat{\pi}_t) dt + \sigma dZ_t$$

$$\hat{\pi}_t = f'X_t$$

where $X_t$ is a vector of factors, following an AR(1):\(^5\)

$$dX_t = -\Phi X_t dt + \sigma X dZ_t$$

and $f$ is a matrix of weights. We call

$$\Sigma^{r,X} = \text{cov}(d\tilde{r}, dX_t) / dt = \sigma \Sigma^{X'}.$$

the matrix of covariance, i.e. $\Sigma^{r,X}_{ij} = \text{cov}(d\tilde{r}_{it}, dX_{jt}) / dt$. We define $\theta_* = \frac{1}{\gamma} \Sigma^{-1} \pi_*$ the portfolio choice in the model with constant variance and expected returns.

Then, the portfolio return is

$$R_t = \frac{1}{2\gamma} (\pi_* + \pi_t)' \Sigma^{-1}_t (\pi_* + \pi_t) = \frac{1}{2\gamma} \pi_* \Sigma^{-1} \pi_* + \theta'_* \hat{\pi}_t + \mathcal{O}(\|X_t\|^2)$$

$$= R_* + \theta'_* \hat{\pi}_t$$

$$= R_* + \theta'_* f'X_t = R_* + b'X_t$$

i.e. the return is augmented by $\theta'_* \hat{\pi}_t$, with

$$b := f \theta_*.$$

Then, the present value of returns (49) is:

$$V_t^R = \frac{R_*}{\mu_*} + b' (\mu_* I + \Phi)^{-1} X_t$$

(50)

where $I$ is the identity matrix of the $X$'s dimension.

For instance, if $X_t$ is a one-dimensional, so that $bX_t = \hat{R}_t := R_t - R_*$, and $\tilde{R}_t := R_* + \frac{\mu_*}{\mu_* + \Phi} \hat{R}_t$.

$$\mu_t = \mu_* + (1 - \psi) \frac{\mu_*}{\mu_* + \Phi} \hat{R}_t$$

(51)

Hence, we obtain a tractable representation of the value function, to the leading order.

\section{5.2 The hedging demand}

We can calculate the hedging demand.\(^5\)

\footnote{Or $X_t$ could be a linearity-generating twisted-AR(1), so that the derivations below can be exact (Gabaix 2009).}
Lemma 3 (Hedging demand, rational) The stock demand is

\[ \theta_t = \frac{1}{\gamma} \Sigma^{-1}_t (\pi_t + H_t) \]  

(52)

where \( H_t \) is the hedging demand premium, equal to (up to second order terms):

\[ H_{it} = (1 - \gamma) \text{cov}(d\hat{r}_i, dV_t^R) \]  

(53)

i.e. \( H_{it} \) is \((1 - \gamma)\) times the covariance between asset \( i \)'s return \((d\hat{r}_i)\) and the present value of future returns \( V_t^R \) (equation 49).

In the AR(1) framework above,

\[ H_t = (1 - \gamma) \Sigma^{-1}_t (\mu_* I + \Phi)^{-1} b. \]  

(54)

Suppose that returns mean-revert, i.e. \( \text{cov}(d\hat{r}_{it}, d\tilde{R}_t) < 0 \). So, if \( \psi < 1 \), then investors load more on stocks because of the hedging demand.

We next state the modification of the value function.

Lemma 4 (Value function with hedging demand, rational) In the hedging demand context, we have:

\[ \mu_t = \psi \rho + (1 - \psi) (\tilde{R}_t + \theta' H_t) \]  

(55)

where \( \tilde{R}_t = \mu_* V_t^R \) is the expected present value of returns, and \( H_t \) is the hedging demand term; they are explicit in (50) and (54).

The intuition for (52) is that \( H_{it} \) is a risk-adjusted risk premium of asset \( i \). This intuition carries over to (55). Compared to (48), the expression for \( \mu(X_t) \) offers one more term, the term \((1 - \psi) \theta' H_t\).

5.2.1 A tractable case

The equity premium \( \pi_t = \bar{\pi} + \hat{\pi}_t \) has a variable part \( \hat{\pi}_t \), which follows

\[ d\hat{\pi}_t = -\phi_R \hat{\pi}_t dt - \chi \sigma dZ_1^1 + \sigma' \sigma dZ_2^2 \]

where the return is \( d\hat{r}_t = (\bar{r} + \pi_t) \) \( dt + \sigma dZ_1^1 \). The parameter \( \chi \geq 0 \) indicates that equity returns mean-revert: good returns today lead to lower returns tomorrow. That will create a hedging demand term.

We call \( \theta_* := \frac{\bar{\pi}}{\sigma^2} \) the standard, myopic demand for stocks.
5.2.2 Sparse agent

We have the following (using the notation $\psi = 1/\gamma$ for the IES):

**Proposition 14** *(Behavioral dynamic portfolio choice)* The fraction of wealth allocated to equities is, with $\theta_* := \frac{\pi}{\gamma \sigma^2}$

$$\theta_*^s = \theta_* + \tau \left( \frac{\hat{\pi}_t}{\gamma \sigma^2}, \kappa \right) + \tau \left( \frac{H_t}{\gamma \sigma^2}, \kappa_\theta \right)$$

while consumption is: $c_*^s = \mu_*^s w_t$ with

$$\mu_*^s = \mu_* + \tau \left( (1 - \psi) \frac{\mu_*}{\mu_* + \Phi} \theta_* \hat{\pi}_t, \kappa_{c/w} \right) + \tau \left( (1 - \psi) \theta_* H_t, \kappa_{c/w} \right)$$

where $H_t$ is the hedging demand term (56)

$$H_t = (1 - \gamma) \text{cov} \left( dr_t, dV_t^R \right) = - (1 - \gamma) \theta_* \frac{1}{\mu_* + \Phi} \sigma^2 \chi_t$$

**Proof** We first calculate the rational values. In that case

$$\bar{R}_t = r_* + \frac{\Lambda_*}{2\gamma} + \theta_* \frac{\mu_*}{\mu_* + \Phi} \hat{\pi}_t$$

$$H_t = (1 - \gamma) \text{cov} \left( dr_t, d \left( \frac{\bar{R}_t}{\mu_*} \right) \right) = - (1 - \gamma) \theta_* \frac{1}{\mu_* + \Phi} \sigma^2 \chi_t$$

(56)

so that

$$\theta_t = \frac{\pi_* + \hat{\pi}_t + H_t}{\gamma \sigma^2}$$

In addition

$$\mu_t = \psi \rho + (1 - \psi) \left( \bar{R}_t + \theta'_* H_t \right) = \mu_* + (1 - \psi) \left( \theta_* \frac{\mu_*}{\mu_* + \Phi} \hat{\pi}_t + \theta'_* H_t \right)$$

As in Proposition 5, with ex-post attention, the BR agent just truncates those terms.

\[\square\]

Proposition 14 predicts the choice of a sparse agent. When $\kappa = 0$, it is the policy of a fully rational agent, e.g. as in Campbell and Viceira (2002). When $\kappa > 0$, it is the policy of a sparse agent. When $\kappa$ is larger, portfolio choice becomes insensitive to the change in the equity premium, $\hat{\pi}_t$, and the agent thinks less about the mean-reversion of asset, the $B\chi$ terms.

In addition, the agents’ consumption function pays little attention to the mean-reversion of assets. [Next iteration should have a calibration.]
6 Behavioral Version of a Few Other Models

To probe the validity of the framework, we study here a few other models.

6.1 Linear-Quadratic models

A lot of economic problems can be conveniently expressed as linear-quadratic (LQ) models (Ljungqvist and Sargent 2012). We show here how to systematically derive a BR version of those models.

We again write $z = (w, x)$, where $w$ is the set of variables known under the default model, and $x$ is the set of variables that are not-considered in the default model. Utility is:

$$u(z, a) := \frac{1}{2} \begin{pmatrix} z \\ a \end{pmatrix}^\prime \begin{pmatrix} U_{zz} & U_{za} \\ U_{az} & U_{aa} \end{pmatrix} \begin{pmatrix} z \\ a \end{pmatrix}$$

and the law of motion is:

$$z' = F^z(z, a) := \Gamma^w z + \Gamma^a a$$

where $U$ and $\Gamma$ are constant matrices. The rational value function is also LQ

$$V(z) = -\frac{1}{2} z' P_{zz} z = -\frac{1}{2} (w' P_{ww} w + 2w' P_{wx} x + x' P_{xx} x)$$

Under the default model, $P_{ww}$ is known, and

$$a^d(w) = A_w w$$

for $A_w$ a constant. Our goal is to find $P_{wx}$, which affects the value function. To do so, we apply from (231).

**Lemma 5** In the linear-quadratic problem, the cross-partial of the value function is

$$P_{wx} = V_{wx} = \left[ 1 - \beta (D_{wx} w') \cdot \Gamma_x \right]^{-1} \left[ U_{wx} + U_{xa} A_w + \beta \Gamma_x w w V_{ww} (D_{wx} w') \right].$$

where $D_{wx} w' = \Gamma_w^w + \Gamma_a^w A_w$. The impact on the action is $a = A_w w + A_x x$, where $A_w$ is the default value, and

$$A_x = -\Psi_a^{-1} \Psi_x,$$  \hspace{1cm} (57)

where

$$\Psi_a = U_{aa} + \beta \Gamma_a^w V_{ww} \Gamma_a^w$$

$$\Psi_x = U_{xa} + \beta V_{wx} \Gamma_a^w$$
This illustrates that the value function can be written:

\[ V(z) = -\frac{1}{2} z' P zz = -\frac{1}{2} w' P_{ww} w + w' P_{wx} x + O(\|x\|^2) \]

with matrix \( P_{wx} \) is expressed in closed form above.

Hence, the BR value function is simply:

\[ V^a(z, m) = -\frac{1}{2} z' P zz = -\frac{1}{2} w' P_{ww} w + w' P_{wx} M (m) x + O(\|x\|^2) \]

for the diagonal attention matrix \( M (m) = \text{diag} (m_x) \).

**Proposition 15** (Behavioral version of linear-quadratic problems) *In a linear-quadratic problem, the optimal attention is\( m_{xi} = A (A_{xi}, \Psi_{xi} A_{xi} \sigma_{xi}^2 / \kappa) \) and the optimal sparse action is\( a = A_{wx} w + A_{xx} x \) where \( M = \text{diag} (m_x) \). Here we use the notations of Lemma 5.*

To study an example, we now turn to the Beker-Murphy model of rational addiction.

### 6.2 The Becker-Murphy model of rational addiction

The Becker-Murphy (1988) model of rational addiction is a peak of the use of rationality in economics. We will give a behavioral version of it. We shall see that the qualitative evidence in favor of the model (the fact that future increase in prices lower consumption today) are also consistent with this BR version: it is shows that agent are at least *partially* rational (as in the present model), not that they are *fully* rational (as assumed by Becker-Murphy). This distinction is important: If people are BR enough, they’d be better off under a high tax, or a ban, of the addictive substance – while the optimal tax is 0 in the Becker-Murphy model. This analysis is in the spirit of Gruber and Koszegi (2001), who study a hyperbolic discounting addict, rather than a boundedly rational one in the sense of this paper.

We call \( c \) the consumption, and \( x \) the level of addition. Utility function is:

\[ u(c, x) = -\frac{1}{2} (c - x - A)^2 - Bx \]

Addition \( x_t \) evolves as:

\[ x_{t+1} = \rho x_t + h c_t. \]

The BR agent has in mind the model:

\[ x_{t+1} = \rho^a x_t + h^a c_t. \]
We posit that in the default model the agent does not perceive any addiction dynamics: he perceives addition as being constant.

$$\rho^d = 1, \quad h^d = 0.$$ 

When the agent has partial attention \( m \) to inattention dynamics, we have:

$$x_{t+1} = (1 - m) x_t + m (\rho x_t + h c_t)$$

so

$$\rho^s = (1 - m) + m \rho, \quad h^s = mh$$

Let us now study the BR dynamics.

**Warm-up: 2 period model**  
As before, it is helpful to study a 2-period model, with \( t = 1, 2 \).

*Behavior at the last period, \( t = 2 \).* The agent should and does consume his optimal consumption

$$c^d (x) = \arg \max_c u (c, x) = x + A$$

We define the resulting utility as \( \bar{u} (x) \)

$$\bar{u} (x) := \max_c u (c, x) = u (c^d (x), x) = -B x.$$  

To, the time-1 value function is

$$V^1 (x) = \bar{u} (x). \quad (59)$$  

*Behavior at period 1, \( t = 1 \).* Given perceived dynamics, the problem is:

$$\max_{c, m} v (c, x, m)$$  

$$v (c, x, m) := u (c, x) + \beta V (\rho^s (m) x + h^s (m) c)$$

which gives:

$$0 = u_c + \beta h^s V' (\rho^s x + h^s c)$$

$$= -c + x + A - \beta h^s B$$

$$c = x + A - \beta h^s B \quad (60)$$

An interesting case is to impose \( c \geq 0 \). Then, first period consumption is \( >0 \) iff \( A - h^s B > 0 \). So, if \( h^s B < A \leq h B \), then the rational agent consumes 0, while the very behavioral agent consumes a positive amount, and gets addicted.

The optimal attention is \( m = A (v_{cc}^2 / \kappa) = A (-u_{cc} B^2 h^2 / \kappa) \).
Infinite horizon model The value function satisfies:

\[ V(x) = \max_{c,m} u(c, x) + \beta V(\rho^s(m)x + h^s(m)c) \]

The FOC is:

\[ u_c(c, x) + \beta V'(\rho^s(m)x + h^s(m)c) h^s(m) = 0 \]

i.e. the agent takes into account only part of the addiction costs, as \( h^s(m) \leq h \). As a result, the agent is more addicted in the steady state. The greater the myopia, the greater the optimal tax.

**Proposition 16** In the Becker-Murphy model with boundedly rational agents, the consumption \( c \) given the stock of addition \( x \) is:

\[ c(x) = x + A + \beta b(m)m^h \]

using \( m = (m^h, m^V) \); the value function is

\[ V(x, m) = a(m) + b(m)x \]

where \( b(m) = -\frac{B}{1-\beta(1+m^V(\rho+h-1))} \) and \( a(m) \) is in the proof. When using the plain (as opposed to iterated) sparse max, \( m^V = 0 \) and attention to addition is \( m^h = A \left( \frac{1}{\alpha} \left( \frac{\beta h}{1-\beta} \right)^2 \right) \).

### 7 Discussion

#### 7.1 Active decision: Consumption or Savings?

Here we assume that the active decision was one of consumption. One could imagine that it would be in savings. Does this matter? First, for many variables, it does matter: the impact of interest rates, future taxes, future income shocks etc. are the same whether a sparse agent uses the consumption frame or saving frame. However, the frame does matter for one variable: current income. Indeed, take the permanent-income setup. \(^6\)

\(^6\)Recall that \( \hat{c}_t^r = \frac{r}{r+\phi}\hat{y}_t \), so

\[ \hat{c}_t^r = \frac{r}{r+\phi}m\hat{y}_t \] under the consumption frame

However, if the consumer chooses savings, \( S_t \), then consumes \( c_t = wy_t - S_t \), the rational amount is \( S_t^r = \hat{y}_t - \hat{c}_t^r \), i.e. \( S_t^r = \frac{\phi}{r+\phi}\hat{y}_t \). Hence, the savings of a sparse agent is \( \hat{S}_t^s = \frac{\phi}{r+\phi}m\hat{y}_t \), and the deviation of consumption is: \( \hat{c}_t^s = \hat{y}_t - \hat{S}_t^s \), i.e.

\[ \hat{c}_t^s = \left( 1 - \frac{m\phi}{r+\phi} \right) \hat{y}_t \] under the savings frame

which is generally not the same as \( \hat{c}_t^r \) under the consumption frame.
Which frame does the agent use? Here, we’ll use the working hypothesis that the agent takes the frame that yields the higher expected utility. We use the following Proposition.

**Proposition 17 (Welfare under the consumption vs savings frame)** The consumption frame yields greater utility than the savings frame if and only if \( \phi > r \), i.e. if income shocks mean-revert not too slowly. More precisely, under the “active consumption” frame, the utility loss from a BR policy is, to the leading order in \( \sigma_y^2 \), \( L^C = A (1 - m)^2 \phi_y^2 \), for \( A = \frac{u''(c^d(\omega))\sigma_y^2}{2(\tau + 2\phi_y)(\tau + \phi_y)^2} \), while under the “active savings” frame, they are \( L^S = A (1 - m)^2 r^2 \).

When \( \phi > r \) (which is probably the relevant case, if business-cycle fluctuations partly mean-revert), the “consumption” frame is indeed better for the agent, at least most of the time. This make sense: savings are there to absorb transitory income shocks, and consumption should be smooth. When the agent chooses consumption in an inattentive manner, it makes consumption quite smooth indeed. However, if the agent chooses savings inattentively, he makes savings smooth, but consumption needs to absorb the shocks, hence it is quite volatile. Hence, generally, to keep consumption smooth, choosing consumption inattentively is better than choosing savings inattentively.

However, when income shocks are a random walk (\( \phi = 0 \)), the savings frame is better. An inattentive agent will keep a constant savings, and let consumption react one for one to income shock, which is the normatively correct behavior when income shocks are completely persistent.

It may be useful to see the effect in a simpler context. Take a 3 period model with \( \beta = R = 1 \), and an income shock with persistence \( \rho \); \( \hat{y}_t = \rho^{t-1} \varepsilon \) for \( t = 0, 1, 2 \), with \( \varepsilon \) a mean-0 shock. Normatively, that should induce the change \( \tilde{c} = (\tilde{c}_t)_{t=0,1,2} = (1, 1, 1) \frac{1}{3} + \rho + \rho^2 \varepsilon \) (indeed, the total value of income has increased by \( (1 + \rho + \rho^2) \varepsilon \)). Let us not consider a BR agent with \( m = 0 \). However, under the consumption frame, \( \tilde{c}^C = (0, \frac{1}{2}, \frac{1}{2} + \rho + \rho^2) \varepsilon \) (as there is no reaction of \( c_0 \), so that time-1 wealth increases by \( \hat{w}_1 = \varepsilon \), of which \( 1/2 \) is consumed at time 1, so \( \tilde{c}^C_t = \frac{\varepsilon}{2} \)). Under the savings frame, we get \( \tilde{c}^S = (1, \rho, \rho^2) \varepsilon \) (savings doesn’t change, consumption absorbs all the shocks). It is easy to verify that for \( \rho \) small, the utility is higher under the consumption frame, while the opposite for large \( \rho \). Indeed, when \( \rho = 0 \), \( \tilde{c}^C = (0, \frac{1}{2}, \frac{1}{2}) \varepsilon \) and \( \tilde{c}^S = (1, 0, 0) \varepsilon \), so there is more smoothing under the consumption frame. Other the other hand, with \( \rho = 1 \), \( \tilde{c}^C = (0, \frac{1}{2}, \frac{5}{2}) \varepsilon \) and \( \tilde{c}^S = (1, 1, 1) \varepsilon \), and there is more smoothing under the savings frame.

### 8 Conclusion

I presented a practical way to do boundedly rational dynamic programming. It is portable and to the first order has just one free continuous parameter, \( \kappa \), the penalty for lack of sparsity, which can also be interpreted as a cost of complexity.

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7 To the leading order, \( \tilde{u} = \frac{1}{2} u''(c^d) E \sum_t c_t^2 \), so \( \tilde{u}^C = \frac{1}{2} u''(c^d) \sigma^2 \left( \frac{1}{4} + \left( \frac{1}{2} + \rho + \rho^2 \right) \right) \) and \( \tilde{u}^S = \frac{1}{2} u''(c^d) \sigma^2 \left( 1 + \rho^2 + \rho^4 \right) \). This yields \( \tilde{u}^C \geq \tilde{u}^S \) if \( \rho < \rho^* \approx 0.32 \).
It allows us to revisit canonical models in economics, and give them a behavioral flavor.

From the micro point of view, we obtain inattention and delayed response. Those are not necessarily very surprising features – however, it is useful to have clean model that generates those things and can be calibrated.

From the macro point of view, the model allows us to think about bounded rationality in general equilibrium. The upshot is that compared to the rational model, sparsity leads to larger and more persistent fluctuations. The reason is that rational actors tend to “dampen” fluctuations. For instance, they consume more when more capital is available. This channel is muted with sparse agents. Hence, fluctuations are more persistent, innovations have a longer-lasting effect, and the average fluctuations (deviations from the mean) are larger.

Given that it seems easy to use and sensible, we can hope that this model may be useful for other extent issues in macroeconomics and finance.
9 Appendix: Attention and Truncation Functions

Here are some good truncation functions. In Gabaix (2014), I study attention functions \( A_{\alpha}(\sigma^2) \): they are weakly increasing, from 0 (complete inattention) to 1 (full attention). Here I defined the related truncation function \( \tau_\alpha \):

\[
\tau_\alpha (b, k) := b A_{\alpha} \left( \frac{b^2}{k^2} \right)
\]

It is the coefficient \( b \), times the attention to the coefficient, divided by the scaled cognition cost \( k \). For instance, for the values \( \alpha = 0, 1, 2 \), we have (Gabaix 2014):

\[
A_0 (\sigma^2) = 1_{\sigma^2 \geq 2}, \quad A_1 (\sigma^2) = \max \left( 1 - \frac{1}{\sigma^2}, 0 \right), \quad A_2 (\sigma^2) = \frac{\sigma^2}{2 + \sigma^2}
\]

hence the truncation functions \( \tau_\alpha (b, k) \):

\[
\tau_0 (b, k) = b \cdot 1_{b^2 \geq 2k^2}, \quad \tau_1 (b, k) = b \max \left( 1 - \frac{k^2}{b^2}, 0 \right), \quad \tau_2 (b, k) = \frac{b^3}{b^2 + k^2}
\]

Figure 5 plots the attention functions, and Figure 6 the corresponding truncation functions.

10 Appendix: More proofs

10.1 Life-cycle example: detailed calculations

We start from the simple life-cycle example. We assume, for simplicity, a stationary environment with no trend growth. The Bellman equation is:
Figure 6: Three truncation functions. Because it gives sparsity and continuity, the $\tau_1$ function is recommended.

\[
V(w, r) = \max_c u(c) + \beta V'((R + r)(w - c) + y', r')
\]  

(62a)

I suppress the expectation operator, as the shocks are assumed to be small. We assume a law of motion:

\[
r' = \rho r + \varepsilon'
\]

Call next-period wealth $w'$:

\[
w' = (R + r)(w - c) + y'
\]

We assume that the agent knows the simple model where the interest rate is always at its average, $r \equiv 0$. As is well-known, the optimal policy is $c = rw + y$, and, with $\bar{R} = 1 + \tau$,

\[
V(w) = A \left( w + w^H \right)^{1-\gamma} / (1 - \gamma), \quad w^H = Y/\tau, \quad A = (\tau/\bar{R})^{-\gamma}
\]

First, we differentiate the Bellman equation with respect to the new variable:

\[
V_r(w, r) = \beta V_{wr}(w', r') \frac{\partial w'}{\partial r} + \beta V_{rr}(w', r') \frac{\partial r'}{\partial r}
\]

\[
V_r(w, r) = \beta V_{wr}'(w', r') (w - c) + \beta V_{rr}'(w', r') \rho
\]  

(63)

Evaluating at $r = 0$, this leads to:

\[
V_r(w, 0) = V_{wr}^d(w) \frac{\beta (w - c)}{1 - \beta \rho}
\]

We now take the total derivative with respect to $w$, $D_w f = \partial_w f + \frac{dw}{dw} \partial_w f$, e.g. the full impact of a change in $w$, including the impact it has on a change in the consumption $c$.  

40
The baseline policy is \( c(w) = \tau w / \bar{R} + \varphi \), so \( D_w c = \tau \), and \( D_w w' = d(\bar{R}(w - c)) / dw = \bar{R} - \bar{R} \tau / \bar{R} = 1 \).

\[
D_w c = \tau / \bar{R} \\
D_w w' = 1 
\]

This means that one extra dollar of wealth received today translates into exactly one dollar of wealth next period: its interest income, \( r \), is entirely consumed.

So differentiate (using the total derivative) equation 63. We obtain:

\[
\beta^{-1} V_{wr}(w, r) = V'_{w'w'}(w', r') (D_w w) \cdot (w - c) + V'_{w'}(w', r') D_w (w - c) + V'_{w'r'}(w', r') \rho D_w w' \\
= V'_{w'w'}(w', r') (w - c) + V'_{w'}(w', r') (1 - \frac{\tau}{\bar{R}}) + V'_{w'r'}(w', r') \rho \\
\]

so, using

\[
V'_{w'w'}(w', r') = -\gamma V'_{w'} \cdot \frac{1}{w + w^H} = -\gamma V'_{w'} \cdot \frac{\tau}{RC} \\
V_{w,r} = \frac{\beta V'_{w'}}{\bar{R}} (1 - \gamma \bar{r} \frac{w - c}{c}) \\
\]

Finally, let’s derive the impact of a change in \( r \) on \( c \): We have

\[
V_w = \beta (\bar{R} + r) V'_{w'}(c) \\
\]

so

\[
\frac{dc}{dr} = \frac{V_{wr}}{u''(c)} = \frac{-1}{u''(c)} \frac{V_w 1 - \gamma \bar{r} \frac{w}{c} - 1}{\bar{R} - \rho r \beta} \\
= \frac{-1}{\gamma u''(c)} \frac{V_w 1 - \gamma \bar{r} \frac{w}{c} - 1}{\bar{R} - \rho r} \\
\frac{dc}{c} = \frac{1}{\bar{R} - \rho r} \left( \frac{w^c}{c} - 1 \right) - \frac{1}{\gamma} dr \\
\]

We note that the result

\[
b_y = \frac{\tau}{\bar{R} (\bar{R} - \rho y)} c_t^d, \quad b_r = \frac{\tau \left( \frac{w}{c_t} - 1 \right) - 1/\gamma}{\bar{R} - \rho r} \\
\]

becomes, in continuous time:

\[
b_y = \frac{\tau}{\bar{r} + \phi_y}, \quad b_r = \frac{\tau \left( \frac{w}{c_t} - 1 \right) - 1/\gamma}{\bar{r} + \phi_r} \\
\text{ (64) } 
\]
10.2 Continuous time

Calculations are typically cleaner in continuous time, so we develop the continuous-time version of the machinery. We take for now problems without stochastic terms (those should be added later).

The laws of motion are:

\[
\dot{w}_t = F^w (w, x, a) \\
\dot{x}_t = F^x (w, x)
\]

and the Bellman equation is:

\[
\rho V (w, x) = u (w, x, a) + V_w (w, x) F^w (w, x, a) + V_x (w, x) F^x (w, x, a)
\]

In the more complex case \(\dot{x}_t = F^w (w, x, a)\), we need to solve for a matrix Ricatti equation — but not here.

Call \(\Delta w = \partial_w + a_w \partial_a\) the “total impact” of a change in \(w\). Then:

\[
\rho V_x = u_x + V_w F^w_x + V_x F^x + V_{xx} F^x
\]

Now, we differentiate and evaluated at \(x = 0\):

\[
\rho V_{wx} = D_w (u_x + V_w F^w_x) + V_{wx} F^x + V_x F^x_{wx}
\]

so

\[
V_x = (\rho - F^x_x)^{-1} [u_x + V_w F^w_x] \tag{66}
\]

\[
V_{wx} = (\rho - F^x_x)^{-1} [D_w (u_x + V_w F^w_x) + V_x F^x_{wx}] \tag{67}
\]

As \(a\) satisfies \(\Psi = 0\) with

\[
\Psi (a, w, x) = u_a + V_w F^w_a
\]

Hence, the impact of \(x\) on the optimal action is

\[
a_x = -\Psi^{-1}_a \Psi_x
\]

\[
\Psi_a = u_{aa} + V_w F^w_{aa} \\
\Psi_x = u_{ax} + V_{wx} F^x_a + V_w F^w_{ax}
\]

Calculation of \(V_{xx}\). We now turn to the more difficult case of \(V_{xx}\). Using \(D_x = \partial_x + a_x \partial_a\) the “total impact” of a change in \(x\), we have:

\[
\rho V_x = D_x u + V_w D_x F^w_x + V_x F^x_x + V_{xx} F^x
\]

\[
= a_x (u_a + V_w F^w_a) + u_x + V_w F^w_x + V_x F^x_x + V_{xx} F^x
\]
Next, differentiating at $x = 0$,

$$
\rho V_{xx} = a_x D_x (u_a + V_w F_a^w) + D_x [u_x + V_w F_x^w + V_z F_x^x] + V_{xx} F_x^x
$$

$$
= a_x [u_{ax} + u_{aa} a_x + V_{wx} F_a^w + V_w F_a^w + V_w F_{aa}^w a_x] \\
+ u_{xx} + u_{xa} a_x + V_{ww} F_x^w + V_{w} D_x F_x^w + 2V_{xx} F_x^x + V_z F_x^x
$$

hence

$$(\rho - 2F_x^x) V_{xx} = a_x [u_{ax} + u_{aa} a_x + V_{wx} F_a^w + V_w F_a^w + V_w F_{aa}^w a_x] \\
+ u_{xx} + u_{xa} a_x + V_{ww} F_x^w + V_{w} D_x F_x^w + V_z F_x^x
$$

This is a bit of a complicated expression. Let us note it can be written

$$(\rho - 2F_x^x) (V_{xx}^s - V_{xx}^r) = a_x A + a_x B a_x + C$$

with $B = u_{aa} + V_w F_{aa}^w$.

We use the following elementary Lemma:

**Lemma 6** Let $f(a) = A a + a' B a + C$, for $B$ symmetric negative definite. Let $a^* = \arg \max_a f(a)$, so $a^* = -\frac{1}{2} B^{-1} A$. Then, for any $a$,

$$f(a) - f(a^*) = (a - a^*) B (a - a^*) \cdot$$

Let’s compare $V_{xx}$ under the sparse vs rational model: the difference is just in the $D_x^r$ vs $D_x^s$ term. Indeed,

$$D_x^s - D_x^r = (a_x^s - a_x^r) \partial a$$

so, using the previous Lemma,

$$V_{xx}^s - V_{xx}^r = (\rho - 2F_x^x)^{-1} (a_x^s - a_x^r) (u_{aa} + V_w F_{aa}^w) (a_x^s - a_x^r) \quad (68)$$

We gather the results.

**Proposition 18** (What are the losses from a suboptimal policy?) Consider the value function $V^r$ under the optimal policy and $V^s$ under a potentially suboptimal policy, and $V^s (w, x) = V^s (w, x) - V^r (w, x)$. Then, evaluating at $x = 0$, we have:

$$V^\delta = 0, V^\delta_w = 0, V^\delta_{ww} = 0, V^\delta_x = 0, V^\delta_{wx} = 0 \quad (69)$$

and

$$V_{xx}^\delta = (\rho - 2F_x^x)^{-1} (a_x^s - a_x^r) (u_{aa} + V_w F_{aa}^w) (a_x^s - a_x^r) \quad (70)$$

43
Equation (70) has an intuitive interpretation. At a point in time, as a function of $a$, present and continuation utility is $v(a) = u(a, w_t) + (1 - \rho dt) V(w_t + F^w(w_t, a_t) dt)$. Hence (omitting for the $dt$ to remove the notational clutter), $v'(a) = u_a + V_w F^w_a$ and $v''(a) = u_{aa} + V_w F^w_{aa}$. Hence, reacting imperfectly to a small $x_t$ (with $\alpha^\delta = \alpha^* - \alpha_t^*$) creates an instantaneous utility loss of $\Lambda_t = -\frac{1}{2} x_t A^\delta x_t$. The full utility loss is the present discounted value of that, i.e.

$$2\Lambda = \int_0^\infty e^{-\rho t} 2\Lambda dt = -\int_0^\infty e^{-\rho t} x_t A^\delta x_t a^\delta x_t x_t = -\int_0^\infty e^{-\rho t} e^{-2\phi t} x_0 a^\delta x_t a^\delta x_0 = \frac{1}{\rho + 2\phi} x_0 a^\delta x_0$$

$$= -x_0 (\rho - 2F_x^{\alpha})^{-1} a^\delta (u_{aa} + V_w F^w_{aa}) a^\delta x_0$$

$$= -x_0 V^\delta x_0.$$ 

It is enough to study the “static” utility losses to derive the dynamic utility losses. This proposition 18 is a dynamic application of the Proposition 26 in Gabaix (2014, online appendix) regarding losses from a suboptimal policy. For convenience, we restate this Proposition here. With static problem $\max u(a, x)$ s.t. $b(a, x) \geq 0$, and a Lagrangian $L(a, x) = u(a, x) + \lambda b(a, x)$, the losses from a suboptimal policy $a^\delta = a - \alpha^*$ (where $\alpha^*$ is the optimal policy) are to the leading order: $rac{1}{2} a^\delta L_{aa} a^\delta$.

Here the Lagrangian is $L = \int e^{-\rho t} [u(a_t, z_t) + \lambda_t (-z_t + F^z(a_t, z_t))] dt$, where $z_t = (w_t, x_t)$ is the state vector. Hence, the loss $\Lambda$ is expressed by (to the leading order)

$$2\Lambda = a^\delta L_{aa} a = \int a^\delta t L_{aa} a_t^\delta = \int e^{-\rho t} a^\delta t [u_{a_t a_t} + \lambda_t F_{a_t a_t}] a_t^\delta dt$$

Suppose that we can linearize, $a_t^\delta = A x_t$, we have

$$2\Lambda = \int e^{-\rho t} x_t^\prime A^\delta t [u_{a_t a_t} + \lambda_t F_{a_t a_t}] A^\delta x_t dt$$

Consider the ergodic limit, where $x_t$ has a distribution independent of $t$. Recall that

$$E x_t^\prime B x_t = \sum_{i,j} E x_i B_{ij} x_j = \sum_{i,j} B_{ij} E [x_i x_j] = Trace (B E [x x^\prime])$$

Hence,

$$2\Lambda = \frac{1}{\rho} Trace (B E [x x^\prime])$$

$$B = A^\delta t [u_{a_t a_t} + \lambda_t F_{a_t a_t}] A^\delta = A^\delta t L_{aa} A^\delta$$
11 Appendix: Proofs

Proof of Proposition 2  The rational reaction function satisfies:

\[ a^* (x) = a^d + \sum b_i x_i + \lambda (x) \]

for a function \( \lambda (x) = O \left( \|x\|^2 \right) \).

So, \( \partial a / \partial x_i = b_i \) and:

\[ m_i^* = \tau \left( 1, \frac{\kappa_a}{\sigma_{x_i} \cdot \partial a / \partial x_i} \right) = \tau \left( 1, \frac{\kappa_a}{\sigma_{x_i} \cdot b_i} \right) \]

We shall use the notation \( \bar{x} (x) := \lambda ((m_i^* x_i)_{i=1}^n) \), which also satisfies \( \bar{x} (x) = O \left( \|x\|^2 \right) \).

The sparse reaction function is:

\[ a^s (x) = \arg \max_a u \left( a, m_1^* x_1, \ldots, m_n^* x_n \right) \]

\[ = a^s \left( m_1^* x_1, \ldots, m_n^* x_n \right) \]

\[ = a^d + \sum b_i m_i^* x_i + \lambda \left( (m_i^* x_i)_{i=1}^n \right) \]

\[ = a^d + \sum b_i \tau \left( 1, \frac{\kappa_a}{\sigma_{x_i} \cdot b_i} \right) x_i + \bar{x} (x) \]

\[ = a^d + \sum \tau \left( b_i, \frac{\kappa_a}{\sigma_{x_i}} \right) x_i + \bar{x} (x) \]

\[ = a^d + \sum \tau \left( b_i, \frac{\kappa_a}{\sigma_{x_i}} \right) x_i + O \left( \|x\|^2 \right) \]

Proof of Proposition 5  Let us consider two functions \( U \) and \( u^s \)

\[ U^* (a, w, x) := u \left( a, w, x \right) + \beta \mathbb{E} V \left( F^w \left( w, x, a \right), F^x \left( w, x, a \right) \right) \]

\[ U^{**} (a, w, x) := u \left( a, w, x \right) + \beta \mathbb{E} V^s \left( F^w \left( w, x, a \right), F^x \left( w, x, a \right) \right) \]

and define the associated optimal actions:

\[ a^* (w, x) := \arg \max_a U^* \left( a, w, x \right), \quad a^{**} (w, x) := \arg \max_a U^{**} \left( a, w, x \right) \]

In \( U^{**} \), there is no inattention: however, the continuation policy \( V^s \) is used: the agent will be inattentive in the future.

First, we will prove:

Lemma 7 Suppose that \( F^x_a = 0 \). We have, at \( x = 0 \), \( \frac{\partial a^* (w, x)}{\partial x} = \frac{\partial a^{**} (w, x)}{\partial x} \)

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Proof. The key fact comes from Proposition 4, and is:

\[ V_w(w, 0) = V^*_w(w, 0) \]
\[ V_{ww}(w, 0) = V^{*_w}_{ww}(w, 0) \]
\[ V_x(w, x)|_{x=0} = V^*_x(w, x)|_{x=0} \]
\[ V_{wx}(w, x)|_{x=0} = V^{*_w}_{wx}(w, x)|_{x=0} \]

and

\[ U^*_a = u_a(a, w, x) + \beta \mathbb{E} [V_w \cdot F^w_a(w, x, a) + V_x \cdot F^x_a(w, x, a)] \]
\[ U^{*_a}_{ax} = u_{ax} + \beta \mathbb{E} [F^w_x \cdot V_{ww} \cdot F^w_a + V_w \cdot F^{*_w}_{ax}] + \beta \mathbb{E} [V^*_x \cdot F^x_a + F^x_x \cdot V_{xx} F^x_a] \]

Likewise, for \( U^{**} \),

\[ U^{**}_{ax} = u_{ax} + \beta \mathbb{E} [F^w_x \cdot V^s_{ww} \cdot F^s_a + V^s_w \cdot F^{**}_{ax}] + \beta \mathbb{E} [V^*_x \cdot F^x_a + V^*_x \cdot F^x_{xx} F^x_a] \]

Hence, we have

\[ U^{**}_{ax} = U^{*_a}_{ax} \text{ at } x = 0 \]

Note that we used \( F^x_a = 0 \). This is necessary, because in general \( V_{xx} \neq V^s_{xx} \).

Likewise,

\[ U^{**}_{aa} = u_{aa}(a, w, x) + \beta \mathbb{E} [F^w_a(w, x, a) \cdot V_{ww} \cdot F^w_a(w, x, a) + V_w \cdot F^{**}_{aa}(w, x, a)] + 2\beta \mathbb{E} [F^s_a(w, x, a) \cdot V_{xx} \cdot F^s_a(w, x, a)] + \beta \mathbb{E} [F^x_a(w, x, a) \cdot V_{xx} \cdot F^x_a(w, x, a) + V_x \cdot F^{**}_{aa}(w, x, a)] \]

and a similar expression for \( U^{**}_{aa} \), which leads to:

\[ U^{**}_{aa} = U^{*_a}_{aa} \text{ at } x = 0 \]

Finally, we have:

\[ \frac{\partial a^{**}(w, x)}{\partial x}|_{x=0} = -U^{**}_{aa}|_{x=0} = -U^{*}_{aa}|_{x=0} = -U^{*}_{ax}|_{x=0} = \frac{\partial a^*(w, x)}{\partial x}|_{x=0}. \]

Given \( a^*(w, x) = a^d(w) + \sum_i b_i(w) x_i + O(x^2), \) we have

\[ \frac{\partial a^*(w, x)}{\partial x_i} = b_i(w) \]
Hence, the lemma gives:

\[
\frac{\partial a^* (w, x)}{\partial x_i} = b_i (w)
\]

so

\[
a^* (w, x) = a^d (w) + \sum_i b_i (w) x_i + O \left( x^2 \right)
\]

Finally,

\[
a^* (x) = a^* (m^*_i x_i)
\]
\[
= a^d (w) + \sum_i b_i (w) m^*_i x_i + O \left( x^2 \right)
\]
\[
= a^d (w) + \sum_i b_i (w) \tau \left( 1, \frac{\kappa_a}{b_i (w) \sigma_{x_i}} \right) x_i + O \left( x^2 \right)
\]
\[
= a^d (w) + \sum_i \tau \left( b_i (w), \frac{\kappa_a}{\sigma_{x_i}} \right) x_i + O \left( x^2 \right).
\]

**Proof of Proposition 12** When \( \phi > 0 \), we saw that

\[
\phi = \left( \frac{\xi}{r + \phi} - \kappa \left( r + \frac{\xi}{r + \phi} \right) \right)^2 r + \phi
\]

Let \( \psi := \frac{r + \phi}{\xi} \neq 0 \). Then

\[
\phi = \psi^{-1} - \kappa(r + \psi^{-1})^2 \psi,
\]

which is equivalent to

\[
\psi(\xi \psi - r) = \psi \phi = 1 - \kappa[(r + \psi^{-1}) \psi]^2
\]
\[
= 1 - \kappa(r \psi + 1)^2
\]
\[
= 1 - \kappa(r^2 \psi^2 + 2r \psi + 1).
\]

Rearranging yields

\[
(\xi + \kappa r^2) \psi^2 + (2 \kappa - 1)r \psi + (\kappa - 1) = 0.
\]

The quadratic formula then gives

\[
\psi = \frac{(1 - 2 \kappa)r \pm \sqrt{\Delta}}{2(\xi + \kappa r^2)},
\]

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where
\[
\Delta = [(2\kappa - 1)\rho^2 - 4(\xi + \kappa \rho^2)(\kappa - 1)]
= r^2 [(2\kappa - 1)^2 - 4\kappa(\kappa - 1)] + 4\xi(1 - \kappa)
= r^2 [(4\kappa^2 - 4\kappa + 1) - (4\kappa^2 - 4\kappa)] + 4\xi(1 - \kappa)
= r^2 + 4\xi(1 - \kappa).
\]

In the case \( \kappa = 0 \), the correct root is the higher one for \( \psi \) (i.e., it’s the higher root of \( \phi = \frac{\xi}{\rho + \sigma} \), the one with the \(+\sqrt{\Delta} \) sign). Hence, \( \psi = \frac{(1 - 2\kappa)r + \sqrt{\Delta}}{2(\xi + \kappa \rho^2)} \)

Finally,
\[
\phi = \xi \psi - r
= \frac{\xi [(1 - 2\kappa)r + \sqrt{\Delta}] - 2(\xi + \kappa \rho^2)r}{2(\xi + \kappa \rho^2)}
= \frac{[\xi(1 - 2\kappa) - 2(\xi + \kappa \rho^2)] r + \xi \sqrt{\Delta}}{2(\xi + \kappa \rho^2)}
= \frac{[2\kappa \rho^2 + 2\xi \kappa + \xi] r + \xi \sqrt{\Delta}}{2(\xi + \kappa \rho^2)}
= \frac{[2\kappa \rho^2 + 2\xi \kappa + \xi] r + \xi \sqrt{\rho^2 + 4\xi(1 - \kappa)}}{2(\xi + \kappa \rho^2)}
\]

**Proof of Lemma 2**  
Here we present a proof sketch, in part because those notions are well-known.

We record the values with a time-discounting of \( D_t \), with \( D_t = e^{-\rho t} \) in the infinite horizon, but \( D_t \) could be different to capture finite-time horizon effects. For instance, with a finite horizon of \( T \), and a terminal weight \( b \) on the last consumption, then \( D_t = ae^{-\rho t}1_{t < T} + b\delta(t - T) \).

First, in the SDF approach, the problem is
\[
\max E \int_0^\infty D_t c_t^{1-\gamma} dt \text{ s.t. } E \int_0^\infty M_t c_t dt = w_0
\]
this leads to \( D_t c_t^{-\gamma} = k' M_t \) and \( c_t = k D_t^{\psi} M_t^{-\psi} \) for constant \( k, k' \). The constant is determined by the budget constraint , \( w_0 = E \int_0^\infty M_t c_t dt = kE \int_0^\infty kD_t^{\psi} M_t^{1-\psi} \). This leads to a utility \( V_w = (\mu_0 w_0)^{-\gamma} \), with
\[
\mu_0^{-1} = E \left[ \int_0^\infty D_t^{\psi} M_t^{1-\psi} dt \right] \tag{71}
\]
When $M_t$ follows (45), routine calculations show that:

$$
\mu_0^{-1} = E \left[ \int_0^\infty D_t^\psi e^{-(1-\psi) \int_0^t \hat{R}_u \, du} \, dt \right]
$$

We next proceed to a Taylor expansion:

$$
\mu_0^{-1} = E \left[ \int_0^\infty D_t^\psi e^{-(1-\psi) \int_0^t (\hat{R}_u + \hat{\nu}_u) \, du} \, dt \right]
= E \left[ \int_0^\infty D_t^\psi e^{-(1-\psi) \hat{R}_u} \left( 1 - (1-\psi) \int_0^t \hat{R}_u \, du \right) \, dt \right]
$$

With an infinite horizon, $D_t = e^{-\mu t}$ and

$$
\mu_0^{-1} = E \left[ \int_0^\infty e^{-\mu^* t} \left( 1 - (1-\psi) \int_0^t \hat{R}_u \, du \right) \, dt \right]
= \frac{1}{\mu^*} - (1-\psi) E \left[ \int_0^\infty e^{-\mu^* t} \int_0^t \hat{R}_u \, du \, dt \right]
= \frac{1}{\mu^*} - (1-\psi) E \left[ \int_0^t \left( \int_0^\infty e^{-\mu^* u} \, du \right) \hat{R}_u \, du \right]
= \frac{1}{\mu^*} - (1-\psi) E \left[ \int_0^t \frac{1}{\mu^*} e^{-\mu^* u} \hat{R}_u \, du \right]
= \frac{1}{\mu^*} - (1-\psi) E \left[ \int_0^t \mu^* e^{-\mu^* u} \hat{R}_u \, du \right]
= \frac{1}{\mu^*} - (1-\psi) \frac{1}{\mu^*} (\hat{R}_0 - \hat{R}_*) \text{ with } \hat{R}_0 - \hat{R}_* = E \left[ \int_0^t \mu^* e^{-\mu^* u} \hat{R}_u \, du \right]
= \frac{1}{\mu^* + (1-\psi) (\hat{R}_0 - \hat{R}_*)} + O \left( \hat{R}_0 - \hat{R}_* \right)^2.
$$

so

$$
\mu_0 = \mu^* + (1-\psi) (\hat{R}_0 - \hat{R}_*) = \psi \rho + (1-\psi) R_* + (1-\psi) (\hat{R}_0 - \hat{R}_*)
= \psi \rho + (1-\psi) \hat{R}_0
$$

When the consumer has a finite horizon and only cares about date $T$ consumption, then $D_t = \delta (t - T)$, and

$$
\mu_0^{-1} = E \left[ \int_0^\infty D_t^\psi e^{-(1-\psi) \hat{R}_u} \left( 1 - (1-\psi) \int_0^t \hat{R}_u \, du \right) \, dt \right]
= e^{-\mu^* T} - e^{-\mu^* T} (1-\psi) E \left[ \int_0^T \hat{R}_u \, du \right]
$$

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so the MPC is 0 but we have

\[ \mu_t^{-1} = e^{-\mu_x(T-t)} \left( 1 - (1 - \psi) E \left[ \int_t^T \tilde{R}_u du \right] \right) \] (72)

so again \( \mu_t \) is related to the present value of future portfolio returns. \( \square \)

**Proof of Lemma 3** In semi-discrete notation the asset demand at time \( t \) comes from:

\[ \max_{\theta} E_t [V \left( w (1 + r_t dt + \theta d\tilde{r}_t, X_t + dX_t) \right] \]

where, with \( \pi_t = \pi_x + f' X_t \),

\[ E_t [V \left( w (1 + r, dt + \theta d\tilde{r}_t, X_t + dX_t) - V \left( w, X_t \right) \right] = V_w w (r_t + \theta' \pi_t) dt + V_{w,X} w \left( \theta' d\tilde{r}_t, dX_t \right) + V_{wX} w^2 \theta' \Sigma_t \theta dt \]

\[ = V_w w \left[ \theta' (\pi_t + H_t) - \frac{\gamma}{2} \theta' \Sigma_t \theta dt \right] dt + \frac{1}{2} Tr \left( V_{XX} \Sigma_{X,X} \right) dt \]

where

\[ \theta' H_t = \frac{V_{w,X}}{V_w} \langle \theta d\tilde{r}_t, dX_t \rangle \]

is the hedging demand premium term. This implies

\[ \theta = \frac{1}{\gamma} \Sigma_t^{-1} (\pi_t + H_t) \]

To calculate \( H_t \) more fully, recall that \( V_w = \mu_x (X_t)^{-\gamma} w^{-\gamma} \), so that \( \ln V_w = -\gamma \mu_x (X_t) - \gamma \ln w \), and

\[ \frac{V_{wX}}{V_w} = -\gamma \frac{\mu_X}{\mu_x} = -\gamma (1 - \psi) \frac{\tilde{R}_X}{\mu_x} = (1 - \gamma) \frac{\tilde{R}_X}{\mu_x} = \mathcal{B}' \]

with

\[ \frac{\tilde{R}_X}{\mu_x} = b' (\mu_x I + \Phi_R)^{-1} \]

Note that \( R_t = r + \frac{1}{2\gamma} \pi_t' \Sigma^{-1} \pi_t \) with \( \pi_t = \pi_x + \hat{\pi}_t \), so, with \( R_* = r + \frac{1}{2\gamma} \pi_*' \Sigma^{-1} \pi_* \)

\[ R_t = R_* + \frac{1}{\gamma} \pi_t' \Sigma^{-1} \hat{\pi}_t = R_* + \theta' \hat{\pi}_t \]

\[ R_t = R_* + \theta' f^t X_t \] (73)

hence

\[ b = f \theta \] (74)
Hence,
\[
\theta' H_t = \frac{V_{wX}}{V_w} \langle \theta d\tilde{r}_t, dX_t \rangle = \sum_{i,j} \theta_i \langle d\tilde{r}_it, B_j dX_j \rangle = \theta_i \Sigma_{ij}^{r,X} B_j
\]
so that
\[
H_t := \Sigma^{r,X} B = (1 - \gamma) \Sigma^{r,X} \frac{\bar{R}_X}{\mu_*} = (1 - \gamma) \text{cov} \left( d\tilde{r}, d\frac{\bar{R}_t}{\mu_*} \right)
\]
\[= (1 - \gamma) \Sigma^{r,X} (\mu_* I + \Phi_R')^{-1} b \]

**Proof of Lemma 4**
Suppose
\[
d\tilde{r} = (r + \pi_* + fX_t) dt + \sigma dZ_t
\]
and that that agents have a constant MPC \( \mu_* : \)
\[
\frac{dw_t}{w_t} = (r - \mu_*) dt + \theta' d\tilde{r}
\]
\[= (r - \mu_* + \theta \pi_* + \theta' f'X_t) dt + \theta \sigma dZ_t \]
\[= (g_* + \theta' f'X_t) dt + \theta \sigma dZ_t \]
\[
\frac{dw_t}{w_t} = (g_* + b'X_t) dt + \theta \sigma dZ_t
\]
with \( b = f\theta \) and
\[
g_* := r + \theta \pi_* - \mu_*
\]
We want to calculate (assuming the policy \( c_t = \mu_* w_t \), which leads only to second order losses)
\[
U = \mathbb{E} \left[ \frac{1}{1 - \gamma} \int_0^\infty e^{-\rho s} c_s^{1-\gamma} ds \right] = \frac{\mu_*^{1-\gamma}}{1 - \gamma} \mathbb{E} \left[ \int_0^\infty e^{-\rho s} c_s^{1-\gamma} ds \right]
\]
Call
\[
m_s = e^{-\rho s} c_s^{1-\gamma}
\]
We calculate:
\[
\frac{dm_t}{m_t} = -\rho + (1 - \gamma) \left( g_* + b'X_t - \frac{\gamma}{2} \|\theta \sigma\|^2 \right) + (1 - \gamma) \theta' \sigma dZ_t
\]
\[= (-a + (1 - \gamma) b'X_t) dt + (1 - \gamma) \theta' \sigma dZ_t \]
\[
a := \rho - (1 - \gamma) \left( g_* - \frac{\gamma}{2} \|\theta \sigma\|^2 \right)
\]
\[= \rho - (1 - \gamma) \left( r + \theta \pi_* - \mu_* - \frac{\gamma}{2} \|\theta \sigma\|^2 \right) = \rho - (1 - \gamma) (R_* - \mu_*)
\]
\[= \mu_*
\]
We calculate LG moments. We assume
\[ \Phi = \Phi \tau - \Phi \Sigma \tau + \sigma \kappa \Phi \tau k^2 \]
so the LG generator (Gabaix, 2009) is
\[ \omega = \begin{pmatrix} \mu & - (1 - \gamma) b' \\ - (1 - \gamma) \Sigma X r \theta & \mu + \Phi \end{pmatrix} \]
Hence, the present value is
\[ V = f \left( 1 + (1 - \gamma) b' (\mu_* + \Phi)^{-1} X_t \right) \]
The value function has the Taylor expansion: 
\[ V (w_t, X_t) = v (X_t) \mu_*^{-\gamma} \frac{w^{1-\gamma}}{1-\gamma} \]
\[ v (X_t) = \frac{1 + (1 - \gamma) b' (\mu_* + \Phi)^{-1} X_t}{\mu_*} \]
\[ \mu_* = \mu_* - (1 - \gamma)^2 b' (\mu_* I + \Phi)^{-1} \Sigma X r \theta \]
\[ = \mu_* - (1 - \gamma)^2 H_t' \theta \text{ using (54), } H_t = \left( 1 - \frac{1}{\psi} \right) \Sigma r, X (\mu_* I + \Phi')^{-1} b \]
Rewrite
\[ V = v (X_t) \mu_*^{1-\gamma} = \frac{1 + K}{\mu_* + L} \mu_*^{1-\gamma} \text{ with} \]
\[ K = (1 - \gamma) b' (\mu_* + \Phi)^{-1} X_t \]
\[ L = - (1 - \gamma) H_t' \theta \]
\[ V = (\mu_* + C)^{-\gamma} = \mu_*^{-\gamma} \left( 1 - \gamma \mu_*^{-1} C \right) \]
\[ = \mu_*^{-\gamma} (1 + K - \mu_*^{-1} L) \]
hence,
\[
\mu_t - \mu^* = C = -\frac{\mu^* K}{\gamma} + \frac{1}{\gamma} L
\]
\[
= (1 - \psi) b' \mu_*(\mu_* + \Phi)^{-1} X_t + \frac{-1}{\gamma} (1 - \gamma) H_t' \theta
\]
\[
= (1 - \psi) b' \mu_*(\mu_* + \Phi)^{-1} X_t + (1 - \psi) H_t' \theta
\]
\[
\mu^t := \frac{1}{\gamma} L = -\frac{1}{\gamma} (1 - \gamma) H_t' \theta = \left(1 - \frac{1}{\psi}\right) H_t' \theta
\]

Intuition: the extra present value of returns is
\[
\bar{\mu}_t - \bar{\mu}^* = \frac{C}{\mu_*} = \frac{C}{(1 - \psi) \mu_*}
\]
\[
= b' (\mu_* + \Phi)^{-1} X_t - \psi (1 - \psi) \frac{1}{\mu_*} b' (\mu_* I + \Phi)^{-1} \theta \Sigma_r X
\]
\[
= b' (\mu_* + \Phi)^{-1} \left(X_t + (1 - \gamma) \frac{1}{\mu_*} \theta \Sigma_r X\right)
\]

**Proof of Proposition 17** We use the content\(^8\) and notations of Proposition 18. We set \(\bar{\mu}_t = \bar{\mu}_t^*\). We have \(F^w(w, x, c) = rw + x_t - c_t\) and \(F^x(w, x) = -\phi x\).

Under the **consumption frame**, \(a_t = c_t\), and \(F_{a_t}^w = 0\), so by Proposition 18, noting \([V_{xx}]^C\) the value of \(V^\delta_{xx}(w, 0)\) under the consumption frame:
\[
[V_{xx}]^C = \frac{u''(c)}{r + 2\phi} \left(c_y^* - c_y^\circ\right)^2
\]
and as \(c_y^* = mc_y^\circ\) with \(c_y^\circ = \frac{x}{r + \phi}\),
\[
[V_{xx}]^C = \frac{u''(c)}{r + 2\phi} (1 - m)^2 \left(\frac{r}{r + \phi}\right)^2
\]
and the expected losses are (with \(\sigma_y^2 = E[\hat{y}_t^2]\)):
\[
L^C = -\frac{1}{2} [V_{xx}]^C \sigma_y^2 = -\frac{1}{2} \frac{u''(c) \sigma_y^2}{r + 2\phi} (1 - m)^2 \left(\frac{r}{r + \phi}\right)^2
\]
\[
= A(1 - m)^2 r^2
\]

Under the savings frame, \(a_t\) is savings, so \(F^w = a_t\), and \(c_t = rw_t + x_t - a_t\). Hence:
\[
[V_{xx}]^S = \frac{u''(c)}{r + 2\phi} (S_y^s - S_y^r)^2
\]

\(^8\)We could also draw on the results in Cochrane (1989), with a variety of adjustments. Proposition 18 extend Cochrane’s results (derived for consumption) to general dynamic problems.
and as $S_y = mS_y^r$, with $S_y^r = 1 - c_y^r = \frac{\phi}{r+\phi},$
\[
[V^\delta]_y^S = \frac{u''(c)}{r + 2\phi} (1 - m)^2 \left( \frac{\phi}{r + \phi} \right)^2
\]
and expected losses are:
\[
L^S = -\frac{1}{2} [V^\delta]_y^S \sigma_y^2 = A (1 - m)^2 \phi^2
\]

**Losses from a general variable $x$.** Using the same reasoning, the losses from not paying attention to a variable $x$ is:
\[
L^x = -\frac{u''(c)}{r + 2\phi} \sigma_x^2 (c_x^r - c_x^{rat})^2 = -\frac{u''(c)}{r + 2\phi} \sigma_x^2 c_x^2 (1 - m_x)^2
\]

We parametrize the losses by the “equivalent permanent tax” $\lambda^x$ such that $L^x = E \int_0^\infty e^{-\mu t} [u(c_t) - u(c_t)] dt$. Hence, using a Taylor expansions, $\lambda^x = \frac{L}{w(c)/c}$. This gives:
\[
\lambda^x = \frac{1}{2} \frac{u''(c)}{r + 2\phi} \sigma_x^2 c_x^2 (1 - m_x)^2
\]
i.e., using $\gamma = \frac{-c u''(c)}{u(c)}$,\n\[
\lambda^x = \frac{1}{2} \frac{r \gamma}{r + 2\phi} \left[ \frac{c_x \sigma_x}{c} (1 - m_x) \right]^2 \tag{77}
\]

**Proposition 19** The losses from paying only attention $m_x$ to variable $x$, expressed in terms of an “equivalent proportional losses in consumption”, $\lambda^x$ are:
\[
\lambda^x = \frac{1}{2} \frac{r \gamma}{r + 2\phi_x} \left[ \frac{c_x \sigma_x}{c} (1 - m_x) \right]^2 \tag{78}
\]
where $\sigma_x$ is the standard deviation of $x$, and $c_x = \frac{\partial u}{\partial c_x}$.

The calibration gives:
\[
\lambda^r = (1 - m_r)^2 \times 0.03\%, \quad \lambda^g = (1 - m_y)^2 \times 3.0\% \tag{79}
\]
Proof of Proposition 10  Taxes lower the present value of his income by \( H e^{-r(T-t)} \), so the consumer’s response is:

\[
\hat{c}_t = r \hat{\omega}_t - \tau \left( H e^{-r(T-t)}, \kappa \right)
\]

so wealth accumulation is:

\[
\frac{d}{dt} \hat{\omega}_t = r \hat{\omega}_t - \hat{c}_t = \tau \left( H e^{-r(T-t)}, \kappa \right).
\]

The consumer starts thinking about it at a time \( s \) s.t. \( H e^{-r(T-s)} = \kappa \) (assuming that the solution is in \((0,T)\)), i.e.

\[
s = \max \left( 0, \min \left( T, \frac{1}{r} \ln \frac{\kappa}{H e^{-rT}} \right) \right) \tag{80}
\]

First, consider the case: \( s < T \).

Then, for \( t \in [s, T) \),

\[
\frac{d}{dt} \hat{\omega}_t = \tau \left( H e^{-r(T-t)}, \kappa \right) = H e^{-r(T-t)} - \kappa
\]

\[
\hat{\omega}_t = \int_s^t \left( H e^{-r(T-t')} - \kappa \right) dt'
\]

\[
\hat{\omega}_t = \frac{H}{r} e^{-rT} \left( e^{rt} - e^{rs} \right) - \kappa (t-s)
\]

\[
\hat{c}_t = r \hat{\omega}_t - \tau \left( H e^{-r(T-t)}, \kappa \right)
\]

\[
= r \left( \frac{H}{r} e^{-rT} \left( e^{rt} - e^{rs} \right) - \kappa (t-s) \right) - \left( H e^{-r(T-t)} - \kappa \right)
\]

\[
\hat{c}_t = -H e^{-r(T-s)} + \kappa (1-r(t-s)) \tag{81}
\]

So at \( t = T \)

\[
\hat{\omega}_T = \frac{H}{r} \left( 1 - e^{-r(T-s)} \right) - \kappa (T-t)
\]

At \( T \), the tax \( H \) is enacted, so that for \( t \geq T \), the agent is aware of it. This yields:

\[
\frac{d}{dt} \hat{\omega}_t = r \hat{\omega}_t - H - \hat{c}_t = \text{investment income} - \text{taxes} - \text{consumption change}
\]

\[
= 0
\]

hence for \( t > T \), \( \hat{\omega}_t = \hat{\omega}_T \), and \( \hat{c}_t = r \hat{\omega}_T - H \).

We conclude that consumption is:

\[
\hat{c}_t = \begin{cases} 
0 & \text{for } t < s \\
-H e^{-r(T-s)} + \kappa (1-r(t-s)) & \text{for } s \leq t < T \\
r \hat{\omega}_T - H & \text{for } t \geq T
\end{cases}
\]

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and wealth is
\[
\hat{w}_t = \begin{cases} 
0 & \text{for } t < s \\
\frac{H}{r} e^{-rT} (e^{rt} - e^{rs}) - \kappa (t - s) & \text{for } s \leq t \leq T \\
\frac{H}{r} (1 - e^{-r(T-t)}) - \kappa (T - s) = \hat{w}_T & \text{for } t \geq T
\end{cases}
\]

**Proof of Proposition 16** We’re looking for a solution of the form: \( V(x) = a + bx \), for \( a, b \) to be determined. The FOC is: \( u_c + \beta V_x h^s = 0 \), i.e. \(- (c - x - A) + \beta h^s = 0 \) and
\[
c = x + A + \beta b^s h^s
\]
\[
u (c(x), x) = -\frac{1}{2} (\beta b^s h^s)^2 - Bx
\]
The self-consistency condition is:
\[
V(x) = u(c(x), x) + \beta V(\rho x + hc)
\]
i.e.
\[
a + bx = -\frac{1}{2} (\beta b^s h^s)^2 - Bx + \beta [a + b (\rho x + h (x + A + \beta b^s h^s))]
\]
This gives:
\[
b = \frac{-B}{1 - \beta \bar{A}}
\]
\[
a = \frac{hA + \beta b^s h^s - \frac{1}{2} \beta (b^s h^s)^2}{1 - \beta}
\]
When the agent perceives \( \rho' = 1 - m^V + m^V \rho \) and \( h' = m^V h \) when forming the value function, we have the same expressions,
\[
b^s(m) = \frac{-B}{1 - \beta (\rho' + h')} = \frac{-B}{1 - \beta (1 + m^V (\rho + h - 1))}
\]
\[
a(m) = \frac{hA + \beta b^s(m) h^s - \frac{1}{2} \beta (b^s(m) h^s)^2}{1 - \beta}
\]
To determine optimal attention \( m \), observe that in the 1-step smax, at the beginning, \( m^V = 0 \), so the perceived value function is
\[
V(x, m^V = 0) = u(c(x), x) + \beta V(x, m^V = 0)
\]
so
\[
V(x, m^V = 0) = \frac{u(c(x), x)}{1 - \beta} = -\frac{1}{2} (\beta b^s h^s)^2 - Bx
\]

This implies: \( b^* = -\frac{B}{1-\beta} \), and

\[
c = x + A + \beta b^* \left( m^V = 0 \right) h m = x + A - \frac{\beta B}{1-\beta} h m
\]

so that the impact of thinking more about \( h \), while keeping the future value function constant is:

\[
\frac{\partial c}{\partial m} = \beta b^* \left( m^V = 0 \right) h = -\frac{\beta B h}{1-\beta}
\]

Hence, optimal attention is:

\[
m = A \left( \frac{\left( \frac{\partial c}{\partial m} \right)^2 u_{ee}}{\kappa} \right) = A \left( \frac{1}{\kappa} \left( \frac{\beta B h}{1-\beta} \right)^2 \right)
\]
References


