We study a supply chain where an upstream supplier auctions his inventory or capacity as a bundle. The importance of this setting is two-fold: From a practical point of view, there are several examples, specially in service industries (e.g., auctioning the capacity of a stadium to hold events, or auctioning the sponsorship of a website), where a supplier’s capacity is sold as a single piece; from a theoretical side, it highlights the information asymmetry introduced on the downstream supply chain parties when the auction result is disclosed.

We formulate the problem as a two-stage supply chain comprising a single supplier and two resellers. Each reseller receives a signal of the consumer demand, and bids for the capacity of the supplier. The supplier announces the winner as well as the auction price. Both resellers can get additional units in a procurement market, and then engage in a Cournot competition in the consumer market.

We analyze the impact of the information elicited by the supplier in the early stage of the game. We characterize sufficient conditions for the existence of equilibrium behavior, derive the equilibrium bidding functions under both first- and second-price auctions, and show that they are lower than the corresponding ones for a single-shot auction without resale. Our computational experiments indicate that both the supplier and resellers are better off by running a second-price auction, and that consumers benefit if the resellers have very different signals on the total demand. Overall, our results suggest that traditional auctions may have a profound impact in the context of a supply chain because of the information disclosure in the upstream stages.

Keywords: supply chain, auctions, information asymmetry, demand effort

1. Introduction

Auctions conform an alternative pricing scheme to the traditional (fixed or dynamic) list price, and the negotiations. Formally speaking, an auction is simply a set of rules defining a mechanism for how products are awarded to buyers, how information is revealed between sellers and buyers, and what payments are made based on the revealed information. Two of the standard (sealed-bid) auction mechanisms are the first- and second-price (Vickrey) auctions.

The use of auctions, jointly with the advances in information technology, constitutes two of the most significant innovations incorporated in supply chains since the middle nineties. The advent of Internet, the consequent reduction in transaction costs, and the easier acces-
sibility to more participants have contributed to the explosive growth that online auctions\(^1\) have exhibited during the last several years (e.g., see the online auction overview by Pinker et al. (2003)). Nowadays, the wide spread of auctions in both virtual and traditional markets creates rich opportunities for firms to reinvent their procurement and selling practices, and in general to redesign the interface through which they interact with the market, from the acquisition of raw material or input services to the delivery of finished goods or services. Auctions take place among suppliers themselves, including both manufacturers and service providers, between suppliers and resellers, and between resellers and final consumers.

In this paper, we study a supply chain where an upstream supplier auctions his inventory or capacity as a bundle. The importance of this setting is two-fold: From a practical point of view, there are several relevant examples spanning both manufacturing and service industries where a supplier’s capacity is sold as a single piece; from a theoretical side, it highlights the information asymmetry introduced on the downstream supply chain parties when the auction result is disclosed.

Regarding the former, in manufacturing industries for instance, consider a small winery that seeks to sell its production through retailers. For a small production lot of a high quality wine, the winery is often interested in dealing with one exclusive retailer, as opposed to arranging deals with several of them. In service industries, consider for instance the auction of the capacity of a stadium to hold a show on a particular date. Certainly, this capacity can not be split, and the loser must consider an alternative location (or date) for her event. Auctions of arenas to host sport teams also take place. Tickets are then offered to final consumers for different events (e.g., see Sharrow (2006)).

Online retailers like Amazon or Buy.com can auction a space in their webpage to advertise a credit card, but they would not split the sponsored space between two competing credit cards. In this case, the capacity is given by the potential number of visitors to the website. Other online companies like Yahoo are considering auctions as a way to clear excess inventory of advertising space.\(^2\) In particular, they sell all the “nonpremium inventory” as a bundle (e.g., see Holahan (2006)). Advertising placement companies bid for this space and resell it to final clients.

In a similar spirit, media such as TV channels bid for the right of carrying special events.

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\(^{1}\)In the online practice, open auctions are more common, with the ascending or English auction being by far the most widely used mechanism. The English auction turns out to be strategically equivalent to the second-price auction in our two-bidder model (see discussion in Section 2.2).

\(^{2}\)Typically, these online companies sell most of the available capacity through long-term contracts.
Of course, media firms cannot purchase a partial right of syndicating a show or a sport event. As an example, ITV and BBC bid for the right of carrying Football Association games in England (e.g., see Gibson (2006)). The capacity is defined by the potential number of viewers. The winner then sells advertising time slots to sport-related companies.

Regarding the theoretical relevance, the realization of auctions in a supply chain enforces information revelation among players that may affect the performance of the overall system. In particular, the disclosure of bids at different stages of the chain reveals *ex-post* signals of the demand received by different players. This observation may clearly affect *ex-ante* the strategic behavior of the bidders. If we focus on a supplier/reseller relationship, resellers may obtain some information advantage (or disadvantage) with respect to their competitors in the consumer market, and consequently adopt different purchasing/selling strategies.

The information revealed from an auction critically depends on how the bids are announced. In practice, a variety of auction-related announcement policies are followed; one being the revelation of the closing price. This, however, induces a clear asymmetry between resellers under both first- and second-price auctions: In the first-price auction the winning bid is disclosed, whereas the highest losing bid is revealed in the second-price auction. This difference is irrelevant in the traditional single-shot auction without resale. However, if bidders (i.e., resellers) compete among themselves after the auction, as in our supply chain setting, it becomes crucial. In fact, the threat of disclosing resellers’ bids raises a number of important research questions. How should resellers behave in a procurement auction when running the risk of having their bids disclosed? How much information content should a reseller embed in her bid, knowing that it could be learned ex-post by a direct competitor? Which auction mechanism mitigates or emphasizes more the asymmetry introduced by the disclosure of information? Is there any mechanism under which both suppliers and resellers are better off?

This paper makes some progress in addressing these questions. We consider a single supplier who auctions his capacity to two resellers. He conducts either a first- or second-price auction, and announces the winner and the payment in each case. Both resellers can get additional (or substitutable) units in a procurement market before engaging in a Cournot competition in the consumer market. How much the resellers value the auctioned capacity depends on the expected revenue to be collected from the end consumers. However, the resellers are aware of the information revelation threat: One reseller will have her bid revealed,
and the other one will keep hers hidden after the auction.\footnote{When the supplier is allowed to split his capacity, then retailers’ bids may be revealed if they both receive a portion of it, and the asymmetry is mitigated.} We investigate the impact of information revelation on the downstream resellers’ bidding behavior, on the supplier’s and resellers’ expected payoffs, and on the volume of transactions in the consumer market.

1.1 Literature review

A vast body of work on auction theory has been published since Vickrey (1961), including the influential paper of Milgrom and Weber (1982), the mechanism design approach of Myerson (1981) and Maskin and Riley (1989), the survey by Klemperer (1999), and the recent book by Krishna (2002).

Gupta and Lebrun (1999), Krishna (2002, Section 4.4), and Haile (2003) consider first-price auctions with resale, but the resale occurs among the same set of bidders, and their assumptions on the announcement policies are quite different from ours: After the auction, all bidders’ values are announced in Gupta and Lebrun (1999), whereas in Krishna (2002) all bids are made public. In Haile (2003), bidders receive new signals after the auction and hence all bidders still hold private information. Benoit and Dubra (2006) considers the information revelation issue in auctions as well, but their concentration is on whether bidders are willing to disclose their private information before submitting their bids. Hafalir and Krishna (2007) investigate the impact of asymmetry between bidders when the winner makes a take-it-or-leave-it offer to the loser after the auction. They assume that the winning bid is announced in both the first- and second-price auctions. Zheng (2002) assumes that each bidder is only aware of whether she wins the auction and how much she pays. He focuses on the design of the optimal auction mechanism with resale. Moresi (2000) develops a model of information acquisition prior to an open auction, in which the common value of the item has two distinct components. Each of the two bidders conducts some research and specialize in one component independently, and then decides how much to bid. This model provides a rationale for bidders to differentiate themselves by conducting different lines of research.

Several papers have addressed the impact of a supplier’s announcement policy and the derived information sharing in more traditional supply chain contexts (e.g., see the surveys by Cachon (2003), Chen (2003), and references therein). There are also papers on the competitive interaction between supply chain parties; most of them first show that competition
hurts the supply chain performance and then propose coordination mechanisms that bring the supply chain performance closer to the centralized optimum. Among them, Li (2002) and Padmanabhan and Png (1997) consider horizontal competition among resellers. Different streams of research that relate supply chain management and auction theory are the design of procurement contracts (e.g., see Dasgupta and Spulber (1989), Chen (2004), and the survey by Elmaghraby (2000)), and the structure of joint inventory policies and auction design decisions (e.g., van Ryzin and Vulcano (2004)). However, in all these papers, there is no integration of procurement, auctions, and horizontal competition among resellers.

In our setting, resellers compete horizontally in the consumer market. We consider a linear demand model based on resellers’ promotional effort. In a competitive retailing environment, the demand-enhancing effort of one reseller may increase the demand of other resellers. These spillovers may lead to free riding, where one reseller enjoys a higher demand due to the efforts of others without exerting the own effort. Some papers in the marketing literature have addressed this topic (e.g., see Lal (1990), Nault and Dexter (1994), and Bergen and John (1997)). Hula (1993) develop two profit-maximization models of firm behavior that incorporate industry-demand externality effects of firm price changes, advertising, and research and development expenditures. On the operational side, Krishnan et al. (2004) study mechanisms to achieve coordination in a simple supply chain setting with one manufacturer and one reseller, accounting for the reseller’s promotional effort.

Some research on treasury bill auctions is also related to our work, e.g., see Bikhchandani and Huang (1989), Nyborg and Sundaresan (1996), Chatterjea and Jarrow (1998), and references therein. These papers model the primary market of stock/bonds as an auction, where traders operate with private information about future values of the financial assets. The purpose of trading in the primary market is to resell the securities afterwards to other uninformed traders or market makers. In particular, Bikhchandani and Huang (1989) consider a multi-unit auction setting with a resale market, but the resale price is determined by the expectation of securities’ true values from the viewpoints of the uninformed traders. They discuss the impact of announcing the winning bid and the highest losing bid on the market equilibrium and traders’ behavior. However, no oligopolistic competition is considered in the resale market.

Finally, we complement this review with the computer science literature on trading agent design, a prominent application area in Artificial Intelligence. The main focus of this stream of research is on the automated decision-making processes of a supply chain agent in terms
of procurement, selling, and production/inventory management (e.g., see Buffett and Scott (2004), Pardoe and Stone (2004), and Wellman et al. (2003)).

1.2 Overview of main results

We characterize the resellers’ equilibrium bidding functions under the first- and second-price auctions. We provide sufficient conditions for the existence of an incentive compatible equilibrium (i.e., an equilibrium where each reseller bids according to the demand signal that she got from the market). We show that the threat of revealing private information discourages the resellers from bidding as high as they would bid in the conventional setting (i.e., the auctions without resale). Moreover, under the first-price auction, the condition for the equilibrium bidding function being monotonic in the reseller’s own signal is more restrictive than under the second-price auction.

We also show that the consumers are better off (worse off) if the resellers receive very different (similar) demand signals. In the extreme case where the resellers receive very different signals, the first-price auction raises the loser’s expectation of the consumer demand because the winner’s bid is announced, thereby bringing the loser’s quantity back to the normal level. However, in the second-price auction, the loser has no access to the winner’s signal. Consequently, she underestimates the consumer demand, and chooses an abnormally low quantity.

We verify numerically that the supplier’s expected revenue is lower in the first-price auction than in the second-price auction. This result is in contrast with the Revenue Linkage Principle (see Krishna (2002, Chapter 7)): In a conventional auction, if the signals are independent, and the valuations as function of the signals are interdependent (as assumed here), the supplier should receive the same revenue from both auction formats. In our setting, since the winning bid is announced in the first-price auction, the winner collects on average less profit and hence is unwilling to pay as much as in the second-price auction. In fact, the announcement of the winning bid translates into some information disadvantage; in this sense, winning brings “bad news”. This can be regarded as another sort of the so called winner’s curse— the possibility that the winner pays more than the “real value” of the object (Krishna, 2002, Chapter 6).

We also find that, not only the supplier but also the resellers are better off when the second-price auction is used. An explanation for this is that the combined first-mover and information advantage for the winning reseller mitigates the horizontal competition between
the resellers, and therefore the profit of the entire supply chain is driven up. This demonstrates the traditional auction mechanisms have a great impact on supply chain performance if the subsequent competition is taken into consideration.

The rest of this paper is organized as follows. In Section 2 we describe our model setup, followed by a discussion of the model assumptions. We derive the equilibrium analysis of the two-stage game and relevant payoffs in Section 3. We provide numerical results in Section 4, and present our conclusions in Section 5. All proofs are in the Appendix.

2. Model description

A single supplier runs an auction to sell his capacity $C$ to two competitive resellers who do not possess any capacity endowment. The resellers can get additional units in a procurement market right after the auction closes.

Once the resellers obtain their capacities, they engage in a Cournot (quantity) competition in the consumer market. The inverse demand function is common knowledge, and is described by:

$$P(Q) = \theta - Q,$$

where the intercept $\theta$ is a random variable and $Q = q_1 + q_2$ is the total aggregated supply, i.e. the sum of the continuous quantities $q_i$ provided by resellers $i = 1, 2$. This linear demand model is commonly adopted in the literature of economics, marketing, and operations, e.g., see Dixit (1979), Li (2002), McGuire and Staelin (1983), and Tirole (1995).

Even though the resellers face a stochastic demand, we neglect the nonnegativity constraint over the price for computational convenience. This assumption is plausible when the quantities chosen in equilibrium drive down the likelihood of a negative price to a negligible level (e.g., see Li (2002)).

2.1 Sequence of events

For ease of presentation, we divide the sequence of events into four periods:

**Period 1:** Reseller $i$ receives a private signal $s_i$ regarding the intercept $\theta$. We keep the following assumption throughout:

**Assumption 1.** The maximum possible demand is $\theta = \theta_0 + s_1 + s_2$, where $\theta_0$ is a common knowledge constant, and $s_1, s_2$ are independently and $\text{Unif}[0, 1]$ distributed random variables.
Neither the other reseller nor the supplier knows the true value of reseller $i$'s signal $s_i$, but they do know that it is a draw from a Unif$[0,1]$ distribution. The contribution $s_i$ to the total demand is a proxy for reseller $i$'s demand effort (e.g., it could be related to advertising levels).

**Period 2:** At the beginning of this period, the supplier announces his capacity $C$ and sells it as a bundle through an auction. Therefore, although auctioning multiple objects, the supplier conducts in fact a single-unit auction. He announces the auction type $A$, which can be either the first- or second-price auction (i.e. $A = I$ or $A = II$ respectively). Then, the auction takes place. Both resellers participate in it, each one submitting a bid $b_i$, $i = 1, 2$. We assume that the individual rationality constraint of the resellers is satisfied (i.e. we are implicitly assuming that by not participating, a reseller can guarantee herself a null payoff).

In Section 3, we verify that this is indeed the case.

**Period 3:** The auction closes and the supplier announces who the winner is and how much she should pay for the capacity $C$. Therefore, in the first-price auction, the winner’s bid is made available to the public,\(^4\) while in the second-price auction, the losing bid is revealed.

**Period 4:** At the beginning of this period, resellers can pay a constant marginal cost $c$ for additional units from a procurement market with unlimited capacity. Finally, reseller $i$ puts $q_i$ units on the market. The demand is then revealed and the market clears. The price is determined, and the resellers’ payoffs are realized.

The timing of this model is summarized in Figure 1. The information flow is presented in Figure 2: The supplier announces the auction type $A$ and the capacity $C$ to the resellers, and then resellers submit bids $b_1, b_2$ to the supplier. The resellers acquire demand information $s_1$ and $s_2$ from the final consumers before the auction, and provide quantities $q_1$ and $q_2$ to the consumer market after the procurement process takes place.

### 2.2 Discussion of the model assumptions

The Cournot competition is merely one possible way to model the retailing game. An alternative would be to explain it in terms of Bertrand (price) competition. However, in a Bertrand game with capacity constraints, we should specify the rationing rule. Otherwise, it is not clear what proportion of consumers would choose reseller 1 over reseller 2 in case\(^4\)Although revealing the price in a first-price auction is not the most usual practice, there are some websites that implement it. Examples are Timeshare Resales International (http://www.tri-timeshare.com) and The Chicago Wine Company (http://www.tcwc.com).
Figure 1: Time flow diagram for the sequence of events.

Figure 2: Information flow diagram of relevant events.
the demand is less than the aggregate capacity. It has been observed that different rationing rules, such as efficient-rationing and proportional-rationing rules, lead to considerably different market equilibria. Moreover, if the efficient-rationing rule is adopted between the two resellers, a pure-strategy equilibrium exists only if the capacity levels are extremely small or extremely large (see Kreps and Scheinkman (1983), and Tirole (1995, Chapter 5)).

Our framework is an interdependent-value model with symmetric bidders. In an auction with \( N \) bidders and interdependent values, a bidder’s valuation may depend on other bidders’ signals. Let \( s_1, \ldots, s_N \) denote the bidders’ signals whose prior symmetric distribution \( F(s_1, \ldots, s_N) \) is common knowledge. If we let \( V_i \) denote bidder \( i \)’s valuation of the auction object, then \( V_i \) can be expressed as \( V_i = v_i(s_1, \ldots, s_N) \). Therefore, until bidder \( i \) receives the signal \( s_i \), she does not know exactly her own valuation, and knowing what other bidders also receive affects the expectation of her own value.\(^5\)

A well-established theorem for revenue comparison of conventional auctions under the interdependent-value setting is the so-called Revenue Linkage Principle (see Milgrom and Weber (1982), and Krishna (2002, Chapter 7)). An application of this principle is that if the signals \( s_1, \ldots, s_N \) are independent (even though the valuations \( V_1, \ldots, V_N \) are interdependent, as in our case with \( N = 2 \)), then the seller’s expected revenue is the same under the first- and second-price conventional auctions. We in Section 4 compare our result to this well-known principle.

We assume that signals are independently distributed (i.e. they are non-affiliated) and additive to the demand. This is a plausible assumption in situations regarding information acquisition (e.g., see Froot et al. (1992) and Moresi (2000)), and its tractability allows us to prove our results analytically. This assumption corresponds to the case in which the resellers put independent efforts to increase the market demand through strategies such as advertising and promotions, and to the fact that the realization of a reseller effort could be observed only by the reseller herself. Thus, the signals capture the resellers’ private information following their private actions.

We consider symmetric uniformly distributed signals. The symmetry in the bidders’ distribution of valuations for the private-value auction model is a strong but common as-

\(^5\)More specifically, if \( V_i = v_i(s_i) \), i.e., bidder \( i \)’s valuation depends only on her own signal, it degenerates to a private value model. If \( V_i = v(s_1, \ldots, s_N) \), we call it a common value model. In this case, if bidders know all signals in the end, their valuations will become the same. An example for the common value model fit is the allocation of U.S. mineral rights. In our case though, a reseller expected payoff will be affected by the other reseller’s signal concerning demand.
sumption in the operations management literature (e.g., see Pinker et al. (2003) and van Ryzin and Vulcano (2004)). Nevertheless, in our interdependent value model, symmetry in the signals is a reasonable assumption for a supply chain setting where resellers have comparable sizes and similar market influence. The uniform distribution is commonly adopted in the existing auction literature, from the early paper by Vickrey (1961), to the more recent works by Pinker et al. (2003) and Krishna (2002, Chapters 6 and 8). The uniform distribution also brings the maximum entropy among the class of all distributions with finite support (see Arndt (2001, Chapter 15)). Therefore, it can be regarded as the worst case belief if the supplier has no other information except that the resellers’ signals have a finite support.

For ease of presentation, we consider only two resellers here to emphasize the different situations that a winner and a loser of the auction will face in the consumer market. If there were more than two participants in the auction, a similar approach would apply since only one bid would be disclosed. We further assume that these resellers do not possess any capacity endowment before participating in the auction, and that their signals have the same precision. Thus, both resellers are ex ante symmetric. If their inventory endowments were different, their bids may depend not only on their signals but also on their inventory levels.

It is known that in auctions with interdependent values, the asymmetry among bidders may destroy the Revenue Linkage Principle; see Krishna (2002, Chapter 8). If the precision of their signals concerning the realized demand varies across bidders, there may not exist a pure-strategy equilibrium; see Engelbrecht-Wiggans et al. (1983). By considering ex-ante symmetric resellers, we isolate the effect of information revelation that arises when they bid for the supplier’s capacity.

In our setting, the supplier sells his capacity as a bundle. As we have mentioned in Section 1, this is appropriate in several relevant practical settings. From a modeling perspective, it stresses the information asymmetry introduced by different announcement policies: If the seller were allowed to split his capacity, then both resellers’ signals would be revealed if both received some units. The paper by Anton and Yao (1989) also touches on the issue of divisible versus indivisible awards, for a complete information procurement auction. They show that under a split award procurement, the two bidders implicitly collude. Therefore, the auctioneer strictly prefers a single source award auction.

In our model, the winning reseller can discard part of her capacity $C$ without any penalty. She is allowed to purchase more from the procurement market with a fixed price $c$ at her own will, but she cannot resell the awarded capacity to the procurement market. This rules out
the possibility of speculation by the resellers (e.g., chasing financial arbitrage opportunities), and focuses resellers’ business on meeting the demand.

The assumption that the procurement market has a given price $c$ is merely made for tractability. It is reasonable if these two resellers are relatively small with respect to the aggregated market for this product. There are several papers in the literature along these lines. In Peleg et al. (2002), a firm is allowed to replenish its inventory via two channels: a long-term contract or an online search. Under the long-term contract, the price is fixed ex ante, while in the online search the price is random, reflecting different market situations. These two procurement alternatives, also labelled as *two-source factor purchasing* in Elmaghraby (2000), are similar to our procurement market and the auction, respectively. In fact, many suppliers provide auctions as a complementary sales channel besides their long-term relationship with resellers (e.g., see Grey et al. (2005)).

Finally, we want to highlight that the procurement market is introduced for generality of the setting. Its existence can be discouraged in the model by setting $c = \infty$. This would introduce a change in the Cournot quantities, but the qualitative aspect of the results that follow still holds.

### 3. Equilibrium analysis of the game

Due to the sequential nature of our four-period game under incomplete information, we will characterize a Perfect Bayesian Equilibrium (PBE) (e.g., see Osborne and Rubinstein (1994)) for more details):

**Definition 1.** In a Perfect Bayesian Equilibrium, players’ strategies and beliefs satisfy the following conditions:

1. **Bayesian updating:** Players have correct initial beliefs. Moreover, after observing players’ actions at a stage, they use Bayes’ rule to update only the corresponding beliefs.

2. **Sequential rationality:** Given their beliefs, the players’ actions must be the best responses.

In the sequel, we use backward induction to analyze the players’ behavior: First, we derive the fourth-period equilibrium in the consumer market, assuming that a strictly increasing equilibrium has been established in the second period (i.e., the auction game). Note that in
the second period, each reseller’s strategy is a bidding function that maps her signal to her
bid, i.e., $\beta : [0, 1] \rightarrow \mathbb{R}^+$. 

**Definition 2.** An equilibrium is said to be strictly increasing if the equilibrium bidding
function is strictly increasing in a bidder’s own signal.

When the bidding function is strictly increasing, as players observe the bid, they can
invert the bidding function to obtain a competitor’s signal. Therefore, it is also fully separ-
ating.

Now let us recall the bid announcement policy under the first- and second-price auctions.
In the first-price auction, the winning bid is announced. Thus, the winning reseller’s private
information becomes public. On the other hand, if a second-price auction is adopted, the
supplier announces the loser’s bid. Table 1 summarizes the information that a bidder pos-
sesses at the end of period 3, where $s_w$ and $s_l$ are the signals received by the winner and the
loser, respectively.

As shown in Table 1, while competing in the consumer market, the loser has information
advantage in the first-price auction; while the winner is the more informed reseller if the
second-price auction is adopted. In what follows, we use superscript $I$ to denote terms
associated with the first-price auction, and superscript $II$ for terms associated with the
second-price auction. The subscripts $w$ and $l$ refer to the variables for the winner and the
loser, respectively.

### 3.1 The consumer market game

We now derive the equilibrium in the consumer market given the two auction formats chosen
by the supplier. Recall that at this stage of the game (period 4), the resellers decide how
many units, $q_1$ and $q_2$, they provide to the consumer market.

#### 3.1.1 Second-price auction procurement case

We first consider the case when the supplier runs a second-price auction. The winning
reseller’s goal is to maximize her expected payoff, given that she knows both signals and

<table>
<thead>
<tr>
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<th>First-price auction</th>
<th>Second-price auction</th>
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<tbody>
<tr>
<td><strong>Winner</strong></td>
<td>${s_w}, {s_l &lt; s_w}$</td>
<td>${s_w}, {s_l}$</td>
</tr>
<tr>
<td><strong>Loser</strong></td>
<td>${s_l}, {s_w}$</td>
<td>${s_l}, {s_w &gt; s_l}$</td>
</tr>
</tbody>
</table>
that she has to pay \( c \) per unit in the procurement market (if her optimal quantity supplied to the consumer market exceeds the auctioned capacity \( C \)). Hence, the winner’s objective is as follows:

\[
\max_{q_w} (\theta - q_w - q_l)q_w - c(q_w - C)^+, \tag{1}
\]

where \( a^+ \equiv \max\{a,0\} \) is the positive part of the real number \( a \). Similarly, the loser’s objective is

\[
\max_{q_l} E_{s_w}[(\theta - q_w - q_l - c)q_l|s_w > s_l, s_l].
\]

Let \( q^I_{w}(s_w, s_l) \) and \( q^I_{l}(s_l) \) denote the optimal quantities chosen by the winner and the loser respectively, where the arguments depict their corresponding information knowledge at the moment of decision. Recalling from Assumption 1 that \( \theta = \theta_0 + s_1 + s_2 \), the following proposition summarizes the equilibrium quantities provided by these two resellers in the consumer market:

**Proposition 1.** Suppose that the second-price auction is used. In the consumer market, there exists a unique Nash equilibrium in which

\[
q^I_{w}(s_w, s_l) = \begin{cases} 
\frac{1}{2} (\theta_0 + s_w + s_l - q^I_{l}(s_l)), & \text{if } s_w < S^I_{1}(s_l) \\
C, & \text{if } S^I_{1}(s_l) \leq s_w \leq S^I_{2}(s_l) \\
\frac{1}{2} (\theta_0 + s_w + s_l - q^I_{l}(s_l) - c), & \text{if } s_w > S^I_{2}(s_l),
\end{cases}
\]

\[
q^I_{l}(s_l) = \frac{1}{2} (E_{s_w} [\theta|s_w > s_l, s_l] - E_{s_w} [q_w(s_w, s_l)|s_w > s_l] - c),
\]

where the two thresholds \( S^I_{1}(s_l) \) and \( S^I_{2}(s_l) \) are

\[
S^I_{1}(s_l) = \min \left\{ \max\{s_l, 2C - \theta_0 - s_l + q^I_{l}(s_l)\}, 1 \right\},
\]

\[
S^I_{2}(s_l) = \min \left\{ \max\{s_l, 2C - \theta_0 - s_l + q^I_{l}(s_l) + c\}, 1 \right\}. \tag{3}
\]

Note that these two thresholds are greater than \( s_l \), and when \( S^I_{2}(s_l) \) does not hit the boundary 1, \( S^I_{2}(s_l) = S^I_{1}(s_l) + c \), i.e. the second threshold is higher. Now we provide the structural property of the points at which the capacity constraint is binding. Figure 3 illustrates the general shape of \( S^I_{1}(s_l) \) and \( S^I_{2}(s_l) \) as functions of \( s_l \).

**Proposition 2.** The threshold function \( S^I_{1}(s_l) \) can be divided into three regions by two values \( s^*_1 \) and \( s^{**}_1 \), where \( s^*_1 \leq s^{**}_1 \).
Figure 3: An example to show the shape of $S_{1}^{II}(s)$ and $S_{2}^{II}(s)$ under the second-price auction procurement case

If $0 \leq s_{l} \leq s_{1}^{*}$, $S_{1}^{II}(s_{l}) = 1$, i.e., the capacity constraint is not binding for all $s_{w}$.

If $s_{1}^{*} \leq s_{l} \leq s_{1}^{**}$, $S_{1}^{II}(s_{l})$ decreases from 1 until it hits the 45-degree line, and

$$S_{1}^{II}(s_{1}) - S_{1}^{II}(s_{2}) < s_{2} - s_{1}, \forall s_{1} < s_{2}. \quad (4)$$

Finally, when $s_{1}^{**} \leq s_{l} \leq 1$, $S_{1}^{II}(s_{l}) = s_{l}$, i.e. $q_{w}^{II}(s_{w}, s_{l}) \geq C$ for all $s_{w}$.

Likewise, there exist two corresponding thresholds $s_{2}^{*}$ and $s_{2}^{**}$ for $S_{2}^{II}(s_{l})$ such that

$$s_{2}^{*} \geq s_{1}^{*}, \ s_{2}^{**} \geq s_{1}^{**}, \ s_{2}^{**} \geq s_{2}^{*}, \ S_{2}^{II}(s_{1}) - S_{2}^{II}(s_{2}) < s_{2} - s_{1}, \forall s_{1} < s_{2}. \quad (5)$$

Note that according to the values of some parameters, a region could be indistinguishable in the graph.

In Figure 3, as $s_{l}$ becomes larger, the threshold function $S_{1}^{II}(s_{l})$ first stays at the upper bound, and then decreases until it hits the 45-degree line. After that, $S_{1}^{II}(s_{l})$ coincides with $s_{l}$. $S_{2}^{II}(s_{l})$ has a similar graph, except that the thresholds $s_{2}^{*}$ and $s_{2}^{**}$ occur at higher values. Note also that within region III, $S_{2}^{II}(s_{l})$ could also stay at 1, depending on the ordering of $s_{2}^{*}$ and $s_{1}^{**}$.
3.1.2 First-price auction procurement case

In the first-price auction, since the winner’s bid is announced, the loser knows both signals, but the winner only knows her own signal. The resellers’ objective functions are respectively

\[ \text{Winner: max}_{q_w} E_{q_l}[(\theta - q_w - q_l)q_w - c(q_w - C)^+]|s_w, s_l < s_w], \]

\[ \text{Loser: max}_{q_l} (\theta - q_w - q_l - c)q_l. \]

After differentiating the objective functions and applying the same argument as in Proposition 1, we obtain the resellers’ best responses. The proof involves routine algebra and hence is omitted.

**Proposition 3.** Suppose the first-price auction is used. Define thresholds \( S_{I1} \) and \( S_{I2} \) as

\[ S_{I1} = \min \left\{ 1, \left(2C - \frac{2}{3}c - \frac{2}{3}\theta_0 \right)^+ \right\}, \]

\[ S_{I2} = \min \left\{ 1, \left(2C + \frac{2}{3}c - \frac{2}{3}\theta_0 \right)^+ \right\}. \]

In the consumer market, there exists a unique Nash equilibrium that satisfies the following:

If \( s_w < S_{I1} \),

\[ q_{Iw}(s_w) = \frac{1}{3}\theta_0 + \frac{1}{2}s_w + \frac{1}{3}c, \]

\[ q_{Il}(s_l, s_w) = \frac{1}{3}\theta_0 + \frac{1}{4}s_w + \frac{1}{2}s_l - \frac{2}{3}c, \forall s_l \leq s_w. \]

If \( S_{I1} \leq s_w \leq S_{I2} \), then

\[ q_{Iw}(s_w) = C, \]

\[ q_{Il}(s_l, s_w) = \frac{1}{2}(\theta_0 + s_w + s_l - C - c), \forall s_l \leq s_w. \]

Finally, if \( s_w > S_{I2} \),

\[ q_{Iw}(s_w) = \frac{1}{3}\theta_0 + \frac{1}{2}s_w - \frac{1}{3}c, \]

\[ q_{Il}(s_l, s_w) = \frac{1}{3}\theta_0 + \frac{1}{4}s_w + \frac{1}{2}s_l - \frac{1}{3}c, \forall s_l \leq s_w. \]

Note that \( S_{I1} \) and \( S_{I2} \) are regulated by 0 and 1 because the signals have finite support \([0, 1]\). In the first case, the winner’s optimum is feasible before hitting \( C \), and the loser chooses the corresponding best response. In the second case, the global optimum of the winner without
considering marginal cost $c$ exceeds her capacity $C$, and the marginal revenue is less than $c$ if we increase $q_w$ above $C$, therefore the equilibrium turns out to be a corner solution. In the third case, the optimal order quantity exceeds $C$.

Observe that in the first case, the loser’s quantity $q^l(s_l, s_w)$ can be expressed as

$$q^l(s_l, s_w) = \frac{1}{3} (\theta_0 + s_w + s_l) + \frac{1}{6} \{(\theta_0 + s_w + s_l) - \mathbb{E}_{s_l}[\theta | s_w, s_l < s_w]\} - \frac{2}{3} c,$$

where $\frac{1}{6} \{(\theta_0 + s_w + s_l) - \mathbb{E}_{s_l}[\theta | s_w, s_l < s_w]\}$ is the adjusted term resulting from the winner’s bias on $\theta$ expectation. In other words, although the loser knows both signals and henceforth has the best prediction of $\theta$, her best response still contains the winner’s bias $\mathbb{E}_{s_l}[\theta | s_w, s_l < s_w]$. Likewise, we observe the same phenomenon in the last case.

We can also see that $q^u(s_w)$ is increasing in $s_w$, and that $q^l(s_l, s_w)$ is increasing in both $s_l$ and $s_w$. This is because the higher the signals are, the higher the demand the resellers can expect, and therefore they will provide higher quantities to the consumer market.

### 3.2 The auction game

Using backward induction, we proceed to the previous stage of the game (Period 2). At this point, the resellers’ decision is about how to bid in the auction. We restrict ourselves to the symmetric (and strictly increasing) equilibrium throughout this section. An equilibrium in the auction mechanism $A$ is said to be symmetric if there exists a bidding function $\beta^A(s)$ such that a reseller who receives a signal $s$ submits the bid $\beta^A(s)$, independent of the reseller’s identity. We establish in the sequel the uniqueness of such an equilibrium under both the first- and second-price auctions.

Suppose a reseller receives a signal $s$. If she participates in auction $A$ and pretends as if her signal were $z$, we denote by $\Pi^A(z | s)$ her ex ante expected payoff. A reseller’s objective is to choose a strategy that maximizes it, and therefore a truth-telling equilibrium requires

$$s \in \arg\max_z \left\{ \Pi^A(z | s) : 0 \leq z \leq 1 \right\}.$$  

In order to verify this fact for both auction mechanisms, we introduce the quantities $\pi^A_{w}(\cdot)$ and $\pi^A_{l}(\cdot)$: the partial payoffs under auction mechanism $A$ in case of winning and losing respectively. By partial payoff, we refer to the expected revenue in the consumer market minus the procurement cost in the procurement market. Thus, we do not account for the procurement cost through the auction.
3.2.1 Second-price auction

Even though *bidding your own value* is a dominant strategy in auctions where bidders possess private values, in an interdependent-value model bidders do consider the bidding function her opponent uses. This is because based on this she may be able to estimate her opponent’s signal and hence the “right value” of the auctioned object. Therefore, strategic interaction scatters away the hope of finding a dominant strategy equilibrium.

If the supplier sells the capacity through a second-price auction, the expected payoff of a type-$s$ reseller who behaves as a type-$z$ reseller, assuming that her opponent adopts the bidding function $\beta^I(y)$, is

$$\Pi^I(z|s) = \int_0^s (\pi^I_w(s,y) - \beta^I(y)) \, dy + \int_s^1 \pi^I_I(z,s,y) \, dy,$$

(6)

where $z$ and $s$ belong to the inspected reseller, and $y$ refers to the opponent’s type. The first term represents the event that she wins the auction, in which case she gets the object and pays the opponent’s bid $\beta^I(y)$. Due to the monotonicity of $\beta^I(y)$, this event happens when the opponent’s signal is less than $z$. Her partial payoff $\pi^I_w(s,y)$ is independent of her own bid $\beta^I(z)$ since in the second-price auction, the supplier does not announce the winning bid. The second term corresponds to the event that she loses in the auction while pretending to be type-$z$: $\pi^I_I(z,s,y)$ is the partial payoff of a loser whose signal is $s$ and plays as a type-$z$, given that her opponent, i.e., the winner, receives signal $y$. Note that if she loses while playing as a type-$z$, her bid $\beta^I(z)$ will be revealed and therefore the opponent will infer a wrong type $z$ and will choose the best response accordingly.

The winning partial payoff $\pi^I_w(s,y)$ is as follows:

$$\pi^I_w(s,y) = (\theta_0 + s + y - q^I_w(s,y) - q^I_I(y)) \, q^I_w(s,y) - c(q^I_w(s,y) - C)^+,$$

where $q^I_w(s,y)$ and $q^I_I(y)$ are given by equation (2). It is simply the product of the realized price and the quantity, minus the purchasing cost in the procurement market.

Now we consider the losing partial payoff $\pi^I_I(z,s,y)$, where now the opponent’s type $y$ is the winning type. From Proposition 1, given $q^I_w(y,z)$, the reseller’s best response is

$$q^I_I(z,s) = \frac{1}{2} \left( E_y[\theta | y > z, s] - \int_z^1 q^I_w(y,z) \frac{1}{1 - z} \, dy - c \right).$$

(7)

The loser does not know the winner’s signal $y$ since only the losing bid $\beta^I(z)$ is announced. Expanding $E_y[\theta | y > z, s]$ in equation (7), we obtain that $q^I_I(z,s) = q^I_I(z) + \frac{1}{2}(s - z)$, where
$q^I_{II}(z)$ is the equilibrium quantity if the bidder’s type is $z$, and the extra term $\frac{1}{2}(s-z)$ captures the noise introduced by the misreported type. The loser’s partial payoff in this case is

$$\pi^I_{II}(z, s, y) = q^I_{II}(z, s) \left( \theta_0 + y + s - q^I_{II}(y, z) - q^I_{II}(z, s) - c \right).$$

Similarly to Milgrom and Weber (1982), we focus on the cases where the expected (total) payoff is unimodal in the reported type. The unimodal pattern of the expected payoff can be rationalized by the following argument: As the bidder increases her bid, her payoff will first increase due to the higher probability of winning. However, the bidder cannot increase her bid unboundedly because at some point she will pay too much to the seller. A sufficient condition to make the expected payoff unimodal is

$$q^I_{II}(s, s) \geq q^I_{II}(s, s), \quad \forall s.$$

This condition simply requires that a truth-telling type-$s$ reseller provide a significant higher quantity when she gets the capacity than when she loses. Note that we could express this condition in terms of the primitive values $C, c$ and $\theta_0$ by using Propositions 1 and 2, but it would become harder to read and less intuitive to understand.

Now we are ready to characterize the equilibrium bidding function of the second-price auction. Assuming that the objective function is differentiable and there exists an interior solution to the first-order condition of $\Pi^I_{II}(z|s)$ in equation (6), we have the following result:

**Theorem 1.** Suppose Assumption 2 holds. The unique symmetric, strictly increasing equilibrium bidding function $\beta^I_{II}(s)$ in the second-price auction is

$$\beta^I_{II}(s) = \pi^I_{II}(s, s) - \pi^I_{II}(s, s, s) + \int_s^1 \frac{\partial}{\partial v} \pi^I_{II}(v, s, y)|_{v=s} dy.$$

Moreover, $\beta^I_{II}(s) \leq \pi^I_{II}(s, s) - \pi^I_{II}(s, s, s)$.

Note that $\pi^I_{II}(s, s) - \pi^I_{II}(s, s, s)$ can be regarded as the equilibrium bidding function in the second-price auction with no resale: A bidder with signal $s$ is asked to bid an amount such that if she were to win the auction with that bid, she would just “break-even” (e.g., see Krishna (2002, Chapter 6)). Theorem 1 tells us that the bidding function here is lower than that in the conventional auction. This is because the bid of a type-$s$ reseller will be announced with probability $1 - P(s)$ (i.e., when she loses), and hence the threat of revealing her private information lowers her bid.
3.2.2 First-price auction

If the supplier uses a first-price auction in period 2, the expected payoff of a type-\(s\) reseller who behaves as a type-\(z\) reseller, assuming that her opponent adopts the same equilibrium bidding function, is

\[
\Pi^I(z|s) = \int_0^z \left( \pi^I_w(z, s, y) - \beta^I(z) \right) dy + \int_z^1 \pi^I_l(s, y) dy, \tag{8}
\]

The first term accounts for the case where the reseller wins the auction. In this case, her payment is her bid \(\beta^I(z)\). She wins because the opponent’s signal \(y\) is less than \(z\), and the opponent believes that she receives the signal \(z\). After the auction, the Cournot game takes place, and \(\pi^I_w(z, s, y)\) depends on the bidder’s reported \(z\) rather than her true signal (which is not disclosed). The second term corresponds to the case in which the reseller loses in the auction. In this case, the partial payoff \(\pi^I_l(s, y)\) is independent of the losing bid since it is not disclosed.

We first present the equilibrium quantities. In all cases, the formulas for \(q^I_l(y, z)\) are obtained from Proposition 3, and following them we calculate \(q^I_w(z, s)\) as the corresponding best responses.

**Proposition 4.** Suppose that the first-price auction is used. If a type-\(s\) bidder that behaves as a type-\(z\) wins the auction, then:

If \(z \leq S^I_1\),

\[
q^I_l(y, z) = \frac{1}{3} \theta_0 + \frac{1}{4} z + \frac{1}{2} y - \frac{2}{3} c,
q^I_w(z, s) = \begin{cases} \frac{1}{3} \theta_0 + \frac{1}{2} s + \frac{1}{3} c, & s \leq 2 \left(C - \frac{1}{3} \theta_0 - \frac{1}{3} c\right), \\ C, & \frac{2}{3} \left(C - \frac{1}{3} \theta_0 - \frac{1}{3} c\right) \leq s \leq 2 \left(C - \frac{1}{3} \theta_0 + \frac{1}{6} c\right), \\ \frac{1}{2} \theta_0 + \frac{1}{2} s - \frac{1}{6} c, & s \geq 2 \left(C - \frac{1}{3} \theta_0 + \frac{1}{6} c\right). \end{cases}
\]

If \(S^I_1 \leq z \leq S^I_2\), then

\[
q^I_l(y, z) = \frac{1}{2} (\theta_0 + z + y - C - c),
q^I_w(z, s) = \begin{cases} \frac{1}{2} \theta_0 + \frac{1}{8} s - \frac{1}{3} z + \frac{1}{3} C + \frac{1}{3} c, & s \leq \frac{3}{2} C - \frac{1}{2} \theta_0 + \frac{1}{3} z - \frac{1}{2} c, \\ C, & \frac{3}{2} \left(C - \frac{1}{2} \theta_0 + \frac{1}{3} z - \frac{1}{2} c\right) \leq s \leq \frac{3}{2} C - \frac{1}{2} \theta_0 + \frac{1}{6} z + \frac{1}{2} c, \\ \frac{1}{4} \theta_0 + \frac{1}{2} s - \frac{1}{3} z + \frac{1}{3} C - \frac{1}{3} c, & s \geq \frac{3}{2} C - \frac{1}{2} \theta_0 + \frac{1}{6} z + \frac{1}{2} c. \end{cases}
\]
Finally, if \( z \geq S_I^2 \),

\[
q^I_w(y, z) = \frac{1}{3} \theta_0 + \frac{1}{4} z + \frac{1}{2} y - \frac{1}{3} c,
\]

\[
q^I_w(z, s) = \begin{cases} 
\frac{1}{3} \theta_0 + \frac{1}{2} s + \frac{1}{6} c, & s \leq 2C - \frac{2}{3} \theta_0 - \frac{1}{3} c, \\
C, & 2C - \frac{2}{3} \theta_0 - \frac{1}{3} c \leq s \leq 2C - \frac{2}{3} \theta_0 + \frac{2}{3} c, \\
\frac{1}{3} \theta_0 + \frac{1}{2} s - \frac{1}{3} c, & s \geq 2C - \frac{2}{3} \theta_0 + \frac{2}{3} c.
\end{cases}
\]

The payoffs \( \pi^I_w(z, s, y) \) and \( \pi^I_l(s, y) \) in both cases can be expressed as follows:

\[
\pi^I_w(z, s, y) = q^I_w(z, s) \left( \theta_0 + s + y - q^I_w(z, s) - q^I_l(y, z) \right) - c(q^I_w(z, s) - C)^+, \\
\pi^I_l(s, y) = q^I_l(s, y) \left( \theta_0 + y + s - q^I_w(y, y) - q^I_l(s, y) - c \right).
\]

In deriving the equilibrium bidding function for the first-price auction, we would like to focus on the case where the expected (total) payoff is unimodal and the bidding function increasing, hence the first-order condition can be applied. We first characterize a sufficient condition similar to Assumption 2:

**Assumption 3.** \( q^I_w(s, s) \geq 4q^I_l(s, s), \forall s \).

Note that the condition is stronger than Assumption 2. In Milgrom and Weber (1982), the sufficient equilibrium condition for the first-price auction is also stronger than the one for the second-price auction. In the second-price auction, they only require the valuation \( v(s, y) \) be strictly increasing in both \( s \) and \( y \), whereas in the first-price auction, they need the signal affiliation to obtain the unimodality of bidders’ payoffs.

The next theorem characterizes the equilibrium bidding function.

**Theorem 2.** Suppose Assumption 3 holds. In the first-price auction, the unique symmetric, strictly increasing equilibrium bidding function is

\[
\beta^I(s) = \frac{1}{s} \int_0^s \left( \pi^I_w(y, y, y) - \pi^I_l(y, y) + \int_0^y \frac{\partial}{\partial u} \pi^I_w(u, y, v)|_{u=y} \, dv \right) \, dy.
\]

Moreover, \( \beta^I(s) \leq \frac{1}{s} \int_0^s \left( \pi^I_w(y, y, y) - \pi^I_l(y, y) \right) \, dy \).

The term \( \frac{1}{s} \int_0^s \left( \pi^I_w(y, y, y) - \pi^I_l(y, y) \right) \, dy \) can be regarded as the equilibrium bidding function in the conventional first-price auction by a slight modification of Milgrom and Weber (1982). Thus, Theorem 2 shows that the winner’s expected payment here is lower than the winner’s expected payment in the conventional counterpart. This is because the bid of a type-s reseller will be announced with probability \( P(s) \), and hence the threat of revealing her private information lowers her bid.
The monotonicity conditions of the bidding functions are also worth noting. \( \pi_w^I(s, s, s) - \pi_w^I(s, s) \) and \( \pi_w^{II}(s, s) - \pi_w^{II}(s, s, s) \) are the differences of equilibrium payoffs of a type-\( s \) reseller between the cases where she wins and where she loses in the auctions, provided that her opponent also receives the same signal. Because our bidding functions \( \beta^{II}(s) \) and \( \beta^I(s) \) have an extra integral term, namely \( \int_s^1 \frac{\partial}{\partial v} \pi_{II}^I(v, s, y)|_{v=s} dy \) and \( \int_0^y \frac{\partial}{\partial u} \pi_{II}^I(u, y, v)|_{u=y} dv \) respectively, it is not enough to ensure that the difference of payoffs is increasing. This is different from Milgrom and Weber (1982). In fact, it constitutes a slight generalization for the type-dependent participation constraint. Put in terms of their notation, \( v_l(s, y) \neq 0 \), where \( s \) is the bidder’s own signal, \( y \) is her opponent’s signal, and \( v_l \) is the utility when the bidder does not win the object. More specifically, when \( v_l(s, y) = 0 \), \( \forall s, \forall y \), Milgrom and Weber (1982) show that \( \beta^{II}(s) = v_w(s, s) \), where \( v_w(s, s) \) is the expected value of the object if both bidders receive signal \( s \). With a \( \text{Unif}[0, 1] \) signal distribution, the equilibrium bidding function under the first-price auction is \( \beta^I(s) = \int_s^y v_w(y, y) \frac{1}{s} dy \). While allowing \( v_l(s, y) \) to depend on the signals as well, it can be verified that \( \beta^{II}(s) = v_w(s, s) - v_l(s, s) \) and \( \beta^I(s) = \int_0^s (v_w(y, y) - v_l(y, y)) \frac{1}{s} dy \). In both auctions, a sufficient condition for the monotonicity of the bidding function is that \( v_w(s, s) - v_l(s, s) \) is increasing in \( s \).

In the first-price auction, we observe in the proof of Theorem 2 that the third term in the integrand is either \( -\frac{1}{2} \int_0^y q_w^I(u, u) du \) or \( -\frac{1}{4} \int_0^y q_w^I(u, u) du \); in either case, it is decreasing in \( y \). This implies that we need a stronger condition to guarantee the monotonicity of \( \beta^I(s) \), i.e., \( \pi_w^I(s, s, s) - \pi_w^I(s, s) \) being strictly increasing does not suffice. More specifically, we require the integrand \( \pi_w^I(y, y, y) - \pi_w^I(y, y) + \int_0^y \frac{\partial}{\partial u} \pi_{II}^I(u, y, v)|_{u=y} dv \) be strictly increasing in \( y \).

On the contrary, in the second-price auction, we can find situations where the integral \( \int_s^1 \frac{\partial}{\partial v} \pi_{II}^I(v, s, y)|_{v=s} dy \) is non-monotonic (in region II for example). Thus, in some cases the extra integral term actually helps to sustain the monotonicity of \( \beta^{II}(s) \). We conclude that it is more difficult to sustain a fully revealing bidding equilibrium in the first-price auction than in the second-price auction.

Finally, note that the resellers’ participation constraint is implicitly taken into consideration under both auction mechanisms: If a reseller does not want to participate, she can always join and submit a type-0 bid (i.e. \( \beta^{II}(0) \) or \( \beta^I(0) \)). In this way, she will lose, and her true signal will not be revealed.

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3.3 Outcome of the game

3.3.1 Supplier’s expected revenue

Under the second-price auction, the supplier’s expected revenue is

$$E[R_{II}] = \int_0^1 \beta_{II}(s) f_{(2)}(s) ds,$$

where $f_{(2)}(s)$ is the probability density function of $\min\{s_1, s_2\}$, since the payment is the highest losing bid. Given the independently uniformly distributed $s_1$ and $s_2$, $f_{(2)}(s) = 2(1 - s)$.

Under the first-price auction, when the payment equals the winning bid, the supplier’s expected revenue is

$$E[R_{I}] = \int_0^1 \beta_{I}(s) f_{(1)}(s) ds,$$

where $f_{(1)}(s)$ is the density function of $\max\{s_1, s_2\}$, and in the context of our model, $f_{(1)}(s) = 2s$.

3.3.2 Quantities in the consumer market

Given the inverse linear demand function (1), the total aggregated consumers’ surplus given auction format $A$ is

$$\Pi_A C(s_1, s_2) = \frac{1}{2} \left( Q_A(s_1, s_2) \right)^2,$$

where $Q_A(s_1, s_2)$ is the total quantity that resellers provide to the consumer market.\(^\text{6}\) Note that for a given pair of signals $(s_1, s_2)$, $\Pi_A(s_1, s_2)$ is strictly increasing in $Q_A(s_1, s_2)$. Therefore, to compare consumers’ surplus in both auctions, it suffices to consider the equilibrium quantities.

According to the auction announcement policy, the equilibrium quantities can be defined as

$$Q^I(s_1, s_2) = q^I_w(s_w) + q^I_l(s_l, s_w) \quad \text{and} \quad Q^{II}(s_1, s_2) = q^{II}_w(s_w, s_l) + q^{II}_l(s_l),$$

where $s_w = \max\{s_1, s_2\}$ and $s_l = \min\{s_1, s_2\}$. We obtain that

**Theorem 3.** If the difference between resellers’ signals is large, consumers are better off in the first-price auction. Otherwise, if the difference between resellers’ signals is small, consumers are better off in the second-price auction.

In the extreme case in which the resellers receive very different signals $(s_1, s_2)$, the real demand is actually close to its mean. According to Theorem 3, in the first-price auction, the disclosure of the winner’s bid raises the loser’s expectation of the demand. Therefore, the loser sets the quantity at the normal level. However, in the second-price auction, the loser has no access to the winner’s signal. Therefore, she underestimates the demand and this results in an abnormally low quantity. In this case, consumers are better off in the first-price auction due to the correction of loser’s belief.

Figure 4 illustrates the quantity difference between the first-price and second-price auctions. The graph is symmetric since these two resellers are symmetric, and the boundary

\(^6\)Note that the for the inverse linear demand function $P(Q) = \theta - Q$, the total aggregated consumers’ surplus is $\int_0^Q P(q)dq - P(Q)Q = \frac{1}{2}Q^2$.\(^2\)
between the two regions in the lower triangle is nondecreasing in $s_1$. Moreover, the closer a point is to the lower-right corner (equivalently, the upper-left corner), the larger the difference between $Q^I(s_1, s_2)$ and $Q^{II}(s_1, s_2)$.

Figure 4: Comparison between aggregated quantities provided to the consumer market under first-price and second-price auctions.

4. Numerical Results

In this section, we provide several numerical examples to illustrate the payoffs, quantities and bidding functions. The parameters $\theta_0, c$ and $C$ are carefully chosen such that the non-negativity of realized prices and quantities is satisfied in all subgames. In particular, unless explicitly mentioned, we fix $\theta_0 = 3$, $C = 1.5$, and $c = 2.5$ in all the following examples. In fact, given $\theta_0 = 3$, we observed similar qualitative results with different values of $c$. Note that in all cases, we need Assumptions 2-4 to hold in order to guarantee the unimodality of the resellers’ payoffs and the monotonicity of the bidding functions.

Figure 5 compares the total quantities $Q^I(s_1, s_2)$ and $Q^{II}(s_1, s_2)$ provided to the consumer market. Since the resellers are symmetric, the graph is symmetric; and $Q^I(s_1, s_2)$ turns out to be higher in the upper-left and lower-right corners. This is consistent with Theorem 3 and Figure 4 since larger quantities in the consumer market corresponds to lower prices for the consumers.
In Figure 6, we plot the bidding function of a type-s reseller. Note that the bidding function in a first-price auction is lower than that in a second-price auction for all signals. Since in the first-price auction the bid is the payment when the reseller wins, she decreases her bid to maintain her rent (similar results are reported in Krishna (2002, Chapter 6)). Moreover, the bidding functions inferred here are lower than the counterparts under the conventional auction (as mentioned in Section 3.2), and the difference becomes larger as the signal is higher.

Figure 7 presents the supplier’s expected revenue under both mechanisms. The supplier’s expected revenue, derived in Section 3.3.1, equals the expected payment from the resellers while taking expectation with respect to their signals. This amount turns out to be higher in the second-price auction than in the first-price auction. Note that this result is in contrast with the Revenue Linkage Principle: In a conventional auction, if the signals are independent as in our case, the Revenue Linkage Principle says that the supplier should receive the same revenue from both the first- and second-price mechanisms. Here, the foreseeable information disadvantage in the consumer market constitutes a sort of winner’s curse in that the winner in the first-price auction collects on average less profit from winning the auction. Therefore, she is unwilling to pay as much as she would pay under the second-price auction. This result directly lowers the supplier’s expected revenue in the first-price auction.

From Figure 7, we also observe that as the supplier increases the capacity level $C$, his expected revenue gets saturated after certain thresholds. This is related to the maximum quantity that the market demand may request (recall the downward sloping demand func-
Figure 6: Comparison of reseller bidding functions under both auction mechanisms.

Figure 7: Comparison of supplier’s revenues $E[R^I]$ and $E[R^{II}]$.
tion in equation (1)), and the resellers are unwilling to pay more for excess capacity because it does not bring more profit. Suppose that the auction mechanism is given. The supplier can decide his optimal capacity, taking into account his own cost of building capacity. Our results further suggest, for example, that providing capacity higher than \( C = 2.5 \) is a weakly dominated strategy, independent of supplier’s cost structure (as long as capacity cost is increasing). The fact that the saturation comes later in the second-price auction (at \( C = 2.5 \) in the figure) is also worth noting: it implies that the resellers are still willing to pay for marginal increase of capacity \( C \) due to the combined first-mover and information advantage.

Next, we investigate the reseller’s willingness-to-pay in both auctions as opposed to the given spot price \( c \), where willingness-to-pay is defined as the expected payment given that the reseller wins. In other words, in the first-price auction, the willingness-to-pay is the bidding function, whereas in the second-price auction, it becomes the expected losing bid. Figure 8 shows that the willingness-to-pay under both auctions increases linearly in \( c \), and runs always below \( c \) (the 45-degree line). The participation of the resellers in the auction can be attributed to two reasons: gathering information about the opponent’s private signal, and lower the expected procurement cost. Thus, a supplier with limited capacity would sell it at a lower price than the unlimited procurement market. This also justifies why dual sourcing is possible: neither source is dominated by the other.

Figure 8: Retailer bidding function under a second-price auction vs spot price \( c \), when \( s_i = 0.5 \).

Figure 9 exhibits the resellers’ expected profit under both auction mechanisms. We draw two curves that come from equations (6) and (8) respectively, when replacing \( z \) by the true type \( s \). The figure shows that a reseller is better off in the second-price auction, and
the gap is expanded as the signal becomes higher. This observation jointly with Figure 7 implies that the second-price auction in fact yields a higher combined payoff for the entire supply chain, because the supplier and the resellers are all better off. Allowing the winner to get both the first-mover and the information advantages reduces the competition tension between downstream resellers, and hence ex ante the entire supply chain benefits. Note also that in the first-price auction, the resellers with low signals receive just their reservation utilities eventually, and hence bidders with low signals do not get information rent.

![Figure 9: Comparison of reseller expected profit under both auction mechanisms.](image)

Finally, we examine the monotonicity of the bidding function. Let us take the parameters \((\theta_0 = 3, C = 1.5, c = 2.5)\) as the reference point, and change them one at a time. Fix \(\theta_0 = 3\) and \(C = 1.5\), the monotonicity holds for both auctions when \(c \in [2.1, \infty)\); fix \(\theta_0 = 3\) and \(c = 2.5\), it holds when \(C \in [0.83, \infty)\); and if \(C = 1.5, c = 2.5\), the bidding functions are monotonic when \(\theta_0 \leq 3.7\). The monotonicity fails as either the gain from the auction becomes small or the value of information advantage in the consumer market becomes large. Naturally, the resellers are unwilling to reveal their signals as the procurement market price is small, as the capacity is small, and as the potential demand is large. Moreover, in all our numerical experiments, as long as the bidding function under the first-price auction is monotonic, then the monotonicity for the bids under the second-price auction holds. This suggests that the separating equilibrium is relatively harder to sustain in the first-price auction, as predicted by our analysis. The numerical experiments support our interpretation of Theorems 1 and 2.
5. Conclusions

Motivated by supply chain settings where a supplier’s capacity is sold as a bundle, especially common in services (e.g., auctioning the capacity of a stadium, or the sponsorship of a website or an event), we analyze a two-stage supply chain model, where two symmetric resellers bid for the capacity of a supplier and then compete in the consumer market. After the auction closes, both resellers can get additional units in a procurement market. We study this model under first- and second-price auctions, and analyze the impact of the information elicited in the first stage of the game.

We characterize sufficient conditions for the existence of monotonic equilibrium bidding functions. These conditions are more restrictive for the first-price auction case. When an increasing equilibrium bidding function exists, the threat of revealing private information induces lower bids than the ones that would be submitted under conventional auctions (i.e., auctions with no resale). The first-price auction makes consumers better off if resellers receive very different demand signals. Under a second-price auction, the supplier collects higher expected revenue, and resellers receive higher expected profits as well. The consumers are also better off if the resellers receive similar signals.

Our model shows that traditional auctions have a significant impact when put to work in the context of a supply chain because of the information asymmetry that may be introduced when announcing their results. It constitutes a starting point to understand the underlying economics in supply chain management using auctions as a means to facilitate transactions.

As a possible extension, we can think of a similar model with budget constraints on the resellers’ side. Note that the equilibrium of a Cournot game like ours is independent of the resellers’ cash endowments. The impact of budget constraint on bidders’ strategies has been studied in Che and Gale (1998). They focus on the conventional auction and show that bids are distorted because of the budget constraint. In our model, a more intriguing trade-off prevails. If a reseller spends too much in acquiring the capacity $C$ from the auction, she may not have enough money to trade in the procurement market after observing the information revealed in the auction. Another extension would be to allow the supplier to split the capacity sold through the auction. When both resellers get part of it, since the supplier announces all the transaction prices, both signals might be revealed. This might mitigate the information asymmetry when the downstream resellers compete afterwards.

Finally, there is an ongoing discussion in the auction theory and experimental economics
communities about whether the game theoretic models, as we adopt in this paper, can be used as valid decision support for the bidders, dating back to the paper by Cox et al. (1982), and including Rothkopf and Harstad (1994), and Lucking-Reiley (1999). It would be interesting to conduct experiments to test how resellers bid in practice, and validate our predictions.

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References


A. Appendix

In this Appendix we extensively use the regions found in Figure 3 and divide our analysis by cases. To avoid redundancy, we present only three of them and refer the other cases to their analogous counterparts. Usually the analogy is made when the terms we consider differ merely in the marginal cost $c$, e.g., regions I and V.

Proof of Proposition 1

We first disregard the capacity constraint of the winner, and derive her first-order conditions given $s_l$.

Let $\Pi_{II}^w \equiv (\theta - q_w - q_l)q_w - c(q_w - C)^+$ denote the winner’s payoff function. By differentiating $\Pi_{II}^w$ with respect to $q_w$ where $q_w \leq C$, we obtain the first-order condition

$$q_w^*(s_w, s_l) = \frac{1}{2} (\theta - q_l(s_l) - c \mathbb{1}\{q_w^*(s_w, s_l) \geq C\})$$,

where $\mathbb{1}\{\cdot\}$ is the indicator function. Similarly, we derive the first-order condition for the loser:

$$q_l^*(s_l) = \frac{1}{2} (E_{s_w}[\theta|s_w > s_l] - E_{s_w}[q_w(s_l)|s_w > s_l] - c)$$.

Observing that the differentiation $\theta - q_l^* - 2q_w - c \mathbb{1}\{q_w \geq C\}$ is decreasing in $q_w$, the marginal change of the winner payoff will look like Figure A.1. Now the optimal quantity $q_{wII}^*(s_w, s_l)$ follows immediately from the comparison between marginal revenue and marginal cost. Note that the discontinuity occurs when capacity $C$ is hit. In Figure A.1, $s_0$ falls into

![Figure A.1](image-url)
the first case of equation (2) since the marginal change becomes negative before hitting $C$. $s_1$ there equals $S^I_1(s_1)$ because the marginal change at $C$ from the right just turns negative. Likewise, $s_2 = S^I_2(s_1)$ because $s_2$ is the largest point whose marginal change is positive for all $q_w \leq C$. By continuity of $q^I_w(s_w, s_1)$, the values of $S^I_1(s_1)$ and $S^I_2(s_1)$ can be obtained by equating the capacity $C$ and $q^I_w(s_w, s_1)$ at the boundary points. \hfill \Box

**Proof of Proposition 2**

The following lemma is needed to show this proposition.

**Lemma 1.** In the equilibrium quantities described in equation (2),

$$0 < q^I_{w_1}(s_2) - q^I_{w_1}(s_1) < s_2 - s_1, \text{ for signals } s_1 < s_2.$$ 

Proof: The proof is by contradiction. Suppose that there exists $s_1$ such that $(q^I_{w_1})'(s_1) = \rho^I_1 > 1$. We can rewrite equation (2) as

$$q^I_{w_1}(s_1) = \frac{1}{2}[\theta_0 + \frac{3}{2}s_l - E_{s_w}[q^I_{w_1}(s_w, s_1)|s_w > s_l] + \frac{1}{2} - c]. \quad (A.2)$$

The coefficient $s_w$ term in $q^I_{w_1}(s_w, s_1)$ is lower bounded by 0 according to equation (2), and hence $(q^I_{w_1})'(s_1) = \rho^I_1 > 1$ implies that there exists a signal $s^1_w$ such that $\frac{\partial}{\partial s_1}(q^I_{w_1})'(s_w, s_1)|_{s_1=s^1_w} < \frac{3}{2} - 2\rho^I_1 < 0$. Since $q^I_{w_1}$ will not change if the capacity is binding, this can happen only in either the first or the third case. But then this implies that $\frac{1}{2}(1 - \rho^I_1) \leq \frac{3}{2} - 2\rho^I_1$, which leads to $\rho^I_1 \leq \frac{2}{3}$, a contradiction.

Similarly, suppose that there exists $s^2_w$ such that $(q^I_{w_1})'(s^2_w) = \rho^I_2 < 0$. Since from equation (2) the coefficient $s_w$ term in $q^I_{w_1}$ is upper bounded by $\frac{1}{2}$, the term associated to $s_w$ in $E_{s_w}[q^I_{w_1}(s_w, s_1)|s_w > s_1]$ contributes at most $\frac{1}{2} + \frac{1}{2} - 3\rho^I_2$. We can show that for some $s^2_w$ whose $(q^I_{w_1}(s^2_w, s^2_w))'$ implies $\frac{1}{2}(1 - \rho^I_2) \geq 1 - 2\rho^I_2 \Rightarrow \rho^I_2 \geq \frac{1}{3}$. Hence we conclude that $0 \leq q^I_{w_1}(s_2) - q^I_{w_1}(s_1) \leq s_2 - s_1, \forall s_1 \leq s_2$. \hfill \Box

Now we prove Proposition 2. We will focus on $S^I_1(s)$; the proof for $S^I_2(s)$ goes along the same argument. The first part of equation (5) follows from the fact that $S^I_2(s) \geq S^I_1(s), \forall s$. In the sequel, $s_1$ and $s_2$ are two distinct signals and $s_1 < s_2$.

**Case a):** $S^I_1(s_2) = 1$

We would like to prove that $S^I_1(s_1) = 1$ as well, i.e., the capacity constraint is never binding when $s_1 = s_1$.

Note that $S^I_1(s_2) = 1$ means that

$$\frac{1}{3}\theta_0 + \frac{1}{2}s_l + \frac{1}{4}s_2 + \frac{1}{3}s_l - \frac{1}{12} \leq C,$$
and therefore
\[
\frac{1}{3} \theta_0 + \frac{1}{2} + \frac{1}{4}s_1 + \frac{1}{3}c - \frac{1}{12} < \frac{1}{3} \theta_0 + \frac{1}{2} + \frac{1}{4}s_2 + \frac{1}{3}c - \frac{1}{12} \leq C,
\]
which implies \( S_1^{II}(s_1) = 1 \).

**Case b):** \( s_1 < S_1^{II}(s_1) < 1 \) and \( s_2 < S_1^{II}(s_2) < 1 \)

Suppose \( s_1 < S_1^{II}(s_1) < 1 \) and \( s_2 < S_1^{II}(s_2) < 1 \). Our goal here is to prove that \( S_1^{II}(s_2) < S_1^{II}(s_1) \). Since in both cases the capacity constraint is not binding for some \( s_w \) and binding for others,

\[
S_1^{II}(s_2) = 2C - \theta_0 - s_2 + q_1^{II}(s_2)
= S_1^{II}(s_1) - (s_2 - s_1) + (q_1^{II}(s_2) - q_1^{II}(s_1)),
\]

and therefore by Lemma 1,

\[
0 < S_1^{II}(s_1) - S_1^{II}(s_2) < s_2 - s_1.
\]

**Case c):** \( S_1^{II}(s_1) = s_1 \).

In this case, we would like to prove that \( S_1^{II}(s_2) = s_2 \) as well. The proof is by contradiction.

First we claim that \( S_1^{II}(s_2) \neq 1 \). If this is not the case, the capacity constraint is never binding when \( s_1 = s_2 \). By the argument in Case a), \( S_1^{II}(s_1) = 1 \) as well since \( s_1 < s_2 \), which contradicts the fact \( S_1^{II}(s_1) = s_1 \).

Therefore, the only possibility is that \( s_2 < S_1^{II}(s_2) < 1 \). In this case, \( S_1^{II}(s_2) = 2C - \theta_0 - s_2 + q_1^{II}(s_2) \), and \( S_1^{II}(s_1) = s_1 \) implies \( \frac{1}{2} (\theta_0 + s_1 + s_1 - q_1^{II}(s_1)) \geq C \). Then we can establish the following inequality:

\[
S_1^{II}(s_2) \leq 2s_1 + \theta_0 - q_1^{II}(s_1) - \theta_0 - s_2 + q_1^{II}(s_2)
\]

Thus,

\[
s_2 < S_1^{II}(s_2) \leq 2s_1 + \theta_0 - q_1^{II}(s_1) - \theta_0 - s_2 + q_1^{II}(s_2),
\Rightarrow 2(s_2 - s_1) < q_1^{II}(s_2) - q_1^{II}(s_1),
\Rightarrow q_1^{II}(s_2) - q_1^{II}(s_1) > 2(s_2 - s_1),
\]

which contradicts Lemma 1. Hence we conclude that \( S_1^{II}(s_2) = s_2 \).

**Proof of Theorem 1**

**Preliminaries**
We start discussing several technical lemmas that lead to Theorem 1 and provide their economic intuition.

If the losing bid $\beta_{II}(z)$ is higher, the winner should expect the demand to be higher, and therefore the quantity she puts in the consumer market $q_{w}^{II}(y, z)$ should also be larger, and the magnitude by which the winning quantity increases should be reasonably bounded. Hence, we have

**Lemma 2.**

\[ 0 \leq q_{w}^{II}(y, z_2) - q_{w}^{II}(y, z_1) < \frac{1}{2}(z_2 - z_1), \quad \forall z_1 < z_2. \]  

**Proof:** We show the monotonicity by dividing the shape of $S_{II}^{I}(z)$ into cases. We write $z_i \in I$ if $z_i$ belongs to region I; $z_i \in II$ and $z_i \in III$ are defined analogously.

**Case 1:** $y \leq S_{II}^{I}(z_1), y \leq S_{II}^{I}(z_2)$

In this case, the capacity constraint is not binding, and therefore

\[ q_{w}^{II}(y, z) = \frac{1}{3} \theta_0 + \frac{1}{2} y + \frac{1}{4} z + \frac{1}{3} c - \frac{1}{12}, \]

which satisfies equation (A.3).

**Case 2:** $(z_1, z_2) \in (II, II)$

Since when $z < S_{II}^{I}(z) < 1$, $S_{II}^{I}(z)$ is decreasing in $z$, the critical point where $q_{w}^{II}(y, z)$ hits the capacity $C$ comes earlier. It suffices to show that while $q_{w}^{II}(y, z)$ is not hitting the capacity, it is increasing in $z$.

Following Proposition 1, we first observe that given $z$, $\frac{\partial}{\partial y} q_{w}^{II}(y, z) = \frac{1}{2}$, which means $q_{w}^{II}(y, z)$ is increasing in $y$ with a fixed rate. Therefore, for all $y$ such that $z_2 \leq y \leq S_{II}^{I}(z_2)$,

\[
q_{w}^{II}(y, z_2) = q_{w}^{II}(S_{II}^{I}(z_2), z_2) - \frac{1}{2}[S_{II}^{I}(z_2) - y]
\]

\[
= C - \frac{1}{2}[S_{II}^{I}(z_2) - y]
\]

\[
\geq C - \frac{1}{2}[S_{II}^{I}(z_1) - y], \quad \text{since } S_{II}^{I}(z_1) > S_{II}^{I}(z_2)
\]

\[
= q_{w}^{II}(S_{II}^{I}(z_1), z_1) - \frac{1}{2}[S_{II}^{I}(z_1) - y], \quad \text{by definition of } S_{II}^{I}(z_1)
\]

\[
= q_{w}^{II}(y, z_1).
\]

Moreover,

\[
q_{w}^{II}(y, z_2) - q_{w}^{II}(y, z_1) = [C - \frac{1}{2}(S_{II}^{I}(z_2) - y)] - [C - \frac{1}{2}(S_{II}^{I}(z_1) - y)]
\]

\[
= \frac{1}{2}(S_{II}^{I}(z_1) - S_{II}^{I}(z_2)) \leq \frac{1}{2}(z_2 - z_1),
\]

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where the last inequality comes from Lemma 1 and equation (3).

Figure A.2: An example to show the relative positions of $z_1, z_2, S_1^{II}(z_2)$, and $S_1^{II}(z_1)$.

Remark: Figure A.2 shows the relationship between $z_1, z_2, S_1^{II}(z_2)$, and $S_1^{II}(z_1)$. Referring to Figure 3, we fix the loser’s signal $s_l$ inside region II. Now we increase $s_w$ along the vertical line that passes $(s_l, 0)$, and draw the winner’s quantity $q_w$ as a function of $s_w$ in Figure A.2.

Case 3: $S_1^{II}(z_1) \leq y \leq S_2^{II}(z_1), S_1^{II}(z_2) \leq y \leq S_2^{II}(z_2)$

In this case, the capacity constraint of the winner is binding for all $z$, i.e., $q_w^{II}(y, z) = C, \forall z$. Thus, the result holds.

Other Cases: The cases $y \geq S_2^{II}(z_1), S_1^{II}(z_2) \leq y \leq S_2^{II}(z_2)$ and $y \geq S_2^{II}(z_1), y \geq S_2^{II}(z_2)$ are very similar to Cases 2 and 3 respectively, and the results follow from analogous derivations. In other cases where $z_1$ and $z_2$ belong to different regions in Figure 3, we can use the triangle inequalities by inserting the boundaries of two regions $s_l^*$ and $s_l^{**}$ respectively.

Lemma 2 confirms the correctness of our intuition. When the losing bid increases, the winner’s expectation of the demand in the consumer market also increases, and therefore she puts more equilibrium quantity. This can be interpreted as the winner’s overestimation of the demand if the loser submits a bid higher than $\beta^{II}(s)$. However, the increment of $q_w^{II}(y, z)$ is bounded above.

Furthermore, if the loser’s signal increases, her optimal quantity should also increase at a reasonable rate:

Lemma 3.

$$\frac{1}{2}(z_2 - z_1) \leq q_l^{II}(z_2) - q_l^{II}(z_1) < \frac{3}{4}(z_2 - z_1), \forall z_1 < z_2.$$  \hspace{1cm} (A.4)
Proof: Recall from equation (2) that

\[ q^I_t(z) = \frac{1}{2} \theta_0 + \frac{3}{4} z - \frac{1}{2} c - \frac{1}{2} E_y[q^0_w(y, z)|y \geq z]. \]

If we increase \( z \) by \( \Delta z \), the term \( \frac{3}{4} z \) will increase by \( \frac{3}{4} \Delta z \). The term \( E_y[q^I_t(y, z)|y \geq z] \) is bounded by 0 and \( \frac{1}{2} \) according to Lemma 2. Therefore, equation (A.4) is valid.

Note that this lemma provides tighter upper and lower bounds for the first-order difference of \( q^I_t(z) \) than Lemma 1. The next lemma says that if a type-\( s \) reseller loses in the auction, her expected partial payoff will be decreasing in her reported type.

Lemma 4. \( \int_s^1 \frac{\partial}{\partial z} \pi^I_t(z, s, y)|_{z=s} dy \leq 0. \)

Proof: Define \( g(s) = \int_s^1 \frac{\partial}{\partial z} \pi^I_t(z, s, y)|_{z=s} dy. \)

\[
g(s) = \int_s^1 \frac{\partial}{\partial z} \left[ q^I_t(z, s)[\theta_0 + y + z - q^I_t(y, z) - q^I_t(z, s) - c]\right]_{z=s} dy
\]

\[
= \int_s^1 \left( \frac{\partial q^I_t(z, s)}{\partial z}[\theta_0 + y + z - q^I_t(y, z) - q^I_t(z, s) - c] + q^I_t(z, s)\left[ -\frac{\partial q^I_t(y, z)}{\partial z} - \frac{\partial q^I_t(z, s)}{\partial z} \right] \right)_{z=s} dy. \tag{A.5}
\]

Recalling that \( q^I_t(z, s) = q^I_t(z) + \frac{1}{2}(s - z) \) and Proposition 1, we have

\[
\frac{\partial q^I_t(z, s)}{\partial z}|_{z=s} = (q^I_t)'(s) - \frac{1}{2},
\]

\[
\frac{\partial q^I_t(y, z)}{\partial z}|_{z=s} = \begin{cases} 
0, & S^I_1(z) \leq y \leq S^I_2(z), \\
\frac{1}{2}[1 - (q^I_t)'(s)], & \text{otherwise}.
\end{cases}
\]

Thus, equation (A.5) can be rewritten as

\[
g(s) = \left( (q^I_t)'(s) - \frac{1}{2} \right) \left( E_y[\theta|s, y \geq s] - E_y[q^I_t(y, s)|s, y \geq s] - q^I_t(s) - c \right)
\]

\[ + q^I_t(s) \left( \frac{1}{2}(q^I_t)'(s) \right) \left( 1 - (S^I_2(s) - S^I_1(s)) \right) - q^I_t(s) \left( (q^I_t)'(s) - \frac{1}{2} \right) \left( S^I_2(s) - S^I_1(s) \right)
\]

\[ = \frac{1}{2} \left( 1 - (S^I_2(s) - S^I_1(s)) \right) q^I_t(s) \left( ((q^I_t)'(s) - 1) \right), \tag{A.6}
\]

where the last equality follows from the fact \( q^I_t(s) = \frac{1}{2} \left( E_y[\theta|s, y \geq s] - E_y[q^I_t(y, s)|s, y \geq s] - c \right). \)

Since \( 1 - (S^I_2(s) - S^I_1(s)) \) and \( q^I_t(s) \) are positive, and \( ((q^I_t)'(s) - 1) \) is negative by Lemma 3, we conclude that \( g(s) < 0 \), which completes the proof.

The average marginal change of a type-\( s \) loser’s partial payoff is obtained by integrating over her opponent’s signal \( y \), with \( y \geq s \). For notational convenience, we assume that
\( \pi^I(z, s, y) \) is differentiable with respect to \( z \) at boundary points. If it is not differentiable, then we should look for the subgradients rather than the gradient from the first-order condition.

Now, we complete the proof of Theorem 1. We will first derive a necessary condition that an equilibrium bidding function must satisfy, and then provide the verification of the proposed bidding function.

1. Necessary Condition

The first-order condition with respect to \( z \) is as follows:

\[
\frac{\partial \Pi^I(z|s)}{\partial z} = [\pi^I_w(s, z) - \beta^I(z) - \pi^I_l(z, s, z)] + \int_z^1 \frac{\partial}{\partial v} \pi^I_l(v, s, y) \bigg|_{v=z} dy.
\] (A.7)

The truth-telling equilibrium requires that the bidder’s optimal strategy is to reveal her own type. Thus, we conjecture the equilibrium bidding function \( \beta^I(s) \) as follows:

\[
\beta^I(s) = \pi^I_w(s, s) - \pi^I_l(s, s, s) + \int_s^1 \frac{\partial}{\partial z} \pi^I_l(z, s, y) \bigg|_{z=s} dy
\] (A.8)

where the last inequality is given by Lemma 4. Note that by Proposition 1, the equilibrium quantities \( q^I_w(z, s) \) and \( q^I_l(y) \) are all uniquely determined for any values \( z, s, y \), and hence the terms \( \pi^I_w(s, s), \pi^I_l(s, s, s), \) and \( \frac{\partial}{\partial z} \pi^I_l(z, s, y) \bigg|_{z=s} \) are all unique since these are generated from \( q^I_w(z, s) \) and \( q^I_l(y) \). Thus, within the class of symmetric equilibria, if there exists an equilibrium bidding function that is strictly increasing in the signal, then it must be uniquely determined by equation (A.8).

2. Verification: Monotonicity of \( \beta^I(s) \)

For the second-price auction, our goal is to show that \( \beta^I(s) \) is strictly increasing in \( s \), i.e., \( \lim_{z \to s} \frac{\beta^I(z) - \beta^I(s)}{z-s} > 0 \), \( \forall s \in [0, 1] \). Except region IV where the capacity constraint is binding for \( q^I_w(y, y) \), we obtain

\[
\pi^I_w(z, z) - \pi^I_l(z, z, z) - \pi^I_l(s, s, s) = (2 - 2\rho_l)q^I_w(s, s) - (1 - \frac{1}{2}\rho_l)q^I_l(s, s) + \rho_l c 1 \{ s \in V \},
\]

where we have ignored the second-order terms and \( \rho_l \) is such that \( q^I_l(z, z) = q^I_l(s, s) + \rho_l(z-s) \). We will divide the analysis according to Figure 3. Note that \( q^I_w(s, s) \) takes values along the 45-degree line.

Case 1: \( s \in I \)
In this case, \( \rho_t = \frac{1}{2} \), \( S_1^{II}(s) = S_2^{II}(s) = 1 \). From equation (A.6),

\[
g(s) = \int_s^1 \frac{\partial}{\partial z} \pi_1^{II}(z, s, y) \big|_{z=s} dy = \frac{1}{2} \left( 1 - (S_2^{II}(s) - S_1^{II}(s)) \right) q_1^{II}(s) \left( (q_1^{II})' (s) - 1 \right),
\]

and hence \( \beta^{II}(z) - \beta^{II}(s) = (z - s)(q_1^{II}(s, s) - 3q_1^{II}(s, s) + \frac{1}{2}c - \frac{1}{8}) + o(z - s) \). In other words,

\[
(\beta^{II})' (z) = q_1^{II}(s, s) - \frac{3}{4} q_1^{II}(s, s) + \frac{1}{2}c - \frac{1}{8} > 0 \text{ by Assumption 2.}
\]

**Case 2:** \( s \in II \)

When \( s \in II \), \( S_2^{II}(s) = 1 \) and \( S_1^{II}(s) = 2C - \theta_0 - s + q_1^{II}(s) \). Therefore,

\[
\beta^{II}(z) - \beta^{II}(s) = (2 - 2\rho_t)q_1^{II}(s, s) - \frac{1}{2}(\rho_t^2 - 3\rho_t + 3)q_1^{II}(s, s) - \frac{1}{2}\rho_t(1 - \rho_t)S_1^{II}(s) + o(z - s).
\]

Note that from Lemma 3, \( \frac{1}{4} \leq \rho_t \leq \frac{3}{4} \), and \( S_1^{II}(s) \leq 1 \). A sufficient condition for monotonicity under these parameters is that \( q_1^{II}(s, s) > \frac{13}{16} q_1^{II}(s, s) + \frac{3}{16} \).

**Case 3:** \( s \in III \)

Here we have \( S_2^{II}(s) - S_1^{II}(s) = 1 - c \), and therefore

\[
\beta^{II}(z) - \beta^{II}(s) = (2 - 2\rho_t)q_1^{II}(s, s) - (1 - \frac{1}{2}\rho_t)q_1^{II}(s, s) - \frac{1}{2}\rho_t(1 - \rho_t) + c(1 - \frac{1}{2}(1 - \rho_t)) + o(z - s).
\]

Note that when \( \frac{1}{2} \leq \rho_t \leq \frac{3}{2} \), \( c(1 - \frac{1}{2}(1 - \rho_t)) \) is always positive. A sufficient condition for \( \beta^{II}(s) \) being increasing is that \( q_1^{II}(s, s) > \frac{5}{4} q_1^{II}(s, s) + \frac{3}{16} \).

**Case 4:** \( s \in IV \)

In this case, the capacity constraint is binding and \( g(s) = -\frac{1}{2}(1 - \rho_t)(1 - s)q_1^{II}(s, s) \).

\[
\beta^{II}(z) - \beta^{II}(s) = (2 - 2\rho_t)q_1^{II}(s, s) - 2q_1^{II}(s, s) - \frac{1}{2}\rho_t(1 - \rho_t) + o(z - s).
\]

\( \frac{1}{2}\rho_t(1 - \rho_t) \) achieves its maximum at \( \rho_t = 2 - \sqrt{2} \), and hence if \( q_1^{II}(s, s) \geq \frac{\sqrt{2}}{2\sqrt{2}} q_1^{II}(s, s) + \frac{(\sqrt{2}-1)(2-\sqrt{2})}{2\sqrt{2}} \), the monotonicity holds. The constant term is roughly 0.086.

**Case 5:** \( s \in V \)

In region V, \( \rho = \frac{1}{2} \), \( S_1^{II} = S_2^{II} = s \), and hence \( g(s) = -\frac{1}{3}q_1^{II}(s, s) \). It can be verified that \( \beta^{II}(z) - \beta^{II}(s) = (z - s)(q_1^{II}(s, s) - \frac{3}{4} q_1^{II}(s, s) - \frac{1}{8}) + o(z - s) \), and therefore monotonicity holds.

We conclude that the bidding function \( \beta^{II}(s) \) is strictly increasing since Assumption 2 is sufficient for all cases.

3. Verification: Incentive compatibility

We will follow Milgrom and Weber (1982) to verify that the proposed bidding function is indeed an equilibrium. The idea is to show that a type-s bidder’s payoff is unimodal in her reported type, and it achieves the maximum at the truth-telling value \( s \).

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Suppose the other player adopts that bidding function. Differentiating the expected payoff with respect to $z$, we obtain
\[
\frac{\partial \Pi_{II}(z|s)}{\partial z} = \left( \pi_w(s, z) - \beta_{II}(z) - \pi_1^{II}(z, s, z) \right) + \int_z^1 \frac{\partial}{\partial v} \pi_{II}(v, s, y) \big|_{v=z} dy = \left( \pi_w(s, z) - \pi_1^{II}(z, s, z) \right) - \left( \pi_w(z, z) - \pi_1^{II}(z, z, z) \right).
\]

(A.9)

Our goal is to show that if $z < s$, then
\[\frac{\partial \Pi_{II}(z|s)}{\partial z} < 0,\]
and if $z > s$, then
\[\frac{\partial \Pi_{II}(z|s)}{\partial z} > 0.\]

We divide the proof into four cases in the sequel.

Case 1: $z, s \leq S_{II}^1(z)$.

Recall that
\[
\pi_w^{II}(s, z) - \pi_1^{II}(z, s, z) = q_w^{II}(s, z) \left( \theta_0 + s + z - q_w^{II}(s, z) - q_l^{II}(z) \right) - q_l^{II}(z, s) \left( \theta_0 + z + s - q_w^{II}(z, z) - q_l^{II}(z, s) - c \right),
\]
and $q_l^{II}(z, s) = q_l^{II}(z, z) + \frac{1}{2}(s-z)$. In this case, the capacity constraint is never binding, and thus $q_w^{II}(s, z) = q_w^{II}(z, z) + \frac{1}{2}(s-z)$. After simple manipulations, we can rewrite equation (A.9) as follows
\[
\frac{\partial \Pi_{II}(z|s)}{\partial z} = \frac{1}{2}(s-z) \left[ q_w^{II}(z, z) - q_l^{II}(z, z) + (\theta_0 + s + z - q_w^{II}(s, z) - q_l^{II}(z)) - (\theta_0 + s + z - q_w^{II}(z, z) - q_l^{II}(z, s) - c) \right] = \frac{1}{2}(s-z) \left[ q_w^{II}(z, z) - q_l^{II}(z, z) + c \right].
\]

The multiplicative term $q_w^{II}(z, z) - q_l^{II}(z, z) + c$ is always positive by Assumption 2. Hence $\partial \Pi_{II}(z|s)/\partial z$ is positive if $s-z > 0$ and negative if $s-z < 0$.

Case 2: $S_{II}^1(z) \leq z, s \leq S_{II}^2(z)$.

In this case, $q_w^{II}(s, z) = q_w^{II}(z, z) = C$.
\[
\frac{\partial \Pi_{II}(z|s)}{\partial z} = \frac{1}{2}(s-z) \left( C - \frac{1}{2}q_w^{II}(z, z) - \frac{1}{2}(\theta_0 + z + s - C - q_l^{II}(z, s) - c) \right) = \frac{1}{2}(s-z) \left( C - \frac{1}{2}(\theta_0 + z + \frac{s+z}{2} - c - C) \right).\]
Note that $C = q''_w(z, z)$ here and $\frac{1}{2} [\theta_0 + z + \frac{z^2}{2} - c - C] \leq \frac{1}{2} [\theta_0 + z + \frac{z^2}{2} - c - C] = q''_I(z)$ since $s \leq 1$. Thus, by Assumption 2, the multiplicative term is positive and the payoff is unimodal.

**Case 3: $z, s \geq S_2^I(z)$**

Mimicking the treatment of Case 1, we get

$$\frac{\partial \Pi^I(z|s)}{\partial z} = \frac{1}{2} (s - z) \left( q''_w(z, z) - q''_I(z, z) \right),$$

where the term in brackets is positive by Assumption 2.

**Case 4: General case.**

Now we consider the case where the capacity constraint is binding only for one of $q''_w(s, z)$ and $q''_I(z, z)$. This includes both $z \leq S_1^I(z) \leq s \leq S_2^I(z)$ and $S_1^I(z) \leq z \leq S_2^I(z) \leq s$. In this case, $q''_w(s, z) - q''_I(z, z) = \frac{1}{2} \rho(s - z)$, where $0 \leq \rho \leq 1$. We can rewrite $\pi^I_w(s, z) - \pi^I_I(z, s, z)$ as follows:

$$\pi^I_w(s, z) - \pi^I_I(z, s, z) = \left( q''_w(z, z) + \frac{1}{2} (S_1^I(z) - z) \right) \left( q''_w(z, z) + (s - z) - \frac{1}{2} (S_1^I(z) - z) \right) - \left( q''_I(z, z) + \frac{1}{2} (s - z) \right)^2$$

$$= \pi^I_w(z, z) - \pi^I_I(z, z, z) + (s - z) \left( q''_w(z, z) - q''_I(z) + \frac{1}{2} \rho \left( 1 - \frac{1}{2} \rho \right)(s - z) - \frac{1}{4}(s - z) \right),$$

and hence

$$\frac{\partial \Pi^I(z|s)}{\partial z} = (s - z) \left( q''_w(z, z) - q''_I(z) + \frac{1}{2} \rho \left( 1 - \frac{1}{2} \rho \right)(s - z) - \frac{1}{4}(s - z) \right).$$

Note that Cases 1 and 2 can be regarded as two special cases of this general case, with $\rho = 1$ and $\rho = 0$ respectively, which bounds $\frac{1}{2} \rho \left( 1 - \frac{1}{2} \rho \right)(s - z)$ from below and above. Thus the multiplicative term is positive according to Cases 1 and 2, and the payoff is unimodal in this case as well.

**Remark:**

It can be verified that the assumption needed to guarantee the monotonicity of bidding function $\beta^I(s)$ is stronger than that for the unimodality of resellers’ expected payoffs (incentive compatibility). This phenomenon does not occur in the conventional auctions (e.g., Milgrom and Weber (1982)). Similar results apply to the first-price auction as well. See the proof of Theorem 2.

**Proof of Theorem 2**

1. **Necessary condition**

The first-order condition of $\Pi^I(z|s)$ is

$$\pi^I_w(z, s, z) - \beta^I(z) - \pi^I_I(z, s) + \int_0^z \pi^I_{w,1}(z, s, y)dy - \int_0^z (\beta^I)'(z)dy = 0.$$
Rewriting the above equation, we find an expression of \((\beta^I)'(z)\) as follows:

\[
(\beta^I)'(z) = \frac{1}{z} \left( \pi^I_w(z, s, z) - \pi^I_l(s, z) + \int_0^z \pi^I_{w,1}(z, s, y) dy - \beta^I(z) \right),
\]

The closed-form solution of \(\beta^I(z)\) can be obtained from this differential equation by the same procedure in Krishna (2002, Proposition 6.3). Note that the proof of Proposition 6.3 in Krishna requires the signal affiliation to guarantee the unimodality for the last steps in his derivation. In our case signals are non-affiliated, but unimodality is guaranteed from Assumption 3.

Together with the incentive compatibility condition, the bidding function is

\[
\beta^I(s) = \int_0^s \left( \pi^I_w(y, y, y) - \pi^I_l(y, y) + \int_0^y \pi^I_{w,1}(y, y, v) dv \right) dL(y|s), \tag{A.10}
\]

where

\[
L(y|s) = \exp \left( -\int_y^s \frac{1}{v} dv \right) = \frac{y}{s}.
\]

Plugging \(L(y|s)\) in equation (A.10), we obtain the bidding function \(\beta^I(s)\). Since Proposition 4 guarantees that \(q^I_w(z, s)\) and \(q^I_l(y, z)\) are unique, all terms in equation (A.10) are known, i.e., there is only one bidding function that satisfies the first-order condition. Thus, if there exists a symmetric, strictly increasing equilibrium, it must be uniquely determined by equation (A.10).

Next, we will show that \(\pi^I_w(z, s, y)\) is decreasing in \(z\), for all pair \((s, y)\). This leads to the conclusion \(\beta^I(s) < \frac{1}{s} \int_0^s (\pi^I_w(y, y, y) - \pi^I_l(y, y)) dy\). Observing equation (9), the differentiation can be expressed as follows:

\[
\frac{\partial}{\partial z} \pi^I_w(z, s, y) = \frac{\partial}{\partial z} q^I_w(z, s) \left( \theta_0 + s + y - q^I_l(z, s) - q^I_l(y, z) \right) - c \frac{\partial}{\partial z} (q^I_w(z, s) - C)^+ \\
+ q^I_l(z, s) \left( -\frac{\partial}{\partial z} q^I_w(z, s) - \frac{\partial}{\partial z} q^I_l(y, z) \right) \tag{A.11}
\]

Note that by Proposition 4, \(-\frac{1}{8} \leq \frac{\partial}{\partial z} q^I_w(z, s) \leq 0\) and \(\frac{1}{4} \leq \frac{\partial}{\partial z} q^I_l(y, z) \leq \frac{1}{2}\). Thus, the last term of equation (A.11) is negative. Observe that the first two terms can be combined into \(q^I_w(z, s) \frac{\partial}{\partial z} q^I_w(z, s)\) regardless of whether \(q^I_w(z, s) > C\) or not. Hence it is negative as well because \(\frac{\partial}{\partial z} q^I_w(z, s) \leq 0\), and we conclude that \(\pi^I_w(z, s, y)\) is decreasing in \(z\).

2. Verification: Monotonicity of \(\beta^I(s)\)
Similar to Milgrom and Weber (1982), a sufficient condition for the monotonicity of bidding function in the first-price auction is that the integrand of $\beta^I(s)$ is increasing, i.e.,

\[
\pi^I_w(y, y, y) - \pi^I_l(y, y) + \int_0^y \frac{\partial}{\partial u} \pi^I_w(u, y, v)|_{u=y}dv
\]

is increasing in $y$. Recall that $\frac{\partial}{\partial u} \pi^I_w(u, y, v)|_{u=y} = -q^I_w(y, y) \frac{\partial}{\partial u}q^I_l(v, u)|_{u=y}$ by equation (A.11), the integral is $-\frac{1}{2}\int_0^y q^I_w(u, u)du$ if $S^I_1 \leq y \leq S^I_2$, and is $-\frac{1}{4}\int_0^y q^I_w(u, u)du$ otherwise. Note also that

\[
\pi^I_w(y, y, y) - \pi^I_l(y, y) = q^I_w(y, y)[\theta_0 + y + y - q^I_w(y, y) - q^I_l(y, y) - c(q^I_w(y, y) - C)^+] - q^I_l(y, y)[\theta_0 + y + y - q^I_w(y, y) - q^I_l(y, y) - c].
\]

Now we will divide our analysis into three cases, depending on the regions to which $y$ belongs.

**Case 1: $y \leq S^I_1$**

We first consider the case $y \in S^I_1$. Let

\[
\beta^I(z) - \beta^I(y) = \pi^I_w(z, z, z) - \pi^I_l(z, z) + \int_0^z \frac{\partial}{\partial u} \pi^I_w(u, z, z)|_{u=z}dv
\]

\[= \pi^I_w(y, y, y) - \pi^I_l(y, y) + \int_0^y \frac{\partial}{\partial u} \pi^I_w(u, y, y)|_{u=y}dv].
\]

Our goal is to show that $\lim_{z \to y} \frac{\beta^I(z) - \beta^I(y)}{z-y} > 0, \forall y$. Note that in this case $q^I_w(z, z) = q^I_w(y, y) + \frac{1}{2}(z-y)$ and $q^I_l(z, z) = q^I_l(y, y) + \frac{3}{4}(z-y)$. Therefore after some algebra, $\beta^I(z) - \beta^I(y)$ can be rewritten as $(z-y)[\frac{2}{3}q^I_w(y, y) - q^I_l(y, y) + \frac{1}{2}c] + o(z-y)$, where $\lim_{x \to 0} o(x)/x = 0$. Hence $\lim_{z \to y} \frac{\beta^I(z) - \beta^I(y)}{z-y} > 0$ by Assumption 3 in this case.

**Case 2: $y > S^I_2$**

Similar to Case 1 except that $\beta^I(z) - \beta^I(y) = (z-y)[\frac{2}{3}q^I_w(y, y) - q^I_l(y, y)] + o(z-y)$, and hence the result holds.

**Case 3: $S^I_1 \leq y \leq S^I_2$**

In this case the capacity constraint is binding, i.e., $q^I_w(z, z) = q^I_w(y, y) = C$, and $q^I_l(z, z) = q^I_l(y, y) + (z-y)$. It can be verified that $\beta^I(z) - \beta^I(y) = (z-y)(\frac{1}{3}q^I_w(y, y) - 2q^I_l(y, y)) + o(z-y)$, which is strictly positive under Assumption 3. The proof of monotonicity is now complete.

3. Verification: Incentive compatibility

We will show that the expected payoff of a type-$s$ bidder is unimodal in the reported type $z$ with maximum achieved at $z = s$. Plugging the bidding function $\beta^I(z)$ into the expected
payoff, we obtain
\[
\frac{\partial}{\partial z} \Pi^f(z|s) = \pi_w^f(z, s, z) - \beta^f(z) - \pi_l^f(s, z) \\
+ \int_0^z \frac{\partial}{\partial v} \pi_w^f(v, s, y)|_{v=z} dy - \int_0^z (\beta^f)'(z) dy \\
= (\pi_w^f(z, s, z) - \pi_l^f(s, z)) - (\pi_w^f(z, z, z) - \pi_l^f(z, z)).
\]

Recall that
\[
\pi_w^f(z, s, z) - \pi_l^f(s, z) = q_w^f(z, s)\left[\theta_0 + s + z - q_w^f(z, s) - q_l^f(z, z) - c(q_w^f(z, s) - C)^+\right] \\
- q_l^f(s, z)\left[\theta_0 + z + s - q_w^f(z, z) - q_l^f(s, z) - c\right],
\]
and
\[
q_l^f(s, z) = q_l^f(z, z) + \frac{1}{2}(s - z).
\]

In the following, we will divide the problem into cases to prove that the differentiation is positive when \(z < s\) and negative if \(z > s\).

**Case 1:** \(q_w^f(z, s) < C\) and \(q_w^f(z, z) < C\)

In this case, the capacity constraint is not binding. Thus
\[
q_w^f(z, s) = q_w^f(z, z) + \frac{1}{2}(s - z).
\]
Plugging it into equation (A.12), we obtain
\[
\frac{\partial}{\partial z} \Pi^f(z|s) = \frac{1}{2}(s - z) \left(q_w^f(z, z) - q_l^f(z, z) + c\right),
\]
where the multiplicative term \(q_w^f(z, z) - q_l^f(z, z) + c\) is positive by Assumption 3.

**Case 2:** \(q_w^f(z, s) = q_w^f(z, z) = C\)

In this case, there is no difference between \(q_w^f(z, s)\) and \(q_w^f(z, z)\). After some algebra, the differentiation becomes
\[
\frac{\partial}{\partial z} \Pi^f(z|s) = \frac{1}{2}(s - z) \left(C - \frac{1}{2}(q_l^f(z, z) + q_l^f(s, z))\right).
\]
Now the multiplicative term is
\[
2[q_w^f(z, z) - q_l^f(z, z) - \frac{1}{4}(s - z)] \geq 2 \left(q_w^f(z, z) - q_l^f(z, z) - \frac{1}{4}(1 - z)\right),
\]
and hence the differentiation is positive if and only if \(z < s\).
Case 3: General case.

We first rewrite the expression for \( \pi_w^I(z, s, z) - \pi^I_w(s, z) \). After some algebra, we obtain that if
\[
S^1_\ell \leq z \leq S^2_\ell, \quad \pi^I_w(z, s, z) = q^I_w(z, s) (q^I_w(z, s) + \frac{3}{4} z); \quad \text{otherwise,} \quad \pi^I_w(z, s, z) = q^I_w(z, s) (q^I_w(z, s) + \frac{3}{4} z).
\]
Thus
\[
\frac{\partial}{\partial z} \Pi^I(z | s) = \begin{cases} 
(s - z) (q^I_w(z, z) - q^I_w(z, z) + (\frac{1}{4} z) (s - z) + \frac{3}{8} \rho z) & \text{if } z \leq S^1_\ell, \\
(s - z) (q^I_w(z, z) - q^I_w(z, z) + (\frac{1}{4} z) (s - z) + \frac{1}{8} \rho z) & \text{otherwise.}
\end{cases}
\]

Let us consider the term \((\frac{1}{2} \rho (1 - \frac{1}{2} \rho) - \frac{1}{4} z) + \frac{1}{8} \rho z\). When \( s - z < 0 \), \((\frac{1}{2} (1 - \frac{1}{2} \rho) - \frac{1}{4} z) > 0\) since \( \frac{1}{2} (1 - \frac{1}{2} \rho) \leq \frac{1}{4} \) for \( \rho \in [0, 1] \). With the last term being positive, we obtain by Assumption 3 that \( \frac{\partial}{\partial z} \Pi^I(z | s) \) is negative for \( z > s \).

It remains to show that this holds for \( z < s \) as well. When \( s - z > 0 \), both \((\frac{1}{2} \rho (1 - \frac{1}{2} \rho) - \frac{1}{4} z) \) and \( \frac{1}{8} \rho z \) are minimized at \( \rho = 0 \), which corresponds to Case 2. Thus, the multiplicative term is bounded below by Case 2 and hence is positive for all cases.

Remark:

From the proofs of Theorems 1 and 2, the monotonicity of bidding functions is relatively hard to guarantee when the capacity constraint is binding in both auctions, i.e., Case III for the first-price auction and region IV for the second-price auction. In these regions, as the signal \( s \) increases, \( \pi^I_w(s, s) \) and \( \pi^I_w(s, s, s) \) are restricted by the capacity constraint, while \( \pi^I(s, s, s) \) and \( \pi^I(s, s) \) keep growing without any constraint. This phenomenon makes it harder to ensure that bidding functions are monotonic.

Proof of Theorem 3

To prove Theorem 3, it suffices to show the following monotonicity result:

Suppose that \( q^I_w(s_w) + q^I(s_w, s_l) \geq q^I_w(s_w, s_l) + q^I(s_l) \). Then for all \( y \geq s_w \),
\[
q^I_w(y) + q^I(y, s_l) \geq q^I_w(y, s_l) + q^I(s_l), \quad \forall y \geq s_w,
\]
and
\[
q^I_w(s_w) + q^I(s_w, z) \geq q^I_w(s_w, z) + q^I(z), \quad \forall z \leq s_l.
\]

Given that \( s_l \) is fixed and \( y \geq s_w \) in inequality (A.13), let \( \rho^I_w, \rho^I \) denote the coefficients of winner’s signals in the first-price and second-price auctions respectively. \( \rho^I_w, \rho^I \) capture the marginal change of the total quantities resulting from one unit change in \( s_w \). The monotonicity is equivalent to the fact \( \rho^I_w \geq \rho^I \), and hence in the sequel we will verify it. According to Lemma 3, we know that \( \frac{1}{2} \leq \rho^I_w \leq \frac{3}{4} \). Since \( q^I(s_l) \) is independent of the
winner’s signal, the bounds of $\rho^I_w$ are $0 \leq \rho^I_w \leq \frac{1}{2}$. Therefore, $\rho^I_w \geq \rho^I_w$ in all cases of inequality (A.13).

Now we focus on inequality (A.14). Given that $s_w$ is fixed and $s_l \geq z$, we compare the coefficients of loser’s signals between these two auctions (similarly to the treatment for inequality (A.13)). Let $\rho^I_l, \rho^I_l$ denote these coefficients. It suffices to show that $\rho^I_l \leq \rho^I_l$ in all cases. First we observe from Proposition 3 that $\rho^I_l = \frac{1}{2}$, $\forall s_w \in [0, 1]$. Combining Proposition 1 and Lemma 3, we obtain

$$0 + \frac{1}{2} \leq \rho^I_l \leq \frac{1}{2} + \frac{3}{4},$$

which implies $\frac{1}{2} \leq \rho^I_l \leq \frac{5}{4}$. Hence, $\rho^I_l \leq \rho^I_l$. In fact it can be shown that $\rho^I_l$ has tighter bounds, i.e., $\frac{1}{2} \leq \rho^I_l \leq \frac{3}{4}$. 

\[\square\]