Robust Pricing in Discrete Time

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We consider the pricing problem faced by a monopolist who sells a product to a population of consumers over a discrete number of periods. Customers are heterogeneous in both the willingness-to-pay for the product and the arrival time during the selling season. We assume that the seller knows only the support of the customers’ valuations and do not make any other distributional assumptions about customers’ willingness-to-pay or arrival times. In this setting, we consider a robust formulation of the seller’s pricing problem which is based on the minimization of her worst-case regret. This regret is defined as the difference between her payoff under full demand information and her realized payoff. We consider two distinct cases of customers’ purchasing behavior: myopic and strategic customers. For each of these demand models, we characterize the optimal pricing strategy and corresponding minimum regret. We also derive a continuous-time pricing approximation as the limit of the discrete-time model when the length of the periods decreases to zero.

A continuous-time model has been considered in Caldentey et al. (2015). Interestingly, in the limit, the pricing strategy of the discrete model presents a different structural property from the strategy presented in Caldentey et al. (2015), which leads to a set of different managerial insights for the firm.

Key words: optimal pricing, strategic consumers, demand uncertainty, robust optimization

History:

1. Introduction

Over the last couple of decades, dynamic pricing has been transformed from a curious and somewhat controversial practice used primarily by upstart airlines into a technique that is widely used in a variety of industries. As technology has evolved and reduced menu costs, retailers of all sorts have adopted intertemporal pricing practices. The key economic driver behind the rapid dissemination of dynamic pricing is demand uncertainty: there is enormous value for a firm in being able to change prices over time in situations where the firm does not know how much customers are willing to pay for its products.

In response to the increasing use of dynamic pricing in practice, academics have proposed a variety of techniques for algorithmically determining pricing policies. However, the vast majority of these approaches require the firm to know the probability distributions of customer valuation and arrival time. Assuming that the firm knows a full probabilistic model of customer valuations and arrival times is problematic for at least two reasons. The first reason is the obvious one:
firms do not have access to such probability distributions; even when they have sales data, they typically do not have access to a dataset that is rich enough to estimate valuation and arrival time distributions. The second reason is less obvious, but is perhaps more fundamental: taking the probability distributions of customer valuations and arrival times as given assumes away a large part of the problem that dynamic pricing is designed to deal with, which is precisely the lack of knowledge about customer valuations and arrival times.

These issues are especially acute for firms introducing new products into the marketplace. When firms launch new products, they usually have very little information about how much customers are willing to pay for them. This great degree of uncertainty makes new products excellent candidates for dynamic pricing strategies. However, the same uncertainty about customer valuations also hobble our ability to use established dynamic pricing techniques since they rely on the firm knowing the probability distribution of customer valuations.

In this paper, we consider the problem of how to set prices over time for a product when there is limited historical demand information available. We approach the pricing problem from the perspective of robust optimization (see Ben-Tal et al. (2009) and Bertsimas and Sim (2004)). Specifically, we assume that the seller only knows the range of customers’ valuation (or willingness to pay) for its product and makes no additional assumptions about their distribution or about the customers’ arrival process. The formulation we propose is quite parsimonious, but still sufficiently rich to give rise to different types of pricing policies for different sets of problem parameters. Our model formulation can be used both in settings with limited information, where the firm has only a very rough guess of the customer’s range of valuations, as well as in an environment that is more data-rich, where the firm can use sales data to estimate an uncertainty set of the customers’ valuations. It is also flexible enough to allow us to study optimal pricing policies for myopic as well as strategic customers, the latter being customers who time their purchases in order to maximize their own discounted utilities.

Under such a minimalist informational structure, the standard expected profit maximization criterium is not applicable as the seller’s objective function. Instead, we consider the seller’s regret, which is defined as the difference between her payoff under full demand information and her realized payoff. In this setting, an optimal pricing strategy is one that minimizes the difference between the seller’s ex-post payoff from those of a clairvoyant who sets prices knowing customers’ type (valuation and arrival time) in advance.
1.1. Contributions

The primary contribution of this paper is the development of a robust optimization methodology to compute intertemporal pricing policies for a firm facing myopic or strategic consumers with uncertain valuations and arrival times. The problem of characterizing such optimal pricing strategies has a long tradition in the economics literature (e.g., Coase conjecture) and, more recently, has received considerably attention from the operations and revenue management communities (see next section). The academic interest stems from the facts that (i) the problem is increasingly relevant from a practical standpoint and (ii) it has proven to be theoretically challenging to solve (i.e., multi-agent stochastic dynamic games with asymmetric information). We believe that our main contribution is to provide an alternative perspective to this problem; one that builds upon a parsimonious robust formulation and leads to tractable solutions and a number of managerial takeaways about the structure of an optimal policy.

In Section 4 we consider the case in which the firm sells to myopic customers. In this setting, we show that an optimal pricing policy is typically not unique. Interestingly, the set of optimal price vectors include maximal and minimal elements, in a component-wise sense. These extreme solutions provide two contrasting recommendations on how to set prices over time. On one hand, the maximal price vector includes an initial full-markup period where prices are set equal to the upper limit of customers’ valuation range (\(\bar{v}\)) followed by a markdown period in which prices drop to the lower limit (\(v\)). In contrast, the minimal price vector has no markup period but markdowns are less significant. Another interesting feature of these extreme solutions is that they coincides towards the end of the selling season. In other words, the seller has some flexibility to set prices either aggressively (maximal solution) or conservatively (minimal solution) during the early stages of the selling season. However, she has no flexibility towards the end and optimal prices are uniquely determined. Our analysis of the myopic case also includes a sensitivity analysis with respect to the length of the selling season. This analysis reveals that there exists a threshold for the value of \(v\) above which a finite-length selling season is optimal but below this threshold the seller is better off extending the selling horizon indefinitely. This result is derived for a discrete-time model. However, we also show that as the length of a period goes to zero, effectively allowing the seller to change prices continuously, then the optimal length of the selling season is uniformly bounded above by \(\ln(3)/r\), where \(r\) is the discount factor. A striking feature of this result is that this upper bound is independent of the customers’ valuations. Another interesting observation that emerges from the continuous-time analysis is that in some cases an optimal price policy can exhibit a discontinuity at the end of the selling season in the form of a fire sale in which the final price drops markedly.
In Section 5, we consider the case in which the market consists of strategic forward-looking consumers. Under our robust formulation, this problem can be viewed as a three player game, with the firm acting first and choosing prices, nature responding and selecting customers’ valuations and arrival times and customers acting last and deciding when and whether to buy the firm’s product. One of the main takeaways of our analysis is that we show that strategic consumers are also acting “myopically” not with respect to prices but with respect to a modified vector of prices that we call threshold valuations (see Definition 2 for details). This similarity with the myopic case allows us to directly extrapolate some key results derived in the model with myopic consumers to the case with strategic consumers. A distinctive feature of threshold valuations is that they capture the slope at which prices decrease over time. We show that there exists an optimal solution under which threshold valuations are decreasing over time and the corresponding prices are $\delta$-weak convex (see Definition 2). As a result of the optimality of $\delta$-weak convex prices, consumers’ utility is unimodal in time a fact that implies that nature can maximize the firms’ worst-case regret by selecting the arrival time of customers to be the first period. In other words, with strategic consumers, a robust optimal pricing strategy is concerned with the case in which consumers arrive at the beginning of the selling season but wait to buy in later periods. We also show that the optimal ($\delta$-weak convex) policy is a markdown policy like the minimal policy for myopic customers, but it is flatter than the policy for the earlier case. The firm starts from a lower price point than it would with myopic customers, but ends with a higher price than it would end otherwise. This is a consequence of the firm’s reduced ability to do price skimming due to the consumers’ strategic behavior. We also show that the firm’s regret is always worse under strategic customers than under myopic customers.

In Section 7, we extend the model to consider the case in which the market includes both myopic and strategic consumers. Interestingly, we conclude that if nature is allowed to select the proportion of each type of consumers then the seller should use the same pricing strategy that she would use if she knew that all consumers were strategic.

The rest of the paper is organized as follows. In Section 2 we position our work with respect to the existing literature on dynamic pricing with strategic consumers and on robust optimization. Section 3 introduces the model including our robust (minimum regret) formulation and consumers’ purchasing behavior. We also show that without loss of generality we can restrict our attention to the case in which the firm faces a single consumer. Section 4 characterizes the firm’s optimal pricing strategies when customers are myopic. In Section 5, we analyze the case where consumers are strategic. Section 6 presents continuous-time approximations for both the myopic and the strategic...
pricing policies. Section 7 extends our base model to the case in which consumers’ valuations are correlated with their arrival times and to the case in which the market includes both myopic and strategic consumers. Finally, Section 8 summarizes our findings and discusses a number of possible extensions.

2. Related Literature

Starting from the seminal paper by Gallego and van Ryzin (1994), the revenue management community has focused its attention on the problem of how to use dynamic pricing for handling uncertain customer valuations and arrival times over a finite selling season. The early literature is vast and we refer readers to surveys by Bitran and Caldentey (2003), Elmaghraby and Keskinocak (2003) and Talluri and Van Ryzin (2005).

The early models in the dynamic pricing literature all assumed that customers were myopic in how they made their decisions, in that the customers would not try to anticipate the firm’s future prices when making their decisions. Recently, there has been a major research drive trying to understand the impact of strategic customer behavior on firms using dynamic pricing strategies. Aviv and Pazgal (2008) showed that ignoring forward-looking customers can be costly for the firm and that committing to a fixed price can potentially be more profitable for the firm even in the face of stochastic demand, a counterintuitive result that builds on the insight of the Coase conjecture (see Coase (1972) and Stokey (1979)). Su (2007), studying a model where customers are heterogeneous in both their valuations and their degree of patience, shows that markup policies are optimal when high-valuation customers are proportionally more strategic, whereas markdowns are optimal if they are proportionally more myopic. Besbes and Lobel (2012), building on the intertemporal price discrimination framework of Conlisk et al. (1984), shows that in an infinite-horizon model with heterogeneous strategic customers, optimal pricing policies are formed by short cycles that typically have a complex structure they call nested sales. Furthermore, recent papers in the economics and operations management literatures by Hendel and Nevo (2013) and Li et al. (2013) have empirically shown that strategic customer behavior is an important issue that should not be ignored when deciding prices in settings such as retail and airline markets.

Several recent papers have also studied the impact of dynamic pricing when customers are strategic not only about prices, but also about product availability. Liu and van Ryzin (2008) show that understocking can be used by the firm to drive early purchases, at higher prices, when customers are forward-looking. Cachon and Swinney (2009) demonstrate that quick response production is especially valuable in the presence of strategic customers. Yin et al. (2009) recommend sellers to
display one item at a time when faced with forward-looking customers in order to increase the sense of product scarcity in the market. Caldentey and Vulcano (2007) and Osadchiy and Vulcano (2010) propose alternative models for selling to strategic customers, such as running an auction in parallel to regular sales channel and selling with binding reservations, respectively.

Determining a good pricing strategy for selling to strategic customers is challenging, so most of the papers above make one or more simplifying assumption on the pricing problem to keep it manageable. Some papers assume there are only two pricing periods and other papers assume there are only two possible customer valuation levels. Some papers that do allow for general valuation models in multi-period models, such as Besbes and Lobel (2012), instead assume there is no uncertainty on customer valuations or arrival times. In contrast to most papers in this literature, we simplify the problem by removing inventory considerations, but offer a multi-period pricing framework that allows for uncertainty on both customer valuations and arrival times. Our framework can be used to generate optimal dynamic pricing policies for both myopic and strategic customers, enabling us to compare and contrast the two.

The approach we develop uses a robust optimization approach (see Ben-Tal et al. (2009) and Bertsimas and Sim (2004)) to model customer valuation and arrival time uncertainty. We consider the problem of finding the policy that minimizes the maximum regret the firm can incur, where regret is defined as the difference between the seller’s payoff under full information and her realized payoff. Our work can be seen as a dynamic extension of the static robust pricing models of Bergemann and Schlag (2008, 2011). Our paper is also related to Eren and Maglaras (2010), who determine how to find a dynamic pricing policy with an optimal competitive ratio for selling to myopic consumers. Our work is also to some extent related to Lobel and Perakis (2010), which combines ideas from data-driven and robust optimization to generate robust dynamic pricing policies.

In the rest of the paper, we use ‘firm’ and ‘seller’ interchangeably and we use ‘consumer’ and ‘customer’ interchangeably. We use the words “increasing” and “decreasing” in a weakly sense.

3. Model Description

We consider the pricing problem faced by a monopolist selling durable products to a population of consumers over a discrete-time horizon with \( N \) periods. Customers are heterogeneous along two dimensions: (1) willingness-to-pay for the product and (2) arrival time during the selling season. We assume that the seller knows only the support \([\underline{v}, \overline{v}]\) of customers’ valuations. We do not make any distributional assumptions about customers’ willingness-to-pay or arrival times. We consider
a robust formulation of the seller’s pricing problem, based on the minimization of her worst-case regret, which is defined as the difference between her payoff under full demand information and her realized payoff. In computing these payoffs, we assume that the seller has unlimited capacity and that there are no holding costs or salvage value for unsold units.

In the rest of this paper, we formulate the seller’s problem for the special case of a single customer. As can be shown in Caldentey et al. (2015) Proposition 1, the seller’s optimal pricing strategy is independent of the number of customers if they are all either myopic and strategic, and if the type of each customer is in the same uncertainty set \( D := \left[ v, \bar{v} \right] \times \{ 1, \ldots, N \} \).

In the single-customer case, demand can be modeled by a pair \((v, \tau)\), where \( v \in [v, \bar{v}] \) is the customer’s willingness-to-pay and \( \tau \) is his arrival period. Without loss of generality, we assume that \( \tau \leq N \), otherwise there would be no demand during the selling season and the seller’s regret would be identically zero. On the supply side, the seller’s pricing strategy can be described by an \( N \)-dimensional vector \( p = (p_1, p_2, \ldots, p_N) \), where \( p_t \) is the price offered during period \( t = 1, \ldots, N \).

In order to compute the seller’s payoffs and corresponding regret, we need to specify how the consumer makes his purchasing decision in response to the seller’s pricing strategy. To this end, we introduce a function \( d(\cdot) \) that maps the state of the market \((v, \tau, p)\) to the period \( d(v, \tau, p) \in \{1, 2, \ldots, N\} \cup \{\infty\} \) when the customer makes the purchase. We use the convention \( d(v, \tau, p) = \infty \) if no purchase is made during the selling season. For completeness, we define \( p_0 = \bar{v} \) and \( p_\infty = 0 \).

We consider two contrasting purchasing behaviors: myopic and strategic.

- **MYOPIC CONSUMER**: Under a myopic purchasing behavior, the consumer will purchase the product as soon as the price falls below his valuation without any consideration of future prices. We denote this myopic purchasing time by \( d_M(v, \tau, p) \), which is given by

\[
d_M(v, \tau, p) := \min_{\tau \leq t \leq N} \{ t | v \geq p_t \}.
\]

If \( v < p_t \) for all \( t \in \{1, 2, \ldots, N\} \), the consumer leaves the market without making any purchase, i.e., \( d_M(v, \tau, p) = \infty \).

- **STRATEGIC CONSUMER**: As opposed to a myopic consumer, a strategic buyer is forward-looking and optimizes the timing of his purchase in order to maximize his net discounted utility. We let \( d_S(v, \tau, p) \) denote the purchasing period of a strategic consumer, which we define as follows:

\[
d_S(v, \tau, p) := \min \arg \max_{\tau \leq t \leq N} \{ \delta^{t-1} (v - p_t) | v \geq p_t \},
\]
where $\delta \in [0, 1]$ is the discount factor. The minimum in the equation above captures the fact that, all else being equal, the consumer would like to get the product as soon as possible. Again, if $v < p_t$ for all $t \in \{1, 2, \ldots, N\}$ then $d_S(v, \tau, p) = \infty$.

As we mentioned above, a distinguishing feature of our model with respect to the existing literature is our prior-free approach, where we assume the seller knows only the domain $\mathcal{D}$ of the consumer’s type $(v, \tau)$. Using the standard expected profit maximization formulation to solve the seller’s pricing problem in such a prior-free model would be problematic unless further assumptions about the distribution of $(v, \tau)$ were made. Instead, we prefer to retain the parsimonious nature of the seller’s information and use robust optimization — in particular, a worst-case regret criterion — to tackle the pricing problem.

For a given a state of the market $(v, \tau, p)$ and a specific consumer’s buying behavior $d(v, \tau, p)$, the seller’s regret is defined by

$$R(v, \tau, p) := \Pi_F(v, \tau) - \Pi(v, \tau, p),$$

which is the difference between her profit with full information $\Pi_F(v, \tau)$ and her realized profit $\Pi(v, \tau, p)$ with limited information. A perfectly informed seller (or clairvoyant) who knows in advance the buyer’s type $(v, \tau)$ is capable of extracting all the consumer’s surplus by charging a price $p_\tau = v$ at the consumer’ arrival time $\tau$ and then charging prices $p_t \geq v$ for all $t > \tau$. It follows that $\Pi_F(v, \tau) = \delta^{\tau-1}v$. On the other hand, the seller’s payoff with limited information depends on the consumer’s purchasing behavior and is equal to $\Pi(v, \tau, p) = \delta^{d(v, \tau, p)-1}p_{d(v, \tau, p)}$. The seller’s worst-case regret problem is then defined as follows:

$$R^* := \inf_{p \geq 0} \sup_{(v, \tau) \in \mathcal{D}} R(v, \tau, p) = \inf_{p \geq 0} \sup_{(v, \tau) \in \mathcal{D}} \delta^{\tau-1}v - \delta^{d(v, \tau, p)-1}p_{d(v, \tau, p)}.$$  (4)

In the remaining of this paper, we characterize the solution to this optimization problem and derive structural properties of the corresponding pricing strategy for various cases in terms of consumers’ buying behavior (myopic and strategic) and market size (number of customers). Before we move into this analysis, a few remarks about our proposed model are in order.

In terms of the specific features of the model, it is worth noticing that the “inf sup” formulation in (4) can be interpreted as a game that the seller plays against nature. In this game, the seller chooses the vector of prices $p$ and then nature picks the customer’s type $(v, \tau)$ so as to maximize the seller’s regret function $R(v, \tau, p)$. Although this game is static in the sense that prices are selected (in an open-loop fashion) at the beginning of the selling season, the solution is indeed a closed-loop solution that captures the evolution of the sales process in the single-customer case. In particular,
price \( p_t \) can be chosen at the beginning of the horizon assuming that no sales has occurred in the first \( t - 1 \) periods. (Otherwise, if the customer purchases the product at some period \( s < t \), then prices after time \( s \) have no impact on the seller’s payoff.)

Another interesting feature of the formulation in (4) —one that will prove useful in the derivation of some of our results— is the decomposition of the seller’s regret into the following two components:

\[
R(v, \tau, p) = \delta^{d(v, \tau, p) - 1}(v - p_{d(v, \tau, p)}) + (\delta^{\tau - 1} - \delta^{d(v, \tau, p) - 1})v
\]

(5)

First, there is a valuation regret generated by the mismatch between the customer’s valuation and the actual price he ends up paying, that is, the discounted payoff that the seller “leaves on the table” by not knowing the customer’s valuation. Second, and due to the multiperiod nature of the model, there is a delay regret that captures the time-value of delaying a sale from the customer’s arrival time \( \tau \) to his actual purchasing time \( d(v, \tau, p) \). By breaking the seller’s regret into these two pieces, one can see that nature has incentives to both postpone and advance the sale (see detailed discussion in Section 4 about the trade-off between valuation and delay regrets).

4. Selling to Myopic Customers

In this section we discuss the case in which the customer uses a myopic strategy to make purchase. According to Eq. (1) a myopic customer buys the product as soon as his valuation exceeds the posted price. From the seller’s perspective, this myopic behavior implies that a skimming price strategy is optimal. We formalize this intuition in Proposition 1 whose statement makes use of the following definition:

**Definition 1.** We define the class \( P_M \) of price vectors \( p = (p_1, \ldots, p_N) \) from \( \{1, 2, \ldots, N\} \) to \([v, \bar{v}]\) which is decreasing over time.

For any \( p \in P_M \), we define \( \bar{t}(p) \triangleq \max\{t | p_t = \bar{v}\} \) and \( \bar{t}(p) \triangleq \min\{t | p_t = v\} \). We set \( \bar{t}(p) = 0 \) if \( p_1 < \bar{v} \) and \( \bar{t}(p) = N + 1 \) if \( p_N > v \). For the rest of this section, we write \( \bar{t} \) and \( \bar{t} \) as shorthand notations for \( \bar{t}(p) \) and \( \bar{t}(p) \) respectively.

**Proposition 1.** Suppose the seller faces a single myopic customer. Then, an optimal price vector can be found in the set \( P_M \).

Proposition 1 allows us to fix a price strategy \( p \in P_M \). Then with non-increasing prices, a \((v, \tau)\)-customer’s purchasing behavior is in one of the following three cases:

(i) Buy immediately: if \((v, \tau) \in [p_t, \bar{v}] \times \{t\}\), the customer arrives and buys in the same period \( t \).
(ii) Wait and buy: if \((v, \tau) \in [p_t, p_{t-1}) \times \{1, 2, \ldots, t-1\}\) for some \(\bar{t} < t \leq t\), it means the customer arrives before period \(t\) and can only afford the product at period \(t\).

(iii) Never buy: if \(p_N > v\) and \((v, \tau) \in [v, p_N) \times \{1, 2, \ldots, N\}\) meaning the customer’s valuation is below all the prices, then such type of customer will leave the market without any purchase.

It is not hard to see that if nature is restricted to selecting \((v, \tau)\) in a way that case (i) above is realized, then she will select \((v, \tau) = (\bar{v}, t)\) to maximize the firm’s valuation regret at time \(t\). Then the supremum valuation regret occurred at time \(t\) is equal to \(\delta^{t-1}(\bar{v} - p_t)\). If nature is restricted to case (ii), it is optimal for her to select \(\tau = 1\) as the optimal arrival time for some \(v \in [p_t, p_{t-1})\). Then the worst-case or the supremum delay regret occurred at time \(t\) is equal to \(p_{t-1} - \delta^{t-1}p_t\).

Similarly, if nature is restricted to case (iii) where low-value customer leaves the market without any purchase, then a delay regret will be created. Nature will again choose \(\tau = 1\) as the optimal arrival time for some \(v \in [v, p_N)\). The worst-case delay regret in this case is then equal to \(p_N\).

Clearly, if the seller tries to reduce the valuation regret by increasing the price path then the delay regret will increase and vice versa. As a result, an optimal price vector must balance these two types of regrets. Based on the previous discussion, we can substantially reduce the complexity of the optimization problem in [equation 4].

**Lemma 1.** Suppose the seller chooses a vector of prices \(p \in P_M\). Then, the seller’s worst-case regret problem can be rewritten as

\[
R^* = \inf_{p \in P_M} \max \left\{ \max_{t \in \{1, 2, \ldots, N\}} \{\delta^{t-1}(\bar{v} - p_t)\}, \max_{t \in \{\bar{t}+1, \ldots, \min\{t, N\}\}} \{p_{t-1} - \delta^{t-1}p_t\}, p_N \cdot \mathbb{1}(t = N + 1) \right\}
\]

where \(\mathbb{1}(\cdot)\) is the indicator function. The result in Lemma 1 is straightforward from the previous discussion, and therefore the proof is omitted.

Lemma 1 and in particular the new representation of the seller’s regret, highlights two important features that an optimal price strategy should consider. On one hand, prices should be set high enough in order to bound the impact of the term \(\delta^{t-1}(\bar{v} - p_t)\) (i.e., to reduce valuation regret). This suggests that aggressive markdowns or sales are not recommended specially for short selling seasons. On the other hand, to limit the effect of the second term \(p_{t-1} - \delta^{t-1}p_t\), prices should not decrease too slowly (i.e., to avoid delay regret).

The last term \(p_N \cdot \mathbb{1}(t = N + 1)\) captures the fact that if \(p_N \geq v\), then the regret could be incurred by the customer with valuation below \(p_N\). The supremum of the regret from these types of customers is equal to \(p_N\). The seller’s optimization problem in Lemma 1 can be rewritten as \(R^* = \min_{p, R} R_t\), subject to the following constraints

\[
\delta^{t-1}(\bar{v} - p_t) \leq R \quad \forall t \in \{1, \ldots, N\}
\]

[1]
Although similar in structure to a linear program, the optimization problem $R^* = \min_{p,R} R$, subject to (7)–(10) is not an LP because the system of constraints in (8) and (9) is parametrized by $\bar{t}$ and $t$. We avoid this issue by breaking this optimization into $N + 1$ subproblems, where each subproblem is restricted to the set of prices $p \in \mathcal{P}_M$ for which $t$ is constant. To this end, let us define $\mathcal{P}_M(\kappa) \triangleq \{p \in \mathcal{P}_M | t = \kappa\}$ and the optimization problem is then $R(\kappa) = \min_{p,R} R$, subject to the following constraints

\begin{align*}
\delta^{t-1}(\bar{v} - p_t) &\leq R \quad \forall t \in \{1, \ldots, \kappa\} \\
p_{t-1} - \delta^{t-1} p_t &\leq R \quad \forall t \in \{1, \ldots, \kappa\} \\
p_\kappa = \bar{v} &\quad \text{if } \kappa \leq N \quad \text{or} \quad p_N = R \quad \text{if } \kappa = N + 1 \\
p &\in \mathcal{P}_M(\kappa).
\end{align*}

Note that constraint (13) can take two different forms depending on whether $\kappa \leq N$ or $\kappa = N + 1$.

For a fixed $\kappa$, the optimization problem $R(\kappa) = \min_{p,R} R$ with constraints (11)-(14) is a linear program. The following proposition solves these $N + 1$ linear programs and uses those solutions to determine an optimal pricing policy.

**Proposition 2.** (Minimax regret and the upper bounds of the optimal prices)

Let $w_t \triangleq \frac{\delta^{t(t-1)}}{2}$. The optimal solution to the optimization problem $R(\kappa) = \min_{p,R} R$ with constraints (11)-(14) is given by

\[
R(\kappa) = \max \left\{ \max_{1 \leq t \leq \kappa} \left\{ \frac{w_t \bar{v} - w_\kappa \bar{v}}{\sum_{r=t-1}^{\kappa} w_r}, (1 - \delta^{\kappa-1}) \bar{v} \right\}, 1 \leq \kappa \leq N \quad \text{and} \right. \\
R(N + 1) = \max \left\{ \max_{1 \leq t \leq N} \left\{ \frac{w_t \bar{v}}{\sum_{r=t-1}^{N} w_r}, \bar{v} \right\} \right. 
\]

The seller’s worst-case regret $R^*$ is equal to $R^* = \min \{ R(\kappa) | \kappa = 1, \ldots, N + 1 \}$. Furthermore, let $\kappa^* \triangleq \arg \min \{ \kappa | R^* = R(\kappa) \}$, then the following price vector $\bar{p} \in \mathcal{P}_M(\kappa^*)$ is an optimal pricing policy

\[
\bar{p}_t = \min \left\{ \bar{v}; \frac{1}{w_{1:t\wedge\kappa^*}} \left[ \sum_{r=t\wedge\kappa^*}^{\kappa^* - 1} w_r R^* + \mathbb{I}(\kappa^* \leq N) w_{\kappa^*} \bar{v} \right] \right\}, \quad t = 1, \ldots, N.
\]

It follows that $\kappa^* = t(\bar{p})$. 
Figure 1 depicts an example of an optimal price path $\bar{p}_t$. In this example, there are $N = 20$ selling periods, customer’ valuation belongs to the interval $[0.5, 1]$ and the optimal worst-case regret is $R^* = 0.2595$. Similarly to the continuous pricing strategy in Caldentey et al. (2015), we decompose the optimal price path in three distinctive regions: an initial Markup period ($t = 1, \ldots, 4$ in Figure 1) during which the seller sets the price equal to the upper limit $\bar{v}$, a Markdown period ($t = 4, \ldots, 16$) during which the price decreases monotonically to the lower limit $v$ and a Clearance period ($t = 17, \ldots, 20$) during which the price is kept constant at the lower limit $v$.

The presence of a Markup period in the optimal path in Figure 1 implies that the seller is effectively delaying sales until after certain period. This feature suggests that the seller’s worst-case regret $R^*$ is not affected by high markup prices at the beginning of the selling season but rather by the choices of prices during the Markdown and Clearance periods.

We will formalize this intuition in the following corollary that shows that optimal prices are not necessarily unique during the first periods of the selling horizon. Furthermore, the corollary reveals that the price $\bar{p}_t$ in Proposition 2 is a point-wise upper bound of the set of optimal price vectors that belong to $\mathcal{P}_M$.

**Corollary 1.** (Lower bound of the optimal prices)

Let $R^*$ be the optimal regret and let $t^* \triangleq \min \left\{ t \mid \delta^{t-1} (\bar{v} - \bar{p}_t) = R^* \right\}$, where $\{\bar{p}_t\}$ is the price vector defined in Proposition 2. Define the vector $\{\underline{p}_t\}$ as follows:

$$
\underline{p}_t = \begin{cases} 
\bar{v} - \delta^{1-t} R^* & \text{if } t = 1, \ldots, t^*-1, \\
\bar{p}_t & \text{if } t = t^*, \ldots, N.
\end{cases}
$$

Then, $\{\underline{p}_t\}$ is an optimal price vector. Furthermore, any optimal price vector $p \in \mathcal{P}_M$ satisfies $\underline{p}_t \leq p_t \leq \bar{p}_t$ for all $t = 1, \ldots, N$. In particular, $p_t = \bar{p}_t$ for $t \geq t^*$. 

Figure 2 depicts the upper and lower bound prices in Corollary 1 for the example in Figure 1 (in this example case $t^* = 14$).

![Figure 2](image)

**Figure 2** Upper bound $\bar{p}_t$ and lower bound $p_t$ price paths for $\bar{v} = 1$, $v = 0.5$, $N = 20$ and $\delta = 0.956$. In this example, $R^* = 0.2595$ and $t^* = 14$.

A few remarks about the results in Corollary 1 are in order. As mentioned, the corollary shows that the seller has a significant amount of flexibility in choosing prices in the periods $t = 1, \ldots, t^* - 1$. Indeed, not only $\{\bar{p}_t\}$ and $\{p_t\}$ are optimal solutions but *any convex combination of these two vectors* is optimal, which follows from the convexity of the feasible region of problem defined by (11)-(14). On the other hand, prices from period $t^*$ onwards are uniquely determined. This result implies that the seller must be careful in selecting the final markdown and clearance prices since her worst-case regret is defined by her pricing strategy in this portion of the selling season.

Furthermore, by construction, $\{\bar{p}_t\}$ is the vector of prices that makes constraint (12) binding. In other words, the fact that prices are uniquely determined by $\bar{p}_t$ for $t \geq t^*$ implies that the seller’s regret is achieved when a myopic customer with low valuation arrives at the very beginning of the selling season and has to wait until prices are sufficiently low (after $t^*$) to make a purchase. The seller’s optimal pricing strategy in this myopic case is then designed to minimize the regret associated with this type of events.

The fact that the optimal price vector is not unique raises the question of how to select a particular one. From the definition of the lower bound, we have $p_1 < \bar{v}^T$ which is, there always exists an optimal price vector with no Markup period. Hence, we can view the lower bound $\{p_t\}$ as a conservative pricing option that charges low prices throughout the selling season and, therefore, requires less markdowns. On the opposite extreme, the upper bound $\{\bar{p}_t\}$ is the most aggressive

1 Unless $t^* = 1$ and $\bar{p}_1 = \bar{v}$ which can only happens if $R^* = 0$ or equivalently $\bar{v} = v$. 
pricing option charging high prices but also imposing significant markdowns. By taking a convex combination of \( \{ \bar{p}_t \} \) and \( \{ p_t \} \), the seller can balance these extreme pricing strategies and impose her particular preferences.

Our next result investigates the relationship between the optimal regret \( R^*_N \) and the number of periods \( N \). In particular, we are interested in characterizing the optimal (minimum) number of periods that the seller should use to minimize her regret. Recall that \( \kappa^*_N \) is the first period at which the optimal price \( \bar{p}_t \) equals the lower level \( v \) (e.g., \( \kappa^*_N = 17 \) in Figure 1). In the discussion that follows, we find convenient to make explicit the dependence of \( R^*_N \) and \( \kappa^*_N \) on the number of periods \( N \) as well as the lower valuation \( v \). We then write \( R^*_N(v) \) and \( \kappa^*_N(v) \). We also define

\[
R^*_\infty(0) \triangleq \lim_{N \to \infty} \max_{1 \leq t \leq N} \left\{ \frac{w_t \bar{v}}{\sum_{r=t-1}^{N} w_r} \right\}
\]

which is the seller’s minimax regret when the sales season is infinitely long and the lower bound of customers’ valuations is 0.

**Proposition 3.** (Optimal length of selling horizon)

The minimax regret \( R^*_N(v) \) is decreasing in \( N \) and \( \kappa^*_N(v) \) is increasing in \( N \).

a) For all \( v \in [0, R^*_N(0)) \), \( R^*_N(v) = R^*_N(0) \) and \( \kappa^*_N(v) = N + 1 \). For \( v \geq R^*_N(0) \), \( R^*_N(v) \) strictly decreases in \( v \).

b) If \( v \in [0, R^*_\infty(0)] \), then \( R^*_N(v) \) strictly decreases and \( \kappa^*_N(v) \) strictly increases in \( N \). They satisfy

\[
R^*_\infty(v) \triangleq \lim_{N \to \infty} R^*_N(v) = R^*_\infty(0) \quad \text{and} \quad \lim_{N \to \infty} \kappa^*_N(v) = \infty.
\]

c) If \( v > R^*_\infty(0) \), then there is an \( N \) such that \( R^*_N(v) = R^*_\infty(v) \) and \( \kappa^*_N(v) = \kappa^*_\infty(v) \) for all \( n \geq N \).

Proposition 3 highlights some useful properties of the seller’s regret for purpose of optimizing the length of the selling season. First of all, part (a) states that if \( v \) is small, that is, if the valuation uncertainty is large, then firm should price the low-value customers out of the market. It is suboptimal to decrease the price to \( v \) since the firm would protect herself from losing too much revenue from the high-value customers. The minimax regret and pricing policy are the same for all \( v \in [0, R^*_N(0)) \). In the example of the left panel of Figure 5, \( R^*_N(v) = R^*_N(0) = 0.4052 \) for all \( v \leq 0.4052 \). \( R^*_N(v) \) strictly decreases for \( v > 0.4052 \) and \( R^*_N(v) = 0 \) at \( v = 1 \), when there is no uncertainty in the customer’s valuation.

Second, Proposition 3 states that, *ceteris paribus*, the seller would like to have longer selling seasons. This is not an obvious result if we consider that by increasing the selling horizon the seller is also increasing the the uncertainty set \( D = [\bar{v}, v] \times \{1, 2, \ldots, N\} \) over which nature maximizes her
regret (see equation (4)). According to the result in Part (b), if the lower valuation \( v \) is sufficiently small then the seller is always better off if she could expand the selling horizon, so it is optimal to have an infinitely long selling horizon. However, as the seller extends the sales season, she will not benefit too much since her regret will converge to \( R^*_\infty(0) \), which is a positive number. For example, in the right panel of Figure 5, \( R^*_N(v) \) is very close to \( R^*_\infty(0) \) at period 3.

On the other hand, Part (c) implies that if \( v \) is large then the selling horizon is longer than needed, since the amount of the valuation uncertainty is small, the seller does not need much time to explore the uncertainty. There exists an optimal finite length for the selling horizon that minimizes the seller’s regret.

![Figure 3](image)

Figure 3  The left panel depicts \( R^*_N(v) \) as a function of \( v \), in this example, \( N = 20 \). The right panel depicts \( R^*_N(v) \) as a function of \( N \), in this example, \( v = 0.2 \). In both examples, \( \delta = 0.4 \).

5. Selling to Strategic Customers

We now consider case where the firm sells to customers who strategically time their purchases. In this case, customers are forward-looking in anticipating the prices the seller will offer and time their purchases to maximize their net utility. We know from equation (4) that the seller’s problem is to pick a vector of prices \( p \) with the goal of minimizing her regret

\[
\tilde{R}^* = \min_{p \geq 0} \max_{(v, \tau) \in D} R(v, \tau, p) = \delta^{-1}v - \delta^d_S(v, \tau, p)^{-1} p d_S(v, \tau, p),
\]

where \( d_S(v, \tau, p) \) is the purchase time chosen by a strategic customer with valuation \( v \), arriving at time \( \tau \) when prices are given by \( p \), i.e., \( d_S(v, \tau, p) = \min \arg \max_{\tau \leq t \leq N} \{\delta^{t-1}(v - p_t)|v \geq p_t\} \).

The problem of selling to a strategic customer can be interpreted as a three player game, where the firm moves first selecting a pricing policy \( p \), nature responds by selecting the value \( v \) and
arrival time $\tau$, and the customer moves last by selecting whether and when to purchase the firm’s product. We solve this game backward looking first at the customer’s best response for a given triplet $(v, \tau, p)$. To this end, suppose the customer is at time $t \geq \tau$ and must decide whether to buy the product now or wait another period. Given his valuation $v$ and the vector of prices $p$, the customer will buy at $t$ if and only if the net utility of buying now exceeds the discounted net utility of buying in any of the future periods, that is,

$$v - p_t \geq \delta^{k-t} (v - p_k), \quad \text{for all } k = t+1, \ldots, N. \quad (16)$$

We will rewrite this condition in an alternative form that will prove useful for the analysis that follows but before we need to introduce the notion of threshold valuations.

**Definition 2.** For a given vector of prices $p = (p_1, \ldots, p_N)$, we define recursively the vector of threshold valuations $\tilde{v}(p) = (\tilde{v}_1, \ldots, \tilde{v}_N)$ as follows: $\tilde{v}_N \triangleq p_N$ and

$$\tilde{v}_t \triangleq \frac{p_t - \delta p_{t+1}}{1 - \delta}, \quad t = N - 1, \ldots, 1. \quad (17)$$

We also set $\tilde{v}_0 = \bar{v}$. A price vector $p_t$ is $\delta$-convex if $\tilde{v}_t$ decreases in $t$. □

With this definition, we can rewrite condition (16) in the following equivalent way

$$v \geq \max_{t+1 \leq k \leq N} \left\{ \frac{\tilde{v}_t + \delta \tilde{v}_{t+1} + \cdots + \delta^{k-t-1} \tilde{v}_{k-1}}{1 + \delta + \cdots + \delta^{k-t-1}} \right\}. \quad (18)$$

Note that the argument inside the maximization is a convex combination of the $(\tilde{v}_t, \ldots, \tilde{v}_k)$.

To proceed with our characterization of the consumer’s purchasing strategy, we further show that there always exists an optimal price path under which the threshold valuation $\tilde{v}_t$ is strictly decreasing in $t$ up to some $\kappa \leq N$. Indeed, we define prices with such property to be $\delta$-convex prices. It is worth noticing that the notion of a $\delta$-convex policy is a relaxation of the concept of strict convexity. A policy that is 1-weak convex is actually a strictly convex policy. However, for $\delta < 1$, a $\delta$-convex policy might not be convex but does satisfy $\frac{p_{t-1} + \delta p_t}{1 + \delta} > p_t$ up to some $\kappa \leq N$. Let us denote by $\mathcal{P}_C$ the set of $\delta$-convex prices under which $\tilde{v}_1 \leq \bar{v}$.

We show in Proposition 4 that restricting our search to the $\delta$-convex prices is without loss of optimality.

**Proposition 4.** There always exists an optimal price vector $p^*$ that is $\delta$-convex.

The concept of a $\delta$-convex policy is useful because it ensures the customer’s problem is “easy”. To see this, note that for a decreasing vector of threshold valuation, the right-hand side in equation (18) is equal to $\tilde{v}_t$. Indeed, if $p \in \mathcal{P}_C$ then consumers act “myopically” with respect to the vector of
threshold valuation \( \tilde{v}_t \) in the sense that they buy the product as soon as their valuation exceeds the corresponding threshold valuation of that period.

To analyze the consumer’s purchase behavior, we now draw a parallel with Section 4 and fix a price strategy \( p \in \mathcal{P}_C \). A \((v, \tau)\)-customer’s behavior is then one of the following three cases:

(i) **Buy immediately**: if \((v, \tau) \in [\tilde{v}_t, \bar{v}_t] \times \{t\}\), the customer arrives and buys in the same period \( t \).

(ii) **Wait and buy**: if \((v, \tau) \in [\tilde{v}_t, \tilde{v}_{t-1}] \times \{1, 2, \ldots, t-1\}\) for some \( t \), then the customer arrives before period \( t \) and can only afford the product at period \( t \).

(iii) **Never buy**: if \( p_N > v \) and \((v, \tau) \in [v, p_N] \times \{1, 2, \ldots, N\}\), then such type of customer will leave the market without any purchase.

Similar to the previous section, we define \( \bar{t}(\tilde{v}) \triangleq \max \{t | \tilde{v}_t = \bar{v}_t\} \) and \( t(\tilde{v}) \triangleq \min \{t | \tilde{v}_t = \tilde{v}_t\} \). We set \( \bar{t}(\tilde{v}) = 0 \) if \( \tilde{v}_1 < \tilde{v} \) and \( t(\tilde{v}) = N + 1 \) if \( \tilde{v}_N > \bar{v} \). We write \( t \) and \( \bar{t} \) as shorthand notations for \( t(\tilde{v}) \) and \( \bar{t}(\tilde{v}) \) respectively. Now we can formalize the seller’s problem.

**Lemma 2.** Suppose the seller chooses a price vector \( p \in \mathcal{P}_C \). Then, the seller’s worst-case regret problem can be rewritten as

\[
\tilde{R}^* = \min_{p \in \mathcal{P}_C} \max \left\{ \max_{t \in \{1, \ldots, N\}} \{ \delta^{t-1}(\bar{v} - p_t) \}, \max_{t \in \{1, \ldots, \min\{\bar{t}, N\}\}} \{ \tilde{v}_{t-1} - \delta^{t-1}p_t \}, p_N \cdot 1_{\{t = N+1\}} \right\}. \tag{19}
\]

We can simplify even further the optimization in equation (19) by noticing that with \( \delta \)-convex prices, it is optimal for nature to send customer at the beginning of the selling horizon, regardless of the valuation \( v \). This is formalized in the following corollary.

**Corollary 2.** Suppose the seller chooses a vector of \( \delta \)-convex prices \( p \in \mathcal{P}_C \). Then, it is optimal for nature to have strategic customers arriving at the beginning of the selling season, that is, \( \tau^* = 1 \).

Therefore we can rewrite the seller’s optimization problem in (19) by decomposing it into the following subproblems, parametrized by \( \kappa \), \( \kappa \in \{1, 2, \ldots, N+1\} \).

\[
\tilde{R}(\kappa) = \min_{p, R} R \tag{20}
\]

subject to \( \bar{v} - p_1 \leq R \) \tag{21}

\[
\frac{p_{t-1} - \delta p_t}{1 - \delta} - \delta^{t-1}p_t \leq R \quad \forall t \in \{2, \ldots, \min\{\kappa, N\}\} \tag{22}
\]

\[
p_{\kappa} = v \quad \text{if} \quad \kappa \leq N \quad \text{or} \quad p_N \leq R \quad \text{if} \quad \kappa = N + 1 \tag{23}
\]

\[
p \in \mathcal{P}_C(\kappa). \tag{24}
\]

where \( \mathcal{P}_C(\kappa) \) is defined as \( \mathcal{P}_C(\kappa) \triangleq \{ p \in \mathcal{P}_C | \bar{t}(\tilde{v}) = \kappa \} \).

A direct implication of this formulation, in particular of constraint (22) evaluated at \( t = 1 \), is that as long as the optimal regret is strictly positive (which is typically the case except for some trivial
instances) the initial price satisfies $p_1 < \bar{v}$. Hence, and in contrast to the myopic customer case, an optimal price vector will not contain an initial *markup period* in which prices are temporarily equal to the upper bound $\bar{v}$.

We now use the mathematical programming characterization of $\tilde{R}(\kappa)$ to solve it explicitly and, in this way, determine the optimal pricing policy in the presence of strategic customers. In this case, the solution is rather straightforward after noticing that equations (22)-(23) form a linear system of $\min\{\kappa,N\}+1$ equations in $\min\{\kappa,N\}+1$ unknowns $(R,p_1,\ldots,p_\kappa)$.

**Proposition 5.** Let $a_{i,j} \triangleq \prod_{k=i}^{j}(\delta+(1-\delta)\delta^k)$ for any two values of $i$ and $j$. The firm’s regret is then given by

$$\tilde{R}^* = \min_{1 \leq \kappa \leq N+1} \left\{ \max \left\{ \frac{\bar{v} - \underline{v} a_{0,\kappa-1}\mathbb{1}(\kappa \leq N)}{1 + (1-\delta)\sum_{i=0}^{\min\{\kappa,N\}-2} a_{0,i} + a_{0,N-1}\mathbb{1}(\kappa > N)} , \underline{v}(1-\delta^{\kappa-1}\mathbb{1}(\kappa \leq N)) \right\} \right\}.$$  

Let $\tilde{\kappa}^*$ be the argument that minimizes the equation above, with ties broken in favor of a smaller $\tilde{\kappa}^*$. If $\tilde{\kappa}^* \leq N$, then optimal prices are equal to $\tilde{p}_t^* = \underline{v} a_{t,\tilde{\kappa}^*-1} + (1-\delta)\tilde{R}^*(\sum_{i=t}^{\tilde{\kappa}^*-2} a_{t,i} + 1)$ for all $t < \tilde{\kappa}^*$ and equal to $\underline{v}$ otherwise. If $\tilde{\kappa}^* = N+1$, then prices equal to $\tilde{p}_t^* = \tilde{R}^*(a_{t,N-1} + (1-\delta)(\sum_{i=t}^{N-2} a_{t,i} + 1))$ for $t < N$ and $\tilde{p}_N^* = \tilde{R}^*$ are optimal. The optimal price vector $\tilde{p}^*$ belongs to the set $P_C$.

A distinctive feature of the optimal pricing policy for strategic customers is its relative flatness. This is a direct consequence from the forward-looking behavior of the customers, who tend to delay their purchases, and therefore the highest valuation that makes a strategic customer to buy at price $p_t$ now becomes $\bar{v}_{t-1} = \frac{\delta}{1-\delta}(p_{t-1} - p_t) + p_{t-1}$, instead of $p_{t-1}$ in the myopic customer case.

Accordingly, the firm tends to choose a price path that has less aggressive markdowns to reduce the delay regret defined in (22). From the fact that the optimal price is $\delta$-convex, we know that the markdown size of $p_t - p_{t+1}$ is bounded above by $\frac{p_t - p_{t+1}}{\delta}$.

Figure 4 shows how the optimal pricing policy for selling to strategic customers compares to the upper and lower bound optimal prices for selling to myopic customers. In the instance plotted, strategic prices are initially lower than myopic prices, but since markdowns occur at a slower rate, strategic prices eventually become higher than myopic ones.

Another consequence from the forward-looking behavior of the customers is that the firm’s regret is always larger when selling to strategic customers than selling to myopic ones. This is formalized in the following proposition.

**Proposition 6.** The worst-case regret from selling to strategic customers is higher than the worst-case regret from selling to myopic customers, i.e., $\tilde{R}^* \geq R^*$. 
It is indeed intuitive that the firm is better off when facing myopic consumers, who buy the product at soon as they can. The strategic customer tends to buy the product later with a lower price. It follows that for any price vector $p$ and any pair $(v, \tau)$

$$e^{-rd_M(v, \tau, p)} p_{d_M(v, \tau, p)} \geq e^{-rd_S(v, \tau, p)} p_{d_S(v, \tau, p)}.$$ 

That is, the firm can always extract more rents from a myopic consumer than from a strategic one.

We conclude this section with a result that mimics Proposition 3 in the previous section regarding the sensitivity of an optimal solution with respect to the length of the selling season. To emphasize the dependence of $\tilde{R}^*_N(v)$ and $\tilde{\kappa}^*_N(v)$ on the number of periods $N$ as well as the lower valuation $v$. We then write $\tilde{R}^*_N(v)$ and $\tilde{\kappa}^*_N(v)$. We also define $\tilde{R} \triangleq \lim_{N \to \infty} \tilde{R}^*_N(0)$ which is the regret from selling to strategic customers with valuation in $[0, \bar{v}]$ when $N$ goes to infinity.

**Proposition 7.** (Optimal length of selling horizon)

The minimax regret $\tilde{R}^*_N(v)$ is decreasing in $N$ and $\tilde{\kappa}^*_N(v)$ is increasing in $N$.

a) For all $v \in [0, \bar{R}^*_N(0))$, $\tilde{R}^*_N(v) = \bar{R}^*_N(0)$ and $\tilde{\kappa}^*_N(v) = N + 1$. For $v \geq \bar{R}^*_N(0)$, $\tilde{R}^*_N(v)$ strictly decreases in $v$.

b) If $v \in [0, \bar{R}]$, then $\tilde{R}^*_N(v)$ strictly decreases and $\tilde{\kappa}^*_N(v)$ strictly increases in $N$. They satisfy

$$\tilde{R}^*_\infty(v) \triangleq \lim_{N \to \infty} \tilde{R}^*_N(v) = \bar{R} \quad \text{and} \quad \lim_{N \to \infty} \tilde{\kappa}^*_N(v) = \infty.$$ 

c) If $v > \bar{R}$, then there is an $N$ such that $\tilde{R}^*_n(v) = \tilde{R}^*_N(v)$ and $\tilde{\kappa}^*_n(v) = \tilde{\kappa}^*_N(v)$ for all $n \geq N$.

The result above shows that when the ratio $v/\bar{v}$ is relative high, and thus there is relatively little valuation uncertainty, prices decrease to $v$ in a fairly small number of periods and the seller cannot take advantage of the entire selling horizon. In contrast, when $v/\bar{v}$ is low, prices are strictly
decreasing and never reach \( v \), regardless of the selling horizon \( N \). Similar to the case with myopic customers, if \( v \) is less than \( \tilde{R}_N(0) \), then the particular value of \( v \) will play no role in determining the optimal price policy or the worst-case regret. This can be seen from the left panel of Figure 5. In this case, \( \tilde{R}_N(0) = 0.4212 \), for \( v \leq 0.4212 \), the regret is a constant in \( v \). The right panel of Figure 5 shows that when \( v < \tilde{R} \) where \( \tilde{R} = 0.387 \), the regret strictly decreases in \( N \) and converges to \( \tilde{R} \).

![Figure 5](image-url)  
*Figure 5*  
The left panel depicts \( \tilde{R}_N(v) \) as a function of \( v \), in this example, \( N = 4 \). The right panel depicts \( R_N(v) \) as a function of \( N \), in this example, \( v = 0.2 \). In both examples, \( \delta = 0.8 \).

6. Continuous-Time Pricing

In this section, we analyze the limit of the seller’s discrete-time pricing problem as the time between consecutive price changes goes to zero and derive a continuous-time approximation for an optimal pricing strategy and corresponding minimum regret. Although we primarily view our continuous-time approximation as good (and mathematically tractable) abstraction of a discrete-time model where the seller can change prices frequently, we also acknowledge the fact that the rapid growth of information technologies and online commerce has allowed many companies today to changes prices in an almost “continuous” way.

We consider a sequence of discrete-time instances of the seller’s problem indexed by \( n \in \mathbb{N} \), in which the selling horizon \([0, T]\) is divided into \( 2^n \) periods of equal length \( \Delta_n = T/2^n \). We assume that the selling horizon \( T \) is a fixed calendar time independent of \( n \). For a given instance \( n \), we let \( p^n = \{p^n_i : i = 1, \ldots, 2^n\} \) denote the seller’s pricing strategy, where \( p^n_i \) is the price charged in period \( i \). To be more specific, let us define

\[
T_n \triangleq \left\{ \frac{iT}{2^n} : i = 1, \ldots, 2^n \right\},
\]
to be the partition of the interval \([0, T]\) at which prices are changed, that is, at time \(i \Delta_n\) the seller sets the price \(p^n_i\) for \(i = 1, \ldots, 2^n\). We represent the continuous-time discount rate by \(r\). In each discrete instance \(n\), the discount factor is then given by \(\delta_n = e^{-r \Delta_n}\). This scaling of the discount factor ensures that when we increase \(n\), we simply add more times at which the seller can change her price, rather than extend the fixed calendar duration \([0, T]\) of the selling season.

In what follows, we study the limit of the seller’s discrete-time optimal pricing strategy \(p^n\) along the sequence \(T_n\) and characterize its limit in \(T_\infty\). It is worth noticing that the sequence \(\{T_n\}\) is monotonically increasing in the sense that \(T_n \subseteq T_{n+1}\). Furthermore, the limit \(T_\infty \triangleq \bigcup_n T_n\) is dense in \([0, T]\), which is a fact that we will use to extend the limiting price sequence \(\{p_\infty(t) : t \in T_\infty\}\) to the entire interval \([0, T]\) in an obvious way. As before, we analyze the cases of myopic and strategic customers separately.

### 6.1. Myopic Customers

Let us denote by \(\bar{p}_n \triangleq \{\bar{p}_n^i | i = 1, \ldots, 2^n\}\) the optimal price vector (upper bound) derived in Proposition 2. Similarly, we denote by \(R^*_n\) the corresponding optimal regret. We also define \(\kappa^*_n = t(\bar{p}^n) \Delta_n\) which is the first calendar time when the price vector \(\bar{p}^n\) equals the lower valuation \(v\). In the case where the price path \(\bar{p}^n\) never hits \(v\), we write \(\kappa^*_n = T^+\).

The following proposition characterizes the limiting values of the triplet \((R^*_n, \kappa^*_n, \bar{p}^n)\) as \(n\) goes to infinity.

**Proposition 8.** (Myopic Continuous-Time Pricing)

Consider a fixed selling season of length \(T\), then \((R^*_n, \kappa^*_n)\) converges to \((R^*_\infty, \kappa^*_\infty)\) given by

- **Case 1:** If \(\bar{v} \leq 2v\) then
  \[
  R^*_\infty = \max \left\{ e^{-rT} (\bar{v} - v) \frac{v}{\bar{v}} (\bar{v} - v) \right\} \quad \text{and} \quad \kappa^*_\infty = \min \left\{ T ; \frac{1}{r} \ln \left( \frac{\bar{v}}{v} \right) \right\}.
  \]

- **Case 2:** If \(2v \leq \bar{v} \leq 4v\) then
  \[
  R^*_\infty = \max \left\{ \min \left\{ e^{-rT} \frac{\bar{v}}{1 + e^{-rT}} ; e^{-rT} (\bar{v} - v) \right\} ; \frac{\bar{v}}{4} \right\}
  \]
  and
  \[
  \kappa^*_\infty = \begin{cases} T^+ & \text{if } T \leq \frac{1}{r} \ln \left( \frac{\bar{v} - v}{2v} \right) \\ \min \left\{ T, \frac{1}{r} \ln \left( \frac{4(\bar{v} - v)}{v} \right) \right\} & \text{if } T \geq \frac{1}{r} \ln \left( \frac{\bar{v} - v}{2v} \right). \end{cases}
  \]

- **Case 3:** If \(4v \leq \bar{v}\) then
  \[
  R^*_\infty = \max \left\{ \frac{e^{-rT} \bar{v}}{1 + e^{-rT}} ; \frac{\bar{v}}{4} \right\} \quad \text{and} \quad \kappa^*_\infty = T^+.
  \]
In addition, the optimal pricing vector \( \bar{p}^n \triangleq \{ \bar{p}_i^n | i = 1, \ldots, 2^n \} \) converges to \( \bar{p}_\infty \triangleq \{ \bar{p}_\infty(t) | t \in T_\infty \} \) given by
\[
\bar{p}_\infty(t) = \min \left\{ \max \left\{ \frac{R^*_\infty}{1-e^{-rt}}, \bar{v} \right\}, \bar{v} \right\} \quad \text{if} \quad t < T
\]
and
\[
\bar{p}_\infty(T) = R^*_\infty \mathbb{1}((\bar{v}, T) \in \mathcal{A}) + \bar{v} \mathbb{1}((\bar{v}, T) \notin \mathcal{A}),
\]
where \( \mathcal{A} \triangleq \{ (\bar{v}, T) | (i) \ 4\bar{v} \leq \bar{v} \quad \text{or} \quad (ii) \ 2\bar{v} \leq \bar{v} \leq 4\bar{v} \quad \text{and} \quad rT \leq \ln(\bar{v} - \bar{v}) - \ln \bar{v} \} \).

The previous proposition provides a compact characterization of the optimal minimum regret and corresponding optimal price strategy (upper bound). For the most part (i.e., except when the price is equal to one of the boundaries \( \bar{v} \) and \( \bar{v} \)), the optimal price \( \bar{p}_\infty(t) \) is equal to \( R^*_\infty/(1 - \exp(-rt)) \), which is a decreasing and convex function of \( t \). That is, an optimal markdown policy should be “decelerating” over time in the sense that the magnitude of the markdowns should be decreasing. It is interesting to note that this recommendation is in contrasts with conventional pricing strategies in which the speed of markdowns tends to increase towards the end of the selling season. At the same time, another interesting feature of the continuous-time pricing approximation is that in certain instances the optimal price path can have a discontinuity at \( t = T \) (downward jump). In other words, although the magnitude of markdowns are decreasing over time, it might be optimal to introduce a fire sale at the very end of the season. Note that a discontinuous fire sale emerges in this model purely as a consequence of a price skimming strategy, even in a model without inventory considerations.

Another important takeaway from Proposition 8 relates to the limiting value of \( \kappa^*_\infty \). Recall that this quantity is the time at which the price reaches the lower bound \( \bar{v} \), that is, the time at which the firm liquidates the product in what we have called a clearance period in Figure 1. Hence, if \( \kappa^*_\infty < T \) then marginally increasing or decreasing the length of the selling season \( T \) will have no impact on the firm’s regret. On the other hand, if \( \kappa^*_\infty = T \) then marginally increasing \( T \) will reduce the firm’s regret. With this in mind, we define an optimal length \( T^*_\infty \) for the selling season which is the minimum \( T \) that will achieve the minimum possible regret. Let us denote by \( R^*_\infty(T) \) the firm’s regret if the selling season has length \( T \) and \( R^*_\infty \triangleq \inf_T \{ R^*_\infty(T) \} \). Then, \( T^*_\infty \triangleq \inf \{ T \geq 0 : R^*_\infty(T) = R^*_\infty \} \).

The following result characterizes the value of \( T^*_\infty \) as a function of the model input parameters \((\bar{v}, \bar{v}, r)\). (The proof follows directly from Proposition 8 and is omitted.)

**Corollary 3.**

\[
T^*_\infty = \frac{1}{r} \begin{cases} 
\ln(3) & \text{if} \quad 4\bar{v} \leq \bar{v} \\
\ln(4(\bar{v} - \bar{v})) - \ln(\bar{v}) & \text{if} \quad 2\bar{v} \leq \bar{v} \leq 4\bar{v} \\
\ln(\bar{v}) - \ln(\bar{v}) & \text{if} \quad \bar{v} \leq 2\bar{v}.
\end{cases}
\]
One direct consequence of this result is that $T^*_\infty$ is uniformly bounded above by $\ln(3)/r$. In other words, independently of the seller's prior believes about the range of consumers' valuations $[v, \tilde{v}]$, a sufficient condition to minimize the seller’s regret is to choose a selling season of length $T = \ln(3)/r$.

We conclude this section by extending the result in Corollary 4 to the continuous-time approximation. In stating this result (and the corollaries that follow) we find convenient to introduce the following partition of the space $[0, \tilde{v}] \times \mathbb{R}_+$.

$$A_1 \triangleq \{(v,T)| 4v \leq \tilde{v} \text{ and } rT \geq \ln 3\}$$
$$A_2 \triangleq \{(v,T)| 2v \leq \tilde{v} \leq 4v \text{ and } rT \geq \ln 4(\tilde{v} - v) - \ln \tilde{v}\}$$
$$A_3 \triangleq \{(v,T)| \tilde{v} \leq 2v \text{ and } rT \geq \ln \tilde{v} - \ln v\}$$
$$A_4 \triangleq \{(v,T)| 4v \leq \tilde{v} \text{ and } rT \leq \ln(\tilde{v} - v) - \ln v\}$$

$$A_5 \triangleq [0, \tilde{v}] \times \mathbb{R}_+ \backslash \{A_1 \cup A_2 \cup A_3 \cup A_4\}.$$

**Corollary 4.** Let

$$t^* = \frac{1}{r} \begin{cases} 
\ln 2 & \text{if } (v,T) \in A_1 \cup A_2, \\
\ln(\tilde{v}) - \ln(v) & \text{if } (v,T) \in A_3, \\
rT & \text{if } (v,T) \in A_4 \cup A_5.
\end{cases}$$

The lower bound of optimal pricing vector $p^n \triangleq \{p^n_i | i = 1, \ldots, 2^n\}$ converges to $p^\infty \triangleq \{p^\infty_{\infty}(t) | t \in T_{\infty}\}$ given by

$$p^\infty_{\infty}(t) \triangleq \begin{cases} 
\tilde{v} - e^{rt} R^*_\infty & \text{if } t \leq t^*, \\
\tilde{p}_{\infty}(t) & \text{if } t \geq t^*.
\end{cases}$$

We now want to draw a parallel between the optimal pricing strategy in this model and the continuous pricing strategy in [Caldentey et al. (2015)](Caldentey et al. (2015)). The minimax regret and the upper bound of the prices as well as the optimal length of selling horizon derived in Proposition 8 and Corollary 3 are the same as those in [Caldentey et al. (2015)](Caldentey et al. (2015)). In the discrete-time model, as we have shown, there is a unique price path after time $t^*$, which is also true in the asymptotic case as stated in Corollary 4. However, in the continuous-time case in [Caldentey et al. (2015)](Caldentey et al. (2015)), there is infinitely many price paths during time $[t^*, \kappa^*_\infty]$, which implies that the firm enjoys more flexibility in choosing pricing policy if she could change price in a continuous-time manner.

Figure 6 depicts the upper and lower bound approximations for the example in Figure 2. The dotted lines are the discrete-time solutions and the continuous lines are the corresponding continuous-time approximations. On the left panel the selling horizon is divided into $N = 20$ periods while in the right panel $N = 50$. Interestingly, the continuous-time lower bound price function $\tilde{p}^\infty_{\infty}(t)$ is very accurate even for $N = 20$. We also note that the upper bound continuous-time approximation $\tilde{p}_{\infty}(t)$ is an upper bound of the discrete-time solutions.
Figure 6  Comparison of the discrete-time (dotted lines) and continuous-time (solid lines) pricing policy for the cases in which $N = 20$ (left panel) and $N = 50$ (right panel). In this example, $v = 0.5$, $\bar{v} = 1$, $T = 9$ and $r = 0.1$.

6.2. Strategic Customers

In the case with strategic customers, the pricing policy and minimax regret in Caldentey et al. (2015) can be shown as the optimal solutions for the asymptotic case for the discrete time model. We formalize the results as follows.

Let $(\tilde{R}_n^*, \tilde{\kappa}_n^*, \tilde{p}_n^*)$ be the optimal solution derived in Proposition 5 for the model with strategic consumers. The following proposition characterizes the limiting values of this triplet as $n$ goes to infinity. Using a similar convention as in the myopic case, we set $\tilde{\kappa}_n^* = T^+$ if the optimal price path $\tilde{p}_n^*$ never hits $v$. Our characterization of the continuous-time limiting solution in the proposition below uses the following definitions:

$$\tilde{T} \triangleq -\frac{1}{r} \ln (1 + \ln (v / \bar{v})),$$

$$\tilde{R} \triangleq \begin{cases} 
\tilde{v} \exp (e^{-rT} - 1) - v \exp (-rT) & \text{if } T \leq \tilde{T} \\
\bar{v} (\ln(\tilde{v}) - \ln(v)) & \text{if } T \geq \tilde{T}
\end{cases}$$

and

$$\tilde{R}_+ \triangleq \frac{\tilde{v} \exp (e^{-rT} - 1)}{1 + \exp (-rT)}.$$

In addition, for the case $v \exp(1) \leq \bar{v}$, we define $\tilde{T}$ as the unique solution of the equation

$$\tilde{v} \exp (e^{-rT} - 1) = v.$$

Proposition 9. Consider a fixed selling season of length $T$. As $n \to \infty$, the discrete-time solution $(\tilde{R}_n^*, \tilde{\kappa}_n^*)$ converges to $(\tilde{R}_\infty^*, \tilde{\kappa}_\infty^*)$ given by

- Case 1: If $\bar{v} \leq 2v$ then $(\tilde{R}_\infty^*, \tilde{\kappa}_\infty^*) = (\tilde{R}, \min\{T, \tilde{T}\}).$
- Case 2: If $2\bar{v} \leq \bar{v} \leq v \exp(1)$ then
  
  \[
  (\tilde{R}_\infty^*, \tilde{\kappa}_\infty^*) = \begin{cases} 
  (\tilde{R}_+, T^+) & \text{if } T \leq \hat{T} \\
  (\tilde{R}, \min\{T, \hat{T}\}) & \text{if } T \geq \hat{T}.
  \end{cases}
  \]

- Case 3: If $\bar{v} \exp(1) \leq \bar{v}$ then $(\tilde{R}_\infty^*, \tilde{\kappa}_\infty^*) = (\tilde{R}_+, T^+)$.  

In addition, the optimal pricing vector $\tilde{p}_n \triangleq \{\tilde{p}_n^i | i = 1, \ldots, 2^n\}$ converges to $\tilde{p}_\infty \triangleq \{\tilde{p}_\infty(t) | t \in T_\infty\}$ given by

\[
\tilde{p}_\infty(t) = \bar{v} \exp\left( rt + e^{-rt} - 1 \right) - \tilde{R}_\infty \exp(rt), \quad t \leq T.
\]

Figure 7 Continuous price paths for strategic customers, upper bound $\tilde{p}_t$ and lower bound $p_t$ price paths for myopic customers, with $\bar{v} = 1$, $v = 0.5$, $T = 9$ and $r = 0.1$.

Figure 7 depicts the continuous-time price paths. The pricing strategies here share the same feature as in the discrete-time model, that is, the firm tends to use a less aggressive markdown policy when customers are strategic. Prices designed for strategic customers typically start lower than prices created for myopic customers, but eventually become higher than prices for myopic ones. A major difference with the solution for myopic customers is the absence of discontinuities. Unlike in the myopic case, there is never a fire sale at the last period of the continuous-time limit. 

With strategic customers, a price discontinuity would create an obvious incentive for customers to wait in the moments preceding a fire sale. We could weakly reduce the regret of any policy that contained a fire sale by reducing the price in those pre-sale moments and, in this way, create a weakly better continuous pricing policy.
7. Extensions

7.1. Time-Dependent Valuations

The robust formulation presented in the previous sections assumes that customers’ valuations are independent of time in the sense that nature is allowed to select customers’ types from the rectangle $D = [\underline{v}, \overline{v}] \times \{1, 2, \ldots, N\}$. In many practical situations, however, we can expect some degree of correlation between a customer’s arrival time and his valuation. For example, customers arriving at the begin of the season could have higher valuation for the product than late-comers.

To capture this correlation between arrival times and valuations, in this section we consider the case in which the upper level of the range of valuations is a decreasing function of time, that is, $\overline{v}(t)$ such that $\underline{v} \leq \overline{v}(t)$ for all $t = 1, 2, \ldots, N$. A more general formulation could also consider the lower limit as a function of time, i.e., $\underline{v}(t)$, however, to ease the exposition we will focus on this special case that captures in simple terms the notion of decreasing valuations. Also, to avoid unnecessary technical details, we will assume that there exists a threshold $\hat{t}$ such that $\overline{v}(1) = \overline{v}(2) = \cdots = \overline{v}(\hat{t}) > \overline{v}(\hat{t} + 1) > \cdots > \overline{v}(N)$.

(Note that our base model assumes $\hat{t} = N$).

In this new setting, nature is now restricted to select customers’ types from the set $\hat{D} = \{(v, \tau) : \underline{v} \leq v \leq \overline{v}(\tau), 1 \leq \tau \leq N\}$. The robust optimization problem in equation (4) becomes

$$\hat{R}^* \triangleq \min_{p \geq 0} \max_{(v, \tau) \in \hat{D}} \delta^{\tau - 1} v - \delta^{d(v, \tau, p)-1} p_d(v, \tau, p).$$

In what follows, we will show how to solve this optimization for the case in which consumers are myopic, that is, $d(v, \tau, p) = d_M(v, \tau, p) = \min_{t \leq \tau \leq N} \{t \mid v \geq p_t\}$. The case with strategic consumers follows similar steps but replacing prices by threshold valuations as we did in Section 5.

The first step is to extend the definition of a decreasing price vector and the set $P_M$ in Definition 1.

**Definition 3.** The price vector $p = (p_1, p_2, \ldots, p_N)$ belongs to $\hat{P}_M$ if $\underline{v} \leq p_t \leq \overline{v}(t)$ for all $t = 1, \ldots, N$ and there exist two thresholds $0 \leq \bar{t} < \hat{t} \leq N + 1$ such that

$$p_t = \overline{v}(t), \quad t = 1, \ldots, \bar{t}$$
$$p_t > p_{t+1} > \cdots > p_{\hat{t}}$$
$$p_t < \overline{v}(t), \quad \forall t > \hat{t}$$
$$p_t = \overline{v}, \quad t = \hat{t}, \ldots, N.$$  

(25)

Next, we extend the result in Lemma 1 to this case with time-dependent valuations.

**Lemma 3.** Suppose the seller chooses a vector of prices $p \in \hat{P}_M$. Then, nature’s best response is to choose one of the following value-time pairs

...
a) \((v, \tau) = (\bar{v}(t), t), t \in \{1, 2, \ldots, N\}\) or
b) \((v, \tau) = (p_t, 1), t \in \{\ell(p), \ldots, \ell(p) - 1\}\).

As a result, the seller’s worst-case regret problem can be rewritten as follows similar steps and is omitted.

\[
\hat{R}^* = \min_{p \in \hat{P}_M} \max \left\{ \max_{t \in \{1, 2, \ldots, N\}} \left\{ \delta^{t-1}(\bar{v} - p_t) \right\}, \max_{t \in \{\ell(p) + 1, \ldots, \ell(p)\}} \left\{ p_{t-1} - \delta^{t-1}p_t \right\} \right\}.
\]  

(26)

The seller’s optimization problem in Lemma 3 can be rewritten in the following alternative form:

\[
\hat{R}^* = \min_{p \in \hat{P}_M} \max \left\{ \delta^{t-1}(\bar{v} - p_t) \right\}, \max_{t \in \{\ell(p) + 1, \ldots, \ell(p)\}} \left\{ p_{t-1} - \delta^{t-1}p_t \right\}.
\]

We can use the same approach that we used in Section 4 to rewrite this optimization problem as a collection of linear programs parametrized by the period \(\kappa\) at which the price path hits the lower bound \(v\), i.e., \(\ell(p) = \kappa\) for details). The following result is a direct analog of Proposition 2 (its proof follows similar steps and is omitted).

**Proposition 10.** Define

\[
\hat{R}(\kappa) = \max_{1 \leq t \leq \kappa} \left\{ \frac{\delta^{(t-1)\kappa-1}}{2} \bar{v}(t) - \frac{\delta^{(t-1)\kappa}}{2} \bar{v} \right\}, \quad (1 - \delta^{\kappa-1})\bar{v} \right\}, \quad 1 \leq \kappa \leq N \quad \text{and}
\]

\[
\hat{R}(N + 1) = \max_{1 \leq t \leq N} \left\{ \frac{\delta^{(t-1)(N+1)-1}}{2} \bar{v}(t) \right\}, \quad \bar{v} \right\}.
\]

Then, the seller’s worst-case regret is equal to \(\hat{R}^* = \min \{\hat{R}(\kappa) \mid \kappa = 1, \ldots, N + 1\}\). Furthermore, let \(\kappa^* = \arg \min \{\kappa \mid \hat{R}(\kappa) = \hat{R}(\kappa)\}\), then the following price vector \(\bar{p} \in \hat{P}_M(\kappa^*)\) is an optimal pricing strategy:

\[
\bar{p}_t = \min \left\{ \bar{v}(t); \frac{\delta^{-(t-1)\kappa^*}}{2} \left[ \sum_{r=1}^{\kappa^*-1} \delta^{(r-1)\kappa^*} R^* + \mathbb{I}(\kappa^* \leq N) \frac{\delta^{\kappa^*\kappa^* - 1}}{2} \bar{v} \right] \right\}, \quad t = 1, \ldots, N.
\]

Furthermore, if \(p^*\) is also an optimal price vector, then \(p^*(t) \leq \bar{p}(t)\), for all \(t = 1, \ldots, N\).

Let us now consider the case with strategic customers, the uncertainty set from which is denoted as \(\hat{D}'\). Suppose that \(D_0 \subset \hat{D}' \subset D\). This condition is reasonable in many practical settings, specially in those in which consumers’ valuations tend to decrease over time.

**Proposition 11.** Suppose customers are strategic and nature is restricted to choose types from a set \(\hat{D}'\) such that \(D_0 \subset \hat{D}' \subset D\). Then, the optimal solution identified in Proposition 3 is still optimal.
It is worth noticing that this result generalizes to an arbitrary set $\hat{D}$. Intuitively, this result highlights a distinctive feature of the case with strategic customers, namely that an optimal price path is one that protects the seller against the event of a consumer arriving at the very beginning of the selling season, i.e., a customer with type in $\mathcal{D}_0$. Hence, as long as $\mathcal{D}_0 \subseteq \hat{D} \subseteq \mathcal{D}$, the solution in Proposition 5 is optimal.

### 7.2. Mix of Myopic and Strategic Consumers

Another important extension of our base model is the case in which the market is populated by both myopic and strategic consumers. So far, we have assumed that there is a single type of consumers and that the seller knows of which type they are. However, as the results in Su (2007) and Mersereau and Zhang (2012) suggest, the structure of an optimal pricing strategy might be sensitive to the proportions of myopic and strategic consumers in the marketplace. To capture this type of market heterogeneity, in this section we extend our robust formulation by allowing nature to select the mix of myopic and strategic consumers.

Let us consider first the case of a single consumer. In this new setting, the type of this consumer is a triplet $(v, \tau, \theta)$, where $v \in [v, \bar{v}]$ is his valuation, $\tau \in \{1, \ldots, N\}$ is his arrival time and $\theta \in \{M, S\}$ is his purchasing behavior, where $M$ stands for Myopic and $S$ for Strategic. We define the set of the consumer’s type to be $\mathcal{D}_\Theta = [v, \bar{v}] \times \{1, 2, \ldots, N\} \times \{M, S\}$. We also define the projections $\mathcal{D}_M \triangleq \mathcal{D}_\Theta \cap \{\theta = M\}$ and $\mathcal{D}_S \triangleq \mathcal{D}_\Theta \cap \{\theta = S\}$, corresponding to the cases when all consumers are myopic or strategic, respectively.

Depending on whether nature can select a consumer’s type from the set $\mathcal{D}_\Theta$, $\mathcal{D}_M$ or $\mathcal{D}_S$, the seller’s minimum regret is given by

$$R^*_j = \min_{p \geq 0} \max_{(v, \tau, \theta) \in \mathcal{D}_j} \delta^{\tau-1} v - \delta^{d(v, \tau, \theta, p)-1} d(p, v, \tau, \theta) \quad \text{for } j = \Theta, M, S,$$

where $d(v, \tau, \theta, p)$ is the purchasing time of a $(v, \tau, \theta)$-consumer that faces the price vector $p$. We also denote by $p^*_j$ an optimal price vector for $j = \Theta, M, S$.

According to Proposition 6 we know that $R^*_S \geq R^*_M$. Also, since $\mathcal{D}_S \subseteq \mathcal{D}_\Theta$, we also have that $R^*_\Theta \geq R^*_S$. As a result, we can rank the seller’s minimum regrets as follows $R^*_\Theta \geq R^*_S \geq R^*_M$. The next result shows that the first inequality is indeed an equality. That is, even if nature is able to select consumers’ types from $\mathcal{D}_\Theta$, the seller can achieve the same minimum regret as if the nature was restricted to select types from the set of strategic consumers $\mathcal{D}_S$.

**Proposition 12.** Suppose nature can select the buyer’s type from the set $\mathcal{D}_\Theta$. Then, an optimal pricing strategy for the seller is to select the vector $p^*_\Theta = p^*_S$. As a result, $R^*_\Theta = R^*_S$. 

Suppose now that instead of a single buyer, the market is populated by \( C \geq 1 \) consumers and nature is capable of selecting the proportion of myopic and strategic consumers. Let us denote by \( R^*_C \), the seller’s minimum regret.

**Proposition 13.** The regret \( R^*_C \) is linear in \( C \), i.e., \( R^*_N = C R^*_\Theta \), where \( R^*_\Theta \) is the optimal regret with a single customer. In addition, any optimal pricing strategy for the single customer case is also optimal for any \( C \in \mathbb{N} \). We conclude from Proposition 14 that \( p^*_S \) (an optimal price vector when there is a single strategic consumer in the market) is optimal.

In summary, the previous result reveals that if the seller is uncertain about the types of consumers in the market then her best strategy is to use the same pricing policy that she would use if she knew that all consumers were strategic.

### 8. Concluding Remarks

We studied in this paper the intertemporal pricing problem faced by a monopolist selling to myopic or strategic customers, where we assume the seller only knows the range of the customers’ valuations and has no information about customer arrival times. With myopic customers, we show that a range of policies is optimal, with some of the policies consisting of three distinct phases: a markup, a markdown and a clearance phase. With strategic customers, we define a property called \( \delta \)-weak convexity and use it to construct optimal policies, which typically involve less aggressive markdowns when compared to the optimal policies for myopic customers. Regardless of whether customers are myopic or strategic, we showed that when the valuations are high, or equivalently, when the demand uncertainty is small, it is optimal to use a “short” selling horizon. It is optimal for the firm to use up the entire sales season only when the demand uncertainty is large. We also show that the firm’s regret is always greater under strategic customers than under myopic ones.

There are many interesting questions that we leave open for future research. One such question is whether a randomized pricing policy could be used to reduce further the seller’s regret. Another interesting follow-up project would be incorporate inventory into our model. Inventory considerations are an important factor in practice, but, as Correa et al. (2012) show, they generally lead to a multiplicity of equilibria in settings with strategic customers. Extending our methodology to more complex uncertainty sets than intervals over the customers’ valuations would also be a valuable extension.

### References


Appendix

Proof of Proposition 1 Consider an arbitrary price path \( p \) from \( \{1, 2, ..., N\} \) to \([\bar{v}, \bar{v}]\) and let \( R(p) := \sup_{(v, \tau) \in \mathcal{D}} R(v, \tau, p) \) be the corresponding seller’s worst-case regret. In what follows, we show that there always exists a (weakly) decreasing price path \( \hat{p} \) such that \( R(\hat{p}) \leq R(p) \).

To do so, we let \( \hat{p} \) be the running minimum price path induced by \( p \) which is given by \( \hat{p} := \min \{p_s : s \in [0, t]\} \) for \( t \in [0, T] \). By construction, \( \hat{p} \) is weakly decreasing.

Let us first partition the customer type’s space \( \mathcal{D} \) into three subsets \( \mathcal{D}_1 := \{(v, \tau) \in \mathcal{D} : p_{\tau} = \hat{p}_\tau, v \geq p_{\tau}\}, \mathcal{D}_2 := \{(v, \tau) \in \mathcal{D} : p_{\tau} \neq \hat{p}_\tau, v \geq \hat{p}_\tau\} \) and \( \mathcal{D}_3 := \{(v, \tau) \in \mathcal{D} : v < \hat{p}_\tau\} \).

It is clear that for \( (v, \tau) \in \mathcal{D}_1 \), the customer will buy the product immediately after arrival, that is, \( d_M(v, \tau, p) = d_M(v, \tau, \hat{p}) = \tau \) and therefore \( R(v, \tau, p) = R(v, \tau, \hat{p}) \) for all \( (v, \tau) \in \mathcal{D}_1 \).

Similarly, for \( (v, \tau) \in \mathcal{D}_3 \), \( d_M(v, \tau, p) = d_M(v, \tau, \hat{p}) = \tau \). Also we have \( p_{d_M(v, \tau, p)} = \hat{p}_{d_M(v, \tau, \hat{p})} \) and therefore \( R(v, \tau, p) = R(v, \tau, \hat{p}) \) for all \( (v, \tau) \in \mathcal{D}_3 \).

For any \( (v, \tau) \in \mathcal{D}_2 \), by definition, there exists a period \( s < \tau \), such that \( p_s = \hat{p}_\tau \). Therefore, we have \( R(v, \tau, \hat{p}) = \delta^{\tau-1}(v - \hat{p}_\tau) < \delta^{s-1}(v - p_s) = R(v, s, p) \). This completes the proof that \( R(\hat{p}) \leq R(p) \).

\( \square \)

Proof of Proposition 2 Let us first consider the optimization problem \( R(\kappa) = \min_{p, R} R \)
subject to constraints (11)-(14) with \( 1 \leq \kappa \leq N \).

For a fixed value of \( R \), we define the sets \( \mathcal{P}_1(R) := \{(p_1, p_2, \ldots, p_r) \mid \delta^{t-1}(\bar{v} - p_t) \leq R\} \) and \( \mathcal{P}_2(R) := \{(p_1, p_2, \ldots, p_r) \mid p_\kappa = \bar{v} \text{ and } p_t - \delta^t p_{t+1} \leq R, t = 1, \ldots, \kappa - 1\} \). It follows that the feasible region for the \( R(\kappa) \) problem above is equal to \( \mathcal{P}_1(R) \cap \mathcal{P}_2(R) \cap \mathcal{P}_M(\kappa) \).

Let us momentarily drop the monotonicity constraint \( p \in \mathcal{P}_M(\kappa) \). For every value of \( R \), such that \( \mathcal{P}_1(R) \cap \mathcal{P}_2(R) \neq \emptyset \), we will find a feasible price vector in \( \mathcal{P}_1(R) \cap \mathcal{P}_2(R) \). First, it is not hard to see that \( \mathcal{P}_2(R) \) admits a maximal element \( \bar{p} = (\bar{p}_1, \ldots, \bar{p}_\kappa) \), that is, \( \bar{p} \geq p \) for all \( p \in \mathcal{P}_\kappa(R) \), where the inequality is component-wise. Indeed, define \( \bar{p} \) recursively as follows: \( \bar{p}_\kappa = \bar{v} \) and \( \bar{p}_t = R + \delta^t \bar{p}_{t+1} \) for \( t = \kappa - 1, \ldots, 2, 1 \). Solving this recursion, we get

\[
\bar{p}_t = s_t \bar{v} + u_t R \quad \text{where} \quad s_t := \frac{w_\kappa}{w_t} \quad \text{and} \quad u_t := \sum_{r=t}^{\kappa-1} \frac{w_r}{w_t}, \quad t = 1, 2, \ldots, \kappa.
\]

Note that \( u_\kappa = 0 \) and \( u_t = 1 + \delta^t u_{t+1} \), \( \forall t < \kappa \).

Our choice of the maximal element \( \bar{p} \) in \( \mathcal{P}_2(R) \) is motivated by the fact that by maximizing component-wise the price in every period \( t \), we are also reducing the left-hand-side of constraint (11) of the \( R(\kappa) \) problem. In other words, if \( \mathcal{P}_1(R) \cap \mathcal{P}_2(R) \neq \emptyset \) then \( \bar{p} \) defined above belongs to
\(\mathcal{P}_1(R) \cap \mathcal{P}_2(R)\). So, the problem of finding the minimum value of \(R\) for which \(\mathcal{P}_1(R) \cap \mathcal{P}_2(R) \neq \emptyset\) is equivalent to finding the minimum \(R\) for which all the constraints \(\delta^{t-1}(\bar{v} - \bar{p}_t) \leq R\) are satisfied for \(t = 1, \ldots, \kappa\). After some straightforward manipulations, this condition is equivalent to

\[
R \geq \frac{w_t \bar{v} - w_{\kappa} \bar{v}}{\sum_{r=t-1}^{\kappa-1} w_r}, \quad \text{for all } t = 1, \ldots, \kappa.
\]

At this point, we would like to set \(R(\kappa)\) as the maximum value of the right-hand side above over \(t = 1, \ldots, \kappa\). However, this value of \(R\) does not guarantee that the resulting price \(\bar{p}\) is in \(\mathcal{P}_M(\kappa)\). From the definition of \(\bar{p}\) above, one can show that \(\bar{p}_t\) is decreasing (weakly) in \(t\) if \(R \geq (1 - \delta^{\kappa-1}) \bar{v}\).

We conclude then that

\[
R(\kappa) = \max \left\{ \max_{1 \leq t \leq \kappa} \left\{ \frac{w_t \bar{v} - w_{\kappa} \bar{v}}{\sum_{r=t-1}^{\kappa-1} w_r} - (1 - \delta^{\kappa-1}) \bar{v} \right\} \right\}.
\]

To complete the proof, we need to derive the optimal regret \(R(N + 1)\). We can repeat the steps above replacing \(\bar{v}\) by \(R\). In this case, \(\bar{p}_t = (s_t + u_t) R\) which is a monotonically decreasing sequence for all \(R\). In other words, the condition \(\bar{p} \in \mathcal{P}_M(N + 1)\) is automatically satisfied for all \(R \geq \bar{v}\). Hence, in order to find the minimum value of \(R\) we only need to impose the constraints \(\delta^{t-1}(\bar{v} - \bar{p}_t) \leq R\) for all \(t = 1, \ldots, N\). After some simple manipulations we get

\[
R_N(N + 1) = \max \left\{ \max_{1 \leq t \leq N} \left\{ \frac{w_t \bar{v}}{\sum_{r=t-1}^{N-1} w_r} , \bar{v} \right\} \right\}. \quad \square
\]

**Proof of Corollary 1** First of all, the existence of \(t^*\) follows from the proof of Proposition 2 from which we can also conclude that the optimal prices are uniquely decided from period \(t^*\) to \(\kappa^*\). Also, by definition, \(p_t = \bar{v}, \forall \kappa^* < t \leq N\). Therefore, \(p_t = \bar{p}_t\), if \(t \in \{t^*, \ldots, N\}\). From condition (11), it is clear that \(p_t \geq \bar{v} - \delta^{t-t^*} R^* = p_{t^*}, \forall t \in \{1, 2, \ldots, t^*-1\}\). It is easy to verify that the vector \(\{\bar{p}_t\}\) is in the set \(\mathcal{P}_M(\kappa^*)\). To show that \(\bar{p}_t > p_{t^*}, \forall t < t^*\), notice that \(R^* > \frac{w_t \bar{v} - w_{\kappa} \bar{v}}{\sum_{r=t-1}^{\kappa-1} w_r} = \frac{\bar{v} - s_t \bar{v}}{u_t + \delta^{t-t^*}}, \forall t < t^*\). Thus, for any \(t < t^*\) we have that \(\bar{p}_t - p_{t^*} = s_t \bar{v} + u_t R^* - (\bar{v} - \delta^{t-t^*} R^*) > 0\). \(\square\)

**Lemma 4.** \(R_N^*(\bar{v})\) is non-increasing in \(\bar{v}\).

**Proof of Lemma 4** Denote \(D(\bar{v}, N) = [\bar{v}, \bar{v}] \times \{1, 2, \ldots, N\}\). Then from equation (4), the minimax regret is given by

\[
R_N^*(\bar{v}) := \min_{p \geq 0} \max_{(\bar{v}, \tau) \in D(\bar{v}, N)} R(v, \tau, p) = \delta^{\tau-1} \bar{v} - \delta^{d_M(\bar{v}, \tau, p) - 1} p_{d_M(\bar{v}, \tau, p)}.
\]
Let \( p^*(v) \in P_M \) denote an optimal price vector under which \( R^*_N(v) \) is attained. If \( v' > v \), then \( D(v', N) \subset D(v, N) \) and so \( \max \{ R(v, \tau, p^*(v)) | (v, \tau) \in D(v', N) \} \leq R^*_N(v) \). We conclude that \( R^*_N(v') \leq R^*_N(v) \). □

**Proof of Proposition 3** We denote by \( R_N(k, v) \), \( \forall k \in [1, N + 1] \), as the function of \( N, k \) and \( v \) to be the solution to \( \min_{p, R} R \) with constraints (11)-(14).

First of all, we prove that \( R^*_N(v) \) weakly decreases in \( N \) and \( \kappa^*_N(v) \) weakly increases in \( N \). Recall from Proposition 2 that \( R^*_N(v) = \min \{ R_N(k, v) | k = 1, \ldots, N + 1 \} \). By definition, \( R_N(k, v) = R_{N+1}(k, v) \) for \( k = 1, \ldots, N \). Hence, in order to prove that \( R^*_N(v) \geq R^*_{N+1}(v) \) it will suffice to show that \( R_N(N + 1, v) \geq R_{N+1}(N + 2, v) \). Furthermore, since \( R_N(N + 1, v) = \max \{ \max_{1 \leq t \leq N} \left\{ \frac{w_N v}{\sum_{r=N-1}^t w_r} \right\} - v \} \), it is sufficient to show the inequality \( \frac{w_N v}{\sum_{r=N-1}^t w_r} \geq \frac{w_{N+1} v}{\sum_{r=N}^t w_r} \) holds, which follows after some straightforward manipulations. Note that the previous argument also implies that \( \kappa^*_N \leq \kappa^*_N \).

We now prove part (a). First show if \( v = R^*_N(0) \), then \( R^*_N(v) = R^*_N(0) \). By Lemma 4 we know that \( R^*_N(v) \) is non-increasing in \( v \). Since \( R^*_N(0) = R_N(N + 1, v) \), it will be equivalent to show that, \( R^*_N(0) \leq R_N(k, v), \forall 1 \leq k \leq N \). Prove by contradiction.

Assume \( v = R^*_N(0) \), and \( R^*_N(0) > R_N(k, v) \) for some \( 1 \leq k \leq N \). Note that the following equations hold

\[
R^*_N(0) = \min \{ R | p_t - \delta^t p_{t+1} = 0, \forall 1 \leq t \leq N; \delta^{-1}(v - p_t) \leq 0, \forall 1 \leq t \leq N \}, \quad \text{and} \\
R_N(k, v) = \min \{ R | p_t - \delta^t p_{t+1} = 0, \forall 1 \leq t \leq k; \delta^{-1}(v - p_t) \leq 0, \forall 1 \leq t \leq k; (1 - \delta^{k-1})v \leq 0 \}
\]

Let \( p^* \in \mathbb{R}^N \) solve \( p_t - \delta^t p_{t+1} = R^*_N(0), \forall 1 \leq t \leq N \), where \( p^*_{N+1} := 0 \); let \( q^* \in \mathbb{R}^k \) solve \( q_t - \delta^t q_{t+1} = R_N(k, v), \forall 1 \leq t \leq k - 1 \), with \( q^*_{N+1} := v \), \( \forall t \in \{k, \ldots, N\} \).

Denote \( z_t := \delta^{-1} q^*_t, 1 \leq t \leq k \), then \( z_t = R_N(k, v) \cdot \sum_{r=1}^{k-1} w_r + z_k, 1 \leq t \leq k - 1 \). Similarly, denote \( y_t := w_t p^*_t, 1 \leq t \leq N \), then \( y_t = R^*_N(0) \cdot \sum_{r=1}^{N-1} w_r + y_N, 1 \leq t \leq N - 1 \).

By assumption, \( R^*_N(0) > R_N(k, v), R^*_N(0) = v \) and \( k \leq N \), we see that \( z_t < y_t \) so \( q^*_t < p^*_t, \forall 1 \leq t \leq k \) and \( p^*_t > R^*_N(0) = v = q^*_t, \forall k + 1 \leq t \leq N - 1 \), it is true that

\[
\max_{1 \leq t \leq N-1} \delta^{-1}(v - p^*_t) < \delta^{-1}(v - v) = \delta^{-1}(v - q^*_t).
\]

Also, \( \delta^{N-1}(v - p^*_N) \leq \delta^{-1}(v - q^*_N) \). Given

\[
R^*_N(0) = \max_{1 \leq t \leq N} \delta^{-1}(v - p^*_t) \quad \text{and} \quad R_N(k, v) = \max \left\{ (1 - \delta^{k-1})v, \max_{1 \leq t \leq k} \delta^{-1}(v - q^*_t) \right\}
\]

we can conclude that \( R_N(k, v) \geq R^*_N(0) \), which contradicts the assumption \( R_N(k, v) < R^*_N(0) \).
Therefore, if \( v = R_N^*(0) \), then \( R_N^*(v) = R_N^*(0) \). Since \( R_N^*(v) \) is non-increasing in \( v \), it is true that \( R_N^*(v) = R_N^*(0) \), for all \( v \in [0, R_N^*(0)] \).

Since \( R_N^*(0) \) strictly decreases in \( N \) and converges to \( R_N^*(0) \), then for all \( v \in [0, R_N^*(0)] \), we have \( R_N^*(v) = R_N^*(0) \), \( \kappa_N^*(v) = N + 1 \), and

\[
R_N^*(v) := \lim_{N \to \infty} R_N^*(v) = R_N^*(0) \quad \text{and} \quad \lim_{N \to \infty} \kappa_N^*(0) = \infty.
\]

Now we prove part (b) by contradiction. Let us now assume that the result in part (b) does not hold. Then there exists an increasing subsequence \( \{N_k \in \mathbb{N} \mid k = 1, 2, \ldots \} \) such that \( \kappa^*_{N_k}(v) \to \infty \) as \( k \) goes to infinity and \( R_{N_k}^*(v) \) is strictly decreasing in \( k \). Then, we have \( R_{N_k}^*(v) \leq R_{N_k}(N_k + 1) = \max \{ R_{N_k}^*(0), v \} \) and \( R_{N_k}^*(v) \geq \left( 1 - \delta_{N_k} \right)^v \). Since \( v > R_{N_k}^*(0) \), the upper bound of \( R_{N_k}^*(v) \) converges to \( v \), i.e.

\[
\lim_{k \to \infty} R_{N_k}^*(v) \leq \lim_{k \to \infty} \max \{ R_{N_k}^*(0), v \} = v.
\]

Similarly, the lower bound of \( R_{N_k}^*(v) \) above also converges to \( v \):

\[
\lim_{k \to \infty} R_{N_k}^*(v) \geq \lim_{k \to \infty} \left( 1 - \delta_{N_k} \right)^v = v
\]

which implies \( R_{N_k}^*(v) = v \). This leads to \( v = R_{N_k}^*(v) \leq R_{N_k}^*(0) \), which contradicts the fact that \( v > R_{N_k}^*(0) \). This completes the proof of Proposition 3. \( \square \)

**Lemma 5.** If the pricing policy \( p \) is \( \delta \)-convex, a customer’s utility is weakly unimodal in his purchasing time. That is, for all \( v \in [\bar{v}, \tilde{v}] \), there exists some \( \hat{t}(v, p) \in \{1, \ldots, N\} \) such that the customer’s utility \( \delta^{t-1}(v - p_t) \) is strictly increasing for \( t \leq \hat{t}(v, p) \) and weakly decreasing for \( t \geq \hat{t}(v, p) \). Furthermore, if the pricing policy \( p \) is \( \delta \)-convex, nature’s optimal arrival time is \( \tau = 1 \).

**Proof of Lemma 5.** Consider any \( v \in [\bar{v}, \tilde{v}] \). If \( \delta^{t-1}(v - p_t) \) is not a strictly increasing function in the entire domain \( t = 1, \ldots, N \), then there must exist some \( t' \) such that \( \delta^{t'-1}(v - p_{t'}) \geq \delta^{t'}(v - p_{t'+1}) \), which implies that \( (1 - \delta)v \geq p_{t'} - \delta p_{t'+1} \). Since the price vector \( p \) is \( \delta \)-convex, we obtain that \( (1 - \delta)v > p_{t'+1} - \delta p_{t'+2} \) and, thus, \( \delta^{t'}(v - p_{t'+1}) > \delta^{t'+1}(v - p_{t'+2}) \), proving the weak unimodality of the function \( \delta^{t-1}(v - p_t) \).

The unimodality of the customer’s utility function implies that, for any given arrival time \( \tau \), his chosen purchasing time will be \( \max \{ \hat{t}, \tau \} \), where \( \hat{t} \) is a shorthand for \( \hat{t}(v, p) \). If nature selects an arrival time \( \tau \in \{ \hat{t}, \ldots, N\} \), then the regret, as given by Eq. (4), will be equal to \( \delta^{\tau-1}(v - p_\tau) \), which is a weakly decreasing function of \( \tau \) by the weak unimodality of the customer utility function. Therefore, nature can restrict itself to arrival times in between 1 and \( \hat{t} \). With \( \tau \leq \hat{t} \), the firm’s regret will be equal to \( \delta^{\tau-1}v - \delta^{\tau-1}p_\tau \), which is a decreasing function of \( \tau \), implying that nature’s regret maximizing choice is \( \tau = 1 \). \( \square \)
Definition 4. (Definition of active period) Given a pricing policy \( p \), a period \( t \) is active if there exists some \( v \in (\underline{v}, \overline{v}) \) and some \( \epsilon > 0 \) such that a customer arriving in period 1 with valuation in \((v - \epsilon, v + \epsilon)\) will choose to buy in period \( t \).

Lemma 6. A pricing policy \( p \) is \( \delta \)-convex if, and only if, all periods up to \( t(p) \) are active and \( p_t = \bar{v} \) for all \( t \geq t(p) \).

Proof of Lemma 6: We first show that \( \delta \)-convexity of the prices implies the first min\{\( t(p), N \)\} are active. Consider an arbitrary \( \delta \)-convex pricing policy \( p \). Consider some period \( t < t(p) \) and some valuation \( v > \frac{p_t - \delta p_{t+1}}{1 - \delta} \). Such a customer will prefer to buy in period \( t \) than in period \( t + 1 \) as \( \delta^{-1}(v - p_t) > \delta(v - p_{t+1}) \). By the modularity of the customer’s utility function from Lemma 5, such a customer will also prefer to buy at period \( t \) than in any other period \( t' > t \). Similarly, if \( v < \frac{p_t - \delta p_{t+1}}{1 - \delta} \), the customer will prefer to buy at period \( t \) than in any period \( t'' < t \). Therefore, all periods leading up to min\{\( t(p), N \)\} are active. Period \( t = \min\{t(p), N\} \) itself is also active with \( \tilde{v}_t = p_t \).

We now prove the converse. Suppose \( p \) is not \( \delta \)-convex, that is, there exists \( t \) such that \( \tilde{v}_{t+1} > \tilde{v}_t \).

If \( v \geq \tilde{v}_t \), then \( p_t - \delta p_{t+1} \leq v - \delta v \) or \( \delta(v - p_{t+1}) \leq v - p_t \), implying that \( p_t \) is preferred by customer with valuation \( v \). If \( v < \tilde{v}_t \), then \( v < \tilde{v}_t < \tilde{v}_{t+1} \) and we have \( v - \delta v < p_{t+1} - \delta p_{t+2} \) or \( v - p_{t+1} < \delta(v - p_{t+2}) \), implying that \( p_{t+1} \) is less preferred. Therefore period \( t + 1 \) is not active. This completes the proof. □

Lemma 7. Consider the model where the strategic customer is restricted to arrive at \( \tau = 1 \). Then there exists an optimal pricing policy \( p^* \) which is \( \delta \)-convex.

Proof of Lemma 7: We first define two new notions: pre-active and semi-active periods. We recursively turn the periods from an arbitrary \( \delta \)-convex pricing policy \( p \) pre-active and, subsequently, semi-active. We label this new pricing policy \( p' \). We then show how to create a new policy \( p'' \) with all periods active up to \( t(p) \) from policy \( p' \). By showing that at each of these steps, the regret of the policy weakly decreases, we demonstrate that there is exists an optimal policy with all periods up to \( t(p) \) active and, by Lemma 6, the existence of an optimal \( \delta \)-convex policy.

A period \( t \) is pre-active if there exists some \( v \in (0, \tilde{v}) \) such that it will be optimal for a customer arriving in period 1 with valuation \( v \) to buy in period \( t \). Similarly, a period \( t \) is semi-active if there exists some \( v \in (0, \tilde{v}) \) and some \( \epsilon > 0 \) such that a customer arriving in period 1 with valuation in \((v - \epsilon, v + \epsilon)\) will choose to buy in period \( t \). The definition of a semi-active period is similar to the one of an active period, but it differs in that a range of customer valuations \((v - \epsilon, v + \epsilon)\) wanting
to purchase at period \( t \) qualifies that period as active only if \( v > \bar{v} \). The definition of a pre-active period is similar to the definition of a semi-active one, except it requires a single customer valuation \( v \) to find the price \( p_s \) optimal, instead of an interval of valuations.

Consider an arbitrary price vector \( p \). We recursively turn the periods of this policy into pre-active ones and, subsequently, into semi-active periods. Let \( s \) be the last non-pre-active period in policy \( p \) and let \( r \) be the last pre-active period before \( s \), where we use the convention \( r = 0 \) if there is no pre-active period before \( s \). Both periods \( r \) and \( s + 1 \) are pre-active (in case \( s = N \), we refer to \( N + 1 \) as if it was a pre-active period) and no periods in between are pre-active. Therefore, there must exist some threshold \( v(p_r, p_{s+1}) \) such that a customer with this valuation will be indifferent between buying in periods \( r \) and \( s + 1 \). Specifically, \( v(p_r, p_{s+1}) \) is determined by the equation \( \delta^r (v - p_r) = \delta^{s+1} (v - p_{s+1}) \), except in the cases where \( r = 0 \) and \( s = N \), where by convention we set \( v(p_0, p_{s+1}) = \bar{v} \) and \( v(p_r, p_{N+1}) = p_r \). Figure 8 shows the utility customers with different valuations earn from buying in periods \( r \), \( s \) and \( s + 1 \) respectively. In particular, the leftmost panel shows the case where period \( s \) is not pre-active and, therefore, the price offered at that period is too high compared to the prices offered in periods \( r \) and \( s + 1 \) for all customer valuations. We can, without loss of optimality, lower price \( p_s \) until period \( s \) becomes pre-active, as shown in the center panel. We call this price that makes period \( s \) pre-active the critical price \( p_s^C \). We now show that we can further lower price \( p_s \) below \( p_s^C \) and make period \( s \) semi-active. This situation is portrayed in the rightmost panel of Figure 8, where the price \( p_s \) has been lowered sufficiently so that its respective line has entered the upper envelope (thick line in each plot in Figure 8) and is optimal for a region of customer valuations. We do so by lowering \( p_s \) to \( p_s^C - \epsilon \) for a small positive \( \epsilon \). With \( \epsilon \) small enough, only the regret associated with a customer purchasing in periods \( s \) and \( s + 1 \) will be changed. Under price \( p_s = p_s^C - \epsilon \), the maximum regret associated with a purchase in period \( s + 1 \) is \( v(p_s^C - \epsilon, p_{s+1}) - \delta^s p_{s+1} \) and is less than the regret before we decrease the price in period \( s \), \( v(p_r, p_{s+1}) - \delta^s p_{s+1} \), as can be verified from the definition of \( v(\cdot, p_{s+1}) \).

Also note that, under \( p_s = p_s^C - \epsilon \), period \( s \) becomes newly semi-active, and the maximum regret associated with a purchase in that period is equal to \( v(p_r, p_s^C - \epsilon) - \delta^{s-1} \cdot (p_s^C - \epsilon) \). Let \( \epsilon_1 \) be the value that solves \( v(p_r, p_s^C - \epsilon) - \delta^{s-1} \cdot (p_s^C - \epsilon) = v(p_r, p_{s+1}) - \delta^s p_{s+1} \). Since \( p_s^C > p_{s+1} \), one can conclude that for any \( \epsilon < \epsilon_1 \), the maximum regret from the new semi-active period \( s \) is reduced under \( p_s^C - \epsilon \).

Next, we construct a policy \( p'' \) where all periods up to \( t(p'') \) are active by letting \( p''_k = \max(p'_k, v) \). The following argument shows that the worst-case regret is weakly lower under \( p'' \) than under \( p' \). Under policy \( p' \), if a customer’s valuation is in \([v, \bar{v}]\), then firm’s worst-case regret is given by:

\[
\max_{v \in [\bar{v}, v]} R(v, p') = \max_{k \in \{0, 1, \ldots, N\}} \{(\tilde{v}_k(p') - \delta^k p_{k+1}) \cdot \mathbb{I}\{\tilde{v}_k(p') > v\}\},
\]
where the indicator function represents the fact that the customer valuation is always above $\bar{v}$. The following three observations about the regret of policies $p'$ and $p''$ hold: (i) the regret associated with a purchase in period $k < t(p')$ is weakly lower under the new policy, i.e., $\bar{v}_k(p'') - \delta^k p''_{k+1} \leq \bar{v}_k(p') - \delta^k p_{k+1}$; (ii) the regret associated with a purchase in period $t(p')$ is also lower under $p''$, i.e., $\bar{v}_{t(p')-1}(p'') - \delta^{t(p')-1} p''_{t(p')-1} \leq \bar{v}_{t-1}(p') - \delta^{t-1} p_{t-1}$; (iii) there is no regret associated with a purchase in the non-active periods $k > t(p')$. In conclusion, $\max_{v \in [\bar{v}, \bar{v}]} R(v, p'') \leq \max_{v \in [\bar{v}, \bar{v}]} R(v, p')$, completing our proof of Lemma 7. □

**Proof of Proposition 4** Let $p^*$ denote the $\delta$-convex optimal price to the model where the strategic customer is restricted to arrive at $t = 1$. Then firm’s optimal regret $\min_p \max_{(v, \tau)} R(v, \tau, p)$ is bounded above by $\max_{(v, \tau)} R(v, \tau, p^*)$. From Lemma 5, we know if the pricing policy $p$ is $\delta$-convex, then nature’s optimal arrival time is $\tau = 1$, and thus $\max_{(v, \tau)} R(v, \tau, p^*) = \max_{(v, 1)} R(v, \tau, p^*)$.

On the other hand, firm’s optimal regret $\min_p \max_{(v, \tau)} R(v, \tau, p)$ is bounded below by $\min_p \max_{(v, 1)} R(v, \tau, p)$, which is also equal to $\max_{(v, 1)} R(v, \tau, p^*)$. Therefore, $\min_p \max_{(v, \tau)} R(v, \tau, p) = \max_{(v, \tau)} R(v, \tau, p^*)$, implying that the $\delta$-convex price $p^*$ is the optimal price vector for the firm. This completes the proof. □

**Proof of Proposition 5** We solve the firm’s problem by solving the $N + 1$ problems described by equations (20)-(24) and selecting the one with lowest regret. For any given $\kappa \in \{1, \ldots, N + 1\}$, we consider the relaxation of this optimization problem where we ignore the constraint $p \in \mathcal{P}_C(\kappa)$. Without the constraint given by Eq. (24), each of the $N + 1$ problems becomes a linear program where it’s easy to observe that all constraints are binding in optimality (see the proof of Proposition
and the optimal solution value of its respective subproblem is $\tilde{\delta}_1 = \max \{ \delta_1, v \}$, which is true whenever $R_v$ prove this, we observe that the threshold valuations are given by $\tilde{\delta}_t < \kappa$ all $R_v$ the solution is $\kappa$ and, for $\kappa = N + 1$, the linear equations are

$$
\begin{align*}
\tilde{v} - p_1 &= R \\
\frac{p_{t-1} - \delta p_t}{1 - \delta} - \delta^{t-1} p_t &= R & \forall t \in \{2, \ldots, \kappa\} \\
p_\kappa &= \bar{v}.
\end{align*}
$$

and, for $\kappa = N + 1$, the linear equations are

$$
\begin{align*}
\tilde{v} - p_1 &= R \\
\frac{p_{t-1} - \delta p_t}{1 - \delta} - \delta^{t-1} p_t &= R & \forall t \in \{2, \ldots, N\} \\
p_N &= R.
\end{align*}
$$

We can thus explicitly compute the optimal solution of each of the relaxed problems. For $\kappa \leq N$, the solution is $R_v^*(\kappa) = \frac{\tilde{v} - a_{0,\kappa-1} \bar{v}}{1 + (1 - \delta) \sum_{i=0}^{\kappa-1} a_{0,i}}$ and $p^*_t(\kappa) = \bar{v} \cdot a_{t,\kappa-1} + (1 - \delta) R_v^*(\kappa) \left( \sum_{i=t}^{\kappa-2} a_{t,i} + 1 \right)$ for all $t < \kappa$. For $\kappa = N + 1$, the solution is $R_v^*(N + 1) = \frac{\tilde{v}}{1+\sum_{i=0}^{N-1} a_{0,i}}$, and $p^*_t(N + 1) = \tilde{R}_v^*(N + 1) \left[ a_{t,N-1} + (1 - \delta) \left( \sum_{i=t}^{N-2} a_{t,i} + 1 \right) \right]$ for all $t < N$ and $p^*_t(N + 1) = R_v^*(N + 1)$.

For the case $\kappa \leq N$, the prices $p^*_t(\kappa)$ satisfy $\delta$-convexity if, and only if, $R_v^*(\kappa) > (1 - \delta^{\kappa-1}) \bar{v}$. To prove this, we observe that the threshold valuations are given by $\tilde{v}_t(p^*_t(\kappa)) = R_v^*(\kappa) + \delta^t p^*_t+1(\kappa)$ – see Eqs. (17) and (22) when binding – and are decreasing whenever the prices $p^*_t(\kappa)$ are decreasing, which is true whenever $R_v^*(\kappa) > (1 - \delta^{\kappa-1}) \bar{v}$. By Lemma 6, decreasing threshold valuations is a characterization of $\delta$-convexity. Therefore, the optimal solution value of the $\kappa^{th}$ subproblem is $\tilde{R}(\kappa) = \max \{ R_v^*(\kappa), \bar{v} (1 - \delta^{\kappa-1}) \}$. Similarly, for $\kappa = N + 1$, the prices $p^*(N + 1)$ are $\delta$-convex if, and only if, $R_v^*(N + 1) > \bar{v}$ and the optimal solution value of its respective subproblem is $\tilde{R}(N + 1) = \max \{ R_v^*(N + 1), \bar{v} \}$. Hence, the optimal regret satisfies $\tilde{R}^* = \min_{1 \leq \kappa \leq N + 1} \max \{ R_v^*(\kappa), \bar{v} (1 - \delta^{\kappa-1}) \} \{ \kappa \leq N \}$. □

**Proof of Proposition 6.** Let $p$ be optimal price vector for selling to strategic customers, as given by Proposition 5. We prove the result by showing that the regret obtained from using the price vector $p$ for selling to myopic customers is weakly lower than the regret obtained from using the same price vector for selling to strategic customers.

Recall from Lemma 1 that, since the price vector $p$ belongs to $P_C \subset P_M$, the regret for selling to myopic customers is given by

$$
\max_{(v, \tau)} R(v, \tau, p) = \max \left\{ \max_{t \in \{1, 2, \ldots, N\}} \{ \delta^{t-1} (\bar{v} - p_t) \}, \max_{t \in \{2, \ldots, \bar{t}(p)\}} - \delta^{t-1} p_t \right\}.
$$
Let \( \kappa^* \) be as defined in Proposition 5. Since Eq. (22) is valid for all \( t \in \{2, \ldots, \kappa^*\} \),

\[
\tilde{R}^*_N \geq \frac{p_{t-1} - \delta p_t}{1 - \delta} - \delta^{t-1} p_t \geq \frac{p_{t-1} - \delta p_{t-1}}{1 - \delta} - \delta^{t-1} p_t = p_{t-1} - \delta^{t-1} p_t,
\]

where the inequality follows from the monotonicity of the price vector \( p \). Thus, the regret from selling to strategic customers is always weakly higher than the last maximization term on the regret from selling to myopic customers.

To complete the proof, we need to show that \( \tilde{R}^*_N \geq \delta^{t-1}(\bar{v} - p_t) \) for all \( t \). For \( t = 1 \), the desired result follows from the fact that Eq. (21) holds. Since \( p \) is \( \delta \)-convex, \( p_1 - \delta p_2 < \bar{v}(1 - \delta) \) and, therefore, \( \tilde{R}^*_N \geq \bar{v} - p_1 > \delta(\bar{v} - p_2) \). Notice that \( \delta^{t-1}(\bar{v} - p_t) \) is the utility of a customer with valuation \( \bar{v} \) buying at time \( t \). By Lemma 5, this utility is unimodal in \( t \). Therefore, \( \tilde{R}^*_N \geq \delta^{t-1}(\bar{v} - p_t) \) for all \( t \).

\[ \Box \]

**Proof of Proposition 7**

a) The value \( \tilde{R}^*_N \) is described by a minimization over \( N + 1 \) numbers in Proposition 5. The first \( N \) of these numbers are also included in the minimization that describes \( \tilde{R}^*_{N+1} \). Therefore, if \( \tau^*_N \leq N \), then the result follows. We now show that if \( \tau^*_N = N + 1 \), then the \((N + 1)\)st term in the minimization of \( \tilde{R}^*_N \) is bigger than or equal to the \((N + 2)\)nd term in the minimization of \( \tilde{R}^*_{N+1} \), i.e.,

\[
\frac{\bar{v}}{1 + (1 - \delta) \sum_{i=0}^{N-1} a_{0,i} + (a_{0,N-1} - (1 - \delta)a_{0,N-1})} \geq \frac{\bar{v}}{1 + (1 - \delta) \sum_{i=0}^{N} a_{0,i} + (a_{0,N} - (1 - \delta)a_{0,N})}.
\]

By canceling terms and using the identity \( a_{0,N-1} = a_{0,N-1} \), the equation above is equal to \( a_{0,N} \geq \delta a_{0,N-1} \), which is true by the definition of \( a_{i,j} \).

b) Proposition 5 gives us that \( \tilde{R}^*_N = \min\{\min_{1 \leq \kappa \leq N} G(\kappa), H(N + 1)\} \) for some functions \( G \) and \( H \). IF \( \bar{v} \) were equal to zero, prices would never reach \( \bar{v} \) and the minimizer of the set above would be the final, or \( H \), term. Therefore, \( H(N + 1) = \tilde{R}^*_N(0) \) where the zero represents the modified \( \bar{v} = 0 \). Now consider any \( \bar{v} \in [0, \tilde{R}^*_N(0)] \). We show that for any \( \bar{v} \) in this range, the minimum of \( \tilde{R}^*_N \) is still equal to \( \tilde{R}^*_N(0) \). Note from the structural form of \( \tilde{R}^*_N \) that it is a weakly decreasing function of \( \bar{v} \). Therefore, to prove our claim, it’s sufficient to show it for \( \bar{v} = \tilde{R}^*_N(0) \), i.e.,

\[
R^*_N(0) = \frac{\bar{v}}{1 + (1 - \delta) \sum_{i=0}^{N-2} a_{0,i} + a_{0,N-1}} \leq \min_{1 \leq \kappa \leq N} \frac{\bar{v}}{1 + (1 - \delta) \sum_{i=0}^{\kappa-2} a_{0,i}}.
\]

The equation above can be rearranged as

\[
\frac{\bar{v}}{1 + (1 - \delta) \sum_{i=0}^{N-2} a_{0,i} + a_{0,N-1}} \leq \min_{1 \leq \kappa \leq N} \frac{\bar{v}}{1 + (1 - \delta) \sum_{i=0}^{\kappa-2} a_{0,i}},
\]
which is a true statement by the definition of $a_{ij}$. The last part of the statement follows from taking the limit of $R^*_n(0)$ as $n$ goes to infinity.

c) We skip the proof of the last part to avoid repetition since it’s virtually identical to the proof of part (c) of Proposition 3. □

The following lemma verifies two properties of the sequence $\{u_t\}$.

**Lemma 8.** For any $t \in \{1, 2, ..., \kappa\}$, (i) $u_t$ strictly decreases in $t$, and (ii) $u_t < \frac{1}{1-\delta t}$.

**Proof of Lemma 8** Show part (i) by induction. Note that $u_{\kappa-1} = 1 > u_\kappa = 0$. Suppose $u_{t+1} > u_{t+2}$, then we have $u_t - u_{t+1} = \delta^t u_{t+1} - \delta^{t+1} u_{t+2} \geq \delta^t u_{t+1} - \delta^t u_{t+2} > 0$. Now prove part (ii) also by induction. Obviously, $u_\kappa = 0 < \frac{1}{1-\delta t}$. Assume $u_{t+1} < \frac{1}{1-\delta t}$, then $u_t = 1 + \delta^t u_{t+1} < 1 + \frac{\delta^t}{1-\delta t} < \frac{1}{1-\delta t}$ which completes the proof. □

**Proof of Proposition 8** The proof is divided in a number of steps.

(1) The following is to show $R^*_n$ converges when $n$ goes to infinity. Given $\hat{p}^n \in \mathcal{T}_n$, let us consider a price vector $\hat{p}^{n+1} \in \mathcal{T}_{n+1}$, where $\hat{p}_1^{n+1} = \hat{v}$ and $\hat{p}_{2k}^{n+1} = \hat{p}_{2k+1}^n$, $\forall k = 1, 2, ..., 2^n$. Define $R_n(p) := \max_{(v, \tau) \in [\hat{v}; \hat{v}] \times \mathcal{T}_n} R(v, \tau, p)$. It can be easily verified that $R^*_n = R_n(\hat{p}_n) \geq R_{n+1}(\hat{p}_{n+1}) \geq R^*_{n+1}$. Since $R^*_n \geq 0, \forall n \in \mathbb{N}$, we can conclude that $R^*_n$ converges when $n \to \infty$, and denote $R^*_\infty = \lim_{n \to \infty} R^*_n$.

(2) Next, we want to find out the closed-form of $R^*_\infty$.

Note that for any $\tau' \in \mathcal{T}_\infty$, there exists $n'$ such that $\tau' \in \mathcal{T}_n$ for all $n \geq n'$. For some $n \geq n'$, define $R_n(\tau')$ as follows

$$R_n(\tau') = \max \left\{ \max_{1 \leq k \leq \frac{\tau'}{\Delta_n}} \left\{ \frac{\hat{v} - s_k \hat{v}}{u_k + \delta^{1-k}_n} \right\}, \left(1 - \delta^\tau_n\right) \hat{v} \right\} ,$$

which is the minimum worst-case regret the firm can get if she is restricted to decrease the price to $\hat{v}$ at time $\tau'$.

Denote $\delta_n = e^{-r \Delta_n}$, and for fixed $t \in \mathcal{T}_n$ such that $t < \tau'$, denote $i_n = \frac{t}{\Delta_n}$. The proofs in (2-1) and (2-2) is to find the limit of $s_k$’s and $u_k$’s at time $t$ given the price first hit $\hat{v}$ at time $\tau'$.

(2-1) Since for any $k$, we have $s_k = \delta^k_n s_{k+1}$, or $\ln s_k - \ln s_{k+1} = -k \Delta_n$, one can see that

$$\ln s_{i_n} = \ln s^\tau_{\Delta_n} - \sum_{k=i_n}^{\frac{\tau'}{\Delta_n}-1} rk \Delta_n = -r \left( \frac{\tau'}{2} - t \right) \left( \frac{\tau' + t}{\Delta_n} - 1 \right)$$

goesto $-\infty$, so $s_{i_n} \to 0$ as $n$ goes to infinity.

(2-2) It remains to prove $u_{i_n}$ also converges. Consider $u^n \in \mathcal{T}_n$ and $u^{n+1} \in \mathcal{T}_{n+1}$, where $u^n_k = 0$ for $k = \frac{\tau'}{\Delta_n}$, and $u^{n+1}_k = 0$ for $k = \frac{\tau'}{\Delta_{n+1}}$. 
First, we want to prove that $u_{i_n} < u_{i_{n+1}}$ by showing $u_k^n < u_{2k+1}^{n+1}$, $\forall k < \frac{r'}{\Delta n}$.

For $k = \frac{r'}{\Delta n} - 1$, we have $u_k^n = 1$; by Lemma 8, $u_{2k+1}^{n+1} > u_{2k+2}^{n+1}$, where $u_{2k+1}^{n+1} = 1$, so $u_{2k+1}^{n+1} > u_k^n$, if $k = \frac{r'}{\Delta n} - 1$. Assume $u_{2k+2}^{n+1} > u_{k+1}^n$, for some $k < \frac{r'}{\Delta n} - 1$, then

$$u_{2k+1}^{n+1} = 1 + \delta_{n+1}^{2k} u_{2k+1}^{n+1} > 1 + \delta_{n+1}^{2k} u_{2k+2}^{n+1} > 1 + \delta_{n+1}^{2k} u_{k+1}^n = 1 + \delta_{n+1}^k u_{k+1}^n = u_k^n$$

from which we can conclude that $u_k^n < u_{2k+1}^{n+1}$, $\forall k < \frac{r'}{\Delta n}$. Therefore $u_{i_n}$ strictly increases in $n$, and from Lemma 8, $u_{i_n}$ is bounded above by $\frac{1}{1-e^{-rt}}$ for all $n$.

The following is to show that $\lim_{n \to \infty} u_{i_n} = \frac{1}{1-e^{-rt}}$.

Let us first consider the value of $u_{i_n} - u_{i_{n+1}}$ for some $t < \frac{r'}{\Delta n} - 1$. By definition of $u_{i_n}$, we have

$$u_{i_n} - u_{i_{n+1}} = \sum_{k=i_n}^{\frac{r'}{\Delta n} - 1} \frac{k(k-1)}{\delta_n^2} \left(1 - \delta_n^{k-i_n} \right) + \frac{\delta_n^{\frac{r'}{\Delta n} - 1} - \delta_n^{\frac{r'}{\Delta n} - 2}}{\delta_n^{i_n(i_n-1)}}$$

After some algebraic manipulation, one can show that

$$u_{i_n} - u_{i_{n+1}} = \sum_{k=1}^{\frac{r'}{\Delta n} - 1} \frac{k^2}{\delta_n^2} \left(1 - \delta_n^{k-1} \right) + \frac{\delta_n^{\frac{r'}{\Delta n} - 1} - \delta_n^{\frac{r'}{\Delta n} - 2}}{\delta_n^{i_n(i_n-1)}}$$

With a slight abuse of notation, let $u_t$ denote the value of $u_{i_n}$ at calendar time $t$, since $\delta_n = e^{-rt}$ and $i_n = \frac{r'}{\Delta n}$. Then (27) becomes

$$u_{i_n} - u_{i_{n+1}} = u_t - u_{i_n} + \Delta_n = \sum_{k=1}^{\frac{r'}{\Delta n} - 1} \frac{k^2}{\delta_n^2} \left(1 - \delta_n^k \right) + \frac{\delta_n^{\frac{r'}{\Delta n} - 1} - \delta_n^{\frac{r'}{\Delta n} - 2}}{\delta_n^{i_n(i_n-1)}}$$

$$= \sum_{k=1}^{\frac{r'}{\Delta n} - 1} e^{-rk(\Delta_n + 2t + \Delta_n)} \left(1 - e^{-rkt} \right) + \frac{\delta_n^{\frac{r'}{\Delta n} - 1} - \delta_n^{\frac{r'}{\Delta n} - 2}}{\delta_n^{i_n(i_n-1)}}$$

(28)

As $\Delta_n$ goes to 0, $\frac{\delta_n^{\frac{r'}{\Delta n} - 1} - \delta_n^{\frac{r'}{\Delta n} - 2}}{\delta_n^{i_n(i_n-1)}}$ will converge to 0. Let $k^* := \frac{\alpha \ln \Delta_n}{rt}$ for some $\alpha > 1$, then (28) implies

$$u_t - u_{i_n} + \Delta_n < \sum_{k=1}^{k^*} \left(1 - e^{-rkt} \right) + \sum_{k=k^*+1}^{\frac{r'}{\Delta n} - 1} e^{-rk(\Delta_n + 2t + \Delta_n)}$$
Since $1 - e^{-rk\Delta_n} \leq rk\Delta_n$ and for any $k > k^*$, we have $e^{\frac{-rk(k\Delta_n+2t+\Delta_n)}{2t}} < e^{-rkt} < \Delta_n^\alpha$, then

$$u_t - u_{t+\Delta_n} < \sum_{k=1}^{k^*} rk\Delta_n + \sum_{k=k^*+1}^{x_n} \Delta_n^\alpha < \frac{1}{2} \Delta_n^\alpha \cdot \frac{\alpha \ln \Delta_n}{rt} \left( \frac{-\alpha \ln \Delta_n}{rt} + 1 \right) + \Delta_n^\alpha \left( \frac{\tau'}{\Delta_n} - t - 2 \right).$$

By L'Hôpital's rule, one can easily verify that $\lim_{\Delta_n \to 0} \Delta_n (\ln \Delta_n)^2 = 0$. Also, since $\alpha > 1$, we see that $\lim_{\Delta_n \to 0} \Delta_n^\alpha \cdot \left( \frac{\tau'-t}{\Delta_n} - 2 \right) = 0$, and therefore $\lim_{\Delta_n \to 0} u_t - u_{t+\Delta_n} \leq 0$. By Lemma 8, we have $u_t > u_{t+\Delta_n}$, which leads to $\lim_{\Delta_n \to 0} u_t - u_{t+\Delta_n} = 0$. From the recursion $u_t = 1 + e^{-rt}u_{t+1}$, we can conclude that as $\Delta_n \to 0$, $\lim_{\Delta_n \to 0} u_t = \frac{1}{1-e^{-rt}}$ for any $t < \tau'$.

Notice that by the definition of $u_{\Delta_n}$ in the discrete model, one should expect $u_{\Delta_n} = 0$, or $u_\cdot = 0$, which introduces the issue of discontinuity of $u_t$ at $t = \tau'$ in our continuous-time approximation. Actually, by the continuity of $\frac{1}{1-e^{-rt}}$ one can see that $\lim_{t \to \tau'} u_t = \frac{1}{1-e^{-rt}}$.

**2-3** Now we want to find out the continuous-time approximation of $R_n(\tau')$. Denote $X_k := \frac{v - s_k u_k}{u_k + \delta_n^k}$. We rewrite $R_n(\tau')$ as follows:

$$R_n(\tau') = \max \left\{ \max_{1 \leq k \leq \lfloor \tau'/\Delta_n \rfloor} \left\{ \frac{v - s_k u_k}{u_k + \delta_n^k} \right\}, \max_{\lfloor \tau'/\Delta_n \rfloor < k \leq \tau'} \left\{ \frac{v - s_k u_k}{u_k + \delta_n^k} \right\}, (1 - \delta_n^{\tau'/\Delta_n} - 1) \cdot \right\}. $$

From part (2-1) and (2-2), we know that for any given $\epsilon \in (0, \tau')$,

$$\lim_{n \to \infty} \max_{1 \leq k \leq \lfloor \tau'/\Delta_n \rfloor} \left\{ \frac{v - s_k u_k}{u_k + \delta_n^k} \right\} = \max_{0 \leq t \leq \tau' - \epsilon} \frac{v}{1-e^{-rt}} + e^{-rt} = \max_{0 \leq t \leq \tau' - \epsilon} (1 - e^{-rt})e^{-rt} \tilde{v},$$

meaning that on the time interval $[0, \tau')$, we can derive the limit of $X_k$ by substituting $u_k$ and $s_k$ by their continuous counterparts. However, for $k \in N_{\tau',\epsilon}$, one needs to be more careful. The following is to show that $X_k$ is unimodal in $k$, for all $k$ in $N_{\tau',\epsilon}$. Then from there, we will obtain the upper bound of $\max_{\lfloor \tau'/\Delta_n \rfloor \leq k \leq \tau'} X_k$.

Without loss of generality, let $\tilde{v} = 1$. We then denote $N_{\tau',\epsilon} = \{k: \tau' - \epsilon \leq k\Delta_n \leq \tau'\}$, and $Y_k = \frac{\delta_n^{k-1}}{s_k}[\delta_n - (1 - \delta_n^k)u_k]$. The following three statements can be easily verified:

(i) $X_k > X_{k+1}$ if and only if $y > Y_k$;
(ii) $Y_k < Y_{k+1}$ if and only if $e^{-rt} < \frac{1}{2}$;
(iii) $X_{N-1} > X_N$ if and only if $e^{-r\tau'} < y$.

If $\epsilon$ is small enough, then in the limit, statement (ii) is equivalent to saying “$Y_k < Y_{k+1}$ if and only if $e^{-r\tau'} < \frac{1}{2}$”, for all $k$ in $N_{\tau',\epsilon}$.

Therefore, if $e^{-r\tau'} \leq \min\{y, \frac{1}{2}\}$, then $X_k$ decreases in $k$, and it is bounded above by $X_{\tau',\epsilon}$, which converges to $(1 - e^{-r(\tau'-\epsilon)})e^{-r(\tau'-\epsilon)}$, as $\Delta_n$ goes to 0.
On the other hand, if \( e^{-r\tau'} \geq \max\{v, \frac{1}{2}\} \), then \( X_k \) increases in \( k \), bounded above by \( X_{\frac{r}{\Delta_n}} \), which converges to \( e^{-r\tau'}(1 - v) \).

Then consider the case where \( v < \frac{1}{2} \), and \( e^{-r\tau'} \in (v, \frac{1}{2}) \). In this regime, \( Y_k \) strictly increases in \( k \) and \( \nu < Y_{N-1} \). Thus \( X_k \) is unimodal and quasi-convex in \( k \), and in limit, \( \max_{\nu \leq \frac{r}{\Delta_n}} X_k \) is bounded above by the maximum of \( (1 - e^{-r\tau'})e^{-r(\tau'-\epsilon)} \) and \( e^{-r\tau'}(1 - v) \).

If \( \nu > \frac{1}{2} \), and \( e^{-r\tau'} \in (\frac{1}{2}, v) \), then \( Y_k \) strictly decreases in \( k \) and \( \nu > Y_{N-1} \). Thus \( X_k \) is quasi-concave in \( k \), and there exists a \( k \) such that \( Y_{k-1} > \nu > Y_k \), implying \( X_k = \max_{k \in N_{\nu', \epsilon}} \{X_k\} \). Now by the definition of \( X_k \), one can see that \( X_k \) is bounded above by \( \frac{1 - s Y_k}{u_k + \delta_{n-k}^1} \). Then, by the definition of \( Y_k \), one can verify that \( \frac{1 - s Y_k}{u_k + \delta_{n-k}^1} = \delta_{n-k}^1(1 - \delta_{n-k}^1) \), which, as \( \Delta_n \) goes to 0, is less than the maximum of \( (1 - e^{-r(\tau'-\epsilon)})e^{-r(\tau'-\epsilon)} \) and \( (1 - e^{-r\tau'})e^{-r\tau'} \).

Then we can conclude that

\[
\lim_{n \to \infty} R_n(\tau') = \max \left\{ \max_{1 \leq k \leq \frac{r'}{\Delta_n}} \left\{ \frac{\bar{v}}{u_k + \delta_{n-k}^1} \right\}, v \right\} = \max \left\{ \max_{1 \leq k \leq \frac{r}{\Delta_n}} \left\{ \frac{\bar{v}}{u_k + \delta_{n-k}^1} \right\}, \max_{1 \leq k \leq \frac{r'}{\Delta_n}} \left\{ \frac{\bar{v}}{u_k + \delta_{n-k}^1} \right\}, v \right\}
\]

Denote \( Z_k := \bar{v}/(u_k + \delta_{n-k}^1) \). The following is to prove that for \( \epsilon \) small enough, and \( \Delta_n \ll \epsilon \), \( Z_k \) is bounded above by \( Z_{\frac{r'}{\Delta_n}} \) for all \( k \in N_{\nu', \epsilon} \). This is equivalent to showing that \( u_k + \delta_{n-k}^1 \geq \frac{1 + e^{-r\tau'} - \Delta_n}{\Delta_n} \) for all \( k \) is \( N_{\nu', \epsilon} \). Since \( u_k \) decreases in \( k \) and \( \delta_{n-k}^{1-k} \) increases in \( k \), we have \( u_k + \delta_{n-k}^{1-k} = u_{\frac{r'}{\Delta_n}-1} + \delta_{n-k}^{1-k} \) for all \( k \in N_{\nu', \epsilon} \). The right-hand side is equal to \( 1 + e^{-r\tau'} + \Delta_n + e^{r(\tau'-\epsilon)} - r\Delta_n \).

Since \( 1 + e^{-r\tau'} + \Delta_n + e^{r(\tau'-\epsilon)} - r\Delta_n \) goes to 0 as \( \epsilon \) goes to 0 and \( \Delta_n \ll \epsilon \), we can conclude that \( u_k + \delta_{n-k}^{1-k} \geq u_{\frac{r'}{\Delta_n}} + \delta_{n-k}^{1-k} = 1 + e^{-r\tau'} - \Delta_n \) for all \( k \in N_{\nu', \epsilon} \), which leads to \( Z_k \) is \( (1 - e^{-r\tau'})e^{-r\tau'} \bar{v} \).

On interval \([0, \tau')\), the limit of \( Z_k \) is \( (1 - e^{-r\tau'})e^{-r\tau'} \bar{v} \). Therefore, we have

\[
\lim_{n \to \infty} R_n(\tau') = \max \left\{ \max_{1 \leq k \leq \frac{r}{\Delta_n}} \left\{ \frac{\bar{v}}{u_k + \delta_{n-k}^1} \right\}, \frac{\bar{v}}{1 + e^{-r\tau'}}, v \right\}.
\]

(3) Hence \( R^*_\infty \) can be formulated as follows: \( R^*_\infty = \min \{ \min_{\tau \leq T} R(\tau), R(T^+) \} \), where

\[
R(\tau) = \max \left\{ \max_{1 \leq t \leq \tau} \left\{ (1 - e^{-t\tau})e^{-t\tau'} \bar{v}, e^{-r\tau'}(\bar{v} - v) \right\}, v \right\}, \forall \tau \leq T \quad \text{and} \quad R(T^+) = \max \left\{ \max_{1 \leq t \leq T} \left\{ (1 - e^{-t\tau})e^{-t\tau'}, \frac{\bar{v}}{1 + e^{r\tau'}} \right\}, v \right\}.
\]

We first discuss the solution of \( \min_{\tau \leq T} R(\tau) \) and then we derive the closed form of \( R^*_\infty \).
We will show that its solution (\(\tilde{\tau}\))

\[ R(\tau) = \begin{cases} e^{-\tau}(\tilde{\tau} - \tau), & \text{if } \tau \leq \frac{1}{r} \ln \left(\frac{\tilde{\tau}}{\tau}\right) \\ (1 - e^{-\tau})\tilde{\tau}, & \text{if } \tau > \frac{1}{r} \ln \left(\frac{\tilde{\tau}}{\tau}\right) \end{cases} \]

and thus \(\min_{\tau \leq T} R(\tau) = e^{-\tau^*}(\tilde{\tau} - \tau)\), with \(\tau^* = \min \left\{ T; \frac{1}{r} \ln \left(\frac{\tilde{\tau}}{\tau}\right) \right\} \).

If \(\tilde{\tau} > \frac{\tilde{\tau}}{2}\), then

\[ R(\tau) = \begin{cases} e^{-\tau}(\tilde{\tau} - \tau), & \text{if } \tau \leq \frac{1}{r} \ln \left(\frac{4(\tilde{\tau} - \tilde{\tau})}{\tau}\right) \\ \geq \frac{1}{4}, & \text{if } \tau > \frac{1}{r} \ln \left(\frac{4(\tilde{\tau} - \tilde{\tau})}{\tau}\right) \end{cases} \]

and we have \(\min_{\tau \leq T} R(\tau) = e^{-\tau^*}(\tilde{\tau} - \tau)\), where \(\tau^* = \min \left\{ T; \frac{1}{r} \ln \left(\frac{4(\tilde{\tau} - \tilde{\tau})}{\tau}\right) \right\} \).

Then after straightforward calculation of comparing \(\min_{\tau \leq T} R(\tau)\) and \(R(T^+)\), we can derive the optimal regret \(R^*_\infty\) and \(\kappa^*_\infty\) as follows

- If \(\tilde{\tau} \geq \frac{\tilde{\tau}}{2}\), then \(R^*_\infty = e^{-\tau^*_\infty}(\tilde{\tau} - \tau)\), where \(\kappa^*_\infty = \min \left\{ T; \frac{1}{r} \ln \left(\frac{\tilde{\tau}}{\tau}\right) \right\} \);
- If \(\frac{\tilde{\tau}}{4} \leq \tilde{\tau} \leq \frac{\tilde{\tau}}{2}\), then \(R^*_\infty = \max \left\{ \frac{1}{r} \ln \left(\frac{4(\tilde{\tau} - \tilde{\tau})}{\tau}\right); \frac{\tilde{\tau}}{4} \right\} \), where \(\kappa^*_\infty = \min \left\{ T; \frac{1}{r} \ln \left(\frac{4(\tilde{\tau} - \tilde{\tau})}{\tau}\right) \right\} \);
- If \(\tilde{\tau} \leq \frac{\tilde{\tau}}{4}\), then \(R^*_\infty = \max \left\{ \frac{\tilde{\tau}}{4}; \frac{\tilde{\tau}}{4} \right\} \), where \(\kappa^*_\infty = T\).

\[ \square \]

**Proof of Proposition 9:** Consider a fixed time time \(\tau \in T_\infty\). Note that for all \(n\) sufficiently large \(\tau \in T_n\).

Let us consider the optimization problem \((20)-(24)\) for period \(\kappa = \tau/\Delta_n\) (recall that \(\Delta_n = T/2^n\)). We will show that its solution \((\tilde{R}_n^*(\tau), \tilde{p}_n^*(\tau))\) converges to \((\tilde{R}_\infty^*(\tau), \tilde{p}_\infty^*(\tau))\) as \(n \to \infty\). First, to prove the convergence of the regret we show that the sequence \(\{\tilde{R}_n^*(\tau)\}\) is decreasing in \(n\).

Given a \(\delta\)-weak convex price \(p^n \in T_n\) that leads to \(\{\tilde{R}_n^*(\tau)\}\), let us consider a price vector \(p^{n+1} \in T_{n+1}\), where \(p_{1}^{n+1} = \tilde{\tau}\) and \(p_{2k}^{n+1} = p_{2k+1}^{n+1} = p_k^n\), \(\forall k = 1, 2, ..., 2^n\). If we consider the relaxed problem where \(\tau = 1\), then \(\max_{v, \tau = 1} R_{n+1}(v, 1, p^{n+1}) = \tilde{R}_n^*(\tau)\), since only Period 2k − 1 is active periods \(\forall k = 1, 2, ..., 2^n\).

By the same argument in proof of Lemma 7 one can find another \(\delta\)-weak convex price vector \(\tilde{p}_n^{n+1} \in T_{n+1}\), such that \(\max_{v, \tau = 1} R_{n+1}(v, 1, \tilde{p}_n^{n+1}) \leq \max_{v, \tau = 1} R_{n+1}(v, 1, p^{n+1}) = \tilde{R}_n^*(\tau)\). Using the same argument in proof of Proposition 4 one can show that \(\max_{v, \tau = 1} R_{n+1}(v, 1, \tilde{p}_n^{n+1}) = \tilde{R}_{n+1}^*(\tau)\). Thus \(\{\tilde{R}_n^*(\tau)\}\) is decreasing in \(n\).

In order to show the convergence of prices, we note that from equations \((22)-(23)\) the price \(\tilde{p}_n^*(\tau)\) satisfy the system of linear equations

\[ \begin{align*}
\sum_{\tau = 1} \sum_{v = 1} \frac{1}{r} &\ln \left(\frac{\tilde{\tau} - \tau}{\tau}\right) = 0 \\
\sum_{\tau = 1} \sum_{v = 1} \frac{1}{r} &\ln \left(\frac{\tilde{\tau} - \tau}{\tau}\right) = 0 \\
\sum_{\tau = 1} \sum_{v = 1} \frac{1}{r} &\ln \left(\frac{\tilde{\tau} - \tau}{\tau}\right) = 0
\end{align*} \]
\[
\frac{\tilde{p}_{t-1} - \delta_n \tilde{p}_t - \delta_n^{t-1} \tilde{p}_t}{1 - \delta_n} = \tilde{R}_n^*(\tau) \quad \forall t \in \{1, \ldots, \min\{\kappa, 2^n\}\}
\]
\[
\tilde{p}_\kappa = \bar{v} \quad \text{if } \kappa = 2^n \quad \text{or} \quad \tilde{p}_\kappa = \tilde{R}_n^*(\tau) \quad \text{if } \kappa = 2^n + 1
\]

Note that there are two border conditions depending on whether at optimality the price reaches \(\bar{v}\) or not. Let us suppose that \(\tilde{p}_\kappa = \bar{v}\). (The case in which \(p_\kappa = \tilde{R}_n^*(\tau)\) follows the same steps). We can rewrite the solution above as the following difference equation:

\[
\tilde{p}_t - \tilde{p}_{t-1} - \delta_n \tilde{p}_{t} = 1 - \delta_n \left[ \tilde{R}_n^*(\tau) - (1 - \delta_n^{t-1}) \tilde{p}_t \right],
\]

with border condition \(\tilde{p}_\kappa = \bar{v}\). Since \(\delta_n = \exp(-r \Delta_n)\), it follows that \((1 - \delta_n)/\Delta_n \to r\) as \(n \to \infty\).

Also, for an arbitrary time \(s \in T_{\infty}\) with \(s < \tau\) let us define \(t(s) = s/\Delta_n\). Then, the function

\[
f_n(s, p) = \tilde{R}_n^*(\tau) - (1 - \exp(-rs)) p
\]

converges pointwise to \(f(s, p) = \tilde{R}_\infty^*(\tau) - (1 - \exp(-rs)) p\), which is a Lipschitz continuous function. As a result, the difference equation above converges to the ordinary differential equation

\[
-\frac{d\tilde{p}_s}{ds} = r \left[ \tilde{R}_\infty^*(\tau) - (1 - \exp(-rs)) p_s \right], \quad \tilde{p}_\tau = \bar{v}.
\]

This is a first-order linear ODE with solution

\[
\tilde{p}(s) = A \exp(rs + e^{-rs}) - \tilde{R}_\infty^* \exp(rs), \quad s \leq \tau,
\]

where \(A\) is a constant of integration. Instead of using the border condition \(\tilde{p}_\tau = \bar{v}\) to determine \(A\), we use note that equation \([20]\) implies that at \(s = 0\) we have \(\bar{v} - \tilde{p}_0 = \tilde{R}_\infty^*(\tau)\). It follows that \(A = \bar{v} \exp(-1)\) and

\[
\tilde{p}(s) = \bar{v} \exp(rs + e^{-rs} - 1) - \tilde{R}_\infty^*(\tau) \exp(rs), \quad s \leq \tau.
\]

Using now the border condition \(\tilde{p}_\tau = \bar{v}\), we can compute the regret

\[
\tilde{R}_\infty^*(\tau) = \bar{v} \exp(e^{-r\tau} - 1) - \bar{v} \exp(-r\tau).
\]

Recall that in this derivation we assumed that the border condition in equation \([23]\) is given by \(\tilde{p}_\kappa = \bar{v}\) instead of \(p_\kappa = \tilde{R}_n^*(\tau)\). However, if we assume that the border condition is \(p_\kappa = \tilde{R}_n^*(\tau)\) the using the same steps we would get

\[
\tilde{R}_\infty^*(\tau) = \bar{v} \exp(e^{-r\tau} - 1) \frac{1}{1 + \exp(-r\tau)}.
\]
Since the our objective is to minimize the regret, we conclude that for a fixed $\tau \in \mathcal{T}_\infty$ the solution to the optimization problem (20)-(24) converges to

$$\tilde{R}_*^\infty(\tau) = \min \left\{ \bar{v} \exp(e^{-r\tau} - 1) - \bar{v} \exp(-r\tau), \frac{\bar{v} \exp(e^{-r\tau} - 1)}{1 + \exp(-r\tau)} \right\}.$$ 

Finally, since $\tau$ is an arbitrary time $t \in [0,T]$ (we are using here the fact that $\mathcal{T}_\infty$ is dense in $[0,T]$), the seller’s optimal regret is given by

$$\tilde{R}_\infty^* = \min_{0 \leq \tau \leq T} \{ \tilde{R}_\infty^*(\tau) \} \quad \text{and} \quad \tilde{\kappa}_\infty^* = \arg \min_{0 \leq \tau \leq T} \{ \tilde{R}_\infty^*(\tau) \}.$$ 

From here, it is not hard to derive the values of $(\tilde{R}_\infty^*, \tilde{\kappa}_\infty^*)$ for the three cases in the Proposition. □

**Proof of Proposition 12.** Because myopic consumers buy at soon as they can (i.e., as soon as the price follows below their valuation), it follows that for any price vector $p$ and any pair $(v,\tau)$

$$\delta^{d(v,\tau,M,p)-1} p_{d(v,\tau,M,p)} \geq \delta^{d(v,\tau,S,p)-1} p_{d(v,\tau,S,p)}.$$ 

That is, the seller, can always extract more rents from a myopic consumer than from a strategic one. Hence, for any $p$ and $(v,\tau)$ nature will select a strategic buyer $\theta = S$, and the result follows. □