Multiperiod Asset Allocation with Dynamic Volatilities

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Abstract
In this paper we extend and generalize the framework of Testing and Valuing Dynamic Correlations for Asset Allocation by focusing on a multistep version of the variance minimization criterion. We consider both a closed and an open loop strategy. It is shown both theoretically and empirically that this approach amplifies the benefits of using dynamic correlations (this has not yet been shown).

This draft: February 25, 2004. Preliminary and Incomplete.

1. INTRODUCTION
This paper is organized as follows. Section 2.1 presents the problem of an investor that at time zero constructs a complete plan of portfolio allocations for the following $T$ periods. This problem provides an intuitive solution, whose importance becomes even more striking in the next section. Section 2.2 describes the optimization plan of an investor that sequentially chooses portfolio weights for one period ahead, trying to minimize the total variance over a multiperiod horizon. The proposed solution highlights the optimizing behavior of the agent that uses the ratio between next period’s variance and total future variance over the period of interest, as a measure of the riskiness of investing tomorrow versus postponing the investment decision. Section 4 provides details to estimate the moments of the distribution of returns that are needed to characterize the investment plans of the problem described in the first two sections. The empirical analysis is offered in section 5. Section ?? comments on the results and concludes the paper.

2. ALLOCATION STRATEGIES
The problem of an investor deciding how to allocate her wealth into a bundle of stocks to minimize variance subject to a required return over a finite horizon can be thought in at least two ways. The first approach is to forecast at the initial time period the whole sequence of future covariance matrices and asset returns, to form a sequence of optimal decisions based on these forecasts and then to implement this strategy in any subsequent period. Another possibility would be to exploit all the information available before making the investment decision in any given period. This would require to update the expectations about future covariances and returns and to choose a one step ahead plan, taking into account that the information set will also be updated in any future period until the end of the investment horizon. In what follows, we will refer to the first strategy as the closed loop and to the second one as the open loop.

2.1 Closed loop strategy
In this section we study a multiperiod portfolio variance minimization, where the weights for all the periods are chosen at time zero. In any subsequent period, the strategy would just require to
rebalance the portfolio according to the initial plan. The problem can be formalized as follows:

\[
\min_{\{w_t\}_{t_0+1}^{T}} \frac{1}{2} \sum_{t=1}^{T} w_t' H_{t/t_0} w_t \\
\text{s.t.} \sum_{t=0+1}^{T} w_t' \mu_{t/t_0} \geq T \mu_0
\]

where \(w_t\) denotes portfolio weights for time \(t\), \(H_{t/t_0}\) is the covariance matrix of time \(t\) conditional on the information set of time zero, \(\mu_{t/t_0}\) is the vector of conditional expectations of returns and \(T \mu_0\) is the required return. In what follows we will assume that \(\mu_{t/t_0} = \mu, \forall t \in \{t_0 + 1, ..., t_0 + T\}\).

The solution to problem (1) is:

\[
w_t = T \mu_0 \frac{H_{t/t_0}^{-1} \mu}{\sum_{i=0+1}^{T} \mu' H_{i/t_0}^{-1} \mu}, \forall t = t_0 + 1, ..., t_0 + T
\]

The economic intuition is quite straightforward: a rational agent would invest a lot at time \(t\) if the forecasted variance is relatively low compared to the total variance expected over the whole sample.

Let’s call \(\Omega_t\) the true conditional covariance matrix. If we knew this matrix, then the vector of weights would be \(w^*_t = T \mu_0 \left(\Omega_{t/t_0}^{-1}\right) \left(\sum_{t=t_0+1}^{t_0+T} \mu' \Omega_{t/t_0}^{-1} \mu\right)^{-1}\). Therefore we have to conditional standard deviations of the portfolio that we can compare:

\[
\frac{V_{t_0}}{T \mu_0} = \sqrt{\frac{\sum_{t=t_0+1}^{t_0+T} w_t' \Omega_{t/t_0} w_t}{\left(\sum_{t=t_0+1}^{t_0+T} \mu' H_{t/t_0}^{-1} \mu\right)^{-2} \left(\sum_{t=t_0+1}^{t_0+T} \mu' H_{t/t_0}^{-1} \mu\right)}}
\]

\[
\frac{V^*_t}{T \mu_0} = \sqrt{\frac{1}{\left(\sum_{t=t_0+1}^{t_0+T} \mu' \Omega_{t/t_0}^{-1} \mu\right)}}
\]

It is very intuitive to conclude that \(V^*_t\) will always be smaller than \(V_{t_0}\). The following theorem formalizes the argument and it will be used as the starting point for the proposed testing strategy, that we discuss later.

**Theorem 1.** If \(\{H_{t/t_0}\}_{t_0+1}^{t_0+T}\) is the sequence of estimated conditional covariance matrices, \(\{\Omega_{t/t_0}\}_{t_0+1}^{t_0+T}\) is the sequence of true covariance matrices, \(\mu \neq 0\) is the vector of returns and \(V_{t_0}\) and \(V^*_t\) are defined as in (3) and (4), then \(V_{t_0} \geq V^*_t, \forall t \in \{H_{t/t_0}\}_{t_0+1}^{t_0+T}\).

**Proof.** See Appendix A.

### 2.2 Open loop strategy

In this section we focus on a more dynamic version of the problem discussed above. We allow the investor to choose portfolio weights for period \(t\) based on the information set at time \(t - 1\), instead of being forced to choose them all at once at time zero. However her objective function will still require to minimize a multiperiod criterion subject to a required return. This amounts to solve \(\forall t = t_0 + 1, ..., t_0 + T\) the following problem:

\[
\min_{w_t} \frac{1}{2} H_{t-1} \left[ \sum_{i=t}^{t_0+T} w_i' H_{i/t-1} w_i \right] \\
\text{s.t.} \sum_{i=t_0+1}^{t_0+T} w_i' \mu_t \geq T \mu_0
\]
where all the variables are defined as in the previous section and $H_{t_t-1}$ is the covariance matrix for time $t$ conditional on the information set of time $t-1$.

Problem (5) can be formulated recursively, using the content of the following theorem and its corollary.

**Theorem 2.** The solution to problem (5) is unique and proportional to the required return.

*Proof.* See Appendix A.

**Corollary 1.** The minimized variance is proportional to the square of the required return.

*Proof.* Follows directly from Theorem 2.

We can now formulate the problem at time zero and solve for $w_1$: 

$$ V = \min_{w_1} \frac{1}{2} w_1^T H_{t_0+1/t_0} w_1 + \frac{1}{2} E_{t_0}[V'] $$ 

$$ = \min_{w_1} \frac{1}{2} w_1^T H_{1/t_0} w_1 + \frac{1}{2} (T \mu_0 - \mu_1^T \mu_1)^2 \Psi_{t_0+1/t_0} $$

whose first order necessary condition implies: 

$$ w_1 = T \mu_0 \left( H_{t_0+1/t_0} + \mu \Psi_{t_0+1/t_0} \mu \right) \Psi_{t_0+1/t_0}^{-1} \left( \Psi_{t_0+1/t_0} \mu \right) \tag{6} $$

Similarly, for all $t = 0, 1, \ldots, t_0 + T - 1$:

$$ w_t = \Psi_{t/t-1} \left( H_{t/t-1} + \mu \Psi_{t/t-1} \mu \right) \Psi_{t/t-1}^{-1} \left( T \mu_0 - \sum_{i=t_0+1}^{t-1} w_i^T \mu \right) \tag{7} $$

that has the structure $w_t = \gamma_t S_t$ used in the proof of theorem 2, with

$$ \gamma_t = \Psi_{t/t-1} \left( H_{t/t-1} + \mu \Psi_{t+1/t} \mu \right) \Psi_{t/t-1}^{-1} \mu $$

Last period’s vector of weights is obtained as a myopic one period allocation with required return $T \mu_0 = \sum_{i=t_0+1}^{t_0+T-1} w_i^T \mu$:

$$ w_T = \frac{H_{t_0+1/T_0+T-1} \mu}{\mu H_{t_0+1/T_0+T-1}} \left( T \mu_0 - \sum_{i=t_0+1}^{t_0+T-1} w_i^T \mu \right) \tag{8} $$

Appendix A shows that the sequence $\{ \Psi_{t/t-1} \}_{t=t_0+1}^{t_0+T-1}$ can be derived recursively as:

$$ \Psi_{t_0+T-1/t_0+T-2} = E_{t_0+T-2} \left[ \frac{1}{\mu (H_{t_0+T/T_0+T-1})^{-1} / \mu} \right] \tag{9} $$

$$ \Psi_{t/t-1} = E_{t-1} \left[ \Psi_{t+1/t} \left( 1 - \mu \Psi_{t+1/t} \left( H_{t+1/t} + \mu \Psi_{t+1/t} \mu \right) \Psi_{t+1/t} \mu \right)^{-1} \mu \right], \forall t = t_0 + 1, \ldots, t_0 + T - 2 $$

*Add an economic intuition.*

Following the same strategy of the previous section, define

$$ V_{t_0} = E_{t_0} \left[ \sum_{t=t_0+1}^{t_0+T} \left( \Psi_{t/t} \right)^2 \right] \tag{11} $$

$$ V_{t_0}^* = E_{t_0} \left[ \sum_{t=t_0+1}^{t_0+T} \left( \Psi_{t/t}^* \right)^2 \right] \tag{12} $$

Where $w_t^*$ is defined as the vector of weights constructed using the true conditional variance sequence. Also in this case it is possible to show that $V_{t_0}^*$ is always smaller than $V_{t_0}$.

**Theorem 3.** If $\{ H_{t/t-1} \}_{t=t_0+1}^{t_0+T}$ is the sequence of estimated conditional covariance matrices, $\{ \Omega_{t/t-1} \}_{t=t_0+1}^{t_0+T}$ is the sequence of true covariance matrices, $\{ \Psi_{t/t} \}_{t=t_0+1}^{t_0+T-1}$ are defined as in (9) and (10), $\mu \neq 0$ is the vector of returns and $V_{t_0}$ and $V_{t_0}^*$ are defined as in (11) and (12), then $V_{t_0} \geq V_{t_0}^*$, and $\{ H_{t/t-1} \}_{t=t_0+1}^{t_0+T}$.

*Proof.* See Appendix A.
3. TESTS

Colacito and Engle (2003) introduced three ways of testing differences between conditional covariance estimators. In this section we extend those methods to the multiperiod problem examined in this paper.

3.1 Comparison of volatilities

As already shown in sections (2.1) and (2.2) an allocation strategy that minimizes variance over a multiperiod horizon will deliver variances \( V_{t\theta} \) and \( V_{t\theta}^* \) as defined in equations (3)-(4) and (11)-(12) respectively.

Use the law of iterated expectations to obtain:

\[
E \left[ \frac{1}{NT} \sum_{t=1}^{NT} (w_t^r r_t)^2 \right] = E \left\{ \frac{1}{NT} \sum_{j=0}^{N-1} E_{jT} \left[ \sum_{t=j+1}^{T+j} (w_t^r r_t)^2 \right] \right\} \\
= E \left[ \frac{1}{NT} \sum_{j=0}^{N-1} V_{jT} \right] \\
= E \left[ \frac{1}{NT} \sum_{j=0}^{N-1} \left( \frac{1}{NT} \sum_{t=1}^{NT} (w_t^r r_t)^2 \right) \right] \\
\tag{13}
\]

Now combine equation (13) with Theorem 1 and Theorem 3 to conclude that:

\[
E \left[ \frac{1}{NT} \sum_{t=1}^{NT} (w_t^r r_t)^2 \right] = E \left[ \frac{1}{NT} \sum_{j=0}^{N-1} V_{jT} \right] \leq E \left[ \frac{1}{NT} \sum_{j=0}^{N-1} V_{jT} \right] = E \left[ \frac{1}{NT} \sum_{t=1}^{NT} (w_t^r r_t)^2 \right]
\]

Under weak assumptions on the dependence of the covariances, the same result will hold with probability one and consequently for volatilities:

\[
p \lim \left[ \sqrt{\frac{1}{NT} \sum_{t=1}^{NT} (w_t^r r_t)^2} \right] \leq p \lim \left[ \sqrt{\frac{1}{NT} \sum_{t=1}^{NT} (w_t^r r_t)^2} \right]
\]

3.2 Other methods

The Diebold and Mariano approach should be easily extendable as well.

4. ESTIMATORS

As already introduced in the previous two sections, the full characterization of the solution \( \{w_t\}_{t=1}^T \) requires the computation variances associated to future positions. This task can be easily attained in the time zero problem with a set of forecast of the covariance matrix for any horizon less than or equal to \( T \), conditional on the information set at time zero. The dynamic program, conversely, requires the computation of more complicated conditional moment of the distribution for which a clear closed form solution cannot be obtained from the equations that describe the law of motion of the conditional covariance matrix.

This section is organized as follows. In the first part we analyze the closed loop problem. Then, we introduce a semi-nonparametric approach to forecast the time series \( \{\Psi_{t/t-1}\}_{t=1}^{T-1} \).

4.1 The closed loop problem

The covariance matrix for any horizon less than or equal to \( T \) conditional on the information set at time zero can be easily computed following the procedure described in Engle (2002).

Let the conditional distribution of the vector of returns be

\[
r_{t|\text{past}} \sim N(0, H_t)
\]

where \( H_t = D_t R_t D_t \) is the conditional covariance matrix, \( D_t = \text{diag} \{ \sqrt{h_{i,t}} \} \) is a diagonal matrix of conditional standard deviations, \( R_t = \text{diag} (Q_t)^{-\frac{1}{2}} Q_t \text{diag} (Q_t)^{-\frac{1}{2}} \) is the symmetric
matrix of correlations and \( Q_t \) is a linear vector time series process. The estimation is based on a two steps procedure. First estimate univariate GARCH models for each of the \( N \) assets and use the estimated \( \{h_{i,t}\}_{i=1}^N \) to construct the matrix \( D_t \). Then, maximize with respect to the parameters of the correlation matrix the likelihood function

\[
L_C = -\frac{1}{2} \sum_t (\log |R_t| + \varepsilon_t'R_t\varepsilon_t)
\]

where \( \varepsilon_t \) is the vector of standardized residuals from the first stage.

Forecasts of univariate variance processes can be easily computed for any horizon. All that is needed is to specify a model for the conditional variance process. Similarly, forecasts of the correlations rely on the specification of the \( Q_t \) process. One of the easiest approaches is the mean reverting model suggested in Engle (2002):

\[
Q_t = \bar{R}(1 - \alpha - \beta) + \alpha \varepsilon_{t-1}' + \beta Q_{t-1}
\]

By taking expectations of (14) and denoting as \( q_{i,j,t+k} \) the element on the \( i^{th} \) row and \( j^{th} \) column of \( Q_{t+k} \)

\[
E_t[q_{i,j,t+k}] = \bar{p}_{i,j} (1 - \alpha - \beta) + \beta E_t[q_{i,j,t+k-1}] + \alpha E_t[\varepsilon_{i,t+k-1}\varepsilon_{j,t+k-1}]
\]

The last expectation is by construction equal to 1 for \( i = j \) and \( E_t[\varepsilon_{i,t+k-1}\varepsilon_{j,t+k-1}] = E_t[p_{i,j,t+k-1}] \), \( \forall i \neq j \). Finally, a first order Taylor expansion of the correlation coefficient about \( Q \) gives

\[
\rho_{i,j,t+k} \approx \frac{q_{i,j,t+k} - \frac{1}{2} \bar{q}_{i,j} (q_{i,i,t+k-1} + q_{j,j,t+k-1})}{\sqrt{q_{i,i}q_{j,j}}}
\]

Hence forecasts of the correlation can be built up by recursively solving equations (15) and (16). Equations (14) and (15) can be modified in order to accommodate for different assumptions on the evolution of the correlation.

4.2 The open loop problem

The dynamic problem described in section 2.2 requires only one step ahead forecasts of the covariance matrices. However a greater difficulty is introduced in this problem by the fact that these forecasts are mixed in a functional form with other random variables, that make it impossible to produce the sequence \( \{\Psi_{i/i-1}\}_{i=1}^{T-1} \) as a simple function of the conditional variances as done in the previous subsection.

(Explain the seminonparametric approach that we follow.)

5. EMPIRICAL ANALYSIS

5.1 A simpler case: one risky and one riskless asset

In this section we focus on a low dimensional case in which the investor can only choose how much of her wealth to invest in one risky and in one riskless asset. We also restrict our attention on a comparison between two alternative estimators of the variance: a constant estimator and a dynamic estimator. The latter is chosen to be the Asymmetric GARCH(1,1) model, that is specified as follows:

\[
y_t = \sqrt{h_t} \varepsilon_t
\]

\[
h_t = \omega + \alpha y_{t-1}^2 + \beta h_{t-1} + \gamma d_{t-1}y_{t-1}^2
\]

Given the linear-quadratic nature of the problems that we want to solve, their solutions will be homogeneous of degree one in the required/expected return ratio. As a consequence, if we

\footnote{This is because \( \varepsilon_t \) is a vector of standardized residuals.}
denote as \( w_t \left( \frac{\mu_0}{\mu} \right) \) the weight at time \( t \) that is optimal for a required/expected return ratio of \( \frac{\mu_0}{\mu} \), it has to be true that:

\[
w_t \left( \frac{\mu_0}{\mu} \right) = \frac{\mu_0}{\mu} w_t (1)
\]

In order to construct the sequence of portfolio allocations in the closed loop strategy, all we need to know is the sequence of variance forecasts up to time \( T \) conditional on the information set at time 0. This is straightforward in the univariate case. Assuming the asymmetric GARCH process reported above, we can construct the whole sequence of variance forecasts recursively as:

\[
E_0 [y^2_t] = h_{1/0} \\
E_0 [y^2_2] = h_{2/0} = \omega + (\alpha + \beta + \frac{\gamma}{2}) h_{1/0}
\]

\[
\vdots
\]

\[
E_0 [y^2_T] = h_{T/0} = \omega + (\alpha + \beta + \frac{\gamma}{2}) h_{T-1/0}
\]

Using this multistep ahead forecast of the variance, we can compute portfolio allocations by simply adapting the general formula reported in the paper

\[
w_{\text{closed}, t} = T \mu_0 \frac{H_{t/0}^{-1} \mu}{\sum_{i=1}^{T} \mu' H_{i/0}^{-1} \mu}, \forall t = 1, \ldots, T
\]

to the univariate case discussed here, also assuming that \( \mu_0/\mu = 1 \)

\[
w_{\text{closed}, t} = T \frac{h_{t/0}^{-1}}{\sum_{i=1}^{T} h_{i/0}^{-1}}, \forall t = 1, \ldots, T
\]

In order to compute the weights for the open loop strategy, we need to construct the sequence \( \{\Psi_{t/1-t=1}\}^{T-1} \). Appendix C shows that the sequence \( \{\Psi_{t/1-t=1}\}^{T-1} \) can be linearly approximated around \( h = 1 \) by the following recursive system\(^2\):

\[
\Psi_{t/1-t-1} \approx \pi_0, t + \pi_{1, t} h_{t/1-t-1}
\]

\[
\pi_{0, t} = \pi_{0, t+1} + \omega \left( \pi^2_{1, t+1} + 2 \pi_{0, t+1} \pi_{1, t+1} + \pi_{1, t+1} + \pi^2_{0, t+1} \right) \left( 1 + \pi_{0, t+1} + \pi_{1, t+1} \right)^2
\]

\[
\pi_{1, t} = \tilde{\omega} \left( \pi^2_{1, t+1} + 2 \pi_{0, t+1} \pi_{1, t+1} + \pi_{1, t+1} + \pi^2_{0, t+1} \right) \left( 1 + \pi_{0, t+1} + \pi_{1, t+1} \right)^2
\]

\[
\pi_{0, T-1} = \omega
\]

\[
\pi_{1, T-1} = \tilde{\omega}
\]

Therefore, assuming that \( \frac{\mu_0}{\mu} = 1 \), the sequence of portfolio weights is simply:

\[
w_{\text{open}, 1} = T \frac{\pi_{0, 1} + \pi_{1, 1} h_{1/0}}{\pi_{0, 1} + (1 + \pi_{1, 1}) h_{1/0}}
\]

\[
\vdots
\]

\[
w_{\text{open}, t} = \left( T - \sum_{i=1}^{t-1} w_{\text{open}, i} \right) \frac{\pi_{0, t} + \pi_{1, t} h_{t/1-t-1}}{\pi_{0, t} + (1 + \pi_{1, t}) h_{t/1-t-1}}
\]

\[
\vdots
\]

\[
w_{\text{open}, T} = T - \sum_{i=1}^{T-1} w_{\text{open}, i}
\]

Myopic asset allocation using only one risky and the risk free asset collapses to a problem in which the estimate of the covariance matrix doesn’t matter and the weight on the risky asset

\(^2\)To simplify the notation, we will denote \( \tilde{\omega} = \alpha + \beta + \frac{\gamma}{2} \).
is simply the required return rescaled by the expected return. Maintaining the assumption that \( \mu_0 / \mu = 1 \), then

\[
w_{\text{myopic}, t} = 1, \quad \forall t
\]

A constant weight is also the optimal solution, when the variance estimator is simply the unconditional mean.

We evaluate efficiency loss by comparing the average realized sample standard error that we get when constructing portfolio weights using the right asymmetric GARCH estimator in each of the three strategies discussed here, as opposed to using a constant estimator of the variance. We begin the illustration of the results, by simulating 100,000 observations from the process (17). Tables 1-4 report the excess volatility an investor that uses constant variance would end up with when the right variance process is the Asymmetric GARCH. As already argued in Colacito and Engle (2003), these numbers can also be interpreted as the extra excess return an informed investor could have required, using only covariance information. The tables are constructed for various investment horizons, spanning from 5 to 20 days, and different parameters’ combinations. Asymmetry \( \gamma \) is fixed at 0.123, while the intercept \( \omega \) varies in such a way that the unconditional variance is always equal to 1. Two effects are the most evident. First of all the gain from both the closed loop and the open loop strategies increases with the investment horizon. Then the gain increases dramatically with \( \alpha \) and it gets as high as 4% in the 20 days simulation.

Next, we report the results obtained on real data. We consider the same SP500 series used in Colacito and Engle (2003). The estimated Asymmetric GARCH parameters are \( \omega = 0.019 \), \( \alpha = 0.001 \), \( \beta = 0.923 \) and \( \gamma = 0.123 \). As it may seem arbitrary to start investing from the first day of the sample, we consider all the possible starting dates. Figure 2 shows the gain from using dynamic volatilities for various investment horizons. It is interesting to notice that the gain from the open loop is on average always greater than the one from the closed loop and the gain is increasing with the investment horizon. This confirms what already emerged in the simulations.
Figure 1: Univariate portfolio allocation. Depending on the time pattern of expected volatility, portfolio choice can be either increasing or decreasing with a multiperiod objective function. The myopic solution is a horizontal line.
Figure 2: Real SP500 series. Efficiency gains from using dynamic variances versus unconditional-constant variances for various investment horizons. For every horizon each point indicates a different starting date of the investment plan.
Table 1: Sensitivity of results to different parameters. The asymmetric term $\gamma$ is fixed at 0.123, while the intercept $\omega$ varies to keep the unconditional variance fixed at 1. For each pair of parameters, the top line is the gain using the open loop and the bottom line is the gain using the closed loop. Investment horizon is 5 days.

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Table 2: Sensitivity of results to different parameters. The asymmetric term $\gamma$ is fixed at 0.123, while the intercept $\omega$ varies to keep the unconditional variance fixed at 1. For each pair of parameters, the top line is the gain using the open loop and the bottom line is the gain using the closed loop. Investment horizon is 10 days.

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<td></td>
<td>0.420</td>
<td>-</td>
<td>-</td>
<td>-</td>
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</tbody>
</table>

Table 3: Sensitivity of results to different parameters. The asymmetric term $\gamma$ is fixed at 0.123, while the intercept $\omega$ varies to keep the unconditional variance fixed at 1. For each pair of parameters, the top line is the gain using the open loop and the bottom line is the gain using the closed loop. Investment horizon is 15 days.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>0.537</td>
<td>0.553</td>
<td>0.496</td>
<td>0.139</td>
<td></td>
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<tr>
<td></td>
<td>0.038</td>
<td>0.052</td>
<td>0.083</td>
<td>0.094</td>
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<tr>
<td></td>
<td>1.489</td>
<td>1.438</td>
<td>0.583</td>
<td>-</td>
<td></td>
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<tr>
<td>0.1</td>
<td>0.136</td>
<td>0.253</td>
<td>0.337</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2.581</td>
<td>0.918</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.543</td>
<td>0.672</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>2.187</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.074</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
</tbody>
</table>
Table 4: Sensitivity of results to different parameters. The asymmetric term $\gamma$ is fixed at 0.123, while the intercept $\omega$ varies to keep the unconditional variance fixed at 1. For each pair of parameters, the top line is the gain using the open loop and the bottom line is the gain using the closed loop. Investment horizon is 20 days.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>(0.6)</th>
<th>(0.7)</th>
<th>(0.8)</th>
<th>(0.9)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>0.552</td>
<td>0.573</td>
<td>0.619</td>
<td>0.289</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.001</td>
<td>0.044</td>
<td>0.088</td>
<td>0.185</td>
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<tr>
<td>0.1</td>
<td>2.067</td>
<td>2.117</td>
<td>0.844</td>
<td>-</td>
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<tr>
<td></td>
<td>0.145</td>
<td>0.317</td>
<td>0.452</td>
<td>-</td>
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</tr>
<tr>
<td>0.2</td>
<td>3.972</td>
<td>2.341</td>
<td>-</td>
<td>-</td>
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</tr>
<tr>
<td></td>
<td>0.701</td>
<td>1.234</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>4.195</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.805</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
</tbody>
</table>

Table 5: Sensitivity of results to different values of asymmetry. $\alpha$ and $\beta$ are respectively equal to 0.001 and 0.92, while the intercept $\omega$ varies to keep the unconditional variance fixed at 1. For each pair of parameters, the top line is the gain using the open loop and the bottom line is the gain using the closed loop. Investment horizon is 20 days.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>(0)</th>
<th>(0.123)</th>
<th>(0.140)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.131</td>
<td>0.156</td>
<td></td>
</tr>
<tr>
<td>-0.030</td>
<td>0.027</td>
<td>0.071</td>
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</tr>
</tbody>
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Appendix A. Proofs of Theorems

Proof of Theorem 1. In what follows we will denote $H_t$ and $\Omega_t$ as being $H_{t/\tau_0}$ and $\Omega_{t/\tau_0}$ respectively. Let $u_t = \mu' H_{t-1} z_t - \left( \mu' H_{t-1} \right) \left( \mu' \Omega_{t-1} \mu \right)^{-1} \mu' \Omega_{t-1} z_t$, where $E[z_t z'_t] = \Omega_t$. Since $E[u^2_t] \geq 0$ and

$$E[u^2_t] = \mu' H_{t-1} \Omega_t H_{t-1} \mu - \left( \mu' H_{t-1} \right)^2 \left( \mu' \Omega_{t-1} \mu \right)^{-1}$$

it follows that

$$\sum_{t=0}^{t_0+T} \mu' H_{t-1} \Omega_t H_{t-1} \mu - \left( \sum_{t=0}^{t_0+T} \mu' H_{t-1} \right)^2 \left( \sum_{t=0}^{t_0+T} \mu' \Omega_{t-1} \mu \right)^{-1} \geq 0 \tag{A.1}$$

Furthermore the fact that both $\{H_t\}_{t=0}^{t_0+T}$ and $\{\Omega_t\}_{t=0}^{t_0+T}$ are positive definite and $\mu \neq 0$ implies that

$$\sum_{t=0}^{t_0+T} \mu' H_{t-1} \Omega_t H_{t-1} \mu - \left( \sum_{t=0}^{t_0+T} \mu' H_{t-1} \right)^2 \left( \sum_{t=0}^{t_0+T} \mu' \Omega_{t-1} \mu \right)^{-1} \geq 0 \tag{A.2}$$

Combining equations (A.1) and (A.2) it follows that

$$\sum_{t=0}^{t_0+T} \mu' H_{t-1} \Omega_t H_{t-1} \mu - \left( \sum_{t=0}^{t_0+T} \mu' H_{t-1} \right)^2 \left( \sum_{t=0}^{t_0+T} \mu' \Omega_{t-1} \mu \right)^{-1} \geq 0$$

and therefore $V_{t_0} \geq V_{t_0}^* \forall \{H_t\}_{t=0}^{t_0+T}$ and $\{\Omega_t\}_{t=0}^{t_0+T}$.

Proof of Theorem 2. Define $S_t = T \mu_0 - \sum_{j=1}^{t-1} w_{j} \mu$ and $S_1 = T \mu_0$. Starting from the last period’s problem and taking as given the sequence $\{w_t\}_{t=1}^{T-1}$, positive definiteness of $H_{T-1}$ implies that the optimal choice $\hat{w}_T$ is unique. Furthermore, this solution can be written as $\hat{w}_T = \gamma_T S_T$, where $\gamma_T$ is a function of conditional moments and expected returns and $S_T$ is defined as above. Use $\tilde{w}_T$ to write the problem at time $T-2$ as:

$$\min_{w_{T-2}} w_{T-2} H_{T-1/2} w_{T-2} + w_{T-2} E_{T-2} \left[ \gamma'_T H_{T-2} \gamma_T \right] w_{T-2}$$

whose first order necessary condition imply $\tilde{w}_{T-1} = \gamma_{T-1} S_{T-1}$. Recursion gives $\hat{w}_t = \gamma_t S_t$, $\forall t = 1, ..., T$. Since $\hat{w}_1$ is proportional to $T \mu_0$, proportionality of any other $\hat{w}_t$ is implied by the definition of $S_t$.

Proof of Theorem 3. To simplify the notation, we will consider the allocation problem from time $t = 1$ to time $t = T$. Furthermore we will denote with a * the variables that make use of the sequence of true covariances and $\Lambda_{T-1} = \left( H_{T-1/2} + \mu \Psi_{T-1/2} \right)$. We need to show that

$$\sum_{t=1}^{T} E_0 \left[ (w'_t r_t)^2 \right] \geq \sum_{t=1}^{T} E_0 \left[ (w'_t r_t)^2 \right]$$

Define $\phi_T = E_0 \left[ (w'_t r_t)^2 \right]$ and focus on $\phi_T$ and $\phi^*_T$:

$$\phi^*_T = E_0 \left[ w'_T E_{T-1} \left( r'_t r_t \right) w_T \right] = E_0 \left[ w'_T \Omega_{T-1} w_T \right] = E_0 \left[ \left( T \mu_0 - \sum_{i=1}^{T-1} w'_i \mu \right) \frac{1}{\mu' \Omega_{T-1} \mu} \right]$$

$$\phi_T = E_0 \left[ \left( T \mu_0 - \sum_{i=1}^{T-1} w'_i \mu \right) \frac{1}{\mu' \Omega_{T-1} \mu} \right]$$

As we have already shown in the proof of Theorem 1

$$\frac{1}{\mu' \Omega_{T-1} \mu} \leq \frac{\mu' H_{T-1} \Omega_{T-1} H_{T-1} \mu}{\left( \mu' H_{T-1} \mu \right)^2}$$

Therefore using the law of iterated expectations and the definition of $\Psi^*_T$ we allow us to rewrite (A.5) as

$$\phi_T = E_0 \left[ \left( T \mu_0 - \sum_{i=1}^{T-1} w'_i \mu \right) \left( \Psi^*_T + \bar{\Delta}_{T-1} \right) \right]$$

(A.6)

$$\phi_T = E_0 \left[ \left( T \mu_0 - \sum_{i=1}^{T-1} w'_i \mu \right) \left( 1 - \mu' \Lambda^{-1} \mu \Psi_{T-1} \right) \left( \Psi^*_T + \bar{\Delta}_{T-1} \right) \right]$$

(A.7)
Now sum up \( \phi_T \) and \( \phi_{T-1} \)
\[
\phi_{T-1} + \phi_T = E_0 \left[ \left(T \mu_0 - \sum_{i=1}^{T-2} w_i' \mu \right)^2 \left( \mu' \Lambda_{T-1}^{-1} \Psi_{T-1/T-2}^{*} \right)^2 \right]
\]
\[
= E_0 \left[ \left(T \mu_0 - \sum_{i=1}^{T-2} w_i' \mu \right)^2 \xi_{T-1} \right]
\]
\[
\phi_{T-1}^* + \phi_T^* = E_0 \left[ \left(T \mu_0 - \sum_{i=1}^{T-2} w_i'' \mu \right)^2 \left( \Psi_{T-1/T-2}^{*} \right)^2 \right]
\]
\[
= E_0 \left[ \left(T \mu_0 - \sum_{i=1}^{T-2} w_i'' \mu \right)^2 \xi_{T-1}^* \right]
\]

Clearly
\[
\xi_{T-1} - \xi_{T-1}^* = \Delta_{T-2} + \mu' \left( \Lambda_{T-1}^{-1} \Psi_{T-1} \left( \Lambda_{T-1}^{-1} \right)^{1/2} - \Psi_{T-1} \left( \Lambda_{T-1}^{-1} \right)^{-1/2} \right)^2 \mu \geq 0
\]

where \( \Delta_{T-2} = \left(1 - \mu' \Lambda_{T-1}^{-1} \mu \Psi_{T-1} \right)^2 \Xi_{T-1} \). Therefore using again the law of iterated expectations and the definition of \( \Psi_{T-2/T-3} \) we can conclude that
\[
\phi_{T-1} + \phi_T = E_0 \left[ \left(T \mu_0 - \sum_{i=1}^{T-2} w_i \mu \right)^2 \left( \Psi_{T-2/T-3}^{*} \right)^2 \right] \quad (A.8)
\]

Notice that equations (A.6) and (A.8) have the same structure. This allows us to repeat the argument recursively to conclude that
\[
\sum_{t=1}^{T} E_0 \left[ \left(w_t r_t \right)^2 \right] = \sum_{t=1}^{T} \phi_t \geq \sum_{t=1}^{T} \phi_t^* = \sum_{t=1}^{T} E_0 \left[ \left(w_t' r_t \right)^2 \right]
\]

\[\square\]

**Appendix B. Recursive derivation of future variances**

**Theorem 4.** If we define \( \Psi_{T-1/T-2} = E_{T-2} \left[ \frac{1}{\mu' (H_{T/T-1})^{-1} \mu} \right] \) as the last element of the sequence of \( \Psi_{i/i-1}, \)
\( \forall i = 1, T-1, \) then
\[
\Psi_{i/i-1} = E_{i-1} \left[ \frac{1}{\mu' (H_{i+1/i}^{-1} \mu) } \right], \quad \forall i = 1, \ldots, T-2 \quad (B.1)
\]

**Proof.** In this proof we simplify the notation as explained in the beginning of the previous proof. Using the definition of leftover variance:
\[
\Psi_{i/i-1} = E_{i-1} \left[ \frac{\sum_{j=i+1}^{T} w_j' H_{j/j-1} w_j}{\left(T \mu_0 - \sum_{i=1}^{T} w_i \mu \right)^2} \right] = \frac{1}{\varphi_i} \sum_{j=i+1}^{T} \Phi_j
\]
where \( \Phi_j = E_{i-1} \left[ w_j' H_{j/j-1} w_j \right] \) and \( \varphi_i = \left(T \mu_0 - \sum_{i=1}^{T} w_i \mu \right)^2 \). Focus on the last term:
\[
\Phi_T = E_{T-1} \left[ w_T' H_{T/T-1} w_T \right]
\]
\[
= E_{T-1} \left[ \frac{\mu' H_{T/T-1}^{-1} \mu}{\mu' H_{T/T-1}^{-1} \mu} \right] \left(T \mu_0 - \sum_{j=1}^{T-2} w_j \mu \right)^2 \left(1 - \mu' \Lambda_{T-1}^{-1} \Psi_{T-1/T-2} \right)^2 \mu \geq 0
\]
now use the law of iterated expectations
\[
= E_{T-1} \left[ E_{T-2} \left[ \frac{1}{\mu' H_{T/T-1}^{-1} \mu} \right] \left(T \mu_0 - \sum_{j=1}^{T-2} w_j \mu \right)^2 \left(1 - \mu' \Lambda_{T-1}^{-1} \Psi_{T-1/T-2} \right)^2 \mu \right]
\]
\[
= E_{T-1} \left[ \Psi_{T-1/T-2} \left(T \mu_0 - \sum_{j=1}^{T-2} w_j \mu \right)^2 \left(1 - \mu' \Lambda_{T-1}^{-1} \Psi_{T-1/T-2} \right)^2 \mu \right]
\]

\[\text{We omit the } "/T-2" \text{ to make the proof more readable.} \]
Next focus on $\Phi_{T-1}$

$$
\Phi_{T-1} = E_{i-1} \left[ \left( \frac{w^i_{T-1} H_{T-1/T-2} w_{T-1}}{\partial h} \right) \right] \\
= E_{i-1} \left[ \left( T \mu_0 - \sum_{j=1}^{T-2} w^i_j \right)^2 (1 - \mu^i \Psi_{T-1/T-2} \Lambda_{T-1}^{-1} \Psi_{T-1/T-2}^{-1}) \right]
$$

Put together $\Phi_{T-1}$ and $\Phi_T$

$$
\Phi_{T-1}^* = \Phi_{T-1} + \Phi_T \\
= E_{i-1} \left[ \left( T \mu_0 - \sum_{j=1}^{T-2} w^i_j \right)^2 \Psi_{T-1/T-2} \left( 1 - \mu^i \Psi_{T-1/T-2} \Lambda_{T-1}^{-1} \right) \right]
$$

Use again the law of iterated expectations and pass the $E_{T-1} \left[ \right]$ operator into the previous formula:

$$
\Phi_{T-1}^* = E_{i-1} \left[ \left( T \mu_0 - \sum_{j=1}^{T-2} w^i_j \right)^2 \Psi_{T-2/T-3} \left( 1 - \mu^i \Psi_{T-1/T-2} \Lambda_{T-1}^{-1} \right) \right]
$$

Repeat the same steps using $\Phi_{T-1}^*$ and $\Phi_{T-2}$, obtain $\Phi_{T-2}$ and iterate recursively. Noticing that $\varphi_t$ cancels in the last iteration, we get

$$
\Psi_{i/T-1} = E_{i-1} \left[ \Psi_{i+1/i} \left( 1 - \mu^i \Psi_{i+1/i} \left( H_{i+1/i} + \mu \Psi_{i+1/i} \right)^{-1} \right) \right], \forall i = 1, ..., T-2
$$

Appendix C. Details of the Taylor expansion

In the univariate case, the sequence $\{\Psi_{i/T-1}\}_{i=1}^{T-1}$ simplifies to:

$$
\Psi_{T-1/T-2} = h_{T/T-2} \\
\Psi_{i/T-1} = E_{i-1} \left[ \Psi_{i+1/i} \left( H_{i+1/i} + \mu \Psi_{i+1/i} \right)^{-1} \right], \forall i = 1, 2, ..., T-2
$$

$\Psi_{T-1/T-2}$ is the two steps ahead forecast of time T variance. This can be immediately computed from equation that defines the TARCH process as

$$
\Psi_{T-1/T-2} = \pi_0 + \pi_1 h_{T-1/T-2} \tag{C.2}
$$

where $\pi_0, \pi_1 = \alpha + \beta + \frac{\gamma}{2}$. Next in the sequence is $\Psi_{T-2/T-3}$. The random variable $\Psi_{T-2}$ is defined as: A first order Taylor expansion of

$$
\Psi_{T-2} = \frac{(\pi_0, T-1 + \pi_1 h_{T-1/T-2} h_{T-1/T-2}) h_{T-1/T-2}}{\pi_0, T-1 + (1 + \pi_1, T-1) h_{T-1/T-2}} \tag{C.3}
$$

Take a first order Taylor expansion of it around $\overline{h}$

$$
\Psi_{T-2} \approx \Psi_{T-2} \frac{\partial \Psi_{T-2}}{\partial h} \left( h_{T-1/T-2} - \overline{h} \right) \tag{C.4}
$$

where

$$
\delta_{0,T-2} = \frac{\partial \Psi_{T-2}}{\partial h} \left( h_{T-1/T-2} - \overline{h} \right) \tag{C.5}
$$

Consider the case in which $\overline{h} = 1$. Then

$$
\delta_{0,T-2} = \frac{\pi_{0,T-1}}{(1 + \pi_{0,T-1} + \pi_{1,T-1})^2} \\
\delta_{1,T-2} = \frac{\pi_{1,T-1} + 2 \pi_{0,T-1} \pi_{1,T-1} + \pi_{1,T-1} + \pi_{2,T-1}}{(1 + \pi_{0,T-1} + \pi_{1,T-1})^2} \\
\delta_{2,T-2} \approx 0 \\
\delta_{3,T-2} \approx 0
$$
Plug these values into (C.4) and take the expectation conditional on the information set at time $T-3$:

$$
\Psi_{T-2/T-3} \approx \delta_{h_{T-2}} + \omega \delta_{1,T-2} + \tilde{\omega} \delta_{1,T-2} h_{T-2/T-3}
$$

Recursions give the solution reported in the main text for the rest of sequence. It can be shown that this approximation is robust to the point about which the expansion is taken. We make use of this result here, by taking the approximation around $h = 1$ even though this is not the unconditional mean of variance. This simplifies computations noticeably.

References


Alexander, C. (2000). A primer on the orthogonal garch models. *manuscript ISMA Center, University of Reading, UK*.


