Problem Set 1 Solution: Time Value of Money and Equity Markets

I. Present Value with Multiple Cash Flows:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>A:</td>
<td>40000</td>
<td>40000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>B:</td>
<td>30000</td>
<td>20000</td>
<td>20000</td>
<td></td>
</tr>
</tbody>
</table>

APR is 16% compounded quarterly;  
Periodic Rate (with quarterly compounding) is \( r_{1/4} = 0.16/4 = 0.04 = 4\% \).  
EAR is \( (1 + r_{1/4})^4 - 1 = 1.04^4 - 1 = 0.16986 = 16.986\% \).

Salary Arrangement A:  
\[ V_A^0 = 40000 \times PVAF_{16.986\%, 2} = 40000 \times \frac{[1-(1.16986)^{-2}]/0.16986}{0.16986} = 63419.66. \]

Salary Arrangement B:  
\[ V_B^0 = 30000 + 20000 \times PVAF_{16.986\%, 2} = 30000 + 20000 \times \frac{[1-(1.16986)^{-2}]/0.16986}{0.16986} = 30000 + 31709.83 = 61709.83. \]

II. Calculating EAR and continuous compounding:

<table>
<thead>
<tr>
<th>APR ( i_{\text{nom}} )</th>
<th>Compound Periods in a Year ( m&lt;\infty )</th>
<th>Periodic Rate ( r_{1/m} )</th>
<th>EAR ( r )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( r_{1/m} = i_{\text{nom}}/m )</td>
<td>( r = (1+r_{1/m})^m - 1 )</td>
<td></td>
</tr>
<tr>
<td>0.04</td>
<td>2</td>
<td>0.02</td>
<td>0.0404</td>
</tr>
<tr>
<td>0.06</td>
<td>4</td>
<td>0.015</td>
<td>0.061364</td>
</tr>
<tr>
<td>0.18</td>
<td>365</td>
<td>0.0004931</td>
<td>0.19716</td>
</tr>
</tbody>
</table>

When the continuously compounded APR \( m=\infty \) is 0.22, then \( \text{EAR} = r = \exp \{0.22\} - 1 = 0.24608 \text{ or } 24.608\% \).

III. EAR vs APR:

Let 1 period be a year. Question says that the effective monthly rate \( r_{1/12} \) is 0.2. So:  
APR \( i_{\text{nom}} \) is 12 x 0.2\% = 2.4 or 240\%.  
EAR \( r \) is \( (1+0.2)^{12} - 1 = 7.9161 \text{ or } 791.61\% \).
IV. Calculating Annuity Payments:

<table>
<thead>
<tr>
<th>Year</th>
<th>35</th>
<th>36</th>
<th>37</th>
<th>65</th>
<th>66</th>
<th>67</th>
<th>75</th>
</tr>
</thead>
<tbody>
<tr>
<td>w:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>V₆₅</td>
<td>10000</td>
<td>10000</td>
<td>10000</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>d:</td>
<td>D</td>
<td>D</td>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>V₃₆</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Told r=0.08.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

A. Define \( V₆₅^{w} \) to be the single sum equivalent at age 65 of the desired withdrawal stream. Now using the present value annuity factor gives the single sum equivalent of the withdrawal annuity stream at age 65 (since the first withdrawal is made at age 66):

\[
V₆₅^{w} = 10000 \times \frac{1-(1.08)^{-10}}{0.08} = 67100.8.
\]

Define \( V₆₅^{d} \) to be the single sum equivalent at age 65 of the necessary deposit stream. For the stream of deposits to be sufficient to allow the stream of withdrawals to be made:

\[
V₆₅^{d} = V₆₅^{w} = 67100.8
\]

But using the future value annuity factor gives the single sum equivalent of the deposit annuity stream at age 65 (since the last deposit is made at age 65):

\[
V₆₅^{d} = D \times (1.08)^{30} - 1)/0.08] = D \times 113.28321
\]
and so \( D = 67100.8/113.2832 = 592.33 \).

B. Using the single sum present value formula,

\[
V₃₆^{d} = V₆₅^{d} \times (1+0.08)^{-29} = 7201.8.
\]

V. Loan Amortization: Consider a 20 year $90000 mortgage loan with monthly payments. Assume an APR of 9% compounded monthly. The $90000 is lent today and the first payment is in one month's time.

A. What is the size of the monthly payments?

90000
APR (\(i_{\text{nom}}\)) is 9% with monthly compounding. Period Rate (\(r\)) is \(i_{\text{nom}}/12 = 9%/12 = 0.75\%\) which is the effective monthly rate.

The present value annuity factor gives the single sum equivalent at time 0 (since the first payment must be made in one month):
\[ V_{0}^{240} = 90000 = C \times PVAF_{0.75\%,240} = C \times \left\{ \frac{1 - (1.0075)^{-240}}{0.0075} \right\} = C \times 111.14495 \]
and so \(C = \frac{90000}{111.14495} = 809.75336\).

B. What is the principal outstanding in 10 years from now (just after the 120th payment)?

\begin{align*}
0 & \quad 1 \quad 2 \quad 119 \quad 120 \quad 121 \quad 240 \\
\hline
809.75 & \quad 809.75 \\
\end{align*}

The balance of the loan outstanding after the 120th payment is just the single sum equivalent at time 120 for the last 120 payments. The present value annuity factor gives the single sum equivalent at time 120 for the last 120 payments (since the first of these payments is made at time 121):
\[ V_{120}^{120} = 809.75336 \times PVAF_{0.75\%,120} = 809.75336 \times \left\{ \frac{1 - (1.0075)^{-120}}{0.0075} \right\} = 809.75336 \times 78.9417 = 63923.301. \]

C. What portion of the 120th payment goes to principal and what goes to interest?

To calculate the interest accruing on the loan during the 120th month, need to calculate the balance outstanding after the 119th payment.

\begin{align*}
0 & \quad 1 \quad 2 \quad 119 \quad 120 \quad 121 \quad 240 \\
\hline
809.75 & \quad 809.75 & \quad 809.75 \\
\end{align*}

As for the previous part, the principal outstanding after the 119th payment is just the single sum equivalent at time 119 for the last 121 payments. The present value annuity factor gives the single sum equivalent at time 119 for the last 121 payments (since the first of these payments is made at time 120):
\[ V_{119}^{121} = 809.75336 \times PVAF_{0.75\%,121} = 809.75336 \times \left\{ \frac{1 - (1.0075)^{-121}}{0.0075} \right\} = 809.75336 \times 79.34659 = 64251.17. \]

Interest = \$64251.17 \times 0.0075 = \$481.8838.
However, it does not matter how much of the 120th payment goes to interest. The balance of the loan still drops from $64251.17 after the 119th payment to $63923.30 after the 120th payment.

VI. **Deferred Annuities:** Consider a single premium deferred annuity (SPDA) which costs $28765.5 and promises yearly payments of $20000 every year beginning 21 years from now. If the advertised EAR for the SPDA is 9%, how many payments must the SPDA make? (use trial and error if you cannot solve it with algebra)

Let N be the number of payments.

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>20</th>
<th>21</th>
<th>22</th>
<th>...</th>
<th>20+N</th>
</tr>
</thead>
<tbody>
<tr>
<td>28765.5</td>
<td>V₀</td>
<td>V₁</td>
<td>...</td>
<td>V₂₀</td>
<td>V₂₁</td>
<td>V₂₂</td>
<td>...</td>
<td>V₂₀+N</td>
</tr>
</tbody>
</table>

**EAR (r)** is 9%.

Using the present value annuity factor,

\[ V_{20} = \frac{20000 \times \text{PVAF}_{9\%,20}}{0.09} \]

Using the present value interest factor for single sums:

\[ 28765.5 = V_0 = V_{20} \times \text{PVIF}_{9\%,20} = V_{20} (1.09)^{-20} \]

But then

\[ 28765.5 = 20000 \times \left[ \frac{1-(1.09)^{-N}}{0.09} \right] (1.09)^{-20} \]

The rest is algebra:

\[ \frac{28765.5}{20000} (1.09)^{20} = \left[ \frac{1-(1.09)^{-N}}{0.09} \right] \]

and

\[ \frac{28765.5}{20000} (1.09)^{20} 0.09 = (1.09)^{-N} \]

and

\[ 1 - (28765.5/20000) (1.09)^{20} 0.09 = (1.09)^{-N} \]

and

\[ \ln [1 - (28765.5/20000) (1.09)^{20} 0.09] = \ln [(1.09)^{-N}] \]

and

\[ \ln [1 - (28765.5/20000) (1.09)^{20} 0.09] = -N \ln [(1.09)] \]

and finally

\[ N = - \ln \left[ \frac{1 - (28765.5/20000) (1.09)^{20} 0.09}{\ln[(1.09)]} \right] = - \left[ -1.292664/0.0861776 \right] = 15. \]
VII. *The Limit-order Book of the NYSE Specialist*: BKM, Chapter 3, Question 7, parts a. and b.

Part a. The market-buy order will be filled at the lowest limit-sell order which is $50.25.

Part b. The next market-buy order will be filled at the lowest remaining limit-sell order which is $51.50.