Problem Set 7 Solution: Fixed Income Valuation.

I. Implied Yield Curve, Forward Rates and No Arbitrage: Consider the following prices for U.S. treasury notes on 2/15/96.

<table>
<thead>
<tr>
<th>Rate</th>
<th>Maturity</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>4½</td>
<td>Aug 96</td>
<td>98:11</td>
</tr>
<tr>
<td>5¼</td>
<td>Feb 97</td>
<td>99:01</td>
</tr>
<tr>
<td>5¾</td>
<td>Aug 97</td>
<td>98:23</td>
</tr>
<tr>
<td>6</td>
<td>Feb 98</td>
<td>98:15</td>
</tr>
</tbody>
</table>

A. What is the implied yield curve (expressed in terms of APRs with semiannual compounding)?

The following formula relating the yield on a discount bond to the relevant discount factor is used throughout this question. Let \( d_t(0) \) be the discount factor for a \( t \)-year discount bond, and \( y_t(0) \) be the yield on a \( t \)-year discount bond expressed as an APR with semiannual compounding:

\[
y_t(0) = 2 \left\{ \left[ \frac{1}{d_t(0)} \right]^{1/(2t)} - 1 \right\} \iff d_t(0) = \frac{1}{\left[ \frac{y_t(0)}{2} \right]^{2t}}.
\]

To obtain the yield on a 6 month discount bond:

a. Can recover the discount factor on a 6-month discount bond using the Aug 96 note and the no-arbitrage formula for pricing coupon bonds:

\[
98.3438 = (100 + [4½/2]) d_{\frac{1}{2}} (Feb 96) \Rightarrow d_{\frac{1}{2}} (Feb 96) = 0.96180.
\]

b. Can convert the 6-month discount bond discount factor into a yield expressed as an APR with semi-annual compounding:

\[
y_{\frac{1}{2}} (Feb 96) = \{[1/d_{\frac{1}{2}} (Feb 96)] -1\} x 2 = \{[1/0.96180] -1\} x 2 = 7.9440\%.
\]

To obtain the yield on a 1-year discount bond:

a. Can recover the discount factor on a 1-year discount bond using the Feb 97 note and the no-arbitrage formula for pricing coupon bonds:

\[
99.03125 = (5¼/2) d_{\frac{1}{2}} (Feb 96) + (100 + [5¼/2]) d_1 (Feb 96) \Rightarrow d_1 (Feb 96) = 0.94038
\]

b. Can convert the 1-year discount bond discount factor into a yield expressed as an APR with semi-annual compounding (using the
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above stated formula):
\[ y_1 \text{ (Feb 96)} = \left\{ \frac{1}{d_1 \text{ (Feb 96)}} \right\}^{0.5} - 1 \times 2 = \left\{ \frac{1}{0.94038} \right\}^{0.5} - 1 \times 2 = 6.2425\% . \]

To obtain the yield on a 1½-year discount bond:

a. Can recover the discount factor on a 1½-year discount bond using the Aug 97 note and the no-arbitrage formula for pricing coupon bonds:

\[
98.71875 = (5\frac{3}{4}/2) \cdot d_{1\frac{1}{2}} \text{ (Feb 96)} + (5\frac{3}{4}/2) \cdot d_1 \text{ (Feb 96)} + (100 + [5\frac{3}{4}/2]) \cdot d_{1\frac{1}{2}} \text{ (Feb 96)}
\]

\[
98.71875 = (5\frac{3}{4}/2) \cdot 0.96180 + (5\frac{3}{4}/2) \cdot 0.94038 + (100 + [5\frac{3}{4}/2]) \cdot d_{1\frac{1}{2}} \text{ (Feb 96)}
\]

\[ \Rightarrow d_{1\frac{1}{2}} \text{ (Feb 96)} = 0.90644. \]

b. Can convert the 1½-year discount bond discount factor into a yield expressed as an APR with semi-annual compounding (using the above stated formula):

\[ y_{1\frac{1}{2}} \text{ (Feb 96)} = \left\{ \frac{1}{d_{1\frac{1}{2}} \text{ (Feb 96)}} \right\}^{0.5} - 1 \times 2 = \left\{ \frac{1}{0.90644} \right\}^{0.5} - 1 \times 2 = 6.6571\% . \]

To obtain the yield on a 2-year discount bond:

a. Can recover the discount factor on a 2-year discount bond using the Feb 98 note and the no-arbitrage formula for pricing coupon bonds:

\[
98.46875 = (6/2) \cdot d_{1\frac{1}{2}} \text{ (Feb 96)} + (6/2) \cdot d_1 \text{ (Feb 96)} + (6/2) \cdot d_{1\frac{1}{2}} \text{ (Feb 96)} + (100 + [6/2]) \cdot d_2 \text{ (Feb 96)}
\]

\[
98.46875 = 3 \times 0.96180 + 3 \times 0.94038 + 3 \times 0.90644 + 103 \times d_2 \text{ (Feb 96)}
\]

\[ \Rightarrow d_2 \text{ (Feb 96)} = 0.87420. \]

b. Can convert the 2-year discount bond discount factor into a yield expressed as an APR with semi-annual compounding (using the above stated formula):

\[ y_2 \text{ (Feb 96)} = \left\{ \frac{1}{d_2 \text{ (Feb 96)}} \right\}^{0.5} - 1 \times 2 = \left\{ \frac{1}{0.87420} \right\}^{0.5} - 1 \times 2 = 6.8364\% . \]

A. What are the implied forward rates for the 6 month periods starting in 6 months, in 1 year and in 18 months (expressed as APRs with semiannual compounding)?

First, calculate \( d_{6,1} \text{ (Feb 96)} \), \( d_{1,1\frac{1}{2}} \text{ (Feb 96)} \), and \( d_{1\frac{1}{2},2} \text{ (Feb 96)} \) using the formula
\[
d_{t,t+\tau}(0) = d_{t+\tau}(0) / d_{t}(0):
\]

a. \( d_{6,1} \text{ (Feb 96)} = d_1 \text{ (Feb 96)} / d_{6}(0) = 0.94038/0.96180 = 0.97773. \)

b. \( d_{1,1\frac{1}{2}} \text{ (Feb 96)} = d_{1\frac{1}{2}} \text{ (Feb 96)} / d_1 \text{ (Feb 96)} = 0.90644/0.94038 = 0.96391. \)

c. \( d_{1\frac{1}{2},2} \text{ (Feb 96)} = d_2 \text{ (Feb 96)} / d_{1\frac{1}{2}} \text{ (Feb 96)} = 0.87420/0.90644 = 0.96443. \)

Then calculate the associated forward rates expressed as APRs with semiannual compounding using the following formula

\[
f_{t,t+\tau}(0) = 2 \left\{ \frac{1}{d_{t,t+\tau}(0)} \right\}^{1/(2\tau)} - 1 \text{ with } \tau = \frac{1}{2}. \]
a. \( f_{\frac{1}{2},1}(\text{Feb 96}) = 2 \left\{ \frac{1}{d_{\frac{1}{2},1}(\text{Feb 96})} - 1 \right\} = 2 \left\{ \frac{1}{0.97773} - 1 \right\} = 4.5554\% \).

b. \( f_{1,1\frac{1}{2}}(\text{Feb 96}) = 2 \left\{ \frac{1}{d_{1,1\frac{1}{2}}(\text{Feb 96})} - 1 \right\} = 2 \left\{ \frac{1}{0.96391} - 1 \right\} = 7.4883\% \).

c. \( f_{1\frac{1}{2},2}(\text{Feb 96}) = 2 \left\{ \frac{1}{d_{1\frac{1}{2},2}(\text{Feb 96})} - 1 \right\} = 2 \left\{ \frac{1}{0.96443} - 1 \right\} = 7.3764\% \).

C. If there are no arbitrage opportunities, what is the price of a Aug 97 U.S. Treasury strip?

Use the discount factor for a 18 month discount bond:
\( P_{\text{Aug 97 strip}}(\text{Feb 96}) = d_{1\frac{1}{2}}(\text{Feb 96}) \times 100 = 0.906440 \times 100 = 90.6440 \).

D. Suppose the price of a Feb 97 U.S. Treasury strip is 94. Is there an arbitrage opportunity? If so, describe a strategy which earns an arbitrage profit.

Use the implied yield on a 1 year discount bond (expressed as an APR with semiannual compounding):
\( P_{\text{Feb 97 strip}}(\text{Feb 96}) = d_{1}(\text{Feb 96}) \times 100 = 0.940831 \times 100 = 94.038 \).

Since the price of the Feb 97 strip implied by the coupon bonds is greater than the strip’s actual price, you want to buy the Feb 97 strip and sell a synthetic Feb 97 strip created using the Aug 96 and Feb 97 coupon bonds.

Let \( a \) be the number of Feb 97 notes that you buy and \( b \) be the number of Aug 96 notes that you buy. Want to choose \( a \) and \( b \) so that the net cash flow at 2/15/97 is zero and the net cash flow at 8/15/96 is zero:
Position 2/15/96 8/15/96 2/15/97

Buy 1
Feb 97 strip -94 100
Buy a x
Feb 97 note -a x 99.03125 a x 2.625 a x 102.625
Buy b x
Aug 96 note -b x 98.34374 b x 102.25
Net -94 - a x 99.03125 0 0
- b x 98.34374

So
a. 100 + a x 102.625 = 0 which implies a = -100/102.625 = -0.97442.
   Thus, the Feb 97 note is sold.
b. a x 2.625 + b x 102.25 = 0 which implies b = -a x 2.625/102.25 = 0.02502.

Thus,

Position 2/15/96 8/15/96 2/15/97

Buy 1
Feb 97 strip -94 100
Sell 0.97442
Feb 97 notes 0.97442 x -0.97442 x 2.625 -0.97442 x 102.625
         99.03125 = -2.558 = -100
         = 96.4980
Buy 0.02502
Aug 96 notes -0.02502 x 0.02502 x 102.25
         98.34374 = -2.4606 = 2.558
Net 0.03744 0 0

E. Suppose the prices for U.S. Treasury notes on 8/15/96 are given by:

<table>
<thead>
<tr>
<th>Rate</th>
<th>Maturity</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>5¼</td>
<td>Feb 97</td>
<td>98:11</td>
</tr>
<tr>
<td>5¾</td>
<td>Aug 97</td>
<td>98:21</td>
</tr>
<tr>
<td>6</td>
<td>Feb 98</td>
<td>98:00</td>
</tr>
</tbody>
</table>

1. What is the return from holding the Aug 97 note from 2/15/96 to 8/15/96?
The return from holding the Aug 97 note from 2/15/96 to 8/15/96 is given by:
\[
\frac{P^{5\%}_{Aug 97} (Aug 96) + C^{5\%}_{Aug 97} (Aug 96) - P^{5\%}_{Aug 97} (Feb 96)}{P^{5\%}_{Aug 97} (Feb 96)} = \frac{98.65625 + 2.875 - 98.71875}{98.71875} = 2.849\%.
\]

2. What is the return from holding the Aug 96 note from 2/15/96 to 8/15/96?

On 2/15/96, the Aug 96 note has an identical payoff to that of a 6 month discount bond. The return from holding the Aug 96 note from 2/15/96 to 8/15/96 is given by the yield on a 6 month discount bond on 2/15/96 expressed as an effective semiannual rate:
\[
y_{\frac{1}{2}}(Feb 96)/2 = 7.9440\%/2 = 3.9720\%.
\]

3. Calculate the implied yield curve (expressed in terms of APRs with semiannual compounding)?

Can recover the discount factor for and yield on a six month discount bond using the Feb 97 note since it has only one payment left on 2/15/97:
\[
d_{\frac{1}{2}} (Aug 96) = \frac{98.34375}{100 + [5\%/2]} = 0.95828; \text{ and,}
\]
\[
y_{\frac{1}{2}} (Aug 96) = \left[\frac{1}{d_{\frac{1}{2}} (Aug 96)}\right] -1 \times 2 = \left[\frac{1}{0.95828}\right] -1 \times 2 = 8.7067\%.
\]

Can recover the yield on a 1 year discount bond by creating a synthetic 1 year discount bond:

\begin{itemize}
  \item (1) More specifically:
  \begin{itemize}
    \item (a) let \( c \) be the number of 5\% Aug 97 notes bought.
    \item (b) let \( b \) be the number of 5\% Feb 97 notes bought.
  \end{itemize}
\end{itemize}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
Position & 8/15/96 & 2/15/97 & 8/15/97 \\
\hline
Buy \( b \) 5\% Feb 97 notes & BUY & \( -b \) 98.34375 & \( b \) 102.625 \\
Buy \( c \) 5\% Aug 97 notes & \text{SYNTHETIC DISCOUNT BOND} & \( -c \) 98.65625 & \( c \) 2.875 & \( c \) 102.875 \\
Net & \( -c \) 98.65625 & 0 & 100 \\
& \( -b \) 98.34375 & & & \\
\hline
\end{tabular}
\end{table}

\begin{itemize}
  \item (2) So \( c 102.875 = 100 \) implies \( c = 0.97205 \).
  \item (3) So \( c 2.875 + b 102.625 = 0 \) implies \( b = -0.02723 \).
  \item (4) Thus, the cost of the synthetic discount bond is \( 0.97205 \times 98.65625 - 0.02723 \times 98.34375 = 93.221 \).
  \item (5) The discount factor for and yield on a 1 year discount bond can then be obtained:
\end{itemize}
Can then recover the yield on a 1½ year discount bond by creating a synthetic 1½ year discount bond:

(1) More specifically:
(a) let \( d \) be the number of 6 Feb 98 notes bought.
(b) let \( c \) be the number of 5¾ Aug 97 notes bought.
(c) let \( b \) be the number of 5¼ Feb 97 notes bought.

<table>
<thead>
<tr>
<th>Position</th>
<th>8/15/96</th>
<th>2/15/97</th>
<th>8/15/97</th>
<th>2/15/98</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy ( b )</td>
<td>-( b ) 98.34375</td>
<td>( b ) 102.625</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5¼ Feb 97</td>
<td></td>
<td></td>
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<td></td>
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<tr>
<td>notes</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Buy ( c )</td>
<td>-( c ) 98.65625</td>
<td>( c ) 2.875</td>
<td>( c ) 102.875</td>
<td></td>
</tr>
<tr>
<td>5¾ Aug 97</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>notes</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Buy ( d )</td>
<td>-( d ) 98</td>
<td>( d ) 3</td>
<td>( d ) 3</td>
<td>( d ) 103</td>
</tr>
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<td>6 Feb 98</td>
<td></td>
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<td>notes</td>
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<tr>
<td>Net</td>
<td>-( d ) 98</td>
<td>0</td>
<td>0</td>
<td>100</td>
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<tr>
<td></td>
<td>-( c ) 98.65625</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-( b ) 98.34375</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(2) So \( d \) 103 = 100 implies \( d = 0.97087 \)
(3) So \( d \) 3 + \( c \) 102.875 = 0 implies \( c = -0.02831 \).
(4) So \( d \) 3 + \( c \) 2.875 + \( b \) 102.625 = 0 implies \( b = -0.02759 \).
(5) Thus, the cost of the synthetic discount bond is
\[
0.97087 \times 98 - 0.02831 \times 98.65625 - 0.02759 \times 98.34375 = 89.639.
\]
(6) The discount factor for and yield on a 1½ year discount bond can then be obtained:

\[
d_{1\frac{1}{2}} (Aug 96) = 89.639/100 = 0.89639; \quad \text{and,}
\]
\[
y_{1\frac{1}{2}} (Aug 96) = \{[1/0.89639]^a -1\} \times 2 = 7.4263%.
\]

4. Consider 2/15/96's forward rate for the period 8/15/96 to 2/15/97. How does it compare to the 6 month interest rate on 8/15/96? If these two rates differ, discuss why.

\[
y_{6} (Aug 96) = 8.7067%; \quad \text{and} \quad f_{n,1} (Feb 96) = 4.5554%.
\]
The two rates are different. But these rates need not be the same and generally will not be. Even if the expectations hypothesis holds and \( f^*_{t+t/2}(0) = E_{t \text{time } 0} [y^*_{t/2}(t)] \), it need not be the case that \( f^*_{t+t/2}(0) = y^*_{t/2}(t) \). The yield on a six month discount bond in Aug 96 depends on economic conditions at that time while the forward rate \( f_{t/2,1}(\text{Feb 96}) \) is set in Feb 96 and depends on expectations in Feb 96 about economic conditions in Aug 96.

II. Forward Rates and the Yield Curve.
   A. To determine the yield on a two year discount bond:
      1. Use the Aug 95 strip to determine the ½-year discount bond discount factor: \( d_{1/2}(2/15/95) = 97.75/100 = 0.9775 \).
      2. Use the Feb 96 strip to determine the 1-year discount bond discount factor: \( d_1(2/15/95) = 93.25/100 = 0.9325 \).
      3. Use the Aug 96 strip to determine the 1½-year discount bond discount factor: \( d_{1.5}(2/15/95) = 90/100 = 0.9 \).
      4. Then use the 5% Feb 97 government bond.
         a. The coupon of 2.5 paid in Aug 95 can be converted to a value today using \( d_{1/2}(2/15/95) \):
            \[ P_{1/2}(2/15/95) = 2.5 \times 0.9775 = 2.44375. \]
         b. The coupon of 2.5 paid in Feb 96 can be converted to a value today using \( d_1(2/15/95) \):
            \[ P_1(2/15/95) = 2.5 \times 0.9325 = 2.33125. \]
         c. The coupon of 2.5 paid in Aug 96 can be converted to a value today using \( d_{1.5}(2/15/95) \):
            \[ P_{1.5}(2/15/95) = 2.5 \times 0.90 = 2.25. \]
         d. The value today of the final cash flow of 102.5 paid in Feb 97 can be obtained by subtracting the values of the earlier coupons from the bond’s price:
            \[ P_2(2/15/95) = 98 - 2.44375 - 2.33125 - 2.25 = 90.975. \]
      5. The coupon of 2.5 paid in Feb 96 can be converted to a value today using \( d_1(2/15/95) \):
         \[ P_1(2/15/95) = 2.5 \times 0.9325 = 2.33125. \]
      6. The coupon of 2.5 paid in Aug 96 can be converted to a value today using \( d_{1.5}(2/15/95) \):
         \[ P_{1.5}(2/15/95) = 2.5 \times 0.90 = 2.25. \]
      7. The value today of the final cash flow of 102.5 paid in Feb 97 can be obtained by subtracting the values of the earlier coupons from the bond’s price:
         \[ P_2(2/15/95) = 98 - 2.44375 - 2.33125 - 2.25 = 90.975. \]
      8. The yield on a two year discount bond expressed as an APR with semiannual compounding can be obtained:
         \[ y_2(2/15/95) = \{[(102.5/90.975)^{1/2} - 1]\} \times 2 = 0.06054 = 6.054\%. \]
   B. Use the following formula to obtain the forward contract discount factor available today for the one year period starting in 6 months expressed as an EAR:
      \[ d_{0.5,1.5}(2/15/95) = d_{1.5}(2/15/95)/d_{0.5}(2/15/95) = 0.90/0.9775 = 0.92072. \]
      Then express this forward contract discount factor as an APR with semiannual compounding by using:
      \[ f_{0.5,1.5}(2/15/95) = \{[1/d_{0.5,1.5}(2/15/95)]^{0.5} - 1\} \times 2 = \{[1/0.92072]^{0.5} - 1\} \times 2 = 8.433\%. \]
C. Now $d_{0.5,1.5}(2/15/95)$ refers to the forward price at 2/15/95 for delivery of a 1 year-discount bond in 6 months. So $d_{0.5,1.5}(2/15/95)$ refers to the forward price today of an Aug 96 strip to be delivered in 6 months time. The forward price at time 2/15/95 for a $100 face value strip can be calculated:

$$d_{0.5,1.5}(2/15/1995) \times 100 = 0.92072 \times 100 = 92.0717.$$